1 Local existence and uniqueness of analytic solutions: contractive mapping approach

Consider the system of equations (or one vector equation if you prefer)

$$y' = F(x, y); \quad y(x_0) = y_0 \tag{1}$$

where $y \in \mathbb{C}^n$, $x \in \mathbb{C}$. The second condition, the initial value, makes (1) an *initial value problem, IVP*. You see that by taking $y = \tilde{y} + y_0$, $x = \tilde{x} + x_0$ and $\tilde{F}(\tilde{x}, \tilde{y}) = F(\tilde{x} + x_0, \tilde{y} + y_0)$, we can assume, without loss of generality that our IVP is

$$y' = F(x, y); \quad y(0) = 0 \tag{2}$$

We must specify the properties of F. Let where $\mathbb{D}_\epsilon = \{z : |z| < \epsilon\}$. We will assume that $F : \mathbb{D}_\delta \times \mathbb{D}_\epsilon^n \mapsto \mathbb{C}^n$ is analytic in $\mathbb{D}_\delta \times \mathbb{D}_\epsilon^n$ for some $\delta > 0, \epsilon > 0$. This means that F has a convergent Taylor series in $(x, y_1, ..., y_n)$ in $\mathbb{D}_\delta \times \mathbb{D}_\epsilon^n$.

It is known (by Hartog's theorem: google it!) that if F is separately analytic in each variable (thinking therefore of the others as being "frozen"), then it is analytic in the stronger sense above.

By taking a slightly smaller ϵ if needed, we can assume that F is continuous up to the boundary, that is continuous in $\overline{\mathbb{D}_\delta} \times \overline{\mathbb{D}_\epsilon^n}$.

Check that the functions y which are analytic in $\mathbb{D}_\delta$ and continuous in $\overline{\mathbb{D}_\delta}$ endowed with the sup norm, $\|y\|_\infty = \sup_{x \in \overline{\mathbb{D}_\delta}} |y|$ form a Banach space; call this Banach space $\mathcal{B}$.

We now consider a closed subspace of $\mathcal{B}$, the closed ball $B = \{y \in \mathcal{B} : \|y\| \leq \epsilon\}$.

Exercise. Check that the IVP (102) is equivalent to

$$y = \int_0^x F(s, y(s))ds \tag{3}$$

Let ϵ be small enough. How small that is, we'll calculate in a moment. We now consider the *nonlinear* operator $\mathcal{M}$ be defined on B with values in B, given by

$$\mathcal{M}(y) = \int_0^x F(s, y(s))ds \tag{4}$$

For $|\mathcal{M}(y)|$ to be bounded by ϵ, we need that $\delta \max_{\overline{\mathbb{D}_\delta} \times \overline{\mathbb{D}_\epsilon^n}} |F(s, y(s)| < \epsilon$. Check that this can be arranged by taking δ small enough.

For $\mathcal{M}(y)$ to be contractive, check that it suffices to have

$$(n + 1) \max\{|\partial F_j / \partial y_k|, |\partial F_j / \partial x| \delta < \alpha < 1$$

This ensures contractivity and therefore existence and uniqueness of solutions of the IVP.

1.1 Reminder: the exponential and the log of a matrix

Note that $n-$matrices can be identified with vectors in $\mathbb{R}^{n^2}$, and they form a Banach space.

We can consider the sum

$$e^M = \sum_{k=0}^{\infty} M^k/k! \tag{5}$$

Since $\|M^2\| \le \|M\|^2$ (the usual $L^2 \mapsto L^2$ norm, and all norms are equivalent in $\mathbb{R}^n$), and also

$$\sum_{k=0}^{\infty} \|M\|^k/k! \tag{6}$$

evidently converges, it follows that e^M is correctly defined. You can check the usual properties of the exponential. Careful though: $AB \ne BA$ in general, so we can't expect $e^{A+B} = e^A e^B$.

For the log, if M is diagonalizable, then define $\log M$ to be $A[\log \Lambda]A^{-1}$. Here, A is the diagonalization matrix, Λ is diagonal, and so is, by definition $\log \Lambda$, which consists on the diagonal of the logs of the diagonal elements.

Exercise: If M has a nontrivial Jordan normal form, it is enough to define the log block by block. Each block is of the form $\lambda I - N$, $\lambda \in \mathbb{C}$, I the identity matrix and N a nilpotent. Then, the sum

$$I \log \lambda - \sum_{k=1}^{m} (N/\lambda)^k \tag{7}$$

where $N^m = 0$ gives a function with the properties of the log.

2 Reminder: the fundamental solution of a linear system

Consider a linear system of differential equations of the form

$$w' = A(z)w \tag{8}$$

where $w \in \mathbb{C}^n$ and A is analytic near z_0; we first look at (8) near z_0; without loss of generality, we can take $z_0 = 0$. You know already that, in a neighborhood of a regular point (i.e., a point where A analytic) there exist n linearly independent vector solutions, $\{w_j : j = 1, ..., n\}$ of (160). Furthermore, you can choose initial conditions so that $w_j(z = 0) = e_j$, the unit vector in the direction j. If you construct a matrix M having as the $j - th$ row the vector w_j, you can check immediately that

$$M' = A(z)M; \quad M(0) = I \tag{9}$$

Since the rows are linearly independent, you know from elementary algebra that M is invertible. Let $N = M^{-1}$. Since $MN = I$ we have

$$M'N + MN' = 0 \Rightarrow MN' = -AMN = -A \Rightarrow \boxed{N' = -NA};\ N(0) = I$$

so N gives, up to the order of the matrices, the backward evolution $z \to -z$.

Take any z-independent vector w_0 independent, and let $w = Mw_0$. We have

$$(Mw_0)' = M'w_0 = AMw_0 \Rightarrow w' = Aw;\quad w(0) = Iw_0 = w_0 \tag{10}$$

Thus, we see that the solution of the initial value problem $w' = Aw, w(0) = w_0$ is simply Mw_0. We will often work with the fundamental matrix solution M, as it often simplifies the calculations.

Remark 1. *Note also that, if we look at the matrix differential equation*

$$W' = AW \tag{11}$$

the general solution is $W = MC$ where C is any z-independent (that is, constant) matrix. Indeed, since M is invertible, we can define $Q = M^{-1}W$, which we write in the form $W = MQ$. We then have

$$M'Q + MQ' = AMQ \Leftrightarrow MQ' = 0 \Leftrightarrow Q' = 0 \tag{12}$$

(since $M' = AM$) which indeed means that Q is a constant matrix.

3 Isolated singularities of linear systems

Consider the system

$$w' = A(z)w \tag{13}$$

where A is a matrix valued analytic function, but now with *an isolated singularity at z_0*. Clearly, by translating z we can take $z_0 = 0$, and by rescaling z, we can assume that A is analytic in $\mathcal{D} = \mathbb{D} \setminus \{0\}$ where $\mathbb{D}$ is the open unit disk. Though the equation is single-valued in $\mathcal{D}$, since $\mathcal{D}$ is not simply connected, the solutions may not be, as seen by solving the equation $y' = ay/z$ with $a \notin \mathbb{Z}$. We can take $z = e^{\zeta}$ and $\mathcal{D}$ becomes $\{\zeta : \mathrm{Re}\zeta \in \mathbb{R}^-\}$, a half plane. By the standard existence and uniqueness theorems, we find that there is a unique solution of the system, rewritten in ζ, and thus there is a fundamental solution of (160), in the form $M(\ln z)$, which shows once more that, in principle at least, the solution of (160) may not be single-valued.

4 Some general facts about solutions near isolated singularities

In the generality of the singular systems in §3 all we can say now, without a lot more theory, is the way the solution itself can be ramified. Once more, we consider that we rescaled everything so that $z = 0$ is the isolated singularity, and $\mathcal{D} = \mathbb{D} \setminus \{0\}$ is the domain of analyticity of A.

Theorem 1. *The general solution of* (160) *is of the form*

$$M(z) = S(z)z^P \quad (z^P := e^{\ln z\, P}) \tag{14}$$

where P is a constant matrix, and $S(z)$ is analytic in $\mathcal{D}$. With the price of changing the matrix M to MT, with T a constant matrix, we can write

$$MT = S_1 x^J \tag{15}$$

where J is the Jordan normal form of P.

What this theorem says is that the solution itself is single-valued up to multiplication by z^P with P constant. Of course, there is no reason to expect that S is analytic at zero-just that 0 is an isolated singularity.

Lemma 1. *Assume M is any matrix analytic on the universal covering of $\mathcal{D}$ which satisfies*

$$M(ze^{2\pi i}) = MC \quad \text{where } C \text{ is a constant invertible matrix.} \tag{16}$$

Then

$$M(z) = S(z)z^P \tag{17}$$

where P is a constant matrix and $S(z)$ is analytic in $\mathcal{D}$. At the price of altering M by a constant matrix, P can be taken to be in Jordan normal form,

Proof of the lemma. Since C is invertible, we can define P (up to $2\mathbb{Z}\pi i I$) by $C = e^{2\pi i P}$. Let

$$S = Mz^{-P} \tag{18}$$

$$S(ze^{2\pi i}) = Me^{2\pi i P}e^{-P\ln z - 2\pi i P} = Me^{-P\ln z} = S(z) \tag{19}$$

since e^{aP} and e^{bP} commute, if a and b are scalars. Let now T be the change of basis that brings P to its Jordan normal form, that is $T^{-1}PT = J$. We then have

$$MT = STT^{-1}z^P T = STz^J \tag{20}$$

where ST is also ingle valued, as required. □

Proof of the theorem. We only need to show that the assumptions of the lemma above hold. Take $N(z) = M(ze^{2\pi i})$. That is, we use the fact that M exists on the universal covering of $\mathcal{D}$, and look at its value on the second Riemann sheet. We have

$$N(z)' = M'(ze^{2\pi i}) = A(ze^{2\pi i})M(ze^{2\pi i}) = A(z)M(ze^{2\pi i}) = A(z)N \tag{21}$$

where we used the fact that M is already a solution, and A is single-valued. Thus, by Remark 1, we must have $N = MC$ where C is a constant matrix. □

Remark 2. *If S happens to be analytic, note also the emerging noninteger powers of z and $\ln z^j$ through the term z^J.*

Indeed, if J_1 is an elementary Jordan block in J, we have

$$z^J = z^{\lambda I + N} = z^\lambda e^{N\ln z} = z^\lambda(1 + N\ln z + \cdots \ln z^l N^l/l!) \tag{22}$$

where $N^{l+1} = 0$, and thus $l < n$, the degree of the system.

5 Regular singular points of differential equations, nondegenerate case

5.1 Example

$$x(x-1)y'' + y = 0 \tag{23}$$

around $x = 0$. The indicial equation is $r(r-1) = 0$ (a *resonant case*: the roots differ by an integer). Substituting $y_0 = \sum_{k=0}^{\infty} c_k x^k$ in the equation and identifying the powers of x yields the recurrence

$$c_{k+1} = \frac{k^2 - k + 1}{k(k+1)} c_k \tag{24}$$

with $c_0 = 0$ and c_1 arbitrary. By linearity we may take $c_1 = 1$ and by induction we see that $0 < c_k < 1$. Thus the power series has radius of convergence at least 1. The radius of convergence is in fact exactly one as it can be seen applying the ratio test and using (137); the series converges exactly up to the nearest singularity of (136).

Exercise 1. *What is the asymptotic behavior of c_k as $k \to \infty$?*

We let $y_0 = y_0 \int g(s)ds$ and get, after some calculations, the equation

$$g' + 2\frac{y_0'}{y_0} g = 0 \tag{25}$$

and, by the previous discussion, $2y_0'/y_0 = 2/x + A(x)$ with $A(x)$ is analytic. The point $x = 0$ is a regular singular point of (138) and in fact we can check that $g(x) = C_1 x^{-2} B(x)$ with C_1 an arbitrary constant and $B(x)$ analytic at $x = 0$. Thus $\int g(s)ds = C_1(a/x + b\ln(x) + A_1(x)) + C_2$ where $A_1(x)$ is analytic at $x = 0$. Undoing the substitutions we see that we have a fundamental set of solutions in the form $\{y_0(x), B_1(x) + B_2(x)\ln x\}$ where B_1 and B_2 are analytic.

5.2 Singularities of the first kind

Consider the system

$$w' = \frac{1}{z} B w + A_1(z) w; \quad \text{or, in matrix form,} \quad M' = \frac{1}{z} BM + A_1(z) M \tag{26}$$

where B is a constant matrix and A_1 is analytic at zero. Let J be the Jordan normal form of B and $T^{-1}BT = J$. Then, we see that

$$T^{-1}MT = \frac{1}{z} T^{-1}BTT^{-1}MT + T^{-1}A_1(z)TT^{-1}MT \tag{27}$$

with $\tilde{M} = T^{-1}MT$ we see that

$$\tilde{M}' = \frac{1}{z} J\tilde{M} + A_2(z)\tilde{M} \tag{28}$$

where clearly A_2 is also analytic. In other words,

Remark 3. *In (53) we can assume, without loss of generality that B is in its Jordan normal form, J. We will thus study equations of the form*

$$y' = \frac{1}{z}Jy + A(z)y \tag{29}$$

where $A(z)$ is analytic.

5.3 Nondegenerate case

Assumption. No two eigenvalues of B differ by a positive integer.

Theorem 2. *Under the assumption above, (29) has a fundamental matrix solution in the form $M(z) = Y(z)z^J$, where $Y(z)$ is a matrix analytic in $\mathcal{D}$.*

Exercise 2. *Check that, if we had not arranged for B to be in its Jordan normal form, the solution of (53) would be $M(z) = Z(z)z^B$, where $Z(z)$ is a matrix analytic at zero.*

Proof. Clearly, it is enough to prove the theorem for (29). We look for a solution of (29) in the form $M = Yz^J$, where

$$Y(z) = J + zY_1 + z^2Y_2 + \cdots \tag{30}$$

we get

$$Y'z^J + \frac{1}{z}YJz^J = \frac{1}{z}JYz^J + AYz^J \tag{31}$$

Multiplying by z^{-J} we obtain

$$Y' + \frac{1}{z}YJ = \frac{1}{z}JY + AY \tag{32}$$

or

$$Y' = \frac{1}{z}\Big(JY - YJ\Big) + AY \tag{33}$$

Using (62) we get

$$\begin{aligned} Y_1 + 2zY_2 + 3z^2Y_3 + \cdots = \Big[(JY_1 - Y_1J) + z(JY_2 - Y_2J) + \cdots\Big] \\ + A_0J + zA_1J + \cdots + zA_0Y_1 + z^2(A_0Y_2 + A_1Y_1) + \cdots \end{aligned} \tag{34}$$

The associated system of equations, after collecting the powers of z is

$$kY_k = (JY_k - Y_kJ) + A_{k-1}J + \sum_{j=1}^{k-1} Y_jA_{k-j-1}; \quad k \in \mathbb{N} \tag{35}$$

or

$$V_kY_k = A_{k-1}J + \sum_{j=1}^{k-1} Y_jA_{k-j-1}; \quad k \in \mathbb{N} \tag{36}$$

where

$$V_k M := kM - (JM - MJ) \tag{37}$$

is a linear operator on matrices $M \in \mathbb{R}^{n^2}$. As a linear operator on a finite dimensional space, $V_k X = Y$ has a unique solution for every Y iff $\det V_k \neq 0$ or, which is the same, $jX - JX + XJ = 0$ implies $X = 0$. We show that this is the case, by showing that $Xv = 0$ for all generalized eigenvectors of J.

Let v be one of the eigenvectors of J. If $V_k X = 0$ we obtain, since $Jv = \lambda v$,

$$k(Xv) - J(Xv) + X\lambda v = 0 \tag{38}$$

or

$$J(Xv) = (\lambda + k)(Xv) \tag{39}$$

Here we use our assumption: $\lambda + k$ is not an eigenvalue of J. This forces

$$Xv = 0 \tag{40}$$

We let $v_0 = v$ and take the next generalized eigenvector, v_1, in the same Jordan block as v, if any.

We remind that we have the following relations between these generalized eigenvectors:

$$Jv_i = \lambda v_i + v_{i-1} \tag{41}$$

where $v_0 = v$ is an eigenvector and $1 \leq i \leq m - 1$ where m is the dimension of the Jordan block. With $i = 1$ we get

$$k(Xv_1) - J(Xv_1) + X(\lambda v_1 + v_0) = 0 \tag{42}$$

and, using (40) (i.e., $Xv_0 = 0$), we get the same equation (43), now for Xv_1:

$$J(Xv_1) = (\lambda + k)(Xv_1) \tag{43}$$

and thus $Xv_1 = 0$. Inductively, we see that $Xv = 0$ for any generalized eigenvector of J, and thus $X = 0$.

Now, we claim that $V_k^{-1} \leq Ck^{-1}$ for some C. We let $\mathcal{C}$ be the commutator operator, $\mathcal{C}X = JX - XJ$ Now $\|JX - XJ\| \leq 2\|J\|\|X\|$ and thus

$$V_k^{-1} = k^{-1}\left(I - k^{-1}\mathcal{C}\right)^{-1} = k^{-1}(1 + o(1)); \quad (k \to \infty) \tag{44}$$

Therefore, the function kV_k is bounded for $k \in \mathbb{R}^+$.

We rewrite the system (35) in the form

$$Y_k = V_k^{-1} A_{k-1} J + V_k^{-1} \sum_{j=1}^{k-1} Y_j A_{k-j-1}; \quad k \in \mathbb{N} \tag{45}$$

or, in abstract form, with $\mathsf{Y} = \{Y_j\}_{j \in \mathbb{N}}$, $(\mathsf{LY})_k := V_k^{-1} \sum_{l=0}^{k-2} A_l Y_{k-1-l}$, where we regard Y as a function defined on $\mathbb{N}$ with matrix values, with the norm

$$\|\mathsf{Y}\| = \sup_{n \in \mathbb{N}} \|\mu^{-n} \mathsf{Y}(n)\|; \quad \mu > 1 \tag{46}$$

we have

$$\mathsf{Y} = \mathsf{Y}_0 + \mathsf{LY} \tag{47}$$

Exercise 3. *Show that (47) is contractive for μ sufficiently large, in an appropriate ball that you will find.*

The solution of this exercise is given in the appendix. $\square$

6 Changing the eigenvalue structure of J by transformations

To solve the general case, in which eigenvalues *may* differ by positive integers, we find transformations which decrease one eigenvalue by one, leaving all, others the same and without changing the structure of the ODE.

Write J in the form

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \tag{48}$$

where J_1 is the Jordan block we care about, $\dim(J_1) = m \geq 1$, while J_2 is a Jordan matrix, consisting of the remaining blocks. The transformation we are looking for would change J into $J - I_1$ where

$$I_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \tag{49}$$

where I is the identity matrix. That, in turn, would change the fundamental solution to

$$Y z^{J-I_1} \tag{50}$$

This suggests we try this change of variables in our equation. In matrix form,

$$M' = z^{-1} JM + AM \tag{51}$$

where we take $M = M_1 z^{I_1}$.

Exercise 1. *Show that, if $a \in \mathbb{C}$ and P is a projector, $P^2 = P$, then*

$$z^{aP} = Pz^a + (I - P) \tag{52}$$

Is it true that $(z^B)' = z^{-1} B z^B$ for any matrix B?

We have

$$M_1' z^{I_1} + z^{-1} M_1 I_1 z^{I_1} = z^{-1} J M_1 z^{I_1} + A M_1 z^{I_1} \tag{53}$$

We can multiply to the right by z^{I_1} and get

$$M_1' = z^{-1} J M_1 - z^{-1} I_1 M_1 + A M_1 \tag{54}$$

which does not quite work, because of non-commutation. So it is natural to try

$$M = z^{I_1} M_1 = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} M_1$$

since J and z^{I_1} commute. We then have

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} M_1 + \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} M_1' = z^{-1} J \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} M_1 + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} M_1 \quad (55)$$

We multiply to the left by

$$\begin{pmatrix} 1/z & 0 \\ 0 & 1 \end{pmatrix}$$

and get

$$\begin{aligned} M_1' &= z^{-1} J M_1 + \begin{pmatrix} 1/z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{11}z & A_{12} \\ A_{21}z & A_{22} \end{pmatrix} M_1 \\ &= z^{-1} J M_1 + \begin{pmatrix} A_{11} & A_{12}(0)/z + \tilde{A}_{12} \\ A_{21}z & A_{22} \end{pmatrix} M_1 = \frac{1}{z} \begin{pmatrix} J_1 & A_{12}(0) \\ 0 & J_2 \end{pmatrix} + \tilde{A} M \end{aligned} \quad (56)$$

where $\tilde{A}$ is analytic. We have thus obtained

Proposition 2. *By the change of variables $M = z^{I_1} M_1$, the equation for M_1 is of the form*

$$M_1' = z^{-1} R M_1 + \tilde{A} M_1 \quad (57)$$

where R has eigenvalues $\lambda_1 - 1, ..., \lambda_m$.

Exercise 2. Use this procedure repeatedly to reduce any resonant system to a nonresonant one. That is done by arranging that the eigenvalues that differ by positive integers become equal.

Exercise 3. Use Exercise 2 to prove the following result.

Theorem 3. *Any system of the form*

$$y' = \frac{1}{z} B(z) y \quad (58)$$

where B is an analytic matrix at zero, has a fundamental solution of the form

$$M(z) = Y(z) z^{B'} \quad (59)$$

where B' is a constant matrix, and Y is analytic at zero. In the nonresonant case, $B' = B(0)$. In the resonant case, the eigenvalues of R do not differ by integers, and they are a subset of the eigenvalues of $B(0)$, precisely those that do not differ by integers of other eigenvalues, or, in the groups that do, the one which has the smallest real part.

Note that this applies even if $B(0) = 0$.

Exercise 4. *Find B' in the case where only two eigenvalues differ by a positive integer, where the integer is 1.*

6.1 Example

Let's consider again the equation

$$x(x-1)y'' + y = 0 \tag{60}$$

We want to use the theory we have developed this far, to find the shape of the generic solution at $0, 1, \infty$ (the only singular points of the equation).

Let's start with $x = 0$. We can write the equation in system form in the following way:$y_1' = y_2, y_2' = -y_1/(x(x-1))$, or in matrix form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1/(x(x-1)) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \frac{1}{x} \begin{pmatrix} 0 & x \\ -1/(x-1) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{61}$$

or

$$\mathbf{y} = x^{-1} B\mathbf{y} + A\mathbf{y} \tag{62}$$

where, by decomposition by partial fractions,

$$B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad A := \begin{pmatrix} 0 & 1 \\ 1/(1-x) & 0 \end{pmatrix} \tag{63}$$

Clearly, A is analytic at 0. The eigenvalues of B are $0, 0$ (e.g., the determinant and trace are zero). Thus, the eigenvalues are nonresonant.

It follows that the fundamental solution of this equation is

$$M = Y(z) z^B \tag{64}$$

where $Y(z)$ is analytic near zero (in this case, analytic in the unit disk, since $x = 1$ is the singular point closest to the origin (other than the origin itself).

Thus,

$$M = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \left(I + \ln z \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \ln z & 1 \end{pmatrix}$$
$$= \begin{pmatrix} y_{11} + y_{12} \ln z & y_{12} \\ y_{21} + y_{22} \ln z & y_{22} \end{pmatrix} \tag{65}$$

and thus, by applying M to some initial condition (a, b) we get that the general solution of (60) in a neighborhood of 0 is

$$y = a(y_{11} + y_{12} \ln z) + b y_{12} \tag{66}$$

6.2 Example:Bessel functions

The equation

$$f'' + \frac{3}{2x} f' + f = 0 \tag{67}$$

has the general solution

$$C_1 x^{-1/4} J_{1/4}(x) + C_2 x^{-1/4} Y_{1/4}(x) \tag{68}$$

where J and Y are Bessel functions. In matrix form, as in the example before, we have

$$\mathbf{f}' = M\mathbf{f} \tag{69}$$

where

$$M = \begin{pmatrix} 0 & 1 \\ -1 & -\dfrac{3}{2x} \end{pmatrix} = \frac{1}{x}\begin{pmatrix} 0 & x \\ -x & -\dfrac{3}{2} \end{pmatrix} = \frac{1}{x}B + A \tag{70}$$

where

$$B = \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{3}{2} \end{pmatrix}; \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{71}$$

and thus the fundamental matrix is

$$\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} x^{\begin{pmatrix} 0 & 0 \\ 0 & -\frac{3}{2} \end{pmatrix}} = \begin{pmatrix} f_{11} & x^{-3/2} f_{12} \\ f_{21} & x^{-3/2} f_{22} \end{pmatrix} \tag{72}$$

This means that the general solution of the equation is of the form

$$f = C_1 f_1(x) + C_2 x^{-3/2} f_2(x) \tag{73}$$

with f_1 and f_2 analytic.

Exercise 5. *It also follows from the analysis above that $f_2(0) = 0$. Why?*

6.3 A different type of example

Let us now look at (60) near infinity. We make the change of variables $x = 1/t$ to bring infinity to 0, our familiar point of analysis. We get, with $u(x) = f(1/x)$,

$$t^2(t-1)f'' + 2t(t-1)f' + f = 0 \tag{74}$$

In matrix form, this is

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ \dfrac{1}{t^2(t-1)} & -\dfrac{2}{t} \end{pmatrix} \mathbf{y} \tag{75}$$

Because of the $1/t^2$ factor, this does not look like one of the equations that can be brought to the form

$$\mathbf{u}' = \frac{1}{x}B\mathbf{u} + A\mathbf{u} \tag{76}$$

But note that the method of conversion, namely $y_1 = y, y_2 = y'$, is not the only possibility! In fact, if $y \sim t^a$ then $y' \sim at^{a-1}$. Then we should take, to get the right singular behavior, $y_1 = u_1, y_2 = u_2/t$. Then $u_1 = y_1$ and $u_2 = ty_2$; thus $u_1' = y_1' = y_2 = u_2/t$ and

$$u_2' = y_2 + ty_2' = \frac{u_2}{t} + t\left(\frac{y_1}{t^2(t-1)} - \frac{2y_2}{t}\right) = \frac{u_2}{t} + \frac{u_1}{t(t-1)} - \frac{2u_2}{t}$$

that is,

$$\mathbf{u}' = \frac{1}{t}B\mathbf{u} + A\mathbf{u} \tag{77}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}; \quad A = \begin{pmatrix} 0 & 0 \\ \dfrac{1}{t-1} & 0 \end{pmatrix} \tag{78}$$

The eigenvalues of B are $-1/2 \pm i\sqrt{3}/2$.

Exercise 6. *This is just a calculation, but very much worth doing. Write the general solution of* (74) *near* $t = 0$.

Exercise 7. *Redo the analysis in §6.1 with thje type of transformations used in the present section.*

Exercise 8. *Consider the system*

$$\mathbf{y}' = \frac{1}{x}B\mathbf{y} + A\mathbf{y} \tag{79}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad A = \begin{pmatrix} x & 1 \\ x & 1 \end{pmatrix} \tag{80}$$

(a) Write it as a second order equation;

(b) Using the matrix form, find the general solution of the second order equation near 0.

Exercise 9. *Use Abramowitz and Stegun (or your favorite tables) to find the solutions of the equations in the previous two sections, and then find the behavior at the singular points analyzed, using the tables. Which is easier, using tables or direct calculation?*

Important note*: there are advantages to using special functions or other methods that we'll study later. We have not determined here how the solutions behave globally, e.g., how a specific solution of* (60) *calculated in a neighborhood of zero behaves at infinity–we only obtained all its possible behaviors.*

7 Scalar n–th order linear equations

These are equations of the form

$$y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_n(z)y = 0 \tag{81}$$

Such an equation can always be transformed into a system of the form $w' = A(x)w$, and viceversa. There are many ways to do that. The simplest is to take $v_0 = y, ..., v_k = y^{(k)}, ...$ and note that (81) is equivalent to

$$\begin{pmatrix} v_0 \\ v_1 \\ \cdots \\ v_{n-1} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ -a_n(z) & -a_{n-1}(z) & -a_{n-2}(z) & \cdots & -a_1(z) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \cdots \\ v_{n-1} \end{pmatrix} \tag{82}$$

In the other direction, to simplify notation assume $n = 2$. The system is then

$$\begin{aligned} y' &= a(x)y + b(x)w \\ w' &= c(x)y + d(x)w \end{aligned} \tag{83}$$

We differentiate one more time say, the second equation, and get

$$w'' = dw' + (bc + d')w + (ac + c')y \tag{84}$$

If $c \equiv 0$, then (84) is already of the desired form. Otherwise, we write

$$y = \frac{1}{c}w' - \frac{d}{c}w \tag{85}$$

and substitute in (84). The result is

$$w'' = \left(a + d + \frac{c'}{c}\right) w' + \left(c\left(\frac{d}{c}\right)' + [cb - ad]\right) w \tag{86}$$

Note that a and c, by assumptions, have at most first order poles, while c'/c has at most simple poles for any analytic function. Therefore, the emergent second order equation has the general form

$$w'' + a_1(x)w' + a_2(x)w = 0$$

where a_i has a pole of order at most i.

Exercise 1. Generalize this transformation procedure to nth order systems. Show that the resulting nth order equation is of the general form

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \tag{87}$$

where the coefficients a_i are analytic in $\mathbb{D}_\rho \setminus \{0\}$ and have a pole of order at most i at zero.

Definition 3. An equation of the form (87) has a singularity of the first kind at zero if the conditions of $a_i = b_i/z^i$ where b_i are analytic at zero. (Compare with Exercise 1.)

7.1 The indicial equation

Looking for power series solutions $z^\lambda(1+c_1z+c_2z^2+\cdots)$, we insert this into the differential equation and note that $y^{(j)} = \lambda(\lambda-1)\cdots(\lambda-j+1)z^{\lambda-j}(1+o(1))$ and also that $a_j z^{\lambda-n+j}(1+o(1)) = b_j(0)z^{\lambda-n})(1+o(1))$. Thus, the equation for the leading power of z is

$$\lambda(\lambda-1)\cdots(\lambda-n+1) + \lambda(\lambda-1)\cdots(\lambda-n+1)b_1(0) + \cdots + b_n(0) = 0 \tag{88}$$

7.2 Regular singularities and singularities of the first kind

7.2.1 Reformulation as a system

There is a reformulation of (87) as a system of the form (58). Clearly, the transformation leading to (82) produces a system of equations with a singularity of order n. As in §6.3, the natural substitution is

$$\varphi_k = z^{k-1}y^{(k-1)},\ k = 1, 2, ..., n \tag{89}$$

We then have $y^{(k-1)} = z^{-k+1}\varphi_{k-1}$ and

$$\varphi_{l+1} = z^l y^{(l)} = z^l \left(z^{-l+1}\varphi_l\right)' = (1-l)\varphi_l + z\varphi_l' \tag{90}$$

or

$$z\varphi_l' = (l-1)\varphi_l + \varphi_{l+1} \tag{91}$$

while

$$\begin{aligned}\varphi_n' &= (n-1)z^{n-2}y^{(n-1)} + z^{n-1}\left(-a_n y - a_{n-1}y' - ... - a_1 y^{(n-1)}\right)\\ &= (n-1)z^{n-2}z^{n-1}\varphi_n - \frac{1}{z}\left(-b_n\varphi_1 - b_{n-1}\varphi_2 - ... - b_1\varphi_n\right)\end{aligned} \tag{92}$$

where $b_{n-k+1}(z)$ are analytic, where we used Definition 3. In matrix form, the end result is the system

$$\boldsymbol{\varphi}' = z^{-1}B\boldsymbol{\varphi} \tag{93}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 1 & 1 & 1 & \cdots & 0\\ 0 & 0 & 2 & 1 & \cdots & 0\\ 0 & 0 & 0 & 3 & \cdots & 0\\ & & \cdots & & & \\ -b_n(z) & -b_{n-1}(z) & -b_{n-2}(z) & -b_{n-3}(z) & \cdots & (n-1)-b_1(z)\end{pmatrix} \tag{94}$$

or

$$\boldsymbol{\varphi}' = z^{-1}B(0)\boldsymbol{\varphi} + A(z)\boldsymbol{\varphi} \tag{95}$$

where A is analytic at zero.

7.2.2 Eigenvalues of $B(0)$ in (95)

An easy way to determine these is to look at the eigenvalue equation, $(B-\lambda I)x = 0$. If we expand this out as a system, using the explicit form (94), we get

$$(0-\lambda)x_0 + x_1 = 0 \tag{96}$$

$$(1-\lambda)x_1 + x_2 = 0 \tag{97}$$

$$(2-\lambda)x_2 + x_3 = 0 \tag{98}$$

$$... \tag{99}$$

$$-b_n(0)x_0 - b_{n-1}(0)x_1 - \cdots - (b_1 - [n-1-\lambda])x_{n-1} = 0 \tag{100}$$

Without loss of generality we can take $x_1 = 1$. Then

$$x_1 = \lambda,\ x_2 = \lambda(\lambda - 1),\ \cdots, x_{n-1} = \lambda(\lambda - 1)\cdots(\lambda - (n-2))$$

and thus (290) is equivalent to

$$-b_n(0) - \lambda b_{n-1}(0) - \cdots - (b_1 - [(n-1) - \lambda])\lambda(\lambda - 1)\cdots(\lambda - (n-2)) = 0 \quad (101)$$

which is precisely (88). We have shown

Proposition 4. *The eigenvalues of $B(0)$ are precisely the roots of the indicial equation.*

7.2.3 Examples of behavior of solutions for first versus higher kind singular equations

Consider the following simple examples.

(1) $f'' + (1 + 1/z)f' - 2f/z^2 = 0$; (2) $f'' + (1 + 1/z^2)f' - 2f/z^3 = 0$.

What is the expected behavior of the solutions? If we try $f(z) = z^m + \cdots$ in (1) we get $(m^2 - 2)z^{-2} + \cdots = 0$ thus $m = \pm\sqrt{2}$.

We try a solution of the form $f(z) = z^{\sqrt{2}} \sum_{k=0}^{\infty} c_k z^k = z^{\sqrt{2}} g(z)$, and get c_0 arbitrary, so we take, say, c_0, then $c_1 = -\sqrt{2}/(1 + 2\sqrt{2})$, and in general,

$$c_m = -\frac{m + \sqrt{2} - 1}{m(m + 2\sqrt{2})} c_{m-1} \quad (102)$$

It is easy to show that c_m are bounded (in fact $|c_m| \sim 1/(m+1)!$, and then g is analytic.

Let us consider, instead, (2). The same substitution, $z^m + \cdots$ now gives $m = 2$.

We try a power series solution of the form $f(z) = \sum_{k=2}^{\infty} c_k z^k$ Again c_0 is undetermined, say we take it to be one, and in general we have

$$c_m = -(m+1)c_{m-1} - c_{m-2} \quad (103)$$

This time, it is not hard to show, $|c(m)| \sim (m+1)!$, and the series diverges.

7.2.4 Solutions of the original nth order ODE

Exercise 2. *Using Theorem 3 check that for any block of eigenvalues $\lambda, \lambda + p_1, ..., \lambda + p_m$, $0 \le p_1 \le p_2 \le ... \le p_m \in \mathbb{Z}$, $m \ge 0$, there is always a solution of* (87) *of the form*

$$y = z^{\lambda + p_m} \phi(z) \quad (104)$$

where ϕ is analytic at zero, and $\phi(0) = 1$.

8 Some special functions and their regular singular points

Here is a good and up to date online source of information about special functions: http://dlmf.nist.gov/.

8.1 Hypergeometric functions

The general solution of the equation

$$x(x-1)y'' + [(a+b+1)x - c]y' + aby = 0 \tag{105}$$

is

$$y = A \cdot \ {}_2F_1(a,b;c;x) + Bx^{1-c}\ {}_2F_1(a-c+1, b-c+1; 2-c; x), \tag{106}$$

With the substitution

$$y = u, y' = v/x$$

we get

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \frac{1}{x} B \begin{pmatrix} u \\ v \end{pmatrix} + A(x) \begin{pmatrix} u \\ v \end{pmatrix} \tag{107}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 1-c \end{pmatrix} \tag{108}$$

and

$$A = \begin{pmatrix} 0 & 0 \\ -ab/(x-1) & -(a+1+b-c)/(x-1) \end{pmatrix} \tag{109}$$

The eigenvalues of B are clearly 0 and $1-c$. Note that they are resonant when $c \in \mathbb{Z} \setminus \{1\}$.

Exercise 1. *(a) Find the behavior near the origin of the general solution in the nonresonant case.*

(b) In the resonant case, show that there is always a solution of the form $x^{1-c}A(x)$ if $1-c > 0$ and $A(x)$ otherwise, where A is analytic. Use reduction of order (explained in general in the next section) to find the behavior of the second solution. Reduction of order in the first case would mean: look for $y(x)$ in the form $x^{1-c}A(x)g(x)$ where $x^{1-c}A(x)$ is already a solution. Solve the equation for g.

8.2 The exponential integral

This is defined by

$$\mathrm{Ei}_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt \quad (z \neq 0) \tag{110}$$

where the path does not cross the negative real axis or pass through the origin. There is a cut along the interval $(-\infty, 0]$. The function is also defined on $\mathbb{R}^-$, in terms of the principal part of the integral, as a multivalued function, but we will not worry about this now.

We see that

$$\mathrm{Ei}_1(z)' = -\frac{e^{-z}}{z} \tag{111}$$

With the substitution $\mathrm{Ei}_1(z) = g(z)e^{-z}$ we get

$$zg' - zg + 1 = 0 \tag{112}$$

We transform this into a second order homogeneous equation by differentiating once more in z:

$$g'' + (1/z - 1)g' - g/z = 0 \tag{113}$$

Clearly, zero is the only singular point of this equation. We write as before $g = u, g' = v/z$ and we get

$$u' = g' = v/z; \tag{114}$$

We get

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \frac{1}{x}B\begin{pmatrix} u \\ v \end{pmatrix} + A(x)\begin{pmatrix} u \\ v \end{pmatrix} \tag{115}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{116}$$

where clearly the eigenvalues of B are $0, 0$. Note that

$$z^B = \begin{pmatrix} 1 & \ln z \\ 0 & 1 \end{pmatrix} \tag{117}$$

Write the general solution of (113) in a neighborhood of zero. Here, it is easy enough to find the behavior of $\mathrm{Ei}_1(z)$ directly from the integral expression. How?

8.3 Bessel functions

The Bessel functions of the first kind satisfy the equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{118}$$

or, in normal form,

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0 \tag{119}$$

The general solution of this equation is

$$y = C_1J_\nu(x) + C_2Y_\nu(x) \tag{120}$$

In this case, the system is

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \frac{1}{x}B\begin{pmatrix} u \\ v \end{pmatrix} + A(x)\begin{pmatrix} u \\ v \end{pmatrix} \tag{121}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ \nu^2 & 0 \end{pmatrix} \tag{122}$$

with eigenvalues $\nu, -\nu$. In the nonresonant case there are two solutions which behave near zero likr

$$z^{\pm\nu} A_{\pm}(z) \tag{123}$$

with $A_{\pm}$ analytic.

The algorithm is clear. I attach a Maple file with the general procedure, in which, instead of $e1$ you would insert any second order ODE. Finally, let's make a simple connection with equilibria. If we have a system of the form

$$x' = ax + by \tag{124}$$

$$y' = cx + dy \tag{125}$$

and the associated matrix is diagonalizable, then we can bring it to the form

$$u' = \lambda_1 u; \quad v' = \lambda_2 v \tag{126}$$

Of course, this can be easily solved in closed form. But we also note that we can write

$$\frac{dv}{du} = b\frac{v}{u}; \quad b = \frac{\lambda_2}{\lambda_1} \tag{127}$$

which perhaps the simplest case we can think of within Frobenius theory. Suppose first that $b \in \mathbb{R}$, then based on Frobenius theory, it is very easy to draw the phase portrait. Discuss also the case when b is complex, and the case when the Jordan form of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is nontrivial.

8.4 Reduction of order

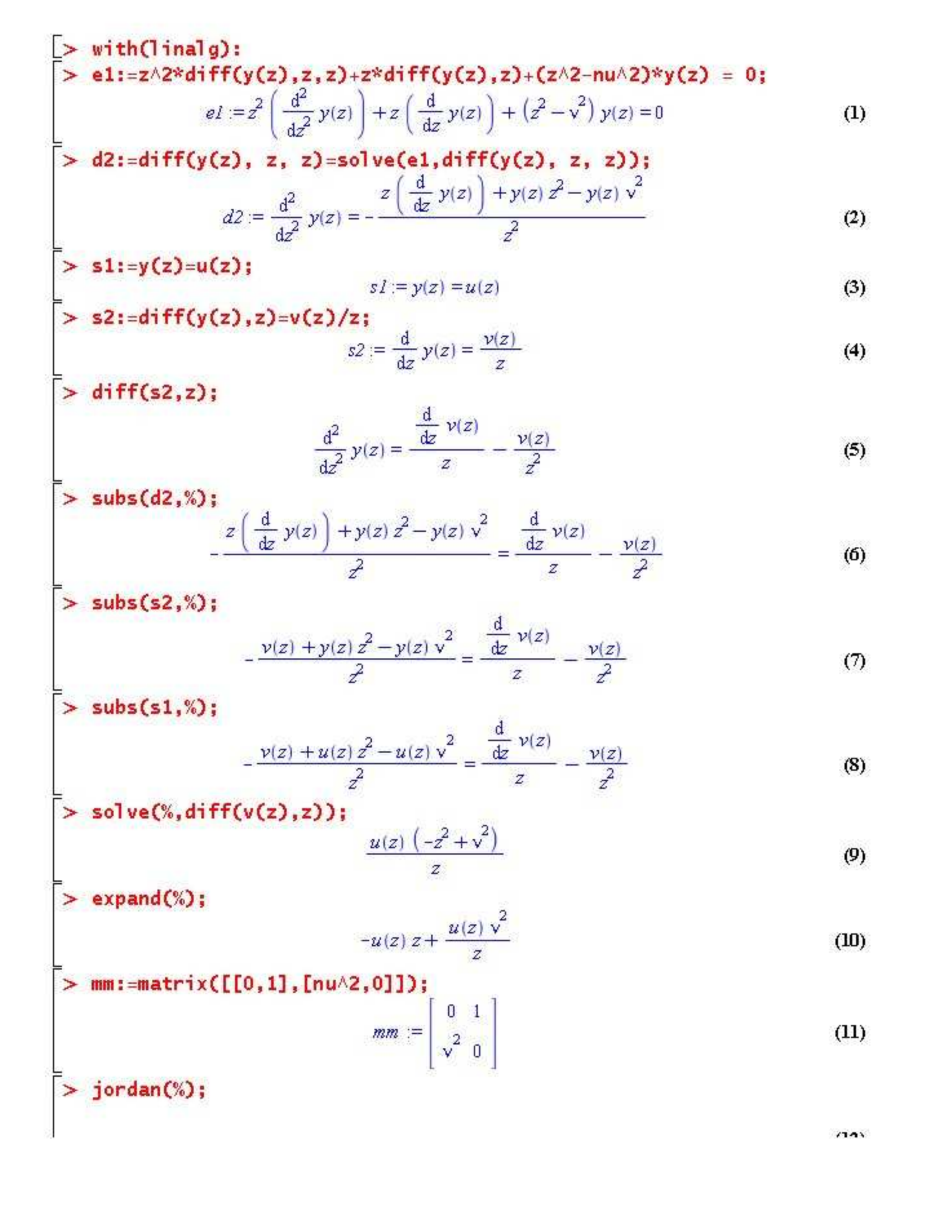
Let λ_1 be a characteristic root such that $\lambda_1 + n$ is not a characteristic root. Then, there is a solution of (87) of the form $y_1 = z^{\lambda_1}\varphi(z)$, where $\varphi(z)$ is analytic and we can take $\varphi(0) = 1$.

We can assume without loss of generality that $\lambda_1 = 0$. Indeed, otherwise we first make the substitution $y = z^{\lambda_1} w$ and divide the equation by z^{λ_1}.

The general term of the new equation is of the form

$$z^{-\lambda_1} b_l z^{-l} (z^{\lambda_1} w)^{n-l} = z^{-\lambda_1} b_l z^{-l} \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} (z^{\lambda_1})^{(n-l-j)}$$

$$= z^{-\lambda_1} b_l z^{-l} \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} z^{\lambda_1 - n + l + j} = b_l \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} z^{-(n-j)} \tag{128}$$



```
> with(linalg):
> e1:=z^2*diff(y(z),z,z)+z*diff(y(z),z)+(z^2-nu^2)*y(z) = 0;
```

$$e1 := z^2\left(\frac{d^2}{dz^2}y(z)\right)+z\left(\frac{d}{dz}y(z)\right)+\left(z^2-\nu^2\right)y(z)=0 \quad (1)$$

```
> d2:=diff(y(z), z, z)=solve(e1,diff(y(z), z, z));
```

$$d2 := \frac{d^2}{dz^2}y(z) = -\frac{z\left(\frac{d}{dz}y(z)\right)+y(z)\,z^2-y(z)\,\nu^2}{z^2} \quad (2)$$

```
> s1:=y(z)=u(z);
```

$$s1 := y(z)=u(z) \quad (3)$$

```
> s2:=diff(y(z),z)=v(z)/z;
```

$$s2 := \frac{d}{dz}y(z)=\frac{v(z)}{z} \quad (4)$$

```
> diff(s2,z);
```

$$\frac{d^2}{dz^2}y(z)=\frac{\frac{d}{dz}v(z)}{z}-\frac{v(z)}{z^2} \quad (5)$$

```
> subs(d2,%);
```

$$-\frac{z\left(\frac{d}{dz}y(z)\right)+y(z)\,z^2-y(z)\,\nu^2}{z^2}=\frac{\frac{d}{dz}v(z)}{z}-\frac{v(z)}{z^2} \quad (6)$$

```
> subs(s2,%);
```

$$-\frac{v(z)+y(z)\,z^2-y(z)\,\nu^2}{z^2}=\frac{\frac{d}{dz}v(z)}{z}-\frac{v(z)}{z^2} \quad (7)$$

```
> subs(s1,%);
```

$$-\frac{v(z)+u(z)\,z^2-u(z)\,\nu^2}{z^2}=\frac{\frac{d}{dz}v(z)}{z}-\frac{v(z)}{z^2} \quad (8)$$

```
> solve(%,diff(v(z),z));
```

$$\frac{u(z)\left(-z^2+\nu^2\right)}{z} \quad (9)$$

```
> expand(%);
```

$$-u(z)\,z+\frac{u(z)\,\nu^2}{z} \quad (10)$$

```
> mm:=matrix([[0,1],[nu^2,0]]);
```

$$mm := \begin{bmatrix} 0 & 1 \\ \nu^2 & 0 \end{bmatrix} \quad (11)$$

```
> jordan(%);
```

Figure 1:

which is of the same type as (87).

Thus we assume $\lambda_1 = 0$ and take $y = \varphi w$. As discussed, we can assume

$\varphi(0) = 1$. The equation for w is

$$\sum_{l=0}^{n} z^{-l} b_l \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} \varphi^{(n-l-j)} = 0 \tag{129}$$

or

$$\sum_{j=0}^{n} w^{(j)} \sum_{l=0}^{n-j} z^{-l} b_l \binom{n-l}{j} \varphi^{(n-l-j)} = 0 \tag{130}$$

or also

$$\sum_{j=0}^{n} w^{(n-j)} \sum_{l=0}^{j} z^{-l} b_l \binom{n-l}{j} \varphi^{(j-l)} = 0 \tag{131}$$

We note that this equation, after division by φ (recall that $1/\varphi$ is analytic) is of the same form as (87). However, now the coefficient of w is

$$\sum_{l=0}^{n} z^{-l} b_l \binom{n-l}{0} \varphi^{(n-l)} = \sum_{l=0}^{n} z^{-l} b_l \varphi^{(n-l)} = 0 \tag{132}$$

since this is indeed the equation φ is solving.

We divide the equation by φ (once more, remember $\varphi(0) = 1$), and we get

$$\sum_{j=0}^{n-1} w^{(1+(n-1-j))} \tilde{b}_j = 0 \tag{133}$$

where

$$\tilde{b}_j = \sum_{l=0}^{j} z^{-l} b_l \binom{n-l}{j} \frac{\varphi^{(j-l)}}{\varphi} \tag{134}$$

has a pole of order at most j, or

$$\sum_{j=0}^{n-1} g^{(n-1-j)} \tilde{b}_j = 0 \tag{135}$$

with $w' = g$. This is an $(n-1)$th order equation for g, and solving the equation for w reduced to solving a lower order equation, and one integration, $w = \int g$.

Thus, by knowing, or assuming to know, one solution of the nth order equation, we can reduce the order of the equation by one. Clearly, the characteristic roots for the g equation are $\lambda_i - \lambda_1 - 1$, $i \neq 1$. We can repeat this procedure until the equation for g becomes of first order, which can be explicitly solved. This shows what to do in the degenerate case, other than, working in a similar (in some sense) way with the equivalent nth order system.

8.4.1 Reduction of order in a degenerate case: an example

Consider the equation

$$z(z-1)y'' + y = 0 \tag{136}$$

This equation can be solved in terms of hypergeometric functions, but it is easier to understand the solutions, at least locally, from the equation. The indicial equation is $r(r-1) = 0$ (a *resonant case*: the roots differ by an integer). Substituting $y_0 = \sum_{k=1}^{\infty} c_k z^k$ in the equation and identifying the powers of z yields the recurrence

$$c_{k+1} = \frac{k^2 - k + 1}{k(k+1)} c_k \tag{137}$$

with c_1 arbitrary, which of course we can take to be 1. By induction we see that $0 < c_k < 1$. Thus the power series has radius of convergence at least 1. The radius of convergence is in fact exactly one as it can be seen applying the ratio test and using (137); the series converges exactly up to the nearest singularity of (136). We knew that we must get an analytic solution, by the general theory. We let $y_0 = y_0 \int g(s)ds$ and get, after some calculations, the equation

$$g' + 2\frac{y_0'}{y_0} g = 0 \tag{138}$$

and, by the previous discussion, $2y_0'/y_0 = 2/z + A(z)$ with $A(z)$ is analytic. The point $z = 0$ is a regular singular point of (138) and in fact we can check that $g(z) = C_1 z^{-2} B(z)$ with C_1 an arbitrary constant and $B(z)$ analytic at $z = 0$. Thus $\int g(s)ds = C_1(a/z + b\ln(z) + A_1(z)) + C_2$ where $A_1(z)$ is analytic at $z = 0$. Undoing the substitutions we see that we have a fundamental set of solutions in the form $\{y_0(z), B_1(z)\ln z + B_2(z)\}$ where B_1 and B_2 are analytic.

8.4.2 Singularities at infinity

An equation has a singularity of first kind at infinity, if after the change of variables $z = 1/\zeta$, the equation in ζ has a singularity of first kind at zero.

For instance, (136) changes into

$$y'' + \frac{2}{\zeta} y + \frac{y}{\zeta^2(1-\zeta)} = 0 \tag{139}$$

As a result, we see that (136) only has singularities of the first kind on the Riemann sphere, $\mathbb{C}_\infty$.

Exercise 2. (i) Show that *any* nonzero solution of (136) has at least one branch point in $\mathbb{C}$. (Hint: Examine the indicial equations at: $0, 1$ and ∞. Alternatively, you can use the indicial equation at ∞ and (137).)

(ii) Use the substitution (**??**) to bring the equation to a system form. What is the matrix B_0', the matrix B' in the notation of Theorem 3 at $z = 0$? What is its Jordan normal form?

(iii) If we write the B_1' corresponding to the singular point $z = 1$, can B_0' and B_1' commute?

9 General isolated singularities

We now take a system of the form

$$y' = By \tag{140}$$

Interpreted as as a matrix equation, we write

$$Y' = BY \tag{141}$$

where, for some $\rho > 0$ the matrix $B(z)$ is analytic in $\mathbb{D}_\rho \setminus \{0\}$. We do not assume anymore that the singularity is a pole. It is clear that (141) has, at any point $z_0 \in \mathbb{D}_\rho \setminus \{0\}$, a fundamental matrix solution Y_0, and that the general matrix solution of (141) is $Y_0 K$ where K is an invertible constant matrix. Indeed, Y_0 is invertible, and if Y is any solution we can thus always write $Y = Y_0 K$, clearly, for $K = Y_0^{-1} Y$. Then, we can check that $Y_0 K' = 0$, or $K' = 0$ which is what we claimed. By our general arguments, Y_0 is analytic (at least) in a disk of radius $|z_0|$. If we take a point $z_1 = z_0 e^{i\phi}$, with ϕ small enough, then the disk $\mathbb{D}_{|z_0|}(z_0)$ and the disk $\mathbb{D}_{|z_0|}(z_1)$ overlap nontrivially, and then $Y_0 = Y_1 K_1$ for some constant matrix K. We see that Y_0 is analytic in $\mathbb{D}_{|z_0|}(z_1)$. It follows that Y_0 is analytic on the Riemann surface of the log at zero, that is, it can be continued along any curve in $\mathbb{D}$ not crossing zero: $Y_0 \to Y_1 K_1 \to Y_2 K_1 K_2 \cdots$. Does this mean that Y_0 is analytic in $\mathbb{D} \setminus \{0\}$? Absolutely not. This is because

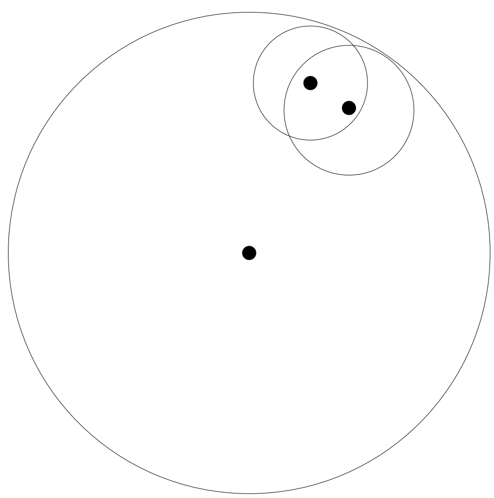


Figure 2:

after one full loop, we may return at z_0 with $Y_n K_n \cdots K_1 = Y_0 K$ for some nontrivial K. To see that, we can just recall the general solution $M(z) z^{B'}$ valid when zB is analytic, or simply look at the solution of the equation $y' = \sqrt{2} y / z$, $y = z^{\sqrt{2}}$. However, note that K is invertible, and thus it can be written in the form e^C. Indeed, the Jordan form J of K is of the type $D + N$, where

D is diagonal with all elements on the diagonal nonzero, and N is a nilpotent commuting with D. We then write $D+N = D(1+D^{-1}N)$ and note that $N_1 := D^{-1}N$ is also nilpotent. We define $\log(D+N) = \log D + \sum_{j=1}^{m}(-1)^{j-1}N_1 j/j$ where m is the size of the largest Jordan block in J. We define $\ln K = U^{-1}\ln JU$ where U is the matrix that brings K to its Jordan form. We can check that $e^{\ln J} = J$ and consequently $e^{\ln K} = K$. If we write $K = e^{2\pi i P}$ we note that the matrix $Y_0 z^{-P}$ is single-valued, thus analytic in $\mathbb{D}_\rho \setminus \{0\}$. Thus we have proved,

Theorem 4. *The matrix equation (141), under the assumptions there, has a fundamental solution of the form $A(z)z^P$ where A is analytic in $\mathbb{D}_\rho \setminus \{0\}$.*

10 Frobenius' theorem

Definition 5. *An equation of the form (87) has a regular singularity at zero if there exists a fundamental set of solutions in the form of finite combinations of functions of the form*

$$y_i = z^{\lambda_i}(\ln z)^{m_i} f_i(z); \quad (\textit{by convention, } f_i(0) \neq 0) \tag{142}$$

where f_i are analytic, $m_i \in \mathbb{N} \cup \{0\}$

Theorem 5 (Frobenius)**.** *An equation of the form (87) has a regular singularity at zero iff the singular point is of the first kind. (Clearly, a similar statement holds at any point.)*

10.1 Converse of Frobenius' theorem

For the proof, we note that we can always change coordinates so that $P = J$ is in Jordan normal form. Then, the equation (87) has the general solution in the form $A(x)x^J K$ where K is a constant matrix. Then, check that there is a solution of the form $x^\lambda y_1(x)$ where y_1 is analytic in $\mathbb{D}_\rho \setminus \{0\}$. By performing a reduction of order on the associated nth order equation and rewriting that as a system, check that we get a system of order $n-1$, otherwise of the same form (87).

Exercise 1. *Use induction on n to complete the proof.*

11 Nonlinear systems

A point, say $z = 0$ is a singular point of the first kind of a nonlinear system if the system can be written in the form

$$y' = z^{-1}h(z,y) = z^{-1}(L(z)y + f(z,y)) \tag{143}$$

where h is analytic in z, y in a neighborhood of $(0,0)$. We will not analyze these systems in detail, but much is known about them, [3] [2]. The problem, in general, is nontrivial and the most general analysis to date for one singular point is in [3], and utilizes techniques beyond the scope of our course now. We present, without proofs, some results in [2], which are more accessible. They apply to several singular points, but we will restrict our attention to just one, in the setting of (143). In the nonlinear case, a "nonlinear nonresonance" condition is needed, namely: if λ_i are the eigenvalues of $L(0)$, we need a *diophantine condition*: for some $\nu > 0$ we have

$$\inf\left\{(|\mathbf{m}|+k)^{\nu}|k+\mathbf{m}\cdot\boldsymbol{\lambda}-\lambda_i| \,\Big|\mathbf{m}\in\mathbb{N}^n, |\mathbf{m}|>1, k\in\mathbb{N}\cup\{0\}; i\le n\right\} > 0 \quad (144)$$

Furthermore, $L(0)$ is assumed to be diagonalizable. (In [3] a weaker nonresonance condition is imposed, known as the Brjuno condition, which is known to be optimal.)

Proposition 6. *Under these assumptions, There is a change of coordinates $y = \Phi(z)u(z)$ where Φ is analytic with analytic inverse, so that the system becomes*

$$u' = z^{-1}h(z,u) = z^{-1}(Bu + f(z,u)) \quad (145)$$

where B is a constant matrix.

Proposition 7. *The system (145) is analytically equivalent* in a neighborhood of $(0,0)$*, that is for small u as well as small z, to its linear part, namely to the system*

$$w' = z^{-1}Bw \quad (146)$$

In terms of solutions, it means that the general *small* solution of (143) can be written as

$$y = H(z, \Phi(z)z^B C) \quad (147)$$

where $H(u,v)$ is analytic as a function of two variables, C is an arbitrary constant vector. The diophantine, and more generally, Brjuno condition is generically satisfied. If the Brjuno condition fails, equivalence is still possible, but unlikely. The structure of y in an equation of the form (147) is

$$y_j(z) = \sum_{m,k} c_{k,\mathbf{m}} z^k z^{\mathbf{m}\cdot\boldsymbol{\lambda}} \quad (148)$$

12 Variation of parameters

As we discussed, a linear nonhomogeneous equation can be brought to a linear homogeneous one, of higher order. While this is useful in a theoretical quest, in practice, it is easier to solve the associated homogeneous system and obtain the solution to the nonhomogeneous one by integration. Indeed, if the matrix equation

$$Y' = B(z)Y \quad (149)$$

has the solution $Y = M(z)$, then in the equation

$$Y' = B(z)Y + C(z) \quad (150)$$

we seek solutions of the form $Y = M(z)W(z)$. We get

$$M'W + MW' = B(z)MW + C(z) \quad \text{or} \quad M(z)W' = C(z) \tag{151}$$

giving

$$Y = M(z) \int_a^z M^{-1}(s)C(s)ds \tag{152}$$

13 Equilibria

We start with the simple example of the harmonic oscillator. It is helpful in a number of ways, since we have a good intuitive understanding of the system. Yet, the ideal (frictionless) oscillator has nongeneric features.

We can use conservation of energy to write

$$\frac{1}{2}mv^2 + mgl(1 - \cos x) = const \tag{153}$$

where x is the angle and $v = dx/dt$, so with $l = 1$ we get

$$x'' = -\sin x \tag{154}$$

13.1 Exact solutions

This equation can be solved exactly, in terms of Weierstrass elliptic functions. Integration could be based on (155), and also by multiplication by x' and integration, which leads to the same.

$$\frac{1}{2}x'^2 - \cos x = C \tag{155}$$

$$\int_0^x \frac{ds}{\sqrt{C + 2\cos s}} = t + t_0 \tag{156}$$

With the substitution $\tan(x/2) = u$ we get

$$\int_0^{\tan(x/2)} \frac{du}{\sqrt{1+u^2}\sqrt{C + 1 + (C-1)u^2}} = t + t_0 \tag{157}$$

Whenever a differential system can be reduced to mere integrations as above, we say that the system is integrable by quadratures. On the other hand, by definition the elliptic integral of the first kind, $F(z,k)$ is defined as

$$F(z,k) = \int_0^z \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{158}$$

and we get, with $K = \sqrt{2}/\sqrt{1+C}$,

$$iKF\left(\cos(z/2), K\right)\Big|_0^x = t + t_0 \tag{159}$$

At this point, we should study elliptic functions to proceed. They are in fact very interesting and worthwhile studying, but we'll leave that for later. For now, it is easier to gain insight on the system from the equation than from the properties of elliptic functions.

13.2 Discussion and qualitative analysis

Written as a system, we have

$$x' = v \tag{160}$$

$$v' = -\sin x \tag{161}$$

The point $(0,0)$ is an equilibrium, and $x = 0, v = 0$ is a solution. So are the points $x = n\pi, v = 0$, $n \in \mathbb{N}$.

Note that (160) is a *Hamiltonian system*, i.e., it is of the form

$$\begin{aligned} x' &= \frac{\partial H(x,v)}{\partial v} \\ v' &= -\frac{\partial H(x,v)}{\partial x} \end{aligned} \tag{162}$$

where $H(x,v) = \frac{1}{2}v^2 + 1 - \cos x$. In all such cases, we see that H is a *conserved quantity*, that is $H(x(t), v(t)) = const$ along a given *trajectory* $\{(x(t), v(t)) : t \in \mathbb{R}\}$. The trajectories are thus the level lines of H, that is

$$H(x,v) = \frac{1}{2}v^2 + 1 - \cos x = C \tag{163}$$

the *trajectories* (we artificially added 1, since H is defined up to an additive constant, to make $H \geq 0$.

We now see the importance of critical points: If H is analytic (in our case, it is entire), at all points where the right side of (162) is nonzero, either $x(y)$ or $v(x)$ are locally analytic, by the implicit function theorem, whereas otherwise, in general, the curves are nonuniquely defined and possibly singular.

We have $H(0,0) = 0$ and we see that $H(x,v) = h$ for $0 < h < 2$ are closed curves.

Indeed, we have in this case,

$$|v| \leq \sqrt{2h+2} \tag{164}$$

$$1 - \cos x < h \tag{165}$$

and thus both x and v are bounded, $(x,v) \in K$, in particular $x \in (-\pi/2, \pi/2)$. Then, $H(x,v) \leq h$ is compact, and since, if $C < 2$ we have $\nabla H = 0$ only at the origin, where H is zero, and H is positive otherwise, its maximum occurs on the boundary of $\{(x,v) : H(x,v) \leq h\}$. Furthermore, $H(x,v) = h$ is an analytic curve, in the sense above, since $\nabla H \neq 0$ in this region.

Physically, for initial conditions close to zero, the pendulum would periodically swing around the origin, with amplitude limited by the total energy.

Fig. 9 represents a numerical contour plot of $v^2/2 - \cos x$. If we zoom in, we see that the program had difficulties at the critical points $\pm\pi$, showing once more that there is something singular there.

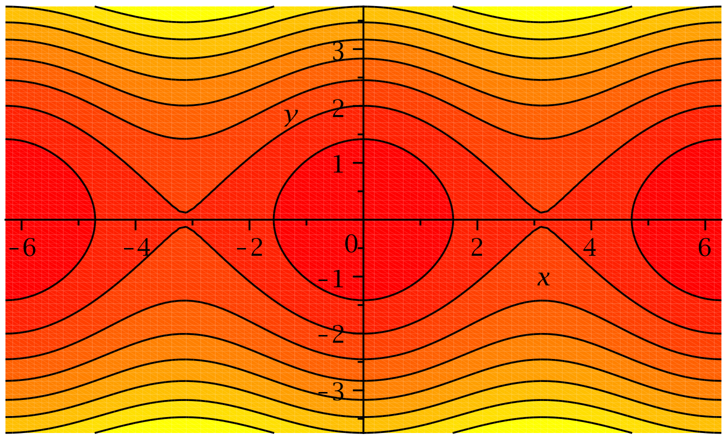


Figure 3: Contour plot of $v^2/2 - \cos x$

13.3 Linearization of the phase portrait

Take

$$(1 - u^2/2) = \cos x; \quad u \in [-2, 2] \tag{166}$$

We can write this as

$$u^2 = 4\sin(x/2)^2 \tag{167}$$

which defines two holomorphic changes of coordinates

$$u = \pm 2\sin(x/2) \tag{168}$$

These are indeed biholomorphic changes of variables until $\sin(x/2)' = 0$ that is, $x = \pm\pi$. With any of these changes of coordinates we get

$$\frac{u}{\sin x} u' = v \tag{169}$$

$$v' = -\sin x \tag{170}$$

or

$$uu' = v \sin x \tag{171}$$

$$v' = -\sin x \tag{172}$$

which would give the same trajectories family as

$$u' = v \tag{173}$$
$$v' = -u \tag{174}$$

for which the exact solution, $A \sin t, A \cos t$ gives rise to circles. The same could have been seen easily seen by making the same substitution, (177) in (163). We note again that in (177) we have $u^2 \in [0, 4]$, so the equivalence does not hold beyond $u = \pm 2$.

What about the other equilibria, $x = (2k + 1)\pi$? It is clear, by periodicity and symmetry that it suffices to look at $x = \pi$. If we make the change of variable $x = \pi + s$ we get

$$s' = v \tag{175}$$
$$v' = \sin s \tag{176}$$

In this case, the same change of variable, $u = 2 \sin(s/2)$ gives

$$u' = v \tag{177}$$
$$v' = u \tag{178}$$

implying $v^2 - u^2 = C$ as long as the change of variable is meaningful, that is, for $u < 2$, or $|s| < \pi$. So the curves associated to (175) are analytically conjugated to the hyperbolas $v^2 - u^2 = C$. The equilibrium is unstable, points starting nearby necessarily moving far away. The point $\pi, 0$ is a saddle point.

The trajectories starting at π are *heteroclinic*: they link different saddles of the system. In general, they do not necessarily exist.

In our case, these trajectories correspond to $H = 2$ and this gives

$$v^2 = 2(1 + \cos(x)) \tag{179}$$

or

$$v^2 = 4 \cos(x/2)^2 \tag{180}$$

that is, the trajectories are given explicitly by

$$v = \pm 2 \cos(x/2) \tag{181}$$

This is a case where the elliptic function solution reduces to elementary functions: The equation

$$\frac{dx}{dt} = 2 \cos(x/2) \tag{182}$$

has the solution

$$x = 2 \arctan(\sinh(t + C)) \tag{183}$$

We see that the time needed to move from one saddle point to the next one is infinite.

13.4 Connection to regularly perturbed equations

Note that at the equilibrium point $(\pi, 0)$ the system of equations is analytically equivalent, insofar as trajectories go, to the system

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \tag{184}$$

The eigenvalues of the matrix are ± 1 with (unnormalized) eigenvectors $(1, 1)$ and $(-1, 1)$. Thus, the change of variables to bring the system to a diagonal form is $x = \xi + \eta, v = \xi - \eta$. We get

$$\xi' + \eta' = \xi - \eta \tag{185}$$

$$\xi' - \eta' = \xi + \eta \tag{186}$$

By adding and subtracting these equations we get the diagonal form

$$\xi' = \xi \tag{187}$$

$$\eta' = -\eta \tag{188}$$

or

$$\frac{d\xi}{d\eta} = -\frac{\xi}{\eta}; \text{ or } \xi_\eta + \frac{1}{\eta}\xi = 0 \tag{189}$$

a standard regularly perturbed equation. Clearly the solutions of (189) are $\xi = C/\eta$ with $C \in (-\infty, \infty)$, and insofar as the phase portrait goes, we could have written $\eta_\xi + \frac{1}{\xi}\eta = 0$, which means that the trajectories are the curves $\xi = C/\eta$ with $C \in [-\infty, \infty]$, hyperbolas and the coordinate axes. In the original variables, the whole picture is rotated by $45°$.

13.5 Completing the phase portrait

We see that, for $H > 2$ we have

$$v = \pm\sqrt{2h + 2\cos(x)} \tag{190}$$

where now $h > 2$. With one choice of branch of the square root (the solutions are analytic, after all), we see that $|v|$ is bounded, and it is an open curve, defined on the whole of $\mathbb{R}$. Note that the explicit form of the trajectories, given by (163) does not, in general, mean that we can solve the second order differential equation. The way the pendulum position depends on time, or the way the point moves along these trajectories, is still transcendental.

13.6 Local and asymptotic analysis

Near the origin, for $C = a^2$ small, we have

$$x' = v \tag{191}$$

$$v' = x - x^3/6 + ... \tag{192}$$

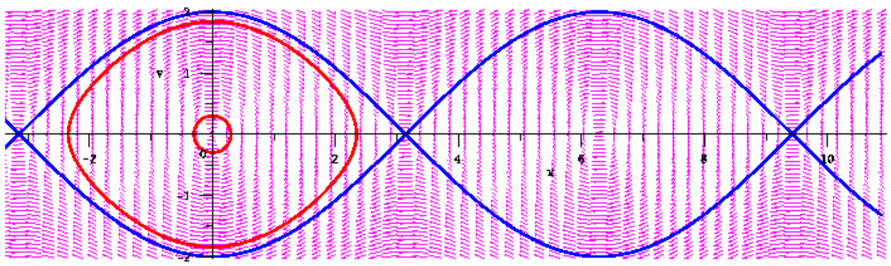


Figure 4: Contour plot of $y^2/2 - \cos x$

implying

$$x' = v \tag{193}$$
$$v' \approx x \tag{194}$$

which means

$$x \approx a \sin t \tag{195}$$
$$v \approx a \cos t \tag{196}$$

For C very large, we have

$$\frac{dx}{\sqrt{C + \cos x}} = dx(C + \cos x)^{-1/2} = dxC^{-1/2}(1 + \cos x/C)^{-1/2}$$
$$= dx(C^{-1/2} - \frac{1}{2}\cos x/C^{-3/2} + \cdots) \tag{197}$$

which means

$$C^{-1/2}x + \frac{1}{2}\sin x/C^{-3/2} + \cdots = t + t_0 \tag{198}$$

or

$$x = C^{1/2}(t + t_0) - \frac{1}{2}\sin(C^{1/2}t)/C^{-3/2} + \cdots = \tag{199}$$

The solutions near the critical point $(\pi, 0)$ can be analyzed similarly.

Local and asymptotic analysis often give sufficient qualitative, and sometimes quantitative information about all solutions of the equation.

14 Equilibria

In [4], Chapter 1.3 about nonlinear systems starts with the words:

"We must start by admitting that almost nothing beyond general statements can be made about most nonlinear systems. In the remainder of this book we will meet some of the delights and horrors about such systems, but the reader must bear in mind that the line of attack we develop in this text is only one and that any other tool in the workshop of applied mathematics, including numerical integration, perturbation methods, and asymptotic analysis, can and should be brought to bear on a specific problem". Since the book was written, there has been substantial progress, especially in using tools of asymptotic analysis, to find the behavior of nonlinear systems. We will see about these later but for now, we start with classical results and tools.

14.1 Flows

Consider the system

$$\frac{dx}{dt} = F(x) \tag{200}$$

where F is smooth enough. Such equations can be considered in $\mathbb{R}^n$ or, more generally, in Banach spaces.

As we know by now, if x_0 is a regular point for F, then there exists a unique local solution of (200) with $x(0) = x_0$.

Remark 4. *(a) Note that equilibria, defined as points where* $F(x_e) = 0$ *are singular points of the* field. Trajectories *can intersect there. But this does not mean that* flows *are singular there. Indeed, if we write*

$$x(t) = x_0 + \int_0^t F(x(s))ds$$

and the field is smooth at x_0*, the map above is contractive, and there is a unique solution. See also Remark 6 below.*

The initial condition x_0 is mapped, by the solution of the differential equation (200) into $x(t)$ where $t \in (-a, b)$.

The map $x(0) \to x(t)$ written as $f^t(x_0)$ is the flow associated to F.

For $t \geq 0$ we note the semigroup property $f^0 = I, f^{s+t} = f^s f^t$. This follows from uniqueness of solutions, giving $x(t+s; x_0) = x(s; x(t; x_0))$.

Fixed points, hyperbolic fixed points in $\mathbb{R}^n$**. Example**. If $F(x) = Bx$ where B **does not depend on** x, then the general solution is

$$x = e^{Bt} x_0 \tag{201}$$

where x_0 is the initial condition at $t = 0$. (Note again that a simple exponential formula does not exist, in general, if B depended on t.)

In this case, the flow f is given by the linear map

$$f^t(x_0) = e^{DF(0)t} x_0 \tag{202}$$

Note that $(D_x f)(0) = e^{Bt}$.

Note 5. Remember that the eigenvalues of $e^{B\alpha}$ are $e^{\lambda_i \alpha}$ where λ_i are the eigenvalues of B.

Definition 8. *The point x_0 is a fixed point of f if $f^t(x_0) = x_0$ for all t.*

Proposition 9. *If f is associated to F, then x_0 is a fixed point of f* iff *$F(x_0) = 0$.*

Proof. Indeed, we have $x(t + \Delta t) = x(t) + F(x_0)\Delta t + O((\Delta t)^2)$ for small Δt. Then $x(t + \Delta t) = x(t)$ implies $F(x_0) + O(\Delta t) = 0$, or, $F(x_0) = O(\Delta t)$. Taking $\Delta t \to 0$, we see that $F(x_0) = 0$. Conversely, it is obvious that $F(x_0) = 0$ implies that $x(t) = x_0$ is a solution of (200), and this solution is unique, by Remark 4. □

Remark 6. *This also shows that if trajectories intersect at an equilibrium, then along any nontrivial trajectory ending at x_0 (that is, a trajectory other that $x(t) = x_0$, we must have $x(t) \neq x_0$ for all $t \in \mathbb{R}$ and thus $x_0 = \lim_{t\to\infty} x(t)$.*

Assume 0 is a fixed point of F, $F(0) = 0$. The flow f depends on two variables, x_0 and t. Since $x(t; x_0) = f^t(x_0)$, we clearly have

$$\frac{\partial f}{\partial t} = F(x(t; x_0)) = F(f_t(x_0)) \tag{203}$$

To see what $\dfrac{\partial f}{\partial x_0}$ is near 0, we see that, if F is differentiable, we have

$$x' = F(x) = F(0) + (DF)(0)x + o(x) = (DF)(0)x + o(x) \tag{204}$$

We thus expect, to leading order, to have

$$x' = (DF)(0)x \Rightarrow x = e^{t(DF)(0)}x_0 \tag{205}$$

This is indeed the case, and it is shown below.

Proposition 10. *If f is associated to the C^1 field F, and x_1 is a fixed point of f, then $D_x f^t|_{x=x_1} = e^{DF(x_1)t}$.*

That is, the flow is tangent to the linear flow.

Proof. Without loss of generality we take $x_1 = 0$. Let t be fixed and take the initial condition $x(t = 0) = x_0$ small enough. Let $DF(0) = B$. We have $F(x) = F(0) + g(x)$ where $g(x) = O(x^2)$ for small x. We have $x' = Bx + g(x)$. Taking $x = e^{Bt}u$ we get

$$e^{Bt}u' + Be^{Bt}u = Be^{Bt}u + g(e^{Bt}u) \tag{206}$$

where $g(x) = o(x)$. Thus

$$u = x_0 + \int_0^t e^{-Bs} g(e^{Bs}u(s))ds \tag{207}$$

or

$$x = e^{Bt}x_0 + \int_0^t e^{B(t-s)} g(x(s))ds \tag{208}$$

Consider the Banach space of functions defined on $[0, T]$ with the sup norm, and the ball B_δ of radius $e^{\|B\|T}\|\delta\|$ for some δ small enough. Consider the neighborhood $N = \{x_0 | \|x_0\| < \delta/2\}$ and assume we are working with initial conditions in N.

We claim that (208) is contractive in B_δ, for all $x_0 \in N$. Indeed, you can easily check that the ball is preserved if δ is small enough, since $g(x) = o(x)$.

To show contractivity, we note that

$$\xi(t) - \eta(t) = \int_0^t e^{B(t-s)}[g(\xi(s)) - g(\eta(s))]ds \tag{209}$$

where we know that

$$\|g(\xi(s)) - g(\eta(s))\| = o(\|\xi(s) - \eta(s)\|) \tag{210}$$

by the definition of differentiability in a neighborhood of zero (since $F(\xi) - F(\eta) = (D(F)(0) + o(1))(\xi - \eta) + o(\xi - \eta))$. The rest of the contractivity proof is straightforward.

Now we see that, for $x_0 < \delta$, for any $\delta_1 > 0$ we can arrange that

$$\left| \|x - e^{Bt}x_0\| \right| \leq \delta_1 \|x\| \tag{211}$$

or

$$\left| \|x\| - \|e^{Bt}x_0\| \right| \leq \delta_1 \|x\| \tag{212}$$

thus $\|x\| \leq \|e^{Bt}x_0\| + \delta_1\|x\|$ or $\|x\| \leq 1/(1-\delta_1)\|e^{Bt}x_0\|$, implying,

$$x = e^{Bt}x_0 + \delta_2; \quad \|\delta_2\| \leq \frac{\delta_1 e^{\|B\|T}}{1-\delta_1}\|x_0\|$$

proving the statement.

Definition 11. • *The fixed point $x = 0$ is hyperbolic if the matrix $D_x f|_{x=0}$ has no eigenvalue on the unit circle.*

• *Equivalently, if f is associated with F, the fixed point 0 is hyperbolic if the matrix $DF(0)$ has no purely imaginary eigenvalues.*

□

14.2 The Hartman-Grobman theorem

The following result generalizes to Banach space settings.

Let U and V be open subsets of $\mathbb{R}^n$. Let f be a diffeomorphism between U and V with a **hyperbolic** fixed point, that is there is $x_0 \in U \cap V$ so that $f(x_0) = x_0$ and $Df(x_0)$ has no spectrum on the unit circle. Without loss of generality, we may assume that $x = 0$.

Theorem 6 (Hartman-Grobman for maps). *Under these assumptions, f and $Df(0)$ are topologically conjugate, that is, there are neighborhoods U_1, V_1 of zero, and a homeomorphism h from U_1 to V_1 so that $h^{-1} \circ f \circ h = Df(0)$.*

The proof is not very difficult, but it is preferable to leave it for later.

Theorem 7 (Hartman-Grobman for flows, [6]). *Let consider $x' = F(t)$ over a Banach space, where F is a C^1 vector field defined in a neighborhood of the origin 0 of E. Suppose that 0 is a hyperbolic fixed point of the flow described by F. Then there is a homeomorphism between the flows of F and $DF(0)$, that is a homeomorphism between a neighborhood of zero into itself so that*

$$f^t = h \circ e^{tDF(0)} \circ h^{-1} \tag{213}$$

See also [5].

The more regularity is needed, the more conditions are required.

Differentiable linearizations

Theorem 8 (Sternberg-Siegel, see [6]). *Assume f is differentiable, with a hyperbolic fixed point at zero, and the derivative Df is Hölder continuous near zero. Assume further that $DF(0)$ is such that its eigenvalues satisfy*

$$\mathrm{Re}\lambda_i \neq \mathrm{Re}\lambda_j + \mathrm{Re}\lambda_k \tag{214}$$

when $\mathrm{Re}\lambda_j < 0 < \mathrm{Re}\lambda_k$. Then the functions h in Theorems 6 and 7 can be taken to be diffeomorphisms.

Smooth linearizations

Theorem 9 (Sternberg-Siegel, see [6]). *Assume $f \in C^\infty$ and the eigenvalues of $Df(0)$ are nonresonant, that is*

$$\lambda_i - \mathbf{k}\boldsymbol{\lambda} \neq 0 \tag{215}$$

for any $\mathbf{k} \in \mathbb{Z}^n$ with $|\mathbf{k}| > 1$. Then the functions h in Theorems 6 and 7 can be taken to be C^∞ diffeomorphisms.

We will prove, in simpler settings, the Hartman-Grobman theorem for flows.

For the analytic case, see Proposition 6.

14.3 Bifurcations

Bifurcations occur in systems depending on a parameter (or more), call it s. Thus, the system is

$$\frac{d}{dt}x(t;s) = F(x;s) \tag{216}$$

A *local* bifurcation at an equilibrium, say $x = 0, F(0) = 0$, may occur when at least one of the eigenvalues of $DF(0)$ becomes purely imaginary. (Otherwise, the linearization theorem shows that the phase portrait is locally similar to that of the linearized system. In this case, the topology does not change unless we indeed go through purely imaginary eigenvalues.) We will explore bifurcation types and prove theorems about some of them, but before that let's see what types of equilibria are possible in linear systems. Those that are associated to hyperbolic fields represent, again by the linearization theorem, the local behavior of general hyperbolic systems.

15 Types of equilibria of linear systems with constant coefficients in 2d

The equation is now

$$x' = Bx \tag{217}$$

where B is a 2×2 matrix with constant coefficients.

15.1 Distinct eigenvalues

In this case, the system can be diagonalized, and it is equivalent to a pair of trivial first order ODEs

$$x' = \lambda_1 x \tag{218}$$

$$y' = \lambda_2 y \tag{219}$$

15.1.1 Real eigenvalues

The change of variables that diagonalizes the system has the effect of rotating and rescaling the phase portrait of (218). The phase portrait of (218) can be fully described, since we can solve the system in closed form, in terms of simple functions:

$$x = x_0 e^{\lambda_1 t} \tag{220}$$

$$y = y_0 e^{\lambda_2 t} \tag{221}$$

On the other hand, we have

$$\frac{dy}{dx} = \frac{\lambda_2}{\lambda_1}\frac{y}{x} = a\frac{y}{x} \Rightarrow y = C|x|^a \tag{222}$$

where we also have as trajectories the coordinate axes: $y = 0$ ($C = 0$) and $x = 0$ ("$C = \infty$"). These trajectories are generalized parabolas. If $a > 0$ then the system is either (i) **a sink**, when both λ's are negative, in which case, clearly, the solutions converge to zero. See Fig. 5, or (ii) **a source**, when both λ's are positive, in which case, the solutions go to infinity.

The other case is that when $a < 0$; then the eigenvalues have opposite sign. Then, we are dealing with a **saddle**. The trajectories are generalized hyperbolas,

$$y = C|x|^{-|a|} \tag{223}$$

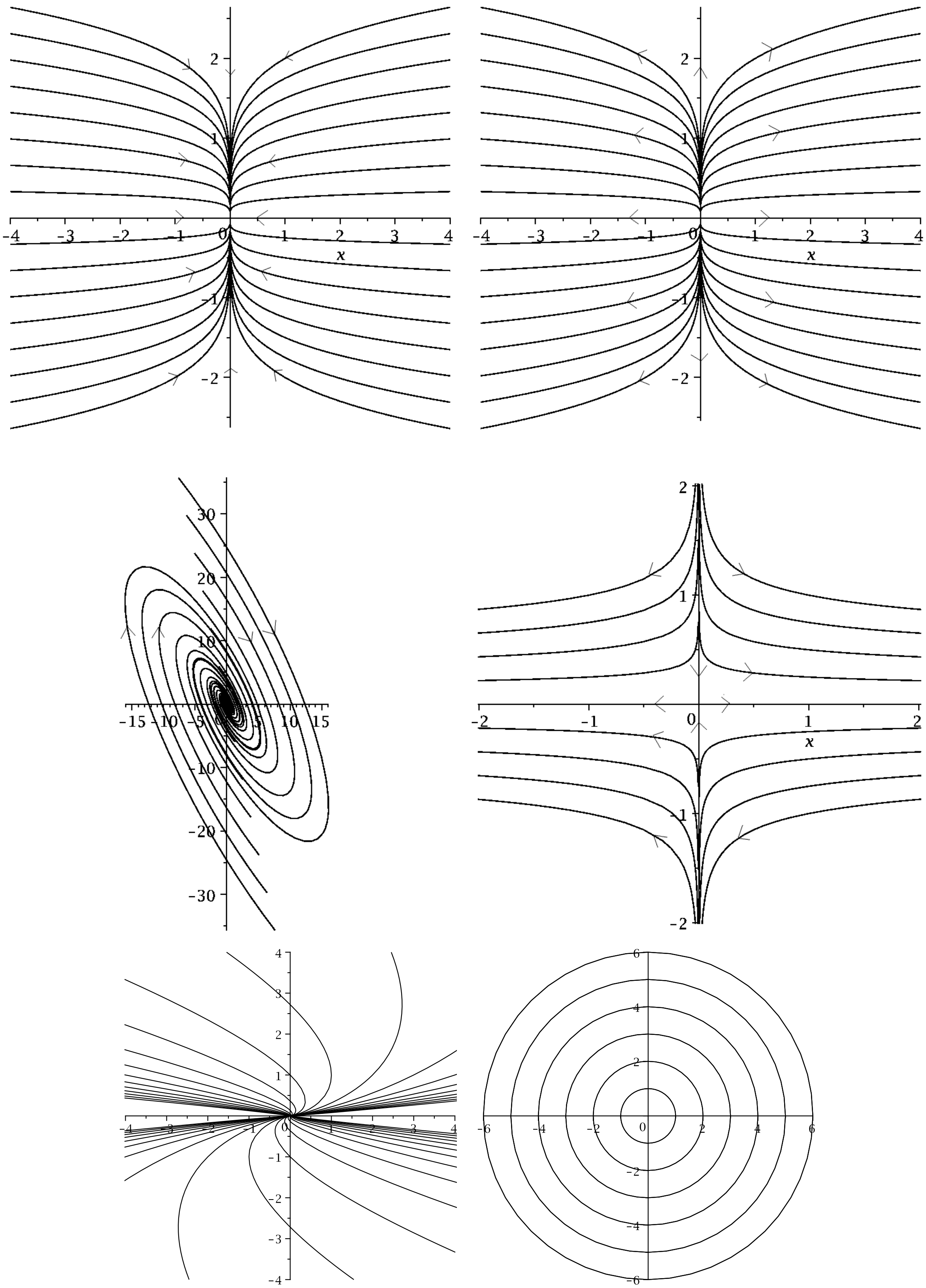


Figure 5: All types of linear equilibria in 2d, modulo euclidian transformations and rescalings: sink, source, spiral sink, saddle, nontrivial Jordan form, center resp. In the last two cases, the arrows point according to the sign of λ or ω, resp.

Say $\lambda_1 > 0$. In this case there is a **stable manifold** the y axis, along which solutions converge to zero, and an **unstable manifold** in which trajectories go to zero as $t \to -\infty$. Other trajectories go to infinity both forward and backward in time. In the other case, $\lambda_1 < 0$, the figure is essentially rotated by $\pi/2$.

15.1.2 Complex eigenvalues

In this case we just keep the system as is,

$$x' = ax + by \tag{224}$$
$$y' = cx + dy \tag{225}$$

We solve for y, assuming $b \neq 0$ (check the case $b = 0$!), introduce in the second equation and we obtain a second order, constant coefficient, differential equation for x:

$$x'' - (a + d)x' + (ad - bc)x = 0 \quad \textit{or} \tag{226}$$
$$x'' - \mathrm{tr}(B)x' + \det(B)x = 0 \tag{227}$$

If we substitute $x = e^{\lambda t}$ in (226) we obtain

$$\lambda^2 - \mathrm{tr}(B)\lambda + \det(B) = 0 \tag{228}$$

and, evidently, since $\lambda_1 + \lambda_2 = \mathrm{tr}(B)$ and $\lambda_1\lambda_2 = \det(B)$, this is the same equation as the one for the eigenvalues of B. The eigenvalues of B have been assumed complex, and since the coefficients we are working with are real, the roots are complex conjugate:

$$\lambda_i = \alpha \pm i\omega \tag{229}$$

The real valued solutions are

$$x = Ae^{\alpha t}\sin(\omega t + \varphi) \tag{230}$$

where A and φ are free constants. Substituting in

$$y = b^{-1}x' - ab^{-1}x \tag{231}$$

we get

$$y(t) = Ae^{\alpha t}b^{-1}[(\alpha - 1)\cos(\omega t + \varphi) - \omega\sin(\omega t + \varphi) \tag{232}$$

which can be written, as usual,

$$y(t) = A_1 e^{\alpha t}\sin(\omega t + \varphi_1) \tag{233}$$

If $\lambda < 0$, then we get the **spiral sink**. If $\alpha > 0$ then we get a spiral source, where the arrows are reverted.

A special case is that when $\alpha = 0$. This is the only non-hyperbolic fixed point with distinct eigenvalues. In this case, show that for some c we have $x^2 + cy^2 = A^2$, and thus the trajectories are ellipses. In this case, we are dealing with a **center**.

15.2 Repeated eigenvalues

In 2d this case there is exactly one eigenvalue, and it must be real, since it coincides with its complex conjugate. Then the system can be brought to a Jordan normal form; this is either a diagonal matrix, in which case it is easy to see that we are dealing with a sink or a source, or else we have

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{234}$$

In this case, we obtain

$$\frac{dx}{dy} = \frac{x}{y} + \frac{1}{\lambda} \tag{235}$$

with solution

$$x = ay + \lambda^{-1} y \ln |y| \tag{236}$$

As a function of time, we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t} = e^{\lambda t} \left[I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \right] \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{237}$$

$$x(t) = (At + B) e^{\lambda t} \tag{238}$$

$$y(t) = A e^{\lambda t} \tag{239}$$

We see that, in this case, only the x axis is a special solution (the y axis is not), and thus, all solutions approach (as $t \to \infty$ or $t \to -\infty$ for $\lambda < 0$ or $\lambda > 0$ respectively) the x axis.

Note 7. The eigenvalues of a matrix depend continuously on the coefficients of the matrix. In two dimensions you can see this by directly solving $\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$. Thus, if a linear or nonlinear system depends on a parameter α (scalar or not) and the equilibrium is hyperbolic when $\alpha = \alpha_0$, then the real part of the eigenvalues will preserve their sign in a neighborhood of $\alpha = \alpha_0$. The type of equilibrium is the same and local phase portrait changes smoothly unless the real part of an eigenvalue goes through zero.

Note 8. When conditions are met for a diffeomorphic local linearization at an equilibrium, then we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi \begin{pmatrix} u \\ v \end{pmatrix} \tag{240}$$

where the equation in (u, v) is linear and the matrix φ is a diffeomorphism. We then have

$$\begin{pmatrix} x \\ y \end{pmatrix} = (D\varphi) \begin{pmatrix} u \\ v \end{pmatrix} + o(u, v) \tag{241}$$

which implies, in particular that the phase portrait very near the equilibrium is changed through a linear transformation.

15.3 Further examples, [5]

Consider the system

$$x' = x + y^2 \tag{242}$$
$$y' = -y \tag{243}$$

The linear part of this system at $(0,0)$ is

$$x' = x \tag{244}$$
$$y' = -y \tag{245}$$

The associated matrix is simply

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{246}$$

with eigenvalues 1 and -1, and the conditions of a differentiable homeomorphism are satisfied.

Locally, near zero, the phase portrait of the system (246) is thus the prototypical saddle.

We will see that, again insofar as the field lines are concerned, this system can be *globally* linearized too.

How about the global behavior? In this case, we can completely solve the system. First, insofar as the field lines go, we have

$$\frac{dx}{dy} = -\frac{x}{y} - y \tag{247}$$

a linear inhomogeneous equation that can be solved by variation of parameters, or more easily noting that, by homogeneity, $x = ay^2$ must be a particular solution for some a, and we check that $a = -1/3$. The general solution of the homogeneous equation is clearly $xy = C$. It is interesting to make it into a homogeneous second order equation by the usual method. We write

$$\frac{1}{y}\frac{dx}{dy} = -\frac{x}{y^2} - 1 \tag{248}$$

and differentiate once more to get

$$\frac{d^2x}{dy^2} = -2\frac{x}{y^2} \tag{249}$$

which is an Euler equation, with indicial equation $(\lambda - 2)(\lambda + 1) = 0$, and thus the general solution is

$$x(y) = ay^2 + \frac{b}{y} \tag{250}$$

where the constants are not arbitrary yet, since we have to solve the more stringent equation (247). Inserting (250) into (247) we get $a = -1/3$. Thus, the general solution of (248) is

$$3xy + y^3 = C \tag{251}$$

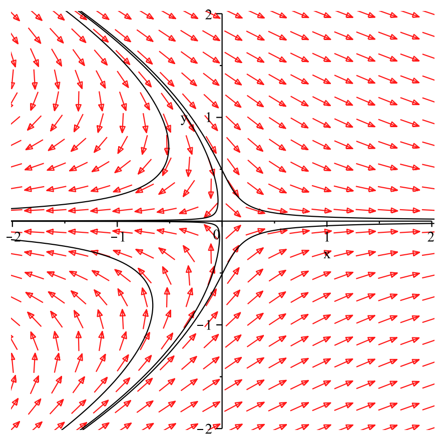


Figure 6: Phase portrait of (242)

which can be, of course, solved for x. The phase can be obtained in the following way: we note that near the origin, the system is diffeomorphic to the linear part, thus we have a saddle there. There is a particular solution with $x = -1/3y^2$ and the field can be completed by analyzing the field for large x and y. This separates the initial conditions for which the solution ends up in the right half plane from those confined to the left half plane.

Global linearization. This is another case of "accidental" analytic linearizability since we can write the *conserved quantity* $3xy(x + y^2/3) = C$, or $(x + y^2/3)y = C$ and thus passing to the variables $u = x + y^2/3$, $v = y$ the system (242) becomes linear, of the form (244) (check!)

Note 9. The change of coordinates is thus

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left(I + \begin{pmatrix} 0 & y^2/3 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} \tag{252}$$

and in particular we see that the transformation is, to leading order, the identity.

Exact solution of the time dependent system. Due to the very special nature of the equation, an exact solution is possible too: we note that the second equation contains y alone, and it gives immediately

$$y = y(0)e^{-t}$$

while x can be either solved from the first equation or, more simply, from (258):

$$x(t) = \frac{c}{y(0)}e^t - \frac{1}{3}y(0)e^{-2t} \tag{253}$$

In the nonlinear system, $y = 0$ is still a solution, but $x = 0$ is not; $x = 0$ is "deformed" into the parabola $x = (-1/3)y^2$.

15.4 Stable and unstable manifolds in 2d

Assume that g is differentiable, and that the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = g \begin{pmatrix} x \\ y \end{pmatrix} \tag{254}$$

has an equilibrium at zero, which is a saddle, that is, the eigenvalues of $(Dg)(0)$ are $-\mu$ and λ, where λ and μ are positive. We can make a linear change of variables so that $(Dg)(0) = \mathrm{diag}(-\mu, \lambda)$. Consider the linearization tangent to the identity, that is, with $Dg(0) = I$. We call the linearized variables (u, v).

Theorem 10. *Under these assumptions, in a disk of radius $\epsilon > 0$ near the origin there exist two functions $y = f_+(x)$ and $x = f_-(y)$ passing through the origin, tangent to the axes at the origin and so that all solutions with initial conditions $(x_0, f_+(x_0))$ converge to zero as $t \to \infty$, while the initial conditions $(f_-(y_0), y_0)$ converge to zero as $t \to -\infty$. The graphs of these functions are called the* **stable and unstable manifolds, resp**. *All other initial conditions necessarily leave this disk as time increases, or decreases.*

Proof. We show the existence of the curve f_+, the proof for f_- being the same, by reverting the signs. We have

$$\begin{aligned} x(t) &= \varphi_1(u(t), v(t)) \\ y(t) &= \varphi_2(u(t), v(t)) \end{aligned} \tag{255}$$

where (u, v) satisfy $u' = -\mu u$ and $v' = \lambda v$.

Consider a point $(\varphi_1(u_0, 0), \varphi_2(u_0, 0))$. There is a unique solution passing through this point, namely $(\varphi_1(u_+(t), 0), \varphi_2(0_+(t), 0))$ where $u_+(0) = u_0, v_+(0) = 0$. Since $u_+(t) \to 0$ as $t \to \infty$ and φ is continuous, we have

$$(\varphi_1(u_+(t), 0), \varphi_2(u_+(t), 0)) \to 0$$

as $t \to \infty$. We now write $(u, v) = \Phi(x, y)$. Along the decaying solution, we have $v = 0$. Since $\Phi = I + o(1)$, we have $\partial \Phi_2 / \partial y = 1$ at $(0, 0)$, and the implicit function theorem shows that $\varphi_2(x, y) = 0$ defines a differentiable function $y = f(x)$ near zero, and $y'(0) = 0$ (check). For other solutions we have, from (255), that x, y exits any small enough disk (check). $\square$

15.5 A limit cycle

Up to now we looked at equilibria, fixed points of the flow, which, along some direction(s), attract solutions as $t \to \infty$ or $t \to -\infty$. Fixed points are of course special, degenerate, trajectories. In nonlinear systems, solutions may be attracted by more structured trajectories: limit cycles.

We follow again [5], but with a different starting point. Let's look at the simple system

$$r' = r(1 - r^2)/2 \tag{256}$$

$$\theta' = 1 \tag{257}$$

which we will ultimately interpret as a system of equations in polar coordinates. Obviously, we can solve this in closed form. The flow clearly has no fixed point, since the field never vanishes. To solve the first equation, note that if we multiply by $2r$ we get

$$2rr' = r^2(1-r^2) \tag{258}$$

or, with $u = r^2$,

$$u' = u(1-u) \tag{259}$$

The exact solution is

$$r = \pm(1+Ce^{-t})^{-1/2};\ \text{and also } r = 0;\ \pm 1, \text{ as special constant solutions} \tag{260}$$

$$\theta = t + t_0 \tag{261}$$

We see that all solutions that start away from zero converge to one as $t \to \infty$. We now interpret r and θ as polar coordinates and write the equations for x

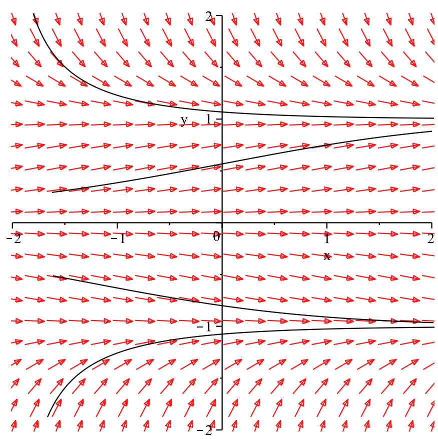


Figure 7: Phase portrait of (256)

and y. We get

$$x' = r'\cos\theta - r\sin\theta\theta' = \frac{1}{2}r(1-r^2)\cos\theta - r\sin\theta$$
$$= \frac{1}{2}x - y - \frac{1}{2}(x^3 + xy^2) \tag{262}$$

$$y' = r'\sin\theta + r\cos\theta\theta' = \frac{1}{2}r(1-r^2)\sin\theta + r\cos\theta$$
$$= x + \frac{1}{2}y - \frac{1}{2}(x^2y + y^3) \tag{263}$$

thus the system

$$x' = \frac{1}{2}x - y - \frac{1}{2}(x^3 + xy^2) \tag{264}$$

$$y' = x + \frac{1}{2}y - \frac{1}{2}(x^2y + y^3) \tag{265}$$

which looks rather hopeless, but we know that it can be solved in closed form.

To analyze this system, we see first that at the origin the matrix is

$$\begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \tag{266}$$

with eigenvalues $1/2 \pm i$. Thus the origin is a spiral source. There are no other equilibria (why?)

Now we know the solution globally, by looking at the solution of (256) and/or its phase portrait. We note that $r = 1$ is a solution of (256), thus the unit circle

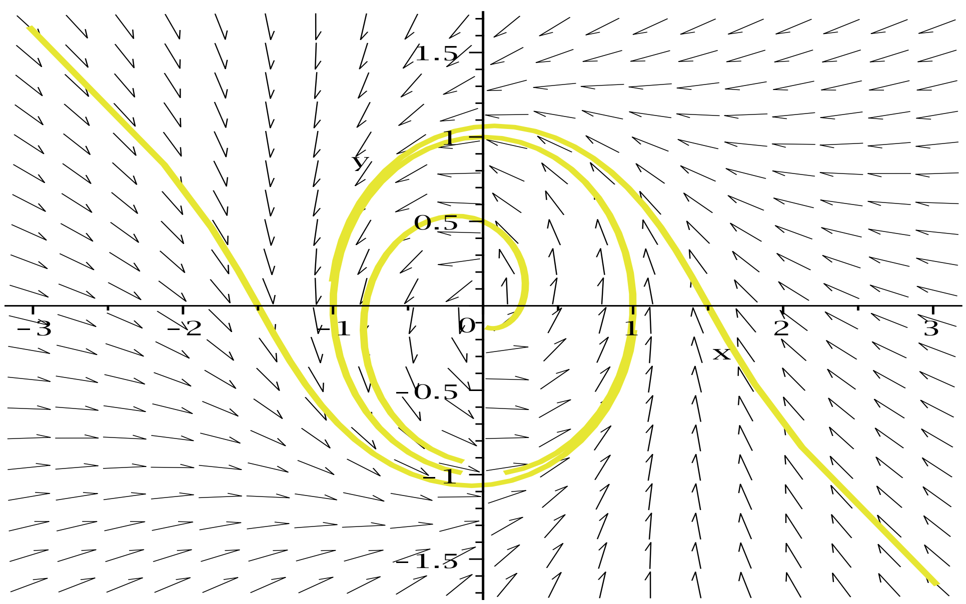


Figure 8: Phase portrait of (256)

is a trajectory of the system (264). It is a closed curve, all trajectories tend to it asymptotically. This is a limit cycle.

15.6 Application: constant real part, imaginary part of analytic functions

Assume for simplicity that f is entire. The transformation $z \to f(z)$ is associated with the planar transformation $(x, y) \to (u(x, y), v(x, y))$ where $f = u + iv$.

The grid $x = const, y = const$ is transformed into the grid $u = const, v = const$. We can first look at what this latter grid is transformed back into, by the transformation. The analysis is more general though, nothing below requires $u + iv$ to be analytic. We use only use this information to shortcut through some calculations.

We take first $v(x(t), y(t)) = const$. We have

$$\frac{\partial v}{\partial x}x'(t) + \frac{\partial v}{\partial y}y'(t) = 0 \tag{267}$$

which we can write, for instance, as the system

$$x' = \frac{\partial v}{\partial y} \tag{268}$$

$$y' = -\frac{\partial v}{\partial x} \tag{269}$$

which, in particular, is a Hamiltonian system. We have a similar system for v. We can draw the curves $u = const, v = const$ either by solving this implicit equation, or by analyzing (268), or even better, by combining the information from both. Let's take, for example $f(z) = z^3 - 3z^2$. Then, $v = 3x^2y - y^3 - 6xy$. It would be rather awkward to solve $v = c$ for either x or y. The system of equations reads

$$x' = -6x + 3x^2 - 3y^2 \tag{270}$$

$$y' = 6y - 6xy \tag{271}$$

Note that $\nabla u = 0$ or $\nabla v = 0$ are equivalent to $z' = 0$. For equilibria, we thus solve $3z^2 - 6z = 0$ which gives $z = 0; z = 2$. Near $z = 0$ we have

$$x' = -6x + o(x, y) \tag{272}$$

$$y' = 6y + o(x, y) \tag{273}$$

which is clearly a saddle point, with x the stable direction and y the unstable one. At $x = 2, y = 0$ we have, denoting $x = 2 + s$,

$$s' = 6s + o(s, y) \tag{274}$$

$$y' = -6y + o(s, y) \tag{275}$$

another saddle, where now $y = 0$ is the unstable direction. We note that $y = 0$ is, in fact, a special trajectory, and it is in the nonlinear unstable/stable manifold at the equilibrium points. Note also that a nonlinear stable manifold exists locally. In this case it changes character as it happens to pass through another equilibrium.

We draw the phase portraits near $x = 0$, near $x = 2$, mark the special trajectory, and look at the behavior of the phase portrait at infinity. Then we "link" smoothly the phase portraits at the special points, and this should suffice for having the phase portrait of the whole system.

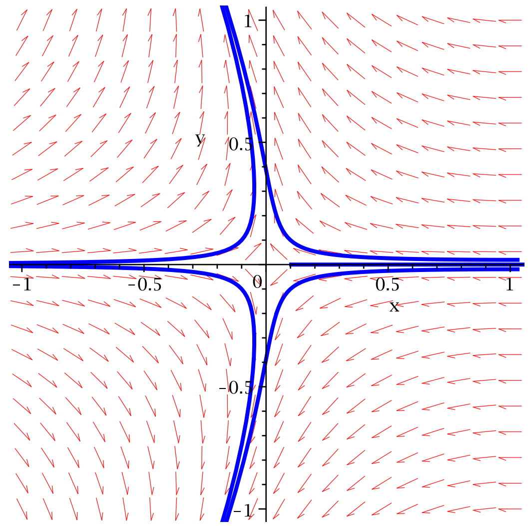


Figure 9: Phase portrait of (270) near $(0, 0)$.

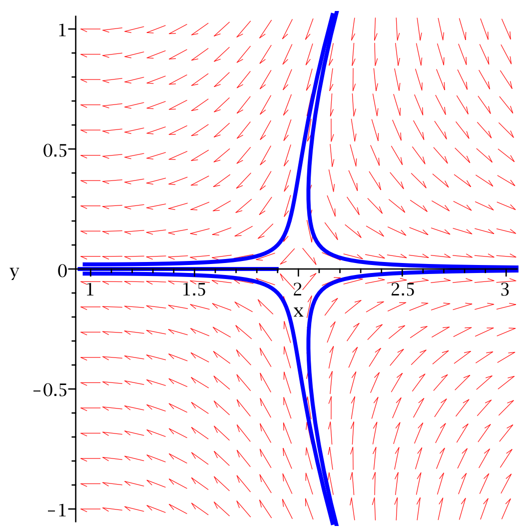


Figure 10: Phase portrait of (270) near $(0, 2)$.

For the behavior at infinity, we note that if we write

$$\frac{dy}{dx} = \frac{y(1 - 6x)}{-6x + 3x^2 - 3y^2} \tag{276}$$

we have the special solution $y = 0$, and otherwise the nonlinear terms dominate and we have

$$\frac{dy}{dx} \approx \frac{-6yx}{3x^2 - 3y^2} \tag{277}$$

By homogeneity, we look for special solutions of the form $y = ax$ (which would be asymptotes for the various branches of $y(x)$. We get, to leading order,

$$a = \frac{-6a}{3 - 3a^2} \tag{278}$$

We obtain

$$a = 0, a = \pm\sqrt{3} \tag{279}$$

We also see that, if $x = o(y)$, then $y' = o(1)$ as well. This would give us information about the whole phase portrait, at least qualitatively.

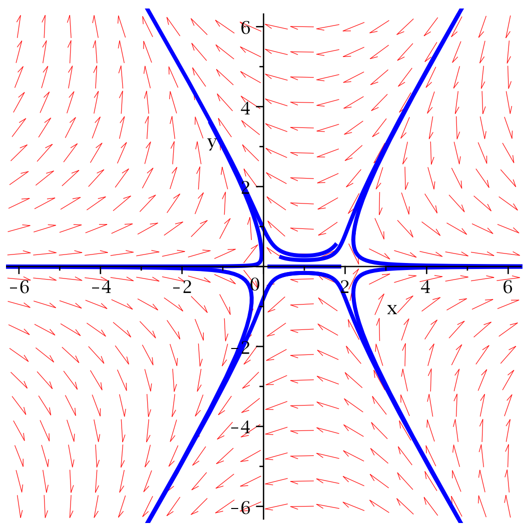


Figure 11: Phase portrait of (270), $v = const$.

Exercise 1. Analyze the phase portrait of $u(x, y) = const$.

The two phase portraits, plotted together give Note how the fields intersect at right angles, except at the saddle points. The reason, of course, is that $f(z)$ is a conformal mapping wherever $f' \neq 0$.

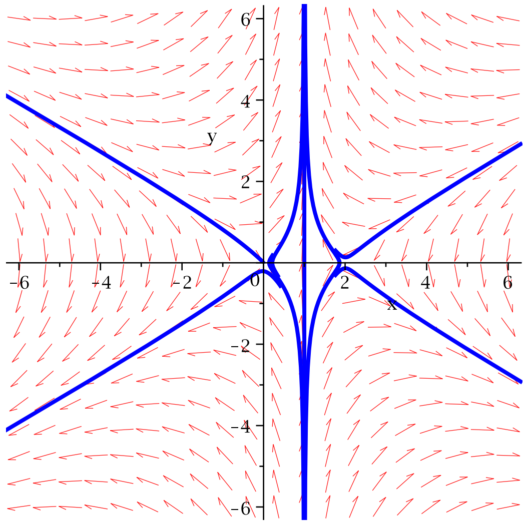


Figure 12: Phase portrait of $u = const$

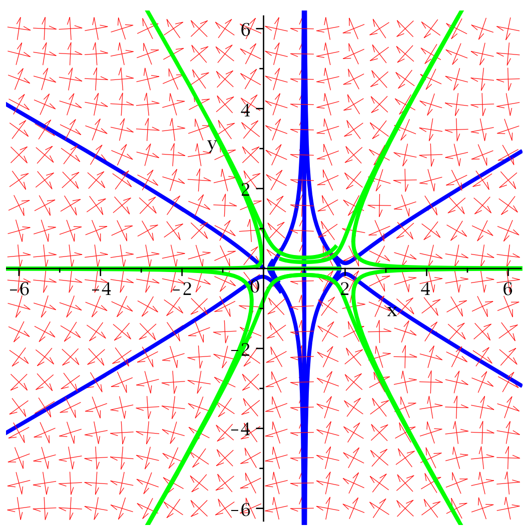


Figure 13: Phase portrait of $u = const$, and $v = const$.

Exercise 2. Draw the global phase portrait of the approximations of the pendulum, $x'' + x - x^3/6 = 0$, $x'' + x - x^3/3 + x^5/5 = 0$. Find the equilibria, local and global behavior. Find out if there are limit cycles. Discuss the conection sith the physical pendulum, $x'' + sin(x) = 0$.

Exercise 3. Draw the global phase portrait of the damped pendulum, $x'' +$

$ax' + \sin x = 0$, where $a > 0$ is the air friction coefficient. Discuss what happens as $a \to 0$ and how this relates to the undamped pendulum, $a = 0$. Discuss also the bifurcation that occurs at $a = 0$ and $x = 0$ (for $a < 0$, the physical interpretation could be that we are looking backwards in time. Also, note any global bifurcations, that is changes in the global topology.

16 Bifurcations

Here, we will speak of local and global bifurcations. Bifurcations occur when a change in a parameter induces topological changes in the phase portrait. These can be local, global (or both).

Local ones refer to the situations when a parameter crosses a value where the stability of a local equilibrium (or coalescing ones) changes.

As we know, the phase portrait of a system depending on a parameter changes structure near an equilibrium only if at least one eigenvalue becomes purely imaginary. This may happen if one eigenvalue (or both, but generically one) becomes zero, or else they pass through a point where they are nonzero, imaginary and complex conjugate to each other (since we are dealing with real-valued equations).

The classification is made by looking at the *normal form.* That is, we discard higher order nonlinearities, keeping the ones that become dominant when the bifurcation occurs. (When one eigenvalue becomes zero or imaginary, the first nonzero nonlinear term takes over, at that point, to control the local behavior).

We will deal later with theorems showing that, under suitable conditions, a nonlinear system is equivalent to this normal form, insofar as the topology of the system is concerned.

For instance,

$$x' = x^2 + a; \quad y' = -y$$

will represent a general system of the form

$$x' = a + bx^2 + O(x^3); \quad y' = -y + O(y^2)$$

16.1 Some types of bifurcations

We will study the following types of normal forms:

$$x' = x^2 + a; \quad y' = -y; \quad \textbf{saddle-node bifurcation}$$

Here, for $a < 0$ there are two equilibria that collide when $a = 0$, while there is no equilibrium when $a > 0$.

$$x' = rx - x^2; \quad y' = -y; \quad \textbf{transcritical bifurcation}$$

Two equilibria collide, and after the collision we still have two equilibria.

$$x' = rx - x^3; \quad y' = -y; \quad \textbf{supercritical pitchfork bifurcation}$$

One unstable equilibrium ($r < 0$) bifurcates into an unstable one and two stable ones.

$$x' = rx + x^3; \quad y' = -y; \quad \textbf{subcritical pitchfork bifurcation}$$

We see that in the pitchfork bifurcation, the quadratic term is absent (in practice, due to some symmetry of the system).

Of course, in these types of equilibria, much of the information is contained in the x part, and we may to some extent ignore the y one; the equations are decoupled.

$$x' = \beta x - y + \sigma x(x^2 + y^2); \quad y' = x + \beta y + \sigma y(x^2 + y^2) \ \textbf{Hopf bifurcation}$$

where $\beta = 0$ is the bifurcation point. Here, eigenvalues become imaginary, but nonzero.

16.2 Normal form of the saddle-node bifurcation

Consider a simple system which illustrates the first case, an eigenvalue going through zero, prototypical for *saddle-node bifurcations*,

$$x' = x^2 + a \tag{280}$$

$$y' = -y \tag{281}$$

Of course, we can solve this explicitly, but we choose not to, because solvable equations are infrequent. We first note that the only possible equilibria are $(\pm\sqrt{-a}, 0)$. Clearly, there are two of them if $a < 0$, one if $a = 0$ and none if $a > 0$. For $a = 0$, the equilibrium is non-hyperbolic and needs to be studied separately. For $a < 0$, at $x = \pm a$, we see that the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \pm 2\sqrt{-a} & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{282}$$

Thus the point $(-\sqrt{-a}, 0)$ is a node, while $(\sqrt{-a}, 0)$ is a saddle.

Let's draw the complete phase portrait in the three regimes, $a < 0$, $a = 0$ and $a > 0$. Again, the portrait is determined by the set of equilibria, limit cycles, and by the behavior at infinity. The three lines $x = \pm a$ and $y = 0$ are special solutions of the system. We see that there are no limit cycles, since trajectories do not cross except at the equilibria, and the lines $x = \pm a$, never crossed, delimit regions where the sign of x' is constant.

Behavior at infinity: for x very large, we have $x' \approx x^2$ and $y' = -y$, and thus

$$\frac{dy}{dx} \approx \frac{-y}{x^2} \tag{283}$$

with the solution $y = Ce^{1/x}$. Thus in the far x field, the trajectories are expected to approach horizontal lines. How do we prove this rigorously? One way is to note that for any $\alpha > 1$ $(x^2 - a)^{-1} \le \alpha x^{-2}$ if x is large enough.

Thus, we can write

$$\frac{y'(x)}{y(x)} \geq -\frac{\alpha}{x^2} \tag{284}$$

where we can integrate both sides and get

$$y(x) \geq C_{x_0,y_0} e^{\alpha/x} \tag{285}$$

where C_{x_0,y_0} is a constant depending on the initial condition (x_0, y_0). Similarly,

$$y(x) \leq C_{x_0,y_0} e^{\alpha'/x} \tag{286}$$

If instead x is bounded and $y \to \infty$, the direction field points straight to the origin, so there the trajectories essentially vertical lines. Piecing all this together, we get the phase portrait depicted below.

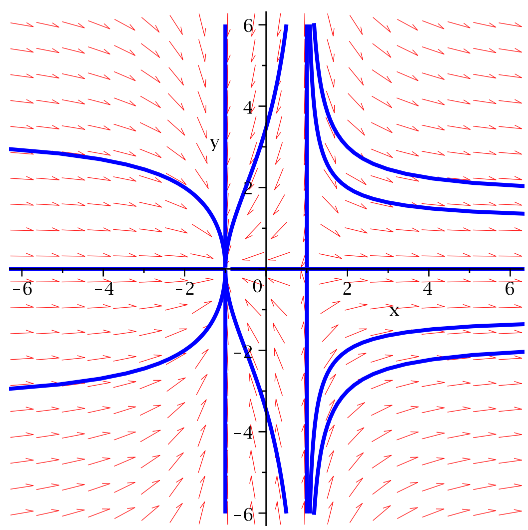


Figure 14: Phase portrait of (287) for $a = -1$

For $a = 0$ the system simply becomes

$$x' = x^2 \tag{287}$$

$$y' = -y \tag{288}$$

Clearly, the line $x = 0$, a special solution, is attracting, while the line $y = 0$ is repelling for $x > 0$ and attracting (since the field points towards the origin) for $x < 0$. So we see that, in some sense, the origin is now half-node, half saddle. All nearby trajectories are attracted to zero if they start in the closed left half plane, and repelled otherwise.

The far-field picture is clearly the same as in the case $a < 0$, so we can piece together these informations to draw the phase portrait. We note that in this

[h!]

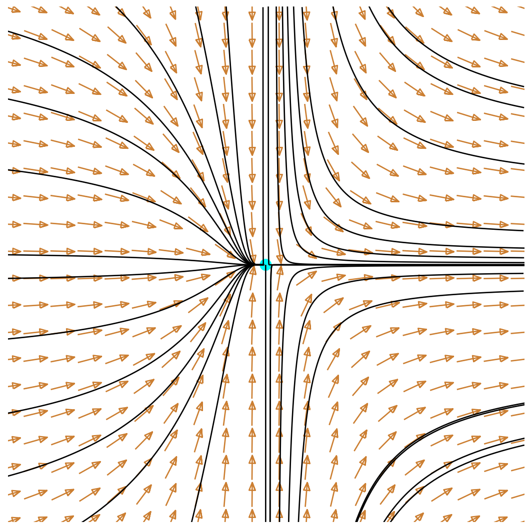


Figure 15: Phase portrait of (287) for $a = 0$

case, of course, the explicit solutions $y = Ce^{1/x}$, $C \in \mathbb{R}$ and $x = 0$, can be easily used to draw the phase portrait. This type of behavior as a function of a, at least in this example, explains the choice of name, saddle-node bifurcation.

Finally, for $a > 0$ there are no equilibria. We see that $x' > 0$ for all x. Trajectories extend from $-\infty$ to $+\infty$ in x. The behavior in the far field is the same as in the previous examples. The trajectories have horizontal lines as asymptotes for $x \to \pm\infty$ and, in the upper half plane, the asymptote for $x < 0$ lies above the one for $x > 0$, since $y' < 0$ there. We can now draw the phase portrait.

As we see, the node in the left half plane approaches the saddle, touches it at which time we have a half-node half-saddle picture, and then the equilibrium vanishes and the curves in the left half plane "spill over" in the right half plane.

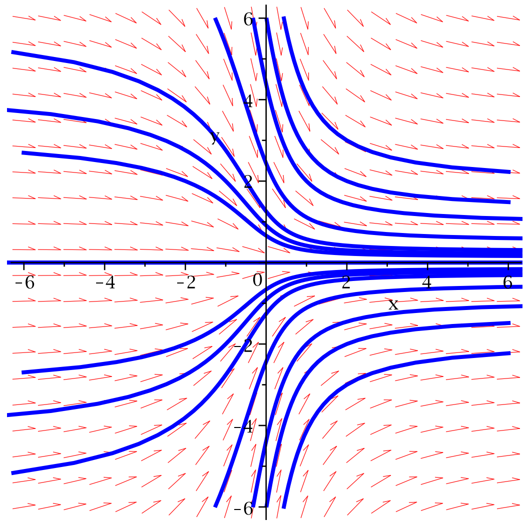


Figure 16: Phase portrait of (287) for $a > 0$

16.3 Transcritical bifurcation

Typically, for this and some other bifurcations, the y part is ignored, and for a good reason, as we mentioned there is effectively no y participation. The reason for which this type of bifurcation is called transcritical is from the way things look as a function of the parameter for the x-only system. To have however a unified picture in mind, and to recall that we are after all dealing with two dimensional systems for which it does *happen* that the normal form makes y "idle" we will look at the two dimensional system,

$$\begin{aligned} x' &= rx - x^2 \\ y' &= -y \end{aligned} \tag{289}$$

For $r \neq 0$ there are two equilibria, and for $r = 0$ only one; the two equilibria collide as before, but the outcome is different.

Take $r < 0$. Clearly, the origin, marked in blue, is a node. The other equilibrium, r is a saddle ($r - 2r = -r > 0$).

The global picture is obtained as before: the rays: $\{(-t, 0) : t < r\}$, $\{(t, 0) : t \in (r, 0)\}$,$\{(t, 0) : t > 0\}$, $\{(0, \pm t^2) : t > 0\}$ are special trajectories; in the far field, the trajectories are almost horizontal.

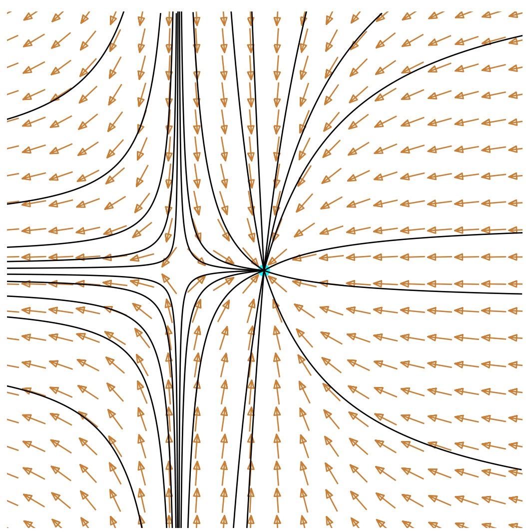


Figure 17: Transcritical phase portrait, $r < 0$

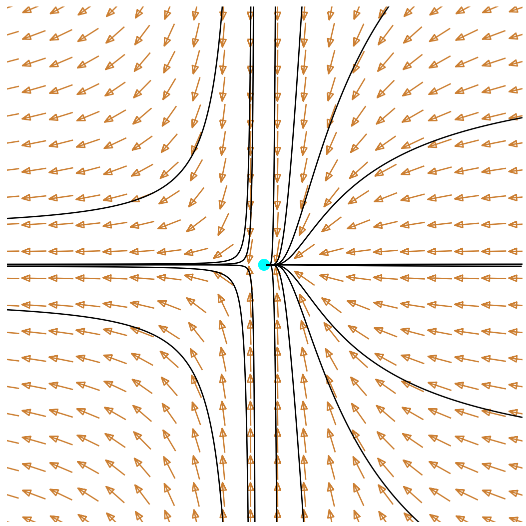


Figure 18: Transcritical phase portrait, $r = 0$

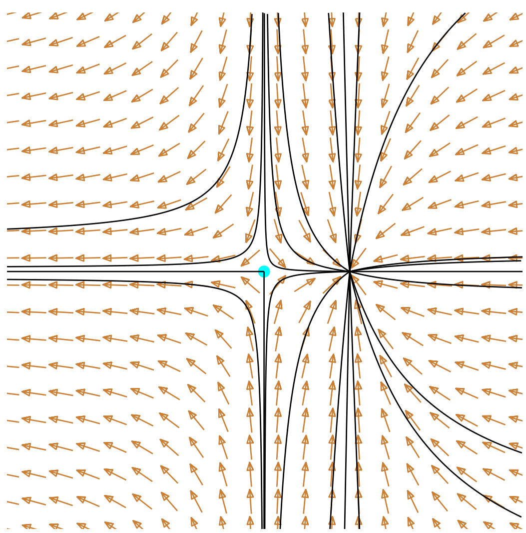


Figure 19: Transcritical phase portrait, $r > 0$

Here, we see that a saddle-node becomes a "half saddle-half node" and then it becomes a node-saddle. The types of equilibria are interchanged.

16.4 Normal form of the pitchfork bifurcation

Here we are dealing with a system with symmetries, so that the normal form is

$$\begin{aligned} x' &= rx - x^3 \\ y' &= -y \end{aligned} \tag{290}$$

The field is an odd function of x and y, and stays odd for all (or only small, maybe) values of r. The name "pitchfork" will become clear in a moment.

In case 1) $r > 0$, we have three equilibria, $x = 0$ and $x = \pm\sqrt{r}$.

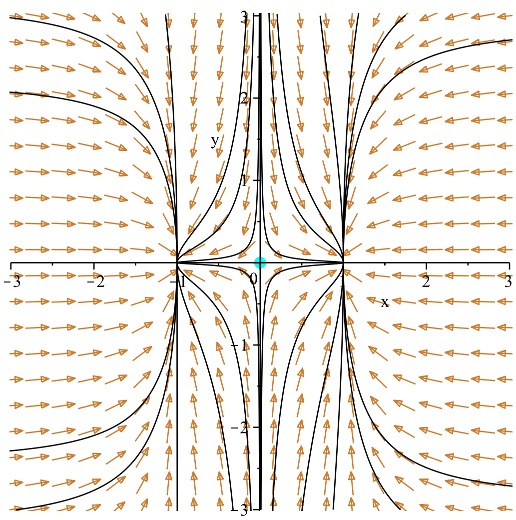


Figure 20: Pitchfork phase portrait, $r < 0$

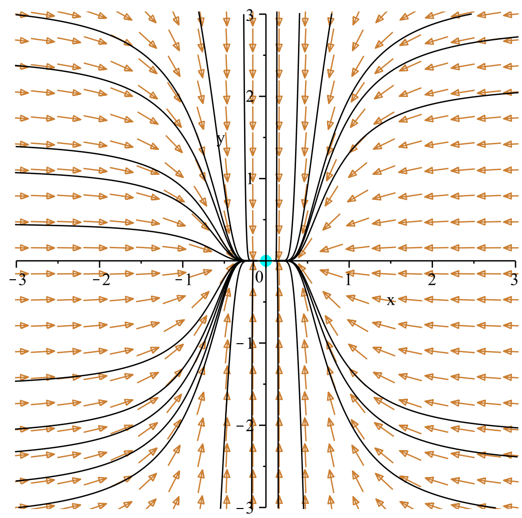


Figure 21: Pitchfork phase portrait, $r = 0$

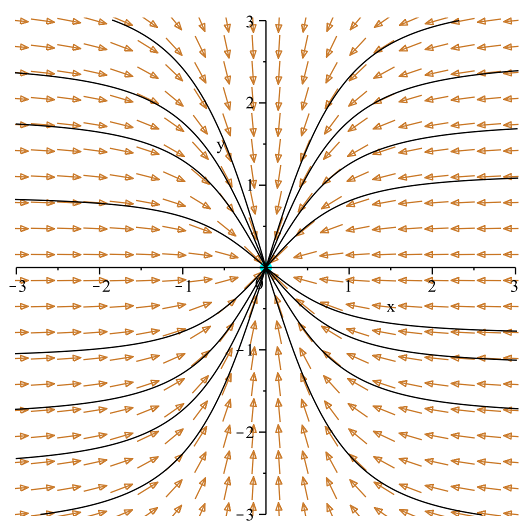


Figure 22: Pitchfork phase portrait, $r > 0$

It is clear that the origin is a saddle whereas the other two equilibria, symmetric, are nodes (sinks).

If, 2), $r = 0$, clearly we only have one equilibrium, and it is a node because $-x^3$ always points towards the origin.

By explicit solution, we see that the trajectories are given by $y = Ce^{-1/(2x^2)}$, which explains the fact that the phase portrait almost seems to have a continuum of nodes near zero.

Finally, in case 1) $r < 0$, we have only one equilibrium and it is a node. the number of equilibria changes.

Note that we can extend artificially the number of variables, to transform the two dimensional parameter-dependent problem into a three-dimensional parameter-free one,

$$x' = rx - x^3 \tag{291}$$

$$y' = -y \tag{292}$$

$$r' = 0$$

Clearly now the change in behavior is seen as a chenge in the 3d phase portrait, as a function of the initial condition in r.

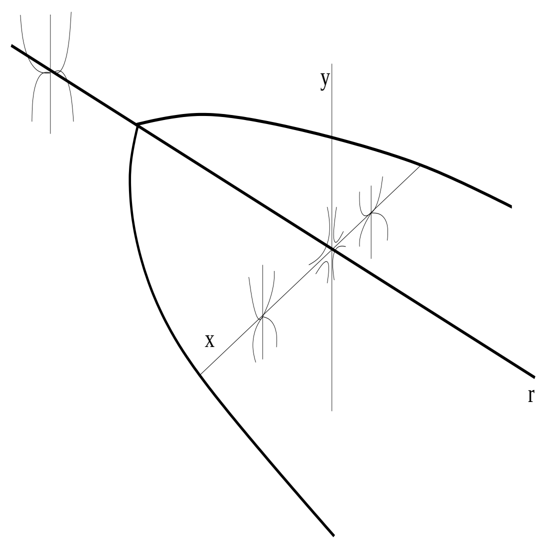


Figure 23: Pitchfork 3d phase portrait.

16.5 Normal form of the Hopf bifurcation

We will In this case, we are looking at a system for which passage through the critical value of the parameter (β) implies nonzero, purely imaginary eigenvalues. Take first $\sigma = -1$:

$$x' = \beta x - y - x(x^2 + y^2) \tag{293}$$

$$y' = x + \beta y - y(x^2 + y^2) \tag{294}$$

The origin is an equilibrium for all β, and it is the only one if we refer to (296, where $\theta' > 0$. At the origin, the linearized system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = B \begin{pmatrix} x \\ y \end{pmatrix}; \quad B = \begin{pmatrix} \beta & -1 \\ 1 & \beta \end{pmatrix} \tag{295}$$

where the equation for the eigenvalues of the matrix B is $(\beta - \lambda)^2 + 1 = 0$, and thus $\lambda_{\pm} = \beta \pm i$. At $\beta \neq 0$ the equilibrium is hyperbolic, a spiral sink if $\beta < 0$ and a spiral source for $\beta > 0$. A change of phase portrait, a bifurcation, should occur at $\beta = 0$.

For a simple analysis of the phase portrait, we rewrite the system in polar coordinates.

$$r' = \beta r - r^3 \tag{296}$$

$$\theta' = 1 \tag{297}$$

For $\beta < 0$, $\beta r - r^3$ has only one solution, $r = 0$. In (x, y), all solutions converge to the origin, while spiraling.

In the far field, we have

$$r' \approx -r^3 \tag{298}$$

$$\theta' = 1 \tag{299}$$

with solution

$$r = (2\theta + 2C)^{-1/2} \tag{300}$$

For r to be very large, we must have θ very close to $-C$. That is, asymptotically the curves in the far field (x, y) plane have radial lines as asymptotes. The spiraling ceases there.

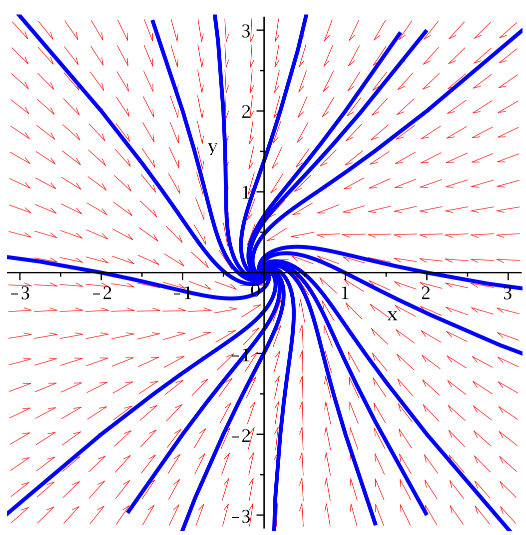


Figure 24: Phase portrait of (295) for $\beta = -1$

When $\beta = 0$, $r = 0$ is still the only solution of $\beta r - r^3 = 0$. Since again $r' < 0$, all trajectories go to the origin. But because as the origin is approached at at very small rate, $O(r^3)$, there is a lot of spiraling going on in that region. We see a tendency of a limit cycle being born.

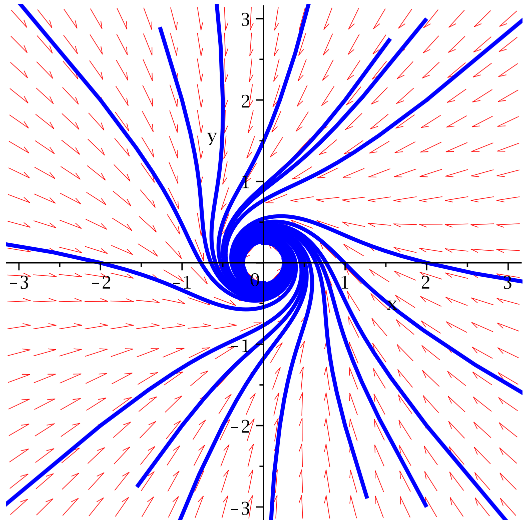


Figure 25: Phase portrait of (295) for $\beta = 0$

For $\beta > 0$, we have three solutions of $\beta r - r^3 = 0$, 0 and $\pm\sqrt{\beta}$ (tghe minus solution is "unphysical" for us. This means, in (x, y) that $x^2 + y^2 = \beta$ is *limit cycle*. We note that it approaches the origin as $\beta \to 0$. The spiral sink changes into a spiral source plus a limit cycle.

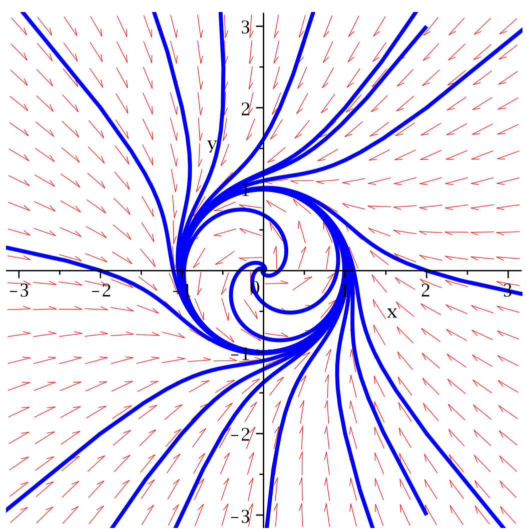


Figure 26: Phase portrait of (295) for $\beta = 1$

17 Bifurcations in more general systems. The central manifold theorem

Here we follow [4]. The setting is that of differential systems depending on a parameter,

$$x' = f_\mu(x), \quad x \in \mathbb{R}^n, \ \mu \in \mathbb{R}^m \tag{301}$$

where we assume sufficient smoothness of f. Equilibrium solutions are given by the (constant) solutions of the equation

$$f_\mu(x) = 0 \tag{302}$$

The equilibrium points depend smoothly on μ, by the implicit function theorem, as long as $D_x f_\mu$ is invertible, that is, as long as it has no zero eigenvalue. If $(\det D_x f_\mu)(x_0, \mu_0) = 0$, several branches of equilibria may form/disappear. These points (x_0, μ_0) are bifurcation points. For example, in the pitchfork bifurcation example, $\mu = r$ and $(0, 0)$ is the only bifurcation point.

We saw that the there equilibria merge into one at that point.

A crucial notion here is that of *transversality*. In one dimension, $y = f(x)$ crosses the x axis transversally at x_0 if $f(x_0) = 0$ and $f'(x_0) \neq 0$. In d dimensions, two manifolds intersect transversally if the tangent spaces at the intersection point span $\mathbb{R}^d$ (there is no loss in dimension). It is clear that transversal intersections are generic. In particular, two manifolds Σ_1 and Σ_2 of dimensions d_1 and d_2 intersect transversally along a manifold of dimension $d_1 + d_2 - d$. Equivalently, the codimension of $\Sigma_1 \cap \Sigma_2$ is $(d - d_1) + (d - d_2)$. Two surfaces in 3d intersect generically along a line, two generic curves do not intersect, and a curve and a manifold generically intersect at a point, etc.

For the vector field $\mu + x^2$ thought of as a family of curves in $\mathbb{R}$, the curve for $\mu = 0$ intersects the x axis non-transversally at $x = 0$.

However, if we lift the number of dimensions to include μ in the picture, we have a *transversal* intersection of the surface $F(x, \mu) = x^2 + \mu$ with the (x, μ) coordinate plane.

Also, it is clear that transversal intersections are stable in the following sense. If two manifolds intersect transversally, then any small perturbation of the manifolds will also have a transversal intersection. On the contrary, if two manifolds intersect non-transversally, then their generic perturbations will intersect transversally. Let f be a C^r vector field on $\mathbb{R}^n$ vanishing at the origin ($f(0) = 0$) and let $A = (DF)(0)$. We denote as usual by $\sigma_{u,c,s}$ the partso of the spectrum (eigenvalues) for which $\mathrm{Re}\lambda > 0, = 0, < 0$ respectively.

Denote the generalized eigenspaces of $\sigma_{u,c,s}$ by $E^{u,c,s}$ respectively. By definition, the stable manifold is a set invariant under the flow which is tangent to E^s, the unstable one is tangent E^u whereas the center manifold is tangent to E^c.

We remember that, in hyperbolic systems (for which therefore the center manifold is absent) the stable/unstable manifolds are unique.

We see that center manifolds need not be unique, on the simple example (287),

$$x' = x^2 \tag{303}$$

$$y' = -y \tag{304}$$

Clearly $(0,0)$ is a non-hyperbolic fixed point, with 0 eigenvalue in the x direction. We have a unique stable manifold at $(0,0)$: here we look for an invariant set tangent to the vertical axis, and in this case it is the vertical axis itself. How about sets tangent to the center direction, $x = 0$? See Figure 16.2. We can solve for the trajectories

$$\frac{dy}{dx} = -\frac{y}{x^2} \tag{305}$$

i.e, $y = Ce^{1/x}$. We see that there is no such trajectory for $x > 0$, but *for all* C, the trajectories in the left half plane are tangent to the real line. Any of these would be a center manifold.

Theorem 11 (Center manifold theorem for flows)**.** *There exist C^r stable and unstable manifolds (invariant under the flow and tangent to E^s, E^u) W^s and W^u respectively, and these are unique. There is a (generally nonunique) center manifold W^c, and it is C^{r-1}.*

Corollary 12. *We can take a set of local coordinates, $\tilde{x}, \tilde{y}, \tilde{z}$, corresponding to the local splitting $\mathbb{R}^d = W^c \times W^s \times W^u$, so that, topologically, the general system is equivalent to*

$$\tilde{x}' = \tilde{f}(\tilde{x}) \tag{306}$$

$$\tilde{y}' = -\tilde{y} \tag{307}$$

$$\tilde{z}' = \tilde{z} \tag{308}$$

Let us see how to determine this equivalent system, in the special case when W^s is empty. We bring the linear part at the equilibrium of our general system to the block diagonal form

$$x' = Cx + f(x, y) \tag{309}$$

$$y' = Hy + g(x, y) \tag{310}$$

where C is the part of the matrix whose eigenvalues have zero real part while H is the rest of the matrix, the "hyperbolic" part. The center manifold is tangent to E^c, and we can thus write it in locally in the form of the graph of a function, $y = h(x)$ (indeed, we write it first in the form $G(x, y) = C$ and we note that $\nabla G \perp \mathbf{e_1}$ and thus the implicit function theorem gives $y = h(x)$). Thus, we can substitute into (309) and get

$$x' = Cx + f(x, h(x)) \tag{311}$$

On the other hand, $h(x) = o(x)$ for small x, since it $Dh = 0$ there. Thus, we expect, and shall prove later, that the flow provided by (311)) is a good approximation of $\tilde{x}' = \tilde{f}(\tilde{x})$, which would evolve inside the center manifold. The following holds.

Theorem 12 (Henry, Carr). *If the origin $x = 0$ of* (311) *is locally asymptotically stable/unstable, then the origin of* (309) *is also locally asymptotically stable/unstable.*

17.1 The saddle-node bifurcation: general case

We follow [4]. We remember that the normal form we were aiming at was $a + x^2$, or, of course, more generally $\mu - \mu_0 \pm (x - x_0)^2$. Consider now the system (301), and assume that at $\mu = \mu_0, x = x_0$ there is an equilibrium in which one eigenvalue is zero and nondegenerate. The center manifold theorem would then allow us to reduce the study to the case where the system is one-dimensional. More precisely, there is a 2d center manifold Σ in $\mathbb{R}^n \times \mathbb{R}$ through (x_0, y_0) so that (1) Σ is tangent to the plane spanned by the 0 eigenvector and the direction of μ,

(2) For any r, Σ is C^r in a neighborhood of (x_0, y_0),

(3) The vector field of (301) is tangent to Σ,

and

(4) There is a neighborhood U of (x_0, y_0) in Σ which is invariant under the flow.

If we restrict (301) to Σ, we get a one-parameter family of equations on the one dimensional curves $\Sigma_\mu := \{z \in \Sigma : \mu = const =: \mu\}$. This is the reduction of the bifurcation problem. We now need to impose conditions that imply that the bifurcation type of this one-dimensional system is the same as that for the normal form $\mu - \mu_0 \pm (x - x_0)^2$. These are: $\frac{\partial f}{\partial \mu}(x_0, \mu_0) \neq 0$ (transversality in the μ direction), and $\frac{\partial^2 f}{\partial x^2}(x_0, \mu_0) \neq 0$, that is the equilibrium is quadratic.

More precisely, the following theorem holds.

Theorem 13. *Consider the setting above, under the following assumptions:*

(SN1) $M = D_x f(x_0, \mu_0)$ has a simple eigenvalue 0 with right eigenvector v and left eigenvector w ($wM = 0 \leftrightarrow M^T w = 0$). M has k eigenvalues with negative real parts and $(n - k - 1)$ with positive real parts.

(SN2) $w \cdot D_\mu f(x_0, \mu_0) \neq 0$.

(SN3) $w \cdot (v \cdot D_x^2 f(x_0, \mu_0) v) \neq 0$. (Note that $v \cdot D_x^2 f(x_0, \mu_0) v$ is a vector since f is a vector.)

Then there is a smooth curve of equilibria in $\mathbb{R}^n \times \mathbb{R}$ passing through (x_0, μ_0) and tangent to the hyperplane $\mathbb{R}^n \times \{\mu_0\}$. Depending on the signs in (SN1), (SN2) there are no equilibria near (x_0, μ_0) for $\mu < \mu_0$ ($\mu > \mu_0$ resp.). The two equilibria near (x_0, μ_0) are hyperbolic, and have stable manifolds of dimension k and $k + 1$, resp. The conditions (SN1) and (SN2) are generic, in the sense of forming an open dense set in the family of vector fields with an equilibrium with zero eigenvalue at (x_0, μ_0).

17.2 Transcritical and pitchfork bifurcations

We need appropriate changes in the assumptions. They are natural, if you think of the shape of the normal form:

(A) Transcritical bifurcation. Here we must have $f_\mu(0) = 0$ for all μ, and thus $D_\mu f$ cannot be nonzero anymore. This condition is replaced by (SN2') $w \cdot (\partial^2 f/\partial\mu\partial x)v \neq 0$ at $\mu = \mu_0$.

(B) Pitchfork bifurcation (one dimension). Here we are dealing with systems with symmetry in which f is odd. Thus, now we cannot have $D_x^2 f \neq 0$. Then, (SN3) is replaced by (SN3'), $D_x^3 f \neq 0$

Under these assumptions a theorem similar to the one in the previous section holds.

17.3 Hopf bifurcations

Consider now a system of the form (301) for which, at some (x_0, y_0) $D_x f$ has exactly one pair of nonzero imaginary eigenvalues, and the systems is hyperbolic otherwise, near (x_0, y_0). Then, by the implicit function theorem, the equilibrium position varies smoothly with μ, unlike in most other bifurcations. We expect however, by looking at what we called the normal form, a qualitative change in the structure of the equilibrium to occur at μ_0: a spiral sink is transformed into a spiral source plus a limit cycle.

By changes of variables (straightforward but rather lengthy [4]), the block affected by the bifurcation can be brought to the form

$$x' = (d\mu + a(x^2 + y^2)x - (\omega + c\mu + b(x^2 + y^2))y + \text{higher order terms} \quad (312)$$

$$y' = (\omega + c\mu + b(x^2 + y^2)x) + (d\mu + a(x^2 + y^2))y + \text{higher order terms} \quad (313)$$

(essentially, the quadratic terms can be eliminated). If we momentarily discard the higher order terms, this takes the following form in polar coordinates

$$r' = (d\mu + ar^2)r \quad (314)$$

$$\theta' = (\omega + c\mu + br^2) \quad (315)$$

The phase portrait of (314) does not differ substantially from the one we used before, where br^2 was missing. If a, d are nonzero, then there are periodic orbits of the (x, y) system lying along the parabola $\mu = -ar^2/d$; the surface of periodic orbits has quadratic tangency with the plane $\mu = 0$ in $\mathbb{R}^2 \times \mathbb{R}$.

The Hopf bifurcation theorem essentially says that the higher order terms do not change this picture locally.

Theorem 14 (Hopf, 1942)**.** *Suppose that the system $x' = f_\mu(x)$, $(x, \mu) \in \mathbb{R}^n \times \mathbb{R}$, has an equilibrium at (x_0, μ_0) so that the following properties are satisfied.*

(H1) $D_x f(\mu_0, x_0)$ has a unique pair of purely imaginary nonzero eigenvalues.

Then, there exists a smooth curve of equilibria $(x(\mu), \mu)$ with $x(\mu_0) = x_0$. The two eigenvalues which are imaginary at (x_0, μ_0), $\lambda(\mu)$ and $\overline{\lambda}(\mu)$ vary smoothly with μ.

Assume furthermore that

$$\frac{d}{d\mu}(\mathrm{Re}(\lambda(\mu))\Big|_{\mu=\mu_0} = d \neq 0 \quad (316)$$

Then, there exists a unique three dimensional center manifold passing through (x_0, μ_0) *in* $\mathbb{R}^n \times \mathbb{R}$*, and a smooth change of coordinates preserving the planes* $\mu = const.$ *for which the Taylor expansion on the center manifold is given by* (312). *If* $a \neq 0$*, then there is a surface of periodic solutions in the center manifold which has quadratic tangency with the eigenspace of* $\lambda(\mu_0)$ *and* $\overline{\lambda}(\mu)$ *agreeing to second order with the paraboloid* $\mu = -(a/d)(x^2 + y^2)$*. If* $a < 0$*, the periodic solutions are repelling.*

18 The Poincaré Bendixson Theorem

Assumptions. Here we state, without proof for now, a few results related to the possible topology of trajectories. The proofs are neither short nor very easy, and we will leave these for later. Consider the autonomous two dimensional system

$$x' = f_1(x, y) \tag{317}$$

$$y' = f_2(x, y) \tag{318}$$

We assume that f_1, f_2 are continuous in a domain $D \in \mathbb{R}^2$. A regular point $P \in D$ of (316) is a point where existence and uniqueness are ensured. We have a studied a number of sufficient conditions on f for this to be the case. In the following, K will be a compact set in D. The theorem in a sense classifies the possible topological behavior of trajectories of planar systems.

18.1 Semiorbits, orbits, limit sets, limit cycles, periodic orbits

A full orbit C is a solution $P(t) = (x(t), y(t))$ defined on the whole of $\mathbb{R}$. A positive semiorbit C^+ is a solution defined for all $t \geq 0$ and correspondingly, a negative semiorbit C^- is a solution defined for all $t \leq 0$. Clearly, a full orbit is the union of a positive and a negative semiorbit.

A limit set $L(C^+)$ of the semiorbit C^+ is the set of all its limit points, that is the set of all $X = (x, y)$ such that $X = \lim_{n\to\infty}(x(t_n), y(t_n))$ for some sequence $\{t_n\}_{n\in\mathbb{N}}$ ($L(C^-)$ is defined similarly). For instance, we see that the circle in Fig. 26 is a limit set for all trajectories other than $(0, 0)$.

ω **and** α **sets.** A more common terminology nowadays is the following. Consider the flow $\Phi(t, P)$ associated with (316). This is nothing more than the solution of (316) with initial condition P.

The ω set of P is exactly the set of all $X = (x, y)$ such that $X = \lim_{n\to\infty}(x(t_n), y(t_n))$ for some sequence $\{t_n\}_{n\in\mathbb{N}}$. This can be empty if the trajectory fails to exist for all $t \geq 0$ or if it goes to infinity. Similarly, α sets are defined in the limit $t \to -\infty$.

Periodic orbit A periodic orbit is a solution $X(t) = (x(t), y(t))$ with the property that there is some $T > 0$ so that $X(t + T) = X(t)$ for all t.

Limit cycle A periodic orbit is a limit cycle if it is the ω or α set of some point not belonging to it.

The question addressed by the Poincaré-Bendixson theorem is the nature of the limit set of a semiorbit which lies in a compact (i.e., bounded) region.

First a few intermediate results.

Proposition 13. *If C^+ is a positive semiorbit contained in a compact set K, then $L(C^+)$ is nonempty, closed and connected.*

Proposition 14. *Let C^+ be a positive semiorbit contained in a compact set K, and assume the point $P = (x, y) \in L(C^+)$ is a regular (nonsingular) point. Then the unique orbit C through P is a full orbit, and $C \subset L(C^+)$.*

A periodic orbit is a solution such that for some T and all t we have $P(t + T) = P(t)$. A limit cycle is a periodic orbit of the form $L(C^+)$ where $C^+ \neq L(C^+)$, that is, $L(C^+)$ is a nontrivial limit set.

Theorem 15 (Poincaré-Bendixson)**.** *Let C^+ be a positive semiorbit contained in K. If all points on $L(C^+)$ are regular, then $L(C^+)$ is a periodic orbit.*

ω set formulation

Theorem 16 (Poincaré-Bendixson)**.** *Assume $\omega(P) \neq \emptyset$ is compact, connected and contains finitely many equilibria. Then there are only three possibilities:*

(i) $\omega(P)$ is an equilibrium (thus P is an equilibrium).

(ii) $\omega(P)$ is periodic orbit consisting in regular points.

(iii) $\omega(P)$ consists of finitely many equilibria $\{x_j\}$ and non-closed orbits $\mathcal{C}(P')$ such that $\omega(P') \in \{x_j\}$ and $\alpha(P') \in \{x_j\}$.

Corollary 15. *Let $\mathcal{C}$ be a closed orbit that forms the boundary of an open set. Then the open set contains at least one equilibrium inside.*

Corollary 16. *Assume (316) has a first integral. If this first integral is not exactly constant on any open set, then (316) has no limit cycles.*

19 Appendix

19.1 Solution to Exercise 3

The equation for Y_k is

$$kY_k + (Y_kJ - JY_k) = R_k + \sum_{j=1}^{k-1} Y_j A_{k-j-1}; \quad R_k = A_{k-1}J \tag{319}$$

We consider the family of Banach spaces indexed by $\mu > 0$,

$$\mathcal{B}_\mu = \{\mathsf{Y} = (Y_l)_{l\in\mathbb{N}} : |||\mathsf{Y}|||_\mu := \sup_{j\in\mathbb{N}} \mu^{-j}\|Y_j\| < \infty\}$$

Note that, since $A(z)$ is analytic, the series $\sum_{l\in\mathbb{N}} A_l z^l$ converges, implying that, for some $C > 0$,

$$\sup_{j\in\mathbb{N}} \|A_j C^{-j}\| < \infty \tag{320}$$

Thus the vector $\mathsf{R} := (R_l)_{l\in\mathbb{N}}$ is in $\mathcal{B}_\mu$ for all $\mu > C$.

The function $\mathcal{C}$ given by $\mathcal{C}X = XJ - JX$ is evidently a linear function on $\mathbb{C}^{n^2}$, thus given by a matrix; since $\|\mathcal{C}X\| \le 2\|J\|\|X\|$ by the triangle inequality, its norm is bounded by

$$\|\mathcal{C}\| \le 2\|J\| \tag{321}$$

The function M_k given by

$$M_k X =: kX + \mathcal{C}X$$

is a linear function on $\mathbb{C}^{n^2}$, and thus it is also given by a matrix. We have shown that M_k is invertible, since $M_k X = 0 \Leftrightarrow X = 0$. Thus, for every k, M_k^{-1} exists (and evidently has finite norm).

We now also note that, if $k > 2\|J\|$ we have

$$\|M_k\|^{-1} \le \frac{1}{k - 2\|J\|} \tag{322}$$

Indeed,

$$M_k^{-1} = k^{-1}(1 - k^{-1}\mathcal{C})^{-1} \tag{323}$$

Thus the series

$$\sum_{l=0}^{\infty} \mathcal{C}^l / k^l \tag{324}$$

converges for all $k > 2\|J\|$. This is called a Neumann series, and you can check that it converges to $(1 - k^{-1}\mathcal{C})^{-1}$.

Thus,

$$\|(1 - k^{-1}\mathcal{C})^{-1}\| \le \sum_{l=0}^{\infty} k^{-l}(2\|J\|)^l = \frac{1}{1 - 2k^{-1}\|J\|} \tag{325}$$

and (321) follows.

Therefore,

$$\begin{aligned} \sup_{k\in\mathbb{N}} \|M_k^{-1}\| &= \max\{\max_{k\le 2\|J\|+1} \|M_k^{-1}\|, \sup_{k\ge 2\|J\|+2} (1 - 2k^{-1}\|J\|)^{-1}\} \\ &= \max\{\max_{k\le 2\|J\|+1} \|M_k^{-1}\|, 1/2\} = a_1 < \infty \end{aligned} \tag{326}$$

Then the operator $\hat{\mathsf{T}}$ defined by

$$(\hat{\mathsf{T}}\mathsf{Y})_j = M_j^{-1} Y_j \tag{327}$$

is bounded in $\mathcal{B}_\mu$, and

$$\|\hat{\mathsf{T}}\| = a_1 \tag{328}$$

We define the (linear) operator $\hat{\mathsf{L}}$ on $\mathcal{B}_\mu$, $\mu > C$, by

$$(\hat{\mathsf{L}}\mathsf{Y})_j = \sum_{j=1}^{k-1} Y_j A_{k-j-1}; \quad j \geq 1 \tag{329}$$

This is well defined on $\mathcal{B}_\mu$ and

$$\|\hat{\mathsf{L}}\| \leq \frac{1}{\mu - C} \tag{330}$$

Indeed, since $\|Y\|_j \leq \mu^j |||Y||| =: N\mu^j$, we have

$$\|\sum_{j=1}^{k-1} Y_j A_{k-j-1}\| \leq N \sum_{j=1}^{k-1} \mu^j C^{k-j-1} \leq NC^{k-1} \frac{\mu^k}{C^k(\mu/C - 1)} = \frac{N\mu^k}{\mu - C} \tag{331}$$

Now, the system (318) can be written compactly as

$$\mathsf{Y} = \hat{\mathsf{T}}\mathsf{A} + \hat{\mathsf{T}}\hat{\mathsf{L}}\mathsf{Y} \tag{332}$$

This is a linear nonhomogeneous equation for Y. For it to be contractive, we need $\|\hat{\mathsf{T}}\hat{\mathsf{L}}\| \leq \|\hat{\mathsf{T}}\| \, \|\hat{\mathsf{L}}\| < 1$.

This is the case if

$$\frac{a_1}{\mu - C} < 1 \tag{333}$$

i.e., if $\mu > \mu_1 = C + a_1$. Thus $\mathsf{Y} \in \mathcal{B}_{\mu_1}$, implying that $\|Y_j\| \leq N\mu_1^j$ for some N and all j, and therefore the series

$$\sum_{j=1}^{\infty} Y_j z^j \tag{334}$$

converges (obviously to an analytic function) for $|z| < 1/\mu_1$, and therefore $Y(z)$ is analytic at zero as required.

19.2 Solution to Exercise 1

The definition of z^{aP} is $\exp(aP \ln z)$ Now, since $P^2 = P$ we have

$$\exp(P \ln z) = I + \sum_{k=1}^{\infty} (a \ln z)^k P^k = I + P \sum_{k=1}^{\infty} (a \ln z)^k / k!$$
$$= I + P(z^a - 1) = Pz^a + (I - P) \tag{335}$$

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