1 Gradient and Hamiltonian systems

1.1 Gradient systems

These are quite special systems of ODEs, Hamiltonian ones arising in conservative classical mechanics, and gradient systems, in some ways related to them, arise in a number of applications. They are certainly nongeneric, but in view of their origin, they are common.

A system of the form

$$X' = -\nabla V(X) \tag{1}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is, say, C^∞, is called, for obvious reasons, a gradient system. A critical point of V is a point where $\nabla V = 0$.

These systems have special properties, easy to derive.

Theorem 1. *For the system (1), if V is smooth, we have (i) If c is a regular point of V, then the vector field is perpendicular to the level hypersurface $V^{-1}(c)$ along $V^{-1}(c)$.*

(ii) A point is critical for V iff it is critical for (1).

(iii) At any equilibrium, the eigenvalues of the linearized system are real.

More properties, related to stability, will be discussed in that context.

Proof. □

(i) It is known that the gradient is orthogonal to level surface.

(ii) This is clear essentially by definition.

(iii) The linearization matrix elements are $a_{ij} = -V_{x_i,x_j}$ (the subscript notation of differentiation is used). Since V is smooth, we have $a_{ij} = a_{ji}$, and all eigenvalues are real.

1.2 Hamiltonian systems

If $\mathbf{F}$ is a conservative field, then $\mathbf{F} = -\nabla V$ and the Newtonian equations of motion (the mass is normalized to one) are

$$q' = p \tag{2}$$

$$p' = -\nabla V \tag{3}$$

where $q \in \mathbb{R}^n$ is the position and $p \in \mathbb{R}^n$ is the momentum. That is

$$q' = \frac{\partial H}{\partial p} \tag{4}$$

$$p' = -\frac{\partial H}{\partial q} \tag{5}$$

where

$$H = \frac{p^2}{2} + V(q) \tag{6}$$

is the Hamiltonian. In general, the motion can take place on a manifold, and then, by coordinate changes, H becomes a more general function of q and p. The coordinates q are called generalized positions, and q are the called generalized momenta; they are canonical coordinates on the phase on the cotangent manifold of the given manifold.

An equation of the form (4) is called a Hamiltonian system.

Exercise 1. *Show that a system $x' = F(x)$ is at the same time a Hamiltonian system and a gradient system iff the Hamiltonian H is a harmonic function.*

Proposition 1. *(i) The Hamiltonian is a constant of motion, that is, for any solution $X(t) = (p(t), q(t))$ we have*

$$H(p(t), q(t)) = const \tag{7}$$

where the constant depends on the solution.

(ii) The constant level surfaces of a smooth function $F(p, q)$ are solutions of a Hamiltonian system

$$q' = \frac{\partial F}{\partial p} \tag{8}$$

$$p' = -\frac{\partial F}{\partial x} \tag{9}$$

Proof. (i) We have

$$\frac{dH}{dt} = \nabla_p H \frac{dp}{dt} + \nabla_q \frac{dq}{dt} = -\nabla_p H \nabla_q + \nabla_q H \nabla_p = 0 \tag{10}$$

(ii) This is obtained very similarly. □

1.2.1 Integrability: a few first remarks

Hamiltonian systems in one dimension are integrable: the solution can be written in closed form, implicitly, as $H(y(x), x) = c$. Note that for an equation of the form $y' = G(y, x)$, this is *equivalent* to the system having a *constant of motion*. The latter is defined as a function $K(x, y)$ defined globally in the phase space, (perhaps with the exception of some isolated points where it may have "simple" singularities, such as poles), and with the property that $K(y(x), x) = const$ for any given trajectory (the constant can depend on the trajectory, but not on x). Indeed, in this case we have

$$\frac{d}{dx} C(y(x), x) = \frac{\partial C}{\partial y} y' + \frac{\partial C}{\partial x} = 0$$

or

$$y' = -\frac{\partial C}{\partial x} / \frac{\partial C}{\partial y}$$

which is equivalent to the system

$$\dot{x} = \frac{\partial C}{\partial y}; \quad \dot{y} = -\frac{\partial C}{\partial x} \tag{11}$$

which is a Hamiltonian system.

1.2.2 Local versus global

It is important to mention that a system is actually called Hamiltonian if the function H is defined over a sufficiently large region, preferably the whole phase space.

Indeed, take any smooth first order ODE, $y' = f(y,x)$ and differentiate with respect to the initial condition (we know already that the dependence is smooth; we let $dy/d(y_0) = \dot{y}$):

$$\dot{y}' = \frac{\partial f}{\partial y}\dot{y} \tag{12}$$

with the solution

$$\dot{y} = \exp\left(\int_{x_0}^{x} \frac{\partial f(y(s),s)}{\partial y} ds\right) \tag{13}$$

and thus, in the local solution $y = G(x; x_0)$ we have $G_{x_0}(x; x_0) \neq 0$, *if* G is smooth –i.e. the field is regular–, and the implicit function theorem provides a local function K so that $x_0 = K(y(x), x)$, that is a constant of motion! The big difference between integrable and nonintegrable systems comes from the possibility to extend K globally.

1.3 Example

As an example for both systems, we study the following problem: draw the contour plot (constant level curves) of

$$F(x,y) = y^2 + x^2(x-1)^2 \tag{14}$$

and draw the lines of steepest descent of F.

For the first part we use Proposition 1 above and we write

$$x' = \frac{\partial F}{\partial y} = 2y \tag{15}$$

$$y' = -\frac{\partial F}{\partial x} = -2x(x-1)(2x-1) \tag{16}$$

The critical points are $(0,0), (1/2,0), (1,0)$. It is easier to analyze them using the Hamiltonian. Near $(0,0)$ H is essentially $x^2 + y^2$, that is the origin is a center, and the trajectories are near-circles. We can also note the symmetry $x \to (1-x)$ so the same conclusion holds for $x = 1$, and the phase portrait is symmetric about 1/2.

Near $x = 1/2$ we write $x = 1/2 + s$, $H = y^2 + (1/4 - s^2)^2$ and the leading Taylor approximation gives $H \sim y^2 - 1/2s^2$. Then, 1/2 is a saddle (check). Now we can draw the phase portrait easily, noting that for large x the curves essentially become $x^4 + y^2 = C$ "flattened circles". Clearly, from the interpretation of the problem and the expression of H we see that *all* trajectories are closed.

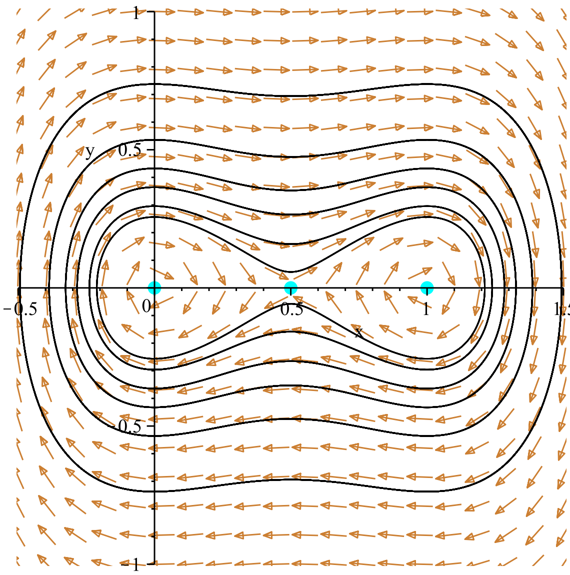

Figure 1:

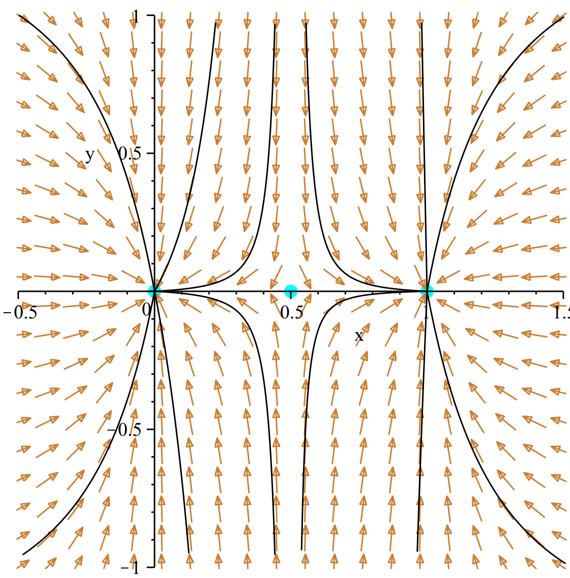

Figure 2:

The perpendicular lines solve the equations

$$x' = -\frac{\partial F}{\partial x} = -2x(x-1)(2x-1) \tag{17}$$

$$y' = -\frac{\partial F}{\partial y} = -2y \tag{18}$$

We note that this equation is separated! In any case, the two equation obviously share the critical points, and the sign diagram can be found immediately from the first figure.

Exercise 2. *Find the phase portrait for this system, and justify rigorously its qualitative features. Find the expression of the trajectories of (17). I found*

$$y = C\left(\frac{1}{(x-1/2)^2} - 4\right)$$

2 Flows, revisited

Often in nonlinear systems, equilibria are of higher order (the linearization has zero eigenvalues). Clearly such points are not hyperbolic and the methods we have seen so far do not apply.

There are no general methods to deal with all cases, but an important one is based on Lyapunov (or Lyapounov,...) functions.

Definition. A flow is a smooth map

$$(X,t) \to \Phi_t(X)$$

A differential system

$$\dot{x} = F(x) \tag{19}$$

generates a flow

$$(X,t) \to x(t;X)$$

where $x(t;X)$ is the solution at time t with initial condition X.

The derivative of a function G along a vector field F is, as usual,

$$D_F(G) = \nabla G \cdot F$$

If we write the differential equation associated to F, (19), then clearly

$$D_F G = \frac{d}{dt} G(x(t))_{|t=0}$$

2.1 Lyapunov stability

Consider the system (19) and assume $x = 0$ is an equilibrium.

Then

1. $x_e = 0$ is Lyapunov stable (or simply stable) if starting with initial conditions near 0 the flow remains in a neighborhood of zero. More precisely, the condition is: for every $\epsilon > 0$ there is a $\delta > 0$ so that if $|x_0| < \delta$ then $|x(t)| < \epsilon$ for all $t > 0$.

2. $x_e = 0$ is asymptotically stable if furthermore, trajectories that start close to the equilibrium converge to the equilibrium. That is, the equilibrium x_e is asymptotically stable if it is Lyapunov stable and if there exists $\delta > 0$ so that if $|x_0| < \delta$, then $\lim_{t\to\infty} x(t) = 0$.

2.2 Lyapunov functions

Let X^* be a fixed point of (19). A Lyapunov function for (19) is a function defined in a neighborhood $\mathcal{O}$ of X^* with the following properties

(1) L is differentiable in $\mathcal{O}$.

(2) $L(X^*) = 0$ (this can be arranged by subtracting a constant).

(3) $L(x) > 0$ in $\mathcal{O} \setminus \{X^*\}$.

(4) $D_F L \leq 0$ in $\mathcal{O}$.

A strict Lyapunov function is a Lyapunov function for which

(4') $D_F L < 0$ in $\mathcal{O}$.

Finding a Lyapunov function is often nontrivial. In systems coming from physics, the energy is a good candidate. In general systems, one may try to find an exactly integrable equation which is a good approximation for the actual one in a neighborhood of X^* and look at the various constants of motion of the approximation as candidates for Lyapunov functions.

Theorem 2 (Lyapunov stability)**.** *Assume X^* is a fixed point for which there exists a Lyapunov function L. Then*

(i) X^* is stable.

(ii) If L is a strict Lyapunov function then X^* is asymptotically stable.

Proof. (i) Consider a small ball $B \ni X^*$ contained in $\mathcal{O}$; we denote the boundary of B (a sphere) by ∂B. Let α be the minimum of L on the ∂B. By the definition of a Lyapunov function, (3), $\alpha > 0$. Consider the following subset:

$$\mathcal{U} = \{x \subset B : L(x) < \alpha\} \tag{20}$$

From the continuity of L, we see that $\mathcal{U}$ is an open set. Clearly, $X^* \subset \mathcal{U}$. Let $X \in \mathcal{U}$. Then $x(t; X)$ is a continuous curve, and it cannot have components outside B without intersecting ∂B. But an intersection is impossible since by monotonicity, $L(x(t)) \leq L(X) < \alpha$ for all t. Thus, trajectories starting in $\mathcal{U}$ are confined to $\mathcal{U}$, proving stability.

(ii)

1. Note first that X^* is the only critical point in $\mathcal{O}$ since $\frac{d}{dt}L(x(t; X_1^*)) = 0$ for any fixed point.

2. Note that trajectories $x(t; X)$ with $X \in \mathcal{U}$ are contained in a compact set, and thus they contain limit points. Any limit point x^* is strictly inside $\mathcal{U}$ since $L(x^*) < L(x(t); X) < \alpha$.

3. Let x^* be a limit point of a trajectory $x(t; X)$ where $X \in \mathcal{U}$, i.e. $x(t_n, X) \to x^*$. Then, by 1 and 2, $x^* \in \mathcal{U}$ and x^* is a regular point of the field.

4. We want to show that $x^* = X^*$. We will do so by contradiction. Assuming $x^* \neq X^*$ we have $L(x^*) = \lambda > 0$, again by (3) of the definition of L.

5. By 3 the trajectory $\{x(t; x^*) : t \geq 0\}$ is well defined and is contained in $\mathcal{B}$.

6. We then have $L(x(t; x^*)) < \lambda \forall t > 0$.

7. The set
$$\mathcal{V} = \{X : L(x(t_{n+1} - t_n; X))\} < \lambda \tag{21}$$
is open, so
$$L(x(t_{n+1} - t_n; X_1)) < \lambda \tag{22}$$
for all X_1 close enough to x^*.

8. Let n be large enough so that $x(t_m; X) \in \mathcal{V}$ for all $m \geq n$.

9. Note that, by existence and uniqueness of solutions at regular points we have
$$x(t_{n+1}; X) = x(t_{n+1} - t_n; x(t_n; X)) \tag{23}$$

10. On the one hand $L(x(t_{n+1})) \downarrow \lambda$ and on the other hand we got $L(x(t_{n+1})) < \lambda$. This is a contradiction.

□

2.3 Examples

Hamiltonian systems, in Cartesian coordinates often assume the form

$$H(q, p) = p^2/2 + V(q) \tag{24}$$

where p is the collection of spatial coordinates and p are the momenta. If this ideal system is subject to external dissipative forces, then the energy cannot increase with time. H is thus a Lyapunov function for the system. If the external force is $F(p, q)$, the new system is generally not Hamiltonian anymore, and the equations of motion become

$$\dot{q} = p \tag{25}$$
$$\dot{p} = -\nabla V + F \tag{26}$$

and thus
$$\frac{dH}{dt} = pF(p, q) \tag{27}$$
which, in a dissipative system should be nonpositive, and typically negative. But, as we see, $dH/dt = 0$ along the curve $p = 0$.

For instance, in the ideal pendulum case with Hamiltonian

$$H = \frac{1}{2}\omega^2 + (1 - \cos\theta) \tag{28}$$

The associated Hamiltonian flow is

$$\theta' = \omega \tag{29}$$
$$\omega' = -\sin\theta \tag{30}$$

Then H is a global Lyapunov function at $(0,0)$ for (31) (in fact, this is true for any system with *nonnegative* Hamiltonian). This is clear from the way Hamiltonian systems are defined.

Then $(0,0)$ is a stable equilibrium. But, clearly, it is not asymptotically stable since $H = const > 0$ on any trajectory not starting at $(0,0)$.

If we add air friction to the system (31), then the equations become

$$\theta' = \omega \tag{31}$$

$$\omega' = -\sin\theta - \kappa\omega \tag{32}$$

where $\kappa > 0$ is the drag coefficient. Note that this time, if we take $L = H$, the same H defined in (28), then

$$\frac{dH}{dt} = -\kappa\omega^2 \tag{33}$$

The function H is a Lyapunov function, but it is not strict, since $H' = 0$ if $\omega = 0$. Thus the system is stable. It is however intuitively clear that furthermore the energy still decreases to zero in the limit, since $\omega = 0$ are isolated points on any trajectory and we expect $(0,0)$ to still be asymptotically stable. In fact, we could adjust the proof of Theorem 2 to show this. However, as we see in (27), this degeneracy is typical and then it is worth having a systematic way to deal with it. This is one application of Lasalle's invariance principle that we will prove next.

3 Some important concepts

We start by introducing some important concepts.

Definition 2. *1. An entire solution $x(t;X)$ is a solution which is defined for all $t \in \mathbb{R}$.*

2. A positively invariant set $\mathcal{P}$ is a set such that $x(t,X) \in \mathcal{P}$ for all $t \geq 0$ and $X \in \mathcal{P}$. Solutions that start in $\mathcal{P}$ stay in $\mathcal{P}$. Similarly one defines negatively invariant sets, and invariant sets.

3. The basin of attraction of a fixed point X^* is the set of all X such that $x(t;X) \to X^*$ when $t \to \infty$.

4. Given a solution $x(t;X)$, the set of all points ξ^* such that solution $x(t_n;X) \to \xi^*$ for some sequence $t_n \to \infty$ is called the set of ω-limit points of $x(t;X)$. At the opposite end, the set of all points ξ^* such that solution $x(-t_n;X) \to \xi^*$ for some sequence $t_n \to \infty$ is called the set of α-limit points. These may of course be empty.

Proposition 3. *Assume $x(t;X)$ belongs to a closed, positively invariant set $\mathcal{P}$ where the field is defined. The ω-limit set is a closed invariant set too. A similar statement holds for the α-set.*

Proof. 1. (Closure) We show the complement is open. Let b be in the complement of the set of ω-limit points. Then $\liminf_{t\to\infty} d(x(t,X),b) > a > 0$ for some a and for all $t > 0$. If b' is close enough to b, then by the triangle inequality, $\liminf_{t\to\infty} d(x(t,X),b') > a/2 > 0$ for all t.

2. (Invariance) Assume $x(t_n,X) \to x^*$. By assumption, the differential equation is well-defined in a neighborhood of any point in $\mathcal{P}$, and since $x^* \in \mathcal{P}$, $x(t;x^*)$ exists for all $t \geq 0$. By continuity with respect to initial conditions and of solutions, we have $x(t;x(t_n)) \to x(t;x^*)$ as $n \to \infty$. Then $x(t;x^*)$ is an omega-point too for any $t \geq 0$ (note that $x(t_n + t, X) \to x(t,x^*)$ by the definition of an ω limit point and continuity .

3. Backward invariance is proved similarly: $x(t_n - t, X) \to x(-t,x^*)$.

□

4 Lasalle's invariance principle

Theorem 3. *Let X^* be an equilibrium point for $X' = F(X)$ and let $L : \mathcal{U} \to \mathbb{R}$ be a Lyapunov function at X^*. Let $X^* \ni \mathcal{P} \subset \mathcal{U}$ be closed, bounded and positively invariant. Assume there is no entire trajectory in $\mathcal{P} - \{X^*\}$ along which L is constant. Then X^* is asymptotically stable, and $\mathcal{P}$ is contained in the basin of attraction of X^*.*

Proof. Since $\mathcal{P}$ is compact and positively invariant, every trajectory in $\mathcal{P}$ has ω-limit points. If X^* is the only limit point, the assumption follows easily (show that all trajectories must tend to X^*). So, we may assume there is an $x^* \neq X^*$ which is also an ω-limit point of some $x(t;X)$. We know that the trajectory $x(t;x^*)$ is entire. Since L is nondecreasing along trajectories, we have $L(x(t;X)) \to \alpha = L(x^*)$ as $t \to \infty$. (This is clear for the subsequence t_n, and the rest follows by inequalities: check!) On the other hand, for any $T \in \mathbb{R}$, positive or negative, $x(T;x^*)$ is arbitrarily close to $x(t_n + T, X)$ if n is large. Since $L(x(T;x^*)) \leq \alpha$ and it is arbitrarily close to $L(x(t,X)) \geq \alpha$, it follows that $L(x(T;x^*)) = \alpha$ for all $t \geq 0$. □

4.1 Example: analysis of the pendulum with drag

Intuitively, it is clear that any trajectory that starts with $\omega = 0$ and $\theta \in (-\pi,\pi)$ should asymptotically end up at the equilibrium point $(0,0)$ (other trajectories, which for the frictionless system would rotate forever, may end up in a different equilibrium, $(2n\pi,0)$. For zero initial ω, the basin of attraction of $(0,0)$ should exactly be $(-\pi,\pi)$. In general, the energy should be less than precisely the one in this marginal case, $H = 1 - \cos(\pi) = 2$. Then, the region $\theta_0 \in (-\pi,\pi)$, $H < 1 - \cos(\pi) = 2$ should be the basin of attraction of $(0,0)$.

So let $c \in (0,2)$, and let

$$\mathcal{P}_c = \{(\theta,\omega) : \text{ and } H(\theta,\omega) \leq c, \quad |\theta| \leq \arccos(1-c) \in (-\pi,\pi)\} \tag{34}$$

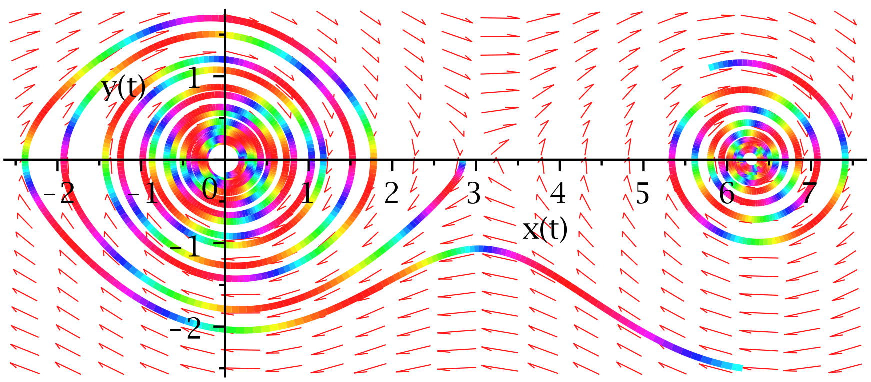

Figure 3:

In H, θ coordinates, this is simply a closed rectangle and since (H, θ) is a continuous map, its preimage in the (p, θ) plane is closed too.

Now we show that $\mathcal{P}_c$ is closed and forward invariant. If a trajectory were to exit $\mathcal{P}_c$, it would mean, by continuity, that for some t we have $H = c + \delta$ for a small $\delta > 0$ (ruled out by $\dot{H} \leq 0$ along trajectories) or that $|\theta| = \arccos(1-c)+\epsilon$ for a small $\epsilon > 0$ which implies, from the formula for H the same thing: $H > c$.

Now there is no nontrivial entire solution (that is, other than $X^* = (0, 0)$) along which $H = const$. Indeed, $H = const$ implies, from (33) that $\omega = 0$ identically along the trajectory. But then, from (30) we see that $\sin\theta = 0$ identically, which, within $\mathcal{P}_c$ simply means $\theta = 0$ identically. Lasalle's theorem applies, and all solutions starting in $\mathcal{P}_c$ approach $(0, 0)$ as $t \to \infty$. The phase portrait of the damped pendulum is depicted in Fig. 3

5 Gradient systems and Lyapunov functions

Recall that a gradient system is of the form (1), that is

$$X' = -\nabla V(X) \tag{35}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is, say, C^∞ and a critical point of V is a point where $\nabla V = 0$. We have the following result:

Theorem 4. *For the system (1): (i) If c is a regular value of V, then the vector field is orthogonal to the level set of $V^{-1}(c)$.*

(ii) The equilibrium points of the system coincide with the critical points of V.

(iii) V is a Lyapunov function for the system, and given a solution x, we have $\frac{d}{dt}V(x(t)) = 0$ iff $x(t) \equiv X^*$, an equilibrium point.

(iv) If a critical point X^* is an isolated minimum *of V, $V(X) - V(X^*)$ is a strict Lyapunov function at X^*, and then X^* is asymptotically stable.*

(v) Any $\alpha-$ limit point of a solution of (1), and any $\omega-$ limit point is an equilibrium.

(vi) The linearized system at any equilibrium has only real eigenvalues.

Note 1. (a) By (v), any solution of a gradient system tends to a limit point or to infinity.

(b) Thus, descent lines of any smooth manifold have the same property: they link critical points, or they tend to infinity.

(c) We can use some of these properties to determine for instance that a system is not integrable. We write the associated gradient system and determine that it fails one of the properties above, for instance the linearized system at a critical point has an eigenvalue which is not real. Then there cannot exist a smooth H so that $H(x, y(x))$ is constant along trajectories.

Proof. We have already shown (i) and (ii), which are in fact straightforward from the definition.

For (iii) we see that

$$\frac{d}{dt}V(x(t)) = \nabla V \frac{dx}{dt} = -|\nabla V|^2 \leq 0 \tag{36}$$

whereas, if $\nabla V = 0$ for some point of the trajectory, then of course that point is an equilibrium, and the whole trajectory is that equilibrium.

(iv) If an equilibrium point is isolated, then $\nabla V \neq 0$ in a set of the form $|X - X^*| \in (0, a)$. Then $-|\nabla V|^2 < 0$ in this set. Furthermore, $V(X) - V(X^*) > 0$ for all X with $|X - X^*| \in (0, a)$.

(v) Since V is a Lyapunov function for (1), we have shown in the proof of Lasalle's invariance principle that V is constant along any trajectory starting at a limit point. But we see from (iii) that this implies that the trajectory reduces to a point, which is an equilibrium point.

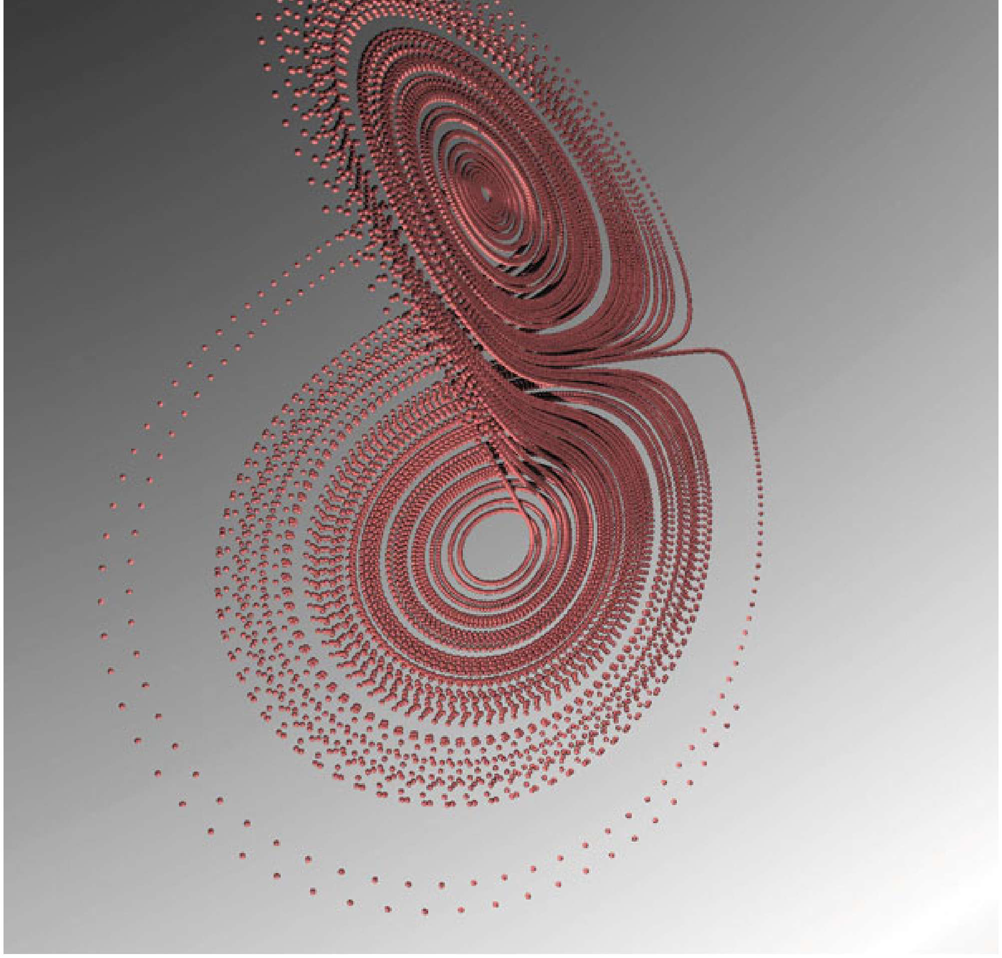

Figure 4: The Lorenz attractor

(vi) Note that for smooth V, the linearization at X^* is simply the matrix

$$A; \quad A_{ij} = \frac{-\partial^2 V(X^*)}{\partial x_i \partial x_j} \tag{37}$$

which is *symmetric*. □

6 Limit sets, Poincaré maps, the Poincaré Bendixson theorem

We shall denote the $\omega-$limit set of a solution starting at X by $\omega(X)$, and likewise, its $\alpha-$limit set by $\alpha(X)$.

In two dimensions, there are typically two types of limit sets: equilibria and periodic orbits (which are thereby limit cycles). Exceptions occur when a limit set contains a number of equilibria, as we will see in examples.

Beyond two dimensions however, the possibilities are far vaster and limit sets can be quite complicated. Fig. 4 depicts a limit set for the Lorenz system, in three dimensions. Note how the trajectories seem to spiral erratically around two points. The limit set here has a fractal structure.

We begin the analysis with the two dimensional case, which plays an important tole in applications.

We have already studied the system $r' = 1/2(r-r^3)$ in Cartesian coordinates. There the circle of radius one was a periodic orbit, and a limit cycle. All trajectories, except for the trivial one $(0,0)$ tended to it as $t \to \infty$.

We have also analyzed many cases of nodes, saddle points etc, where trajectories have equilibria as limit sets, or else they go to infinity.

A rather exceptional situation is that where the limit sets contain equilibria. Here is one example

6.1 Example: equilibria on the limit set

Consider the system

$$x' = \sin x(-\cos x - \cos y) \tag{38}$$

$$y' = \sin y(\cos x - \cos y) \tag{39}$$

The phase portrait is depicted in Fig. 5.

Exercise 1. *Justify the qualitative elements in Fig. 5.*

In the example above, we see that the limit set is a collection of fixed points and orbits, none of which periodic.

6.1.1 Closed orbits

A closed orbit is a solution whose trajectory is a closed curve (with no equilibria on it). Starting at a point X, after a finite time then, the solution returns to X since the absolute value of the velocity along the curve is bounded below. From that time on, the solution must repeat itself identically, by uniqueness of solutions. It then means that the solution is periodic, that is there is a smallest τ so that $\Phi_{t+\tau}(X) = \Phi(X)$. This τ is called the period of the orbit.

Proposition 4. *(i) If X and Z lie on the same solution curve, then $\omega(X) = \omega(Z)$ and $\alpha(X) = \alpha(Z)$.*

(ii) If D is a closed, positively invariant set and $Z \in D$, then $\omega(Z) \subset D$; similarly for negatively invariant sets and $\alpha(Z)$.

(iii) A closed invariant set, and in particular a limit set, contains the $\alpha-$limit and the $\omega-$limit of every point in it.

Proof. Exercise. □

7 Sections; the flowbox theorem

Consider a differential equation $X' = F(X)$ with F smooth, and a point X_0 such that $F(X_0) \neq 0$. Then there is a diffeomorphic change of coordinates *in some neighborhood of* X_0, $X \leftrightarrow X^-$ so that in coordinates X^- the field is simply $\dot{X}^- = \mathbf{e}_1$ where $\mathbf{e}_1 = (1,0)$.

One way to achieve this is the following. Since $F(X_0) \neq 0$ there is a unit vector V_0 at X_0 which is orthogonal to $F(X_0)$, say $(-F_2, F_1)/|F(X_0)|$. we

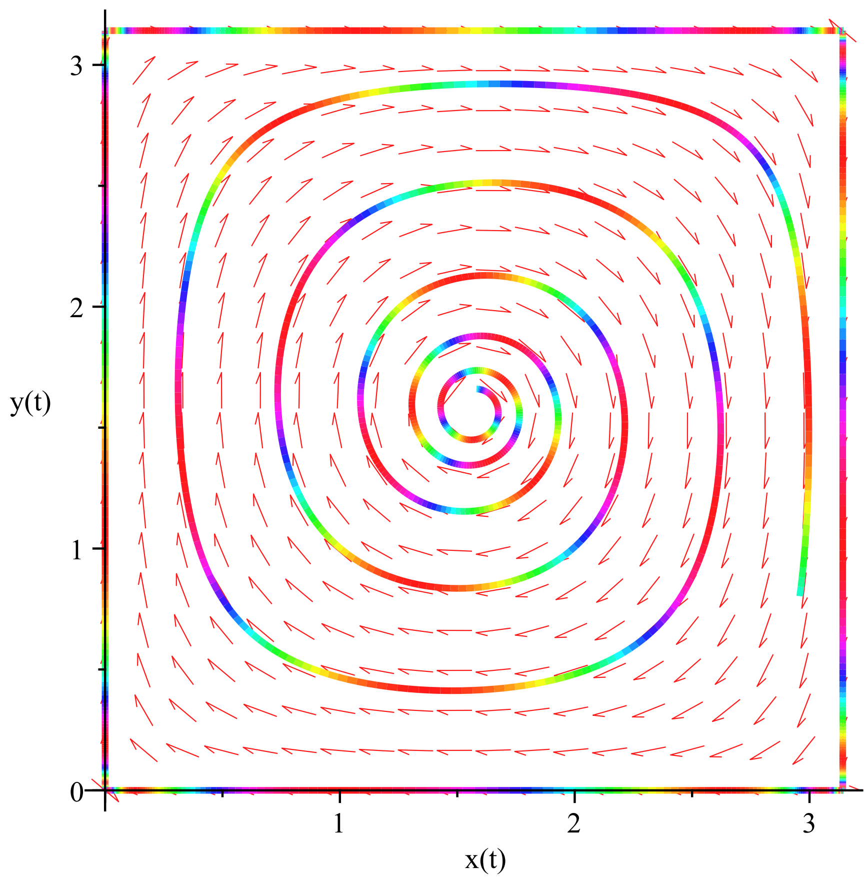

Figure 5: Phase portrait for (38).

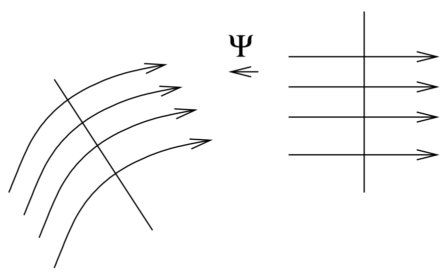

Figure 6: Flowbox and transformation

draw a line segment, $h(u) = X_0 + uV_0$, $u \in (-\epsilon, \epsilon)$. If ϵ is small enough, then $F(h(u)) \neq 0$ for all $u \in (-\epsilon, \epsilon)$ and F is not tangent to u anywhere along the segment $\mathcal{S} = \{X_0 + uV_0 | u \in (-\epsilon, \epsilon)\}$ (that is, $(-F_2, F_1) \cdot V_0 \neq 0$).

Definition 5. *The segment $\mathcal{S}$ defined above is called local section at X_0.*

Note that any solution that intersects $\mathcal{S}$ crosses it, in the same direction relative to $\mathcal{S}$. Indeed, since the field is smooth and $(-F_2, F_1) \cdot V_0 \neq 0$, then $(-F_2, F_1) \cdot V_0$ has constant sign throughout $\mathcal{S}$ and the trajectories flow towards the same side of $\mathcal{S}$.

To straighten the field, we construct the following map, from a neighborhood of X_0 of the form $\mathcal{N} = \{(t, u) : |t| < \delta, u \in \mathcal{S}$. This is a small enough region where the flow is smooth and there are no equilibria.

We consider the function from

$$\Psi(s, u) = \Phi_s(h(u)) := x(s; h(u)) \tag{40}$$

Note that $x(s; h(u)) \in \mathbb{R}^2$ lies in a neighborhood of X_0. Note also that $x(0; h(u)) = h(u)$ and thus the line $(0, u) \in \mathcal{N}$ is mapped onto the section $\mathcal{S}$.

Finally, we note that $x(0, h(u_2)) - x(0, h(u_1)) = h(u_2) - h(u_1) = (u_2 - u_2)V_0$. then Ψ is a diffeomorphism, since the Jacobian of the transformation is

$$\det(J(X_0)) = \det \begin{pmatrix} F_1 & V_1 \\ F_2 & V_2 \end{pmatrix} = |F(X_0)| \tag{41}$$

is nonzero in a neighborhood of X_0, in fact throughout $\mathcal{S}$, and since we have assumed that F does not become tangent to V_0.

We see that the inverse of Ψ takes a neighborhood of X_0 into a neighborhood of X_0.

We saw that $(0, u)$ is mapped onto $\mathcal{S}$. We also see that (s, u_0) is mapped to $x(s; h(u_0))$ which is part of a trajectory. Thus, the inverse image of trajectories through Ψ are straight lines, as depicted. The new field is the trivial flow we have mentioned.

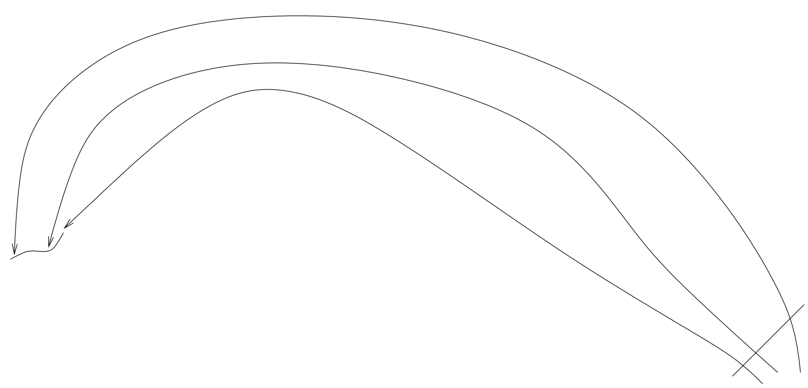

Figure 7: Time of arrival function

7.1 Time of arrival

We consider all solutions in the domain $\mathcal{O}$ where the field is defined. Some of them intersect $\mathcal{S}$. Since the trajectories are continuous, there is a first time of arrival, the smallest t so that $x(t, Z_0) \in \mathcal{S}$.

This time of arrival is continuous in Z_0, as shown in the next proposition.

Proposition 6. *Let $\mathcal{S}$ be a local section at X_0 and assume $\phi_{t_0}(Z_0) = X_0$. Let $\mathcal{W}$ be a neighborhood of Z_0. Then there is an open set $\mathcal{U} \subset \mathcal{W}$ and a diffrentiable function $\tau : \mathcal{U} \to \mathbb{R}$ such that $\tau(Z_0) = t_0$ and*

$$\phi_{\tau(X)}(X) \in \mathcal{S} \tag{42}$$

for each $x \in \mathcal{U}$.

Note 2. *In some sense, a subsegment of the section $\mathcal{S}$ is carried* backwards *smoothly through the field arbitrarily far, assuming that the flow makes sense, and that the subsegment is small enough.*

Proof. A point X_1 belongs to the line ℓ containing $\mathcal{S}$ iff $X = X_0 + uV_0$ for some u, Since V_0 is orthogonal to $F(X_0)$ we see that $X \in \ell$ iff $X \cdot F(X_0) = X_0 F(X_0)$.

We look now at the more general function

$$G(X, t) = x(t; X) \cdot F(X_0) \tag{43}$$

We have

$$G(Z_0, t_0) = X_0 \cdot F(X_0) \tag{44}$$

by construction. We want to see whether we can apply the implicit function theorem to

$$G(X, t) - G(Z_0, t_0) = 0 \tag{45}$$

□

For this we need to check $\frac{\partial}{\partial t} G_{(Z_0, t_0)}$. But this equals

$$x'(t; X) \cdot F(X_0) = |F(X_0)|^2 \neq 0 \tag{46}$$

Then, there is a neighborhood of t_0 and a differentiable function $\tau(X)$ so that

$$G(X, \tau(X)) = G(Z_0, t_0) = X_0 \cdot F(X_0) \tag{47}$$

7.2 The Poincaré map

The Poincaré map is a useful tool in determining whether closed trajectories (that is, periodic orbits) are stable or not. This means that taking an initial close enough to the periodic orbit, the trajectory thus obtained would approach the periodic orbit or not.

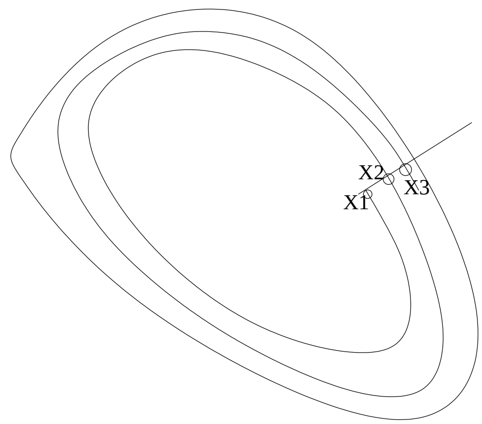

Figure 8: A Poincaré map.

The basic idea is simple, we look at a section containing a point on the periodic orbit, and then follow the successive re-intersections of the perturbed orbit with the section. Now we are dealing with a discrete map $X_{n+1} = P(X_n)$. If $P(X_n) \to X_0$, the point on the closed orbit, then the orbit is asymptotically stable. See Figure 10.

It is often not easy to calculate the Poincaré map, but it is a very useful concept, and it has many theoretical applications.

Let's define the map rigorously.

Consider a periodic orbit $\mathcal{C}$ and a point $X_0 \in \mathcal{C}$. We have

$$x(X_0; T) = X_0 \tag{48}$$

where T is the period of the orbit. Consider a section $\mathcal{S}$ through X_0. Then according to Proposition 6, there is a neighborhood of $\mathcal{U}$ of X_0 and a continuous function $\tau(X)$ so that $x(\tau(t), X) \in \mathcal{S}$ for all $X \in \mathcal{U}$. Then certainly $\mathcal{S}_1 = \mathcal{U} \cap \mathcal{S}$ is an open set in $\mathcal{S}$ in the induced topology. The return map is thus defined on $\mathcal{S}_1$. It means that for each point in $X \in \mathcal{S}_1$ there is a point $P(X) \in \mathcal{S}$, so that $x(\tau(X); X) = P(X)$ and $\tau(X)$ is the smallest time with this property.

This is the Poincaré map associated to $\mathcal{C}$ and to its section $\mathcal{S}$.

This can be defined for planar systems as well as for higher dimensional ones, if we now take as a section a subset of a hyperplane through a point $X_0 \in \mathcal{C}$. The statement and proof of Proposition 6 generalize easily to higher dimensions.

In two dimensions, we can identify the segments $\mathcal{S}$ and $\mathcal{S}_1$ with intervals on the real line, $u \in (-a, a)$, and $u \in (-\epsilon, \epsilon)$ respectively, see also Definition 5. Then P defines an analogous transformation of the interval $(-\epsilon, \epsilon)$, which we still denote by P though this is technically a different function, and we have

$$P(0) = 0$$

$$P(u) \in (-a, a), \quad \forall u \in (-\epsilon, \epsilon)$$

We have the following easy result, the proof of which we leave as an exercise.

Proposition 7. *Assume that $X' = F(X)$ is a planar system with a closed orbit $\mathcal{C}$, let $X_0 \in \mathcal{C}$ and $\mathcal{S}$ a section at X_0. Define the Poincaré map P on an interval $(-\epsilon, \epsilon)$ as above, by identifying the section with a real interval centered at zero. If $|P'(X_0)| < 1$ then the orbit $\mathcal{C}$ is asymptotically stable.*

Example 3. Consider the planar system

$$r' = r(1 - r) \tag{49}$$

$$\theta' = 1 \tag{50}$$

In Cartesian coordinates it has a fixed point, $x = y = 0$ and a closed orbit, $x = \cos t, y = \sin t; x^2 + y^2 = 1$. Any ray originating at $(0, 0)$ is a section of the flow. We choose the positive real axis as $\mathcal{S}$. Let's construct the Poincaré map. Since $\theta' = 1$, for any $X \in \mathbb{R}^+$ we have $x(2\pi; X) = x(0, X)$. We have $P(1) = 1$ since 1 lies on the unit circle. In this case we can calculate explicitly the solutions, thus the Poincaré map and its derivative.

We have

$$\ln r(t) - \ln(r(t) - 1) = t + C \tag{51}$$

and thus

$$r(t) = \frac{Ce^t}{Ce^t - 1} \tag{52}$$

where we determine C by imposing the initial condition $r(0) = x$: $C = x/(x-1)$. Thus,

$$r(t) = \frac{xe^t}{1 - x + xe^t} \tag{53}$$

and therefore we get the Poincaré map by taking $t = 2\pi$,

$$P(x) = \frac{xe^{2\pi}}{1 - x + xe^{2\pi}} \tag{54}$$

Direct calculation shows that $P'(1) = e^{-2\pi}$, and thus the closed orbit is stable. We could have seen this directly from (54) by taking $t \to \infty$.

Note that here we could calculate the orbits explicitly. Thus we don't quite need the Poincaré map anyway, we could just look at (53). When explicit solutions, or at least an explicit formula for the closed orbit is missing, calculating the Poincaré map can be quite a challenge.

8 Monotone sequences in two dimensions

There are two kinds of monotonicity that we can consider. One is monotonicity along a solution: $X_1, ..., X_n$ is monotone along the solution if $X_n = x(t_n, X)$ and t_n is an increasing sequence of times. Or, we can consider monotonicity along a segment, or more generally a piece of a curve. On a piece of a smooth curve, or on an interval we also have a natural order (or two rather), by arclength parameterization of the curve: $\gamma_2 > \gamma_1$ if γ_2 is farther from the chosen endpoint. To avoid this rather trivial distinction (dependence on the choice of endpoint) we say that a sequence $\{\gamma_n\}_n$ is monotone along the curve if γ_n is inbetween γ_{n-1} and γ_{n+1} for all n. Or we could say that a sequence is monotone if it is either increasing or else decreasing.

If we deal with a trajectory crossing a curve, then the two types of monotonicity need not coincide, in general. But for sections, they do.

Proposition 8. *Assume $x(t;X); t \in [0,T]$ is a solution so that F is regular and nonzero in a neighborhood Let $\mathcal{S}$ be a local section for a planar system. Then monotonicity along the solution $x(t;X)$ assumed to intersect $\mathcal{S}$ at $X_1, X_2, ...$ (finitely or infinitely many intersection) and along $\mathcal{S}$ coincide.*

Note that all intersections are supposed to be with $\mathcal{S}$, along which, by definition, they are always transversal.

Proof. We assume we have three successive distinct intersections with $\mathcal{S}$, X_1, X_2, X_3 (if two of them coincide, then the trajectory is a closed orbit and there is nothing to prove).

We want to show that X_3 is not inside the interval (X_1, X_2) (on the section, or on its image on $\mathbb{R}$). Consider the curve $x(t;X_1)$ $t \in [0, t_2)$ where t_2 is the first time of re-intersection of $x(t;X_1)$ with $\mathcal{S}$. By definition $x(t_2;X_1) = X_2$. This is supposed to be a smooth curve, with no self-intersection (since the field is assumed regular along the curve) thus of finite length. If completed with the subsegment $[X_1, X_2] \in \mathcal{S}$ it evidently becomes a closed curve (with a natural parameterization even) $\mathcal{C}$. By Jordan's lemma, we can define the inside int $\mathcal{C}$ and the outside of the curve, $D =$ext $\mathcal{C}$. Note that the field has a definite direction along $[X_1, X_2]$, by the definition of a section. Note also that it points towards ext $\mathcal{C}$, since $x(t;X_1)$ exits int $\mathcal{C}$ at $t = t_2$.

Then, no trajectory can enter int $\mathcal{C}$. Indeed, it should intersect either $x(t;X_1)$ or else $[X_1, X_2]$. The first option is impossible by uniqueness of solutions (solutions do not intersect at regular points). The second case is ruled out since $[X_1, X_2]$ is an exit region, not an entry one. Now we know that $x(t_3, X) = X_3$ where t_3 is the first reintersection time. It must lie in ext $\mathcal{C}$, thus outside $[X_1, X_2$. □

The next result shows points towards limiting points being special: parts of closed curves, or simply infinity.

Proposition 9. *Consider a planar system and $Z \in \omega(X)$ (or $Z \in \alpha(X)$), assumed a regular point of the field. Consider a local section $\mathcal{S}$ through Z. Then either $\{x(t,Z) : t > 0\} \cap \mathcal{S} = Z$ or else $\{x(t,Z) : t > 0\} \cap \mathcal{S} = \emptyset$.*

Proof. Assume there are two distinct intersection points $x(t_1, Z) = Z_1$ and $x(t_2, Z) = Z_2$ in $\mathcal{S}$. Since Z_1 and Z_2 are also in $\omega(X)$, as we have shown, then there are infinitely many points on $x(t, X)$ arbitrarily close to Z_1 and infinitely many others arbitrarily close to Z_1. These nearby points lie on the same section, since the section at Z_1 contains, by definition, an open set around Z_1 and similarly around Z_2. Without loss of generality (rotating and translating the figure) we can assume that $\mathcal{S} = (-a, b) \in \mathbb{R}$ and $[Z_1, Z_2] \subset (-a, b)$. We know that $x(t_j, X)$, where t_j are the increasing times when $x(t_j, X) \in \mathcal{S}$, are monotone. Thus they converge. But then, by definition of convergence, they cannot be arbitrarily close to two distinct points.

□

9 The Poincaré-Bendixson theorem

Theorem 5 (Poincaré-Bendixson)**.** *Let $\Omega = \omega(X)$ be a nonempty compact limit set of a planar system of ODEs, containing no equilibria. Then Ω is a closed orbit.*

Proof. First, recall that Ω is invariant. Let $Y \in \Omega$. Then $x(t; Y) \in \Omega$ for all $t \in \mathbb{R}$. Since Ω is compact, $x(t; Y)$ has (infinitely many) accumulation points in Ω. Let Z be one of them, and let $\mathcal{S}$ be a section through Z. Then $x(t; Y)$ crosses $\mathcal{S}$ infinitely many times. But there is room for only one intersection, by Proposition 9. Thus $x(t_1; Y) = x(t_2, Y)$ for some $t_1 \neq t_2$, and this is enough to guarantee that $x(t; Y)$ is closed. Then, clearly, $\omega(Y) = \{x(t; Y) : t \in [0, T]\}$ where T is the period of the orbit.

Consider a section $\mathcal{S}$ through Y. Since Y is in $\omega(X)$, there is a sequence t'_n so that $x(t'_n, X) \to Y$. If we use the flowbox theorem at Y, we see that $x(t, X)$ crosses $\mathcal{S}$ infinitely often, and arbitrarily close to Y (since this is clear for a straight flow, and initial conditions approaching the image of Y). Denote this of successive intersections of $\mathcal{S}$ by $\{t_n\}_{n \in \mathbb{N}}$.

***

Consider the sequence $x(t_n, X)$ where t_n are the successive intersection times of $x(t, X)$ with $\mathcal{S}$. Since the sequence converges to Y and it is monotone one way or the other by Proposition 8, it can only be monotonic towards Y.

Since the return time τ is continuous, we must have $t_{n+1} - t_n \to T$. By continuity with respect to initial conditions, we have $x(t', X_n) - x(t', Y) \to 0$ for any $t' \leq 2T$ (say). That is, $x(t' + t_n, X) - x(t', Y) \to 0$ as $n \to \infty$ for any fixed $t' \leq 2T$. But since $t_{n+1} - t_n \to T$, any sufficiently large t can be written as $t' + t_n, t' \leq 2T$. Let now Z be any point in $\omega(X)$. Then there is a sequence t''_n so that $x(t''_n, X)$ converges Z. Now, on the one hand we have $x(t''_n, X) - x(t''_n, Y) \to 0$ and on the other hand $x(t''_n, X) - Z \to 0$, and thus $\mathrm{dist}(Z, \omega(Y) = 0$, and since they are both compact sets, we have $Z \in \omega(Y)$ completing the proof.

***

1. Let $\mathcal{C}_Y$ be the trajectory through Y (this is nothing else but $\omega(Y)$, since $x(t; Y)$ is periodic). It is a compact set.

2. We found that $x(t_n; X) \to Y$. Since the return time τ is continuous, we must have $t_{n+1} - t_n \to T$.

3. By continuity with respect to initial conditions,

$$\sup_{\tau\in[0,2T]} d(x(t_n+\tau;X),\ x(\tau;Y)) \to 0$$

as $n\to\infty$.

4. Let t_j' be any increasing sequence, $t_j'\to\infty$. Then for every j there exists $n(j)$ so that $t_j'\in(t_{n(j)},t_{n(j)+1})$. Thus $0\le t_j'-t_{n(j)}\le 2T$ for large j.

5. By 3,

$$d(x(t_j';X),\ x(t_j'-t_{n(j)};Y))\to 0 \ \text{ as } j\to\infty$$

6. Of course, $x(t_j'-t_{n(j)};Y)\in\mathcal{C}_Y$. It follows thus that

$$d(x(t_j';X),\ \mathcal{C}_Y)\le d(x(t_j';X),\ x(t_j'-t_{n(j)};Y))\to 0$$

7. By the above, any sequence $x(t_j')$ which converges, has the limit in $\mathcal{C}_Y$. By definition then, $\omega(X)\subset\mathcal{C}_Y$. Since $\omega(X)$ is invariant and there is no strict subset of $\mathcal{C}_Y$ which is invariant (why?), we have $\omega(X)=\mathcal{C}_Y$.

□

Exercise 1. *Where have we used the fact that the system is planar? Think how crucial dimensionality is for this proof.*

10 Applications of the Poincaré-Bendixson theorem

A limit cycle is a closed orbit γ which is the ω-set of a point $X\notin\gamma$. There are of course closed orbits which are not limit cycles. For instance, the system $x'=-y, y'=x$ with orbits $x^2+y^2=C$ for any C clearly has no limit cycles.

But when limit cycles exist, they have at least one-sided stability.

Corollary 10. *Assume $\omega(X)=\gamma$ and $X\notin\gamma$ is a limit cycle. Then there exists a neighborhood $\mathcal{N}$ of X so that $\forall X'\in\mathcal{N}$ we have $\gamma=\omega(X')$.*

Proof. Let $Y\in\gamma$ and let $\mathcal{S}$ be a section through Y. Then, as we know, there is a sequence $t_n\to\infty$ so that the points $x(t_n,X)\in\mathcal{S}$ and $x(t_n,X)\to Y$. Take for instance a small enough open neighborhood $\mathcal{O}$ of $X_2\in(x(t_2;X),x(t_3,X))$. As we have shown, all points $X''\in\mathcal{O}$ have the property that $x(t'';X'')\in\mathcal{S}$ for some $t''>0$ (which, in fact, is small if $\mathcal{O}$ is small, and, in fact, $x(t'';X'')\in(x(t_2;X),x(t_3,X))$ as well.

We know, by continuity with respect to initial conditions, that $x(-t_2;X'')$ exist, if $\mathcal{O}$ is small enough. Consider the open set (by continuity) $\mathcal{O}_1=\{x(-t_2;X''),\ X''\in\mathcal{O}\}$. Since for any $X_1\in\mathcal{O}_1$ we have $x(t,X_1)\in(x(t_2;X),x(t_3,X))$ for some t, by monotonicity, we have $\omega(X_1)=\omega(X)$.

□

11 The Painlevé property

The classification of equations into integrable and nonintegrable, and in the latter case finding out whether the behavior is chaotic plays a major role in the study of dynamical systems.

As usual, for an $n-$th order differential equation, a constant of motion is a function $\Phi(u_1, ..., u_n, t)$ with a predefined degree of smoothness (analytic, meromorphic, C^n etc.) and with the property that for any solution $y(t)$ we have

$$\frac{d}{dt}\Phi(y(t), y'(t), ..., y^{(n-1)}(t), t) = 0$$

There are multiple precise definitions of integrability, and no one perhaps is comprehensive enough to be widely accepted. For us, let us think of a system as being integrable, relative to a certain regularity class of first integrals, if there are sufficiently many global constants of motion so that a particular solution can be found by knowledge of the values of the constants of motion.

We note once more that an integral of motion needs to be defined in a wide region. The existence of local constants along trajectories follows immediately either from the flowbox theorem, or from the implicit function theorem: indeed, if $\mathbf{Y}' = \mathbf{F}(\mathbf{Y})$ is a system of equations near a regular point, $\mathbf{Y}_0$, then evidently there exists a local solution $\mathbf{Y}(t; \mathbf{Y}_0)$. It is easy to check that $D_{Y_0}\mathbf{Y}_{|t=0} = \mathbf{I}$, so we can write, near $\mathbf{Y}_0, t = 0$, $\mathbf{Y}_0 = \mathbf{\Phi}(\mathbf{Y}, t)$. Clearly $= \mathbf{\Phi}$ is constant along trajectories.

In general, we have an integral of motion $\mathbf{\Phi}(\mathbf{Y}, t) = \mathbf{Y}(-t; \mathbf{Y}(t))$. This brings back the solution to where it started, so it must be a constant along trajectories. Not a very explicit function, admittedly, but smooth, at least locally. Given $\mathbf{Y}(t)$ it asks, where did it start, when t was zero. $\mathbf{\Phi}$ is thus obtained by integrating the equation backwards in time.

Is this an integral of motion?

Not really. This cannot be defined for t which is not small enough, in general since we cannot integrate backwards from any t to zero, without running into singularities. If we think of t in the complex domain, we may think of circumventing singularities, and define $\mathbf{\Phi}$ by analytic continuation around singularities. But what does that mean? If the singularities are always isolated, and in particular solutions are single valued, it does not matter which way we go. But if these are, say, square root branch points, if we avoid the singularity on one side we get $+\surd$ and on the other $-\surd$. There is no consistency.

But we see, if we impose the condition that the equation have only isolated singularities (at least, those depending on the initial condition, or *movable*, then we have a single valued global constant of motion, take away some lower dimensional singular manifolds in $\mathbb{C}^2$.

Such equations are said to have the Painlevé property (PP) and are integrable, at least in the sense above. But it turns out, in those considered so far in applications, that more is true: they were all ultimately re-derived from linear equations.

11.1 The Painlevé equations

11.2 Spontaneous singularities: The Painlevé's equation P_I

Let us analyze local singularities of the Painlevé equation P_I,

$$y'' = y^2 + x \tag{55}$$

In a neighborhood of a point where y is large, keeping only the largest terms in the equation (*dominant balance*) we get $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we may search for a power-like behavior

$$y \sim A(x - x_0)^p$$

where $p < 0$ obtaining, to leading order, the equation $Ap(p-1)x^{p-2} = A^2(x - x_0)^2$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). Let's look for a power series solution, starting with $6(x - x_0)^{-2}$: $y = 6(x-x_0)^{-2} + c_{-1}(x-x_0)^{-1} + c_0 + \cdots$. We get: $c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = -x_0/10, c_3 = -1/6$ and c_4 is undetermined, thus free. Choosing a c_4, all others are uniquely determined. To show that there indeed is a convergent such power series solution we substitute $y(x) = 6(x-x_0)^{-2} + \delta(x)$ where for consistency we should have $\delta(x) = o((x - x_0)^{-2})$ and taking $x = x_0 + z$ we get the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{56}$$

Note now that our assumption $\delta = o(z^{-2})$ makes $\delta^2/(\delta/z^2) = z^2\delta = o(1)$ and thus the nonlinear term in (56) is *relatively* small. Thus, *to leading order*, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximately by a linear equation. It is then natural to separate out the large terms from the small terms and write a fixed point equation for the solution based on this separation. We write (56) in the form

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \tag{57}$$

and integrate as if the right side was known. This leads to an equivalent integral equation. Since all unknown terms on the right side are chosen to be *relatively smaller*, by construction this integral equation is expected to be contractive.

Click here for Maple file of the formal calculation ($y'' = y^2 + x$)

The indicial equation for the Euler equation corresponding to the left side of (57) is $r^2 - r - 12 = 0$ with solutions $4, -3$. By the method of variation of parameters we thus get

$$\delta = \frac{D}{z^3} - \frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta^2(s)ds + \frac{z^4}{7}\int_0^z s^{-3}\delta^2(s)ds$$
$$= -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \tag{58}$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$; C is arbitrary. To find δ formally, we would simply iterate (58) in the following way: We take $r := \delta^2 = 0$ first and obtain $\delta_0 = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4$. Then we take $r = \delta_0^2$ and compute δ_1 from (58) and so on. This yields:

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + ... \tag{59}$$

This series is actually convergent. To see that, we scale out the leading power of z in δ, z^2 and write $\delta = z^2 u$. The equation for u is

$$u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z s u^2(s)ds$$
$$= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \tag{60}$$

It is straightforward to check that, given C_1 large enough (compared to $x_0/10$ etc.) there is an ϵ such that this is a contractive equation for u in the ball

$$y'' = 6y^2 \dots$$
$$y'' = 2y^3 + xy \dots$$
$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 \dots$$
$$y'' = \frac{y'^2}{2y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha)y \dots$$
$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y (y \dots}{y - \dots}$$
$$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)y' + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \frac{\beta x}{y^2} + \frac{\gamma(x-1)}{(y-1)^2} + \frac{\delta x (x - \dots}{(y-x) \dots}\right.$$

Figure 9: The six Painlevé equations, all equations of the form $y'' = R(y', y, x)$, with R rational, having the PP.

The equation $y'' = y^2 + x^2$ does not have the Painlevé property.
Click here for Maple file of the formal calculation, for $y'' = y^2 + x^2$

12 Discrete dynamical systems

The study of the Poincaré map leads naturally to the study of discrete dynamics. In this case we have closed trajectory, x_0 a point on it, $\mathcal{S}$ a section through x_0 and we take a point x_1 near x_0, on the section. If x_1 is sufficiently close to x, it must cross again the section, at x_1', still close to x_1, after the *return time* which

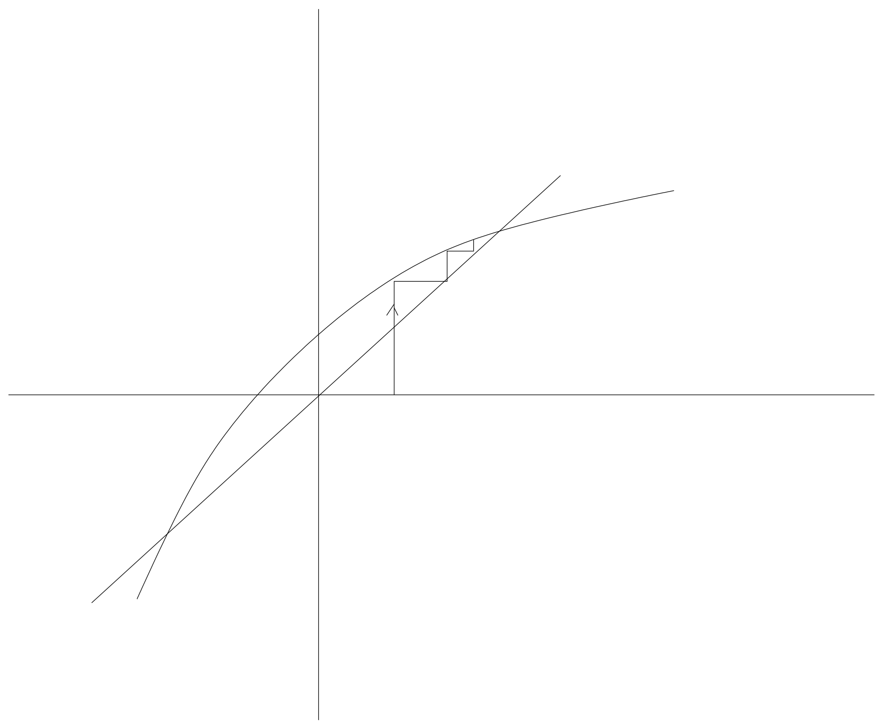

Figure 10:

is then close to the period of the orbit. The application $x_1 \to x_1'$ defines the Poincaré map, which is smooth on the manifold near x_0.

The study of the behavior of differential systems is near closed orbits is often more easily understood by looking at the properties of the Poincaré map.

In one dimension first, we are dealing with a smooth function f, where the iterates of f are what we want to understand.

We write $f^n(x) = f(f(...(f(x))))$ n times. The *orbit* of a point x_0 is the sequence $\{f^n(x_0)\}_{n\in\mathbb{N}}$, assuming that $f^n(x_0)$ is defined for all n. In particular, we may assume that $f : J \to J$, where $J \subset \mathbb{R}$ is an interval, possibly the whole line.

The effects of the iteration are often easy to see on the graph of the iteration, in which we use the bisector $y = x$ to conveniently determine the new point. We have $(x_0, 0) \to (x_0, f(x_0)) \to (f(x_0), f(x_0)) \to (f(x_0), f(f(x_0))$, where the two-dimensionality and the "intermediate" step helps in fact drawing the iteration faster: we go from x_0 up to the graph, horizontally to the bisector, vertically back to the graph, and repeat this sequence.

There are simple iterations, for which the result is simple to understand globally, such as

$$f(x) = x^2$$

where it is clear that $x = 1$ is a fixed point, if $|x_0| < 1$ the iteration goes to zero, and it goes to infinity if $|x_0| > 1$.

Local behavior near a fixed point is also, usually, not difficult to understand, analytically and geometrically.

Theorem 6. *(a) Assume f is smooth, $f(x_0) = x_0$ and $|f'(x_0)| < 1$. Then x_0 is a sink, that is, for x_1 in a neighborhood of x_0 we have $f^n(x_1) \to x_0$.*

(b) If instead we have $|f'(x_0)| > 1$, then x_0 is a source, that is, for x_1 in a small neighborhood $\mathcal{O}$ of x_0 we have $f^n(x_1) \notin \mathcal{O}$ for some n (this does not mean that $f^m(x_1)$ cannot return "later" to $\mathcal{O}$, it just means that points very nearby are repelled, in the short run.)

Proof. We show (a), (b) being very similar. Without loss of generality, we take $x_0 = 0$. There is a $\lambda < 1$ and ϵ small enough so that $|f'(x)| < \lambda$ for $|x| < \epsilon$. If we take x_1 with $|x_1| < \epsilon$, we have $|f(x_1)| = |f'(c)||x_1| < \lambda|x_1|(< \epsilon)$, so the inequality remains true for $f(x_1) : |f(f(x_1))| < \lambda|f(x_1)| < \lambda^2|x_1|$ and in general $f^n(x_1) = O(\lambda^n) \to 0$ as $n \to \infty$.

In fact, it is not hard to show that, for smooth f, the evolution is essentially geometric decay.

□

When the derivative is one, in absolute value, the fixed point is called neutral or indifferent. It does not mean that it can't still be a sink or a source, just that we cannot resort to an argument based on the derivative, as above.

Example 4. *We can examine the following three cases:*

(a) $f(x) = x + x^3$.
(b) $f(x) = x - x^3$
(c) $f(x) = x + x^2$.

It is clear that in the first case, any positive initial condition is driven to $+\infty$. Indeed, the sequence $f^n(x_1)$ is increasing, and it either goes to infinity or else it has a limit. But the latter case cannot happen, because the limit should satisfy $l = l + l^3$, that is $l = 0$, whereas the sequence was increasing.

The other cases are analyzed similarly: in (a), if $x_0 < 0$ then the sequence still diverges. Case (c) is more interesting, since the sequence converges to zero if $x_1 < 0$ is small enough and to ∞ for all $x_1 > 0$. We leave the details to the reader.

It is useful to see what the behavior of such sequences is, in more detail.

Let's take the case (c), where $x_1 < 0$. We have

$$x_{n+1} = x_n + x_n^2$$

where we expect the evolution to be slow, since the relative change is vanishingly small. We then approximate the true evolution by a differential equation

$$(d/dn)x = x^2$$

giving

$$x_n = (C - n)^{-1}$$

We can show rigorously that this is the behavior, by taking $x_n = -1/(n+c_0)+\delta$, $\delta_{n_0} = 0$ and we get

$$\delta_{n+1} - \delta_n = \frac{1}{n^2(n+1)} - \frac{2}{n}\delta_n + \delta_n^2 \tag{61}$$

and thus

$$\delta_n = \sum_{j=n_0}^{n} \left(\frac{1}{j^2(j+1)} - \frac{2}{j}\delta_j + \delta_j^2 \right) \tag{62}$$

Exercise 1. *Show that (62) defines a contraction in the space of sequences with the property* $|\delta_n| < C/n^2$*, where you choose* C *carefully.*

Exercise 2. *Find the behavior for small positive* x_1 *in (b), and then prove rigorously what you found.*

12.1 Bifurcations

The local number of fixed points can only change when $f'(x_0) = 1$. As before, we can assume without loss of generality that $x_0 = 0$.

We have

Theorem 7. *Assume* $f(x, \lambda)$ *is a smooth family of maps, that* $f(0, 0) = 0$ *and that* $f_x(0, 0) \neq 1$*. Then, for small enough* λ *there exists a smooth function* $\varphi(\lambda)$*, also small, so that* $f(\varphi(\lambda), \lambda) = \varphi(\lambda)$*, and the character of the fixed point (source or sink) is the same as that for* $\lambda = 0$*.*

Exercise 3. *Prove the theorem, using the implicit function theorem.*

References

[1] E.A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, (1955).

[2] M.W.Hirsch, S. Smale and R.L. Devaney, *Differential Equations, Dynamical Systems & An Introduction to Chaos*, Academic Press, New York (2004).

[3] D. Ruelle, *Elements of Differentiable Dynamics and Bifurcation theory*, Academic Press, Ney York , (1989)