1 Regular singular points of differential equations, nondegenerate case

Consider the system

$$w' = \frac{1}{z}Bw + A_1(z)w \tag{1}$$

where B is a constant matrix and A_1 is analytic at zero. Assume no two eigenvalues of B differ by an positive integer. Let J be the Jordan normal form of B. As usual, by a linear change of variables, we can bring (1) to the normal form

$$y' = \frac{1}{z}Jy + A(z)y \tag{2}$$

where $A(z)$ is analytic.

Theorem 1. *Under the assumptions above, (1) has a fundamental matrix solution in the form $M(z) = Y(z)z^B$, where $Y(z)$ is a matrix analytic at zero.*

Proof. Clearly, it is enough to prove the theorem for (2). We look for a solution of (2) in the form $M = Yz^J$, where

$$Y(z) = J + zY_1 + z^2Y_2 + \cdots \tag{3}$$

we get

$$Y'z^J + \frac{1}{z}YJz^J = \frac{1}{z}JYz^J + AYz^J \tag{4}$$

Multiplying by z^{-J} we obtain

$$Y' + \frac{1}{z}YJ = \frac{1}{z}JY + AY \tag{5}$$

or

$$Y' = \frac{1}{z}\Big(JY - YJ\Big) + AY \tag{6}$$

Expanding out, we get

$$\begin{aligned} Y_1 + 2zY_2 + 3z^2Y_3 + \cdots = &\Big[(JY_1 - Y_1J) + z(JY_2 - Y_2J) + \cdots\Big] \\ &+ A_0J + zA_1J + \cdots + zA_0Y_1 + z^2(A_0Y_2 + A_1Y_1) + \cdots \end{aligned} \tag{7}$$

The associated system of equations, after collecting the powers of z is

$$kY_k = (JY_k - Y_kJ) + A_kJ + \sum_{l=0}^{k-2} A_lY_{k-1-l}; \quad k \in \mathbb{N} \tag{8}$$

or

$$V_kY_k = A_{k-1}J + \sum_{l=0}^{k-2} A_lY_{k-1-l}; \quad k \in \mathbb{N} \tag{9}$$

where

$$V_k M := kM - (JM - MJ) \tag{10}$$

is a linear operator on matrices $M \in \mathbb{R}^{n^2}$. As a linear operator on a finite dimensional space, $V_k X = Y$ has a unique solution for every Y iff $\det V_k \neq 0$ or, which is the same, $V_k X = 0$ implies $X = 0$.

We show that this is the case.

Let v be one of the eigenvectors of J. If $V_k X = 0$ we obtain,

$$k(Xv) - J(Xv) + X\lambda v = 0 \tag{11}$$

or

$$k(Xv) - J(Xv) + X\lambda v = 0 \tag{12}$$

$$J(Xv) = (\lambda + k)(Xv) \tag{13}$$

Since $\lambda + k$ is not an eigenvalue of J, this forces $Xv = 0$. We take the next generalized eigenvector, w, in the same Jordan block as v, if any.

We remind that we have the following relations between these generalized eigenvectors:

$$Jv_i = \lambda v_i + v_{i-1} \tag{14}$$

where $v_0 = v$ is an eigenvector and $1 \leq i \leq m-1$ where m is the dimension of the Jordan block.

We have

$$k(Xw) - J(Xw) + X(\lambda w + v) = 0 \tag{15}$$

and since $Xv = 0$ we get

$$J(Xw) = (\lambda + k)(Xw) \tag{16}$$

and thus $Xw = 0$. Inductively, we see that $Xv = 0$ for any generalized eigenvector of J, and thus $X = 0$.

Now, we claim that $V_k^{-1} \leq Ck^{-1}$ for some C. We let $\mathcal{C}$ be the commutator operator, $\mathcal{C}X = JX - XJ$ Now $\|JX - XJ\| \leq 2\|J\|\|X\|$ and thus

$$V_k^{-1} = k^{-1}\left(I - k^{-1}\mathcal{C}\right)^{-1} = k^{-1}(1 + o(1)); \quad (k \to \infty) \tag{17}$$

Therefore, the function kV_k is bounded for $k \in \mathbb{R}^+$.

We rewrite the system (8) in the form

$$Y_k = V_k^{-1} A_{k-1} J + V_k^{-1} \sum_{l=0}^{k-2} A_l Y_{k-1-l}; \quad k \in \mathbb{N} \tag{18}$$

or, in abstract form, with $\mathsf{Y} = \{Y_j\}_{j\in\mathbb{N}}$, $(\mathsf{LY})_k := V_k^{-1} \sum_{l=0}^{k-2} A_l Y_{k-1-l}$, where we regard Y as a function defined on $\mathbb{N}$ with matrix values, with the norm

$$\|\mathsf{Y}\| = \sup_{n\in\mathbb{N}} \|\mu^{-n}\mathsf{Y}(n)\|; \quad \mu > 1 \tag{19}$$

we have

$$\mathsf{Y} = \mathsf{Y}_0 + \mathsf{LY} \tag{20}$$

□

Exercise 1. *Show that (20) is contractive for μ sufficiently large, in an appropriate ball that you will find.*

2 Changing the eigenvalue structure of J by transformations

Assume we want to make a transformation so that all eigenvalues of the new system are unchanged, except for one of them which we would like to decrease by one. We write J in the form

$$\begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \tag{21}$$

where J_1 is the Jordan block we care about, $\dim(J_1) = m \geq 1$, while J_2 is a Jordan matrix, consisting of the remaining blocks. Suppose for now that the assumptions of Theorem1 are satisfied. The fundamental solution of $y' = z^{-1}Jy + A(z)y$ is of the form $M = Y(z)z^J$. Suppose we make the transformation

$$M_1 = M \begin{pmatrix} z^{-I} & 0 \\ 0 & I \end{pmatrix} = MS$$

$$S = \begin{pmatrix} z^{-1}I & 0 \\ 0 & I \end{pmatrix} \tag{22}$$

with the diagonal block sizes, counting from above, m and $n-m$. Then,

$$MS = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{13} & Y_{14} \end{pmatrix} \begin{pmatrix} z^{J_1} & 0 \\ 0 & z^{J_2} \end{pmatrix} \begin{pmatrix} z^{-I} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{13} & Y_{14} \end{pmatrix} \begin{pmatrix} z^{J_1 - I} & 0 \\ 0 & z^{J_2} \end{pmatrix} \tag{23}$$

where Y_{ij} are matrices, $\dim(Y_{11}) = \dim(Y_{21}) = m$, $\dim(Y_{12}) = \dim(Y_{22}) = n-m$. The modified matrix, MS, has a structure that would correspond to the Jordan matrix

$$\begin{pmatrix} J_1 - I & 0 \\ 0 & J_2 \end{pmatrix} \tag{24}$$

and thus to a new B with eigenvalues as described at the beginning of the section. The change of variable $Sy = w$, or $y = S^{-1}w = Uw$ where, clearly,

$$U = S^{-1} = \begin{pmatrix} zI & 0 \\ 0 & I \end{pmatrix} \tag{25}$$

is natural, the only question is whether the new equation is of the same type. So, let's check. Let $y = Uw$. Then, we have

$$U'w + Uw' = \frac{1}{z}JUw + AUw \tag{26}$$

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} w + \begin{pmatrix} zI & 0 \\ 0 & I \end{pmatrix} w' = \frac{1}{z}JUw + AUw \tag{27}$$

$$w' = \begin{pmatrix} z^{-1}I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} J_1 - I & \\ 0 & J_2 \end{pmatrix} \frac{1}{z} \begin{pmatrix} zI & 0 \\ 0 & I \end{pmatrix} w$$
$$+ \begin{pmatrix} z^{-1}I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{21} \end{pmatrix} \begin{pmatrix} zI & 0 \\ 0 & I \end{pmatrix} \quad (28)$$

or

$$w' = \frac{1}{z} \begin{pmatrix} J_1 - I & A_{12}(0) \\ 0 & J_2 \end{pmatrix} + A_1 w = \frac{1}{z} B_1 w + A_1 w \quad (29)$$

where A_{ik} are matrices of the same dimensions as Y_{ik}. The eigenvalues of B_1 are now $\lambda_1 - 1, \lambda_2, ...$

Exercise 1. Use this procedure repeatedly to reduce any resonant system to a nonresonant one. That is done by arranging that the eigenvalues that differ by positive integers become equal.

Exercise 2. Use Exercise 1 to prove the following result.

Theorem 2. *Any system of the form*

$$y' = \frac{1}{z} B(z) y \quad (30)$$

where B is an analytic matrix at zero, has a fundamental solution of the form

$$M(z) = Y(z) z^{B'} \quad (31)$$

where B' is a constant matrix, and Y is analytic at zero. In the nonresonant case, $B' = B(0)$.

Note that this applies even if $B(0) = 0$.

Exercise 3. *Find B' in the case where only two eigenvalues differ by a positive integer, where the integer is 1.*

3 Scalar $n-$th order linear equations

These are equations of the form

$$y^{(n)} + a_1(z) y^{(n-1)} + \cdots + a_n(z) y = 0 \quad (32)$$

Such an equation can always be transformed into a system of the form $w' = A(x)w$, and viceversa. There are many ways to do that. The simplest is to take $v_0 = y, ..., v_k = y^{(k)}, ...$ and note that (32) is equivalent to

$$\begin{pmatrix} v_0 \\ v_1 \\ \cdots \\ v_{n-1} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & \cdots & & \\ -a_n(z) & -a_{n-1}(z) & -a_{n-2}(z) & \cdots & -a_1(z) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ \cdots \\ v_{n-1} \end{pmatrix} \quad (33)$$

In the other direction, to simplify notation assume $n = 2$. The system is then

$$\begin{aligned} y' &= a(x)y + b(x)w \\ w' &= c(x)y + d(x)w \end{aligned} \tag{34}$$

We differentiate one more time say, the second equation, and get

$$w'' = dw' + (bc + d')w + (ac + c')y \tag{35}$$

If $c \equiv 0$, then (35) is already of the desired form. Otherwise, we write

$$y = \frac{1}{c}w' - \frac{d}{c}w \tag{36}$$

and substitute in (35). The result is

$$w'' = \left(a + d + \frac{c'}{c}\right) w' + \left(c \left(\frac{d}{c}\right)' + [cb - ad]\right) w \tag{37}$$

Note that a and c, by assumptions, have at most first order poles, while c'/c has at most simple poles for any analytic function. Therefore, the emergent second order equation has the general form

$$w'' + a_1(x)w' + a_2(x)w = 0$$

where a_i has a pole of order at most i.

Exercise 1. Generalize this transformation procedure to nth order systems. Show that the resulting nth order equation is of the general form

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = 0 \tag{38}$$

where the coefficients a_i are analytic in $\mathbb{D}_\rho \setminus \{0\}$ and have a pole of order at most i at zero.

Definition 1. An equation of the form (38) has a singularity of the first kind at zero if the conditions of Exercise 1 are met.

Definition 2. *An equation of the form (38) has a regular singularity at zero if there exists a fundamental set of solutions in the form of finite combinations of functions of the form*

$$y_i = z^{\lambda_i} (\ln z)^{m_i} f_i(z); \quad (\textit{by convention, } f_i(0) \neq 0) \tag{39}$$

where f_i are analytic, $m_i \in \mathbb{N} \cup \{0\}$

Theorem 3 (Frobenius)**.** *An equation of the form (38) has a regular singularity at zero iff the singular point is of the first kind. (Clearly, a similar statement holds at any point.)*

Proof. In one direction, we show that there is a reformulation of (38) as a system of the form (30). The "if" part follows from this and Theorem 2. Clearly, the transformation leading to (33) produces a system of equations with a singularity of order n.

Exercise 2. *If $y = y_i$ is of the form (39), then near zero we have $y'/y = \lambda z^{-1}(1 + o(1))$, $y''/y' = (\lambda - 1)z^{-1}(1 + o(1))$ etc.*

By Exercise 2 we see that in the solution of (33), v_k roughly behaves like $z^{-k}z^{\lambda}\ln(z)^m F_m(z)(1 + o(1))$ with F_m analytic. Instead, if it was a result of solving an equation of the form (30) we should have $v_j \sim c_j z^{\lambda}\ln(z)^m(1 + o(1))$. Then, the natural substitution to attempt is

$$\varphi_k = z^{k-1}y^{(k-1)},\ k = 1, 2, ..., n \tag{40}$$

We then have $y^{(k-1)} = z^{-k+1}\varphi_{k-1}$ and

$$\varphi_{l+1} = z^l y^{(l)} = z^l \left(z^{-l+1}\varphi_l\right)' = (1 - l)\varphi_l + z\varphi_l' \tag{41}$$

or

$$z\varphi_l' = (l - 1)\varphi_l + \varphi_{l+1} \tag{42}$$

while

$$\varphi_n' = z^n y^{(n)} = -\sum_{k=0}^{n-1} z^n z^{-k+1}\varphi_{k-1}a_{n-k+1}(z) = -\sum_{k=0}^{n-1} b_{n-k+1}(z)\varphi_{k-1} \tag{43}$$

where $b_{n-k+1}(z)$ are analytic. n matrix form, the end result is the system

$$\boldsymbol{\varphi}' = z^{-1}B\boldsymbol{\varphi} \tag{44}$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 2 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ & & \cdots & & & \\ -b_n(z) & -b_{n-1}(z) & -b_{n-2}(z) & -b_{n-3}(z)\cdots & (n-1) - b_1(z) & \end{pmatrix} \tag{45}$$

which is exactly of the type (33). From the type of solutions of (33) and backward substitution, the "if" part follows immediately. □

3.0.1 Examples of behavior of solutions for first versus higher kind singular equations

Consider the following simple examples.

(1) $f'' + (1 + 1/z)f' - 2f/z^2 = 0$; (2) $f'' + (1 + 1/z^2)f' - 2f/z^3 = 0$.

What is the expected behavior of the solutions? If we try $f(z) = z^m + \cdots$ in (1) we get $(m^2 - 2)z^{-2} + \cdots = 0$ thus $m = \pm\sqrt{2}$.

We try a solution of the form $f(z) = z^{\sqrt{2}} \sum_{k=0}^{\infty} c_k z^k = z^{\sqrt{2}} g(z)$, and get c_0 arbitrary, so we take, say, c_0, then $c_1 = -\sqrt{2}/(1+2\sqrt{2})$, and in general,

$$c_m = -\frac{m+\sqrt{2}-1}{m(m+2\sqrt{2})} c_{m-1} \tag{46}$$

It is easy to show that c_m are bounded (in fact $|c_m| \sim 1/(m+1)!$, and then g is analytic.

Let us consider, instead, (2). The same substitution, $z^m + \cdots$ now gives $m = 2$.

We try a power series solution of the form $f(z) = \sum_{k=2}^{\infty} c_k z^k$ Again c_0 is undetermined, say we take it to be one, and in general we have

$$c_m = -(m+1)c_{m-1} - c_{m-2} \tag{47}$$

This time, it is not hard to show, $|c(m)| \sim (m+1)!$, and the series diverges.

3.1 Indicial equation

Consider again equation (32). We know from Theorem 3 that there is a solution of the form $y = z^{\lambda} y_{\lambda}(z)$ where $y_{\lambda}(z)$ is analytic at zero, and we can assume, without loss of generality that $y_{\lambda}(0) = 1$. On the other hand, we easily check that $y^{(m)} = z^{\lambda-m}\varphi_m(z)$ where φ is analytic at zero. Thus, inserting $z^{\lambda} y_{\lambda}(z)$ in (32) and dividing by $z^{\lambda-n}$ we get an integer power series identity. The coefficient of the highest power of z clearly depends on λ and it is called the indicial equation. It is also clear that, insofar as this highest power of z is concerned, we would have obtained the same equation if we had substituted instead of $z^{\lambda} y_{\lambda}(z)$ simply z^{λ}, as higher powers in the series of y_{λ} contribute with higher powers of z in the final equation. The contribution of $y^{(n)}$ to the final equation is simply $\lambda(\lambda-1)\cdots(\lambda-n+1)$. The contribution of $y^{(n-1)}$ is $\lambda(\lambda-1)\cdots(\lambda-n+1)(za_1(0) = \lambda(\lambda-1)\cdots(\lambda-n+1)b_1(0)$. Thus, the indicial equation is

$$b_n(0) + \sum_{j=0}^{n-1} \lambda(\lambda-1)\cdots(\lambda-(n-j)+1)b_j(0) = 0 \tag{48}$$

3.1.1 Relation to the matrix form

Let's look at the eigenvalue problem for the matrix form of (32):

$$B(0) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdots \\ x_n \end{pmatrix} \tag{49}$$

where B is given in (45). This amounts to the system:

$$\begin{aligned}
&x_2 = \lambda x_1\\
&x_2 + x_3 = \lambda x_2\\
&2x_3 + x_4 = \lambda x_3\\
&\cdots\\
&-b_n x_1 - b_{n-1}x_2 - \ldots - b_1 x_n = (\lambda - (n-1))x_n
\end{aligned} \tag{50}$$

Inductively we see that $x_l = (\lambda - (l-2))x_{l-1}$ and thus, taking $x_1 = 1$ we have

$$x_j = \lambda(\lambda - 1)\cdots(\lambda - (j-2)) \tag{51}$$

Inserting (51) into the last equation in (50) we obtain exactly (48). Thus, every solution λ of the indicial equation gives rise to a solution of (38) in the form $z^\lambda y_\lambda(z)$ with $y_\lambda(z)$ analytic. Check that, provided the eigenvalues of $B(0)$ do not differ by integers, the other solutions are combinations of the form $z^\lambda y_{\lambda;l}(z)\ln^l z$ where $l \le m-1$ and m is the size of the Jordan block corresponding to λ. If they do differ by integers, then for any root λ of (48) such that $\lambda + m$ is not a root of (48), we still get a family of solutions of the form mentioned. Other solutions are obtained by first arranging nonresonance, or by reduction of order, as shown below.

3.2 Reduction of order

Let λ_1 be a characteristic root such that $\lambda_1 + n$ is not a characteristic root. Then, there is a solution of (38) of the form $y_1 = z^{\lambda_1}\varphi(z)$, where $\varphi(z)$ is analytic and we can take $\varphi(0) = 1$.

We can assume without loss of generality that $\lambda_1 = 0$. Indeed, otherwise we first make the substitution $y = z^{\lambda_1} w$ and divide the equation by z^{λ_1}.

The general term of the new equation is of the form

$$\begin{aligned}
z^{-\lambda_1} b_l z^{-l}(z^{\lambda_1} w)^{n-l} &= z^{-\lambda_1} b_l z^{-l} \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} (z^{\lambda_1})^{(n-l-j)}\\
&= z^{-\lambda_1} b_l z^{-l} \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} z^{\lambda_1 - n + l + j} = b_l \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} z^{-(n-j)}
\end{aligned} \tag{52}$$

which is of the same type as (38).

Thus we assume $\lambda_1 = 0$ and take $y = \varphi w$. As discussed, we can assume $\varphi(0) = 1$. The equation for w is

$$\sum_{l=0}^{n} z^{-l} b_l \sum_{j=0}^{n-l} \binom{n-l}{j} w^{(j)} \varphi^{(n-l-j)} = 0 \tag{53}$$

or

$$\sum_{j=0}^{n} w^{(j)} \sum_{l=0}^{n-j} z^{-l} b_l \binom{n-l}{j} \varphi^{(n-l-j)} = 0 \tag{54}$$

or also

$$\sum_{j=0}^{n} w^{(n-j)} \sum_{l=0}^{j} z^{-l} b_l \binom{n-l}{j} \varphi^{(j-l)} = 0 \tag{55}$$

We note that this equation, after division by φ (recall that $1/\varphi$ is analytic) is of the same form as (38). However, now the coefficient of w is

$$\sum_{l=0}^{n} z^{-l} b_l \binom{n-l}{0} \varphi^{(n-l)} = \sum_{l=0}^{n} z^{-l} b_l \varphi^{(n-l)} = 0 \tag{56}$$

since this is indeed the equation φ is solving.

We divide the equation by φ (once more, remember $\varphi(0) = 1$), and we get

$$\sum_{j=0}^{n-1} w^{(1+(n-1-j))} \tilde{b}_j = 0 \tag{57}$$

where

$$\tilde{b}_j = \sum_{l=0}^{j} z^{-l} b_l \binom{n-l}{j} \frac{\varphi^{(j-l)}}{\varphi} \tag{58}$$

has a pole of order at most j, or

$$\sum_{j=0}^{n-1} g^{(n-1-j)} \tilde{b}_j = 0 \tag{59}$$

with $w' = g$. This is an $(n-1)$th order equation for g, and solving the equation for w reduced to solving a lower order equation, and one integration, $w = \int g$.

Thus, by knowing, or assuming to know, one solution of the nth order equation, we can reduce the order of the equation by one. Clearly, the characteristic roots for the g equation are $\lambda_i - \lambda_1 - 1$, $i \neq 1$. We can repeat this procedure until the equation for g becomes of first order, which can be explicitly solved. This shows what to do in the degenerate case, other than, working in a similar (in some sense) way with the equivalent nth order system.

3.2.1 Reduction of order in a degenerate case: an example

Consider the equation

$$z(z-1)y'' + y = 0 \tag{60}$$

This equation can be solved in terms of hypergeometric functions, but it is easier to understand the solutions, at least locally, from the equation. The indicial equation is $r(r-1) = 0$ (a *resonant case*: the roots differ by an integer). Substituting $y_0 = \sum_{k=1}^{\infty} c_k z^k$ in the equation and identifying the powers of z yields the recurrence

$$c_{k+1} = \frac{k^2 - k + 1}{k(k+1)} c_k \tag{61}$$

with c_1 arbitrary, which of course we can take to be 1. By induction we see that $0 < c_k < 1$. Thus the power series has radius of convergence at least 1. The radius of convergence is in fact exactly one as it can be seen applying the ratio test and using (61); the series converges exactly up to the nearest singularity of (60). We knew that we must get an analytic solution, by the general theory. We let $y_0 = y_0 \int g(s)ds$ and get, after some calculations, the equation

$$g' + 2\frac{y_0'}{y_0}g = 0 \tag{62}$$

and, by the previous discussion, $2y_0'/y_0 = 2/z + A(z)$ with $A(z)$ is analytic. The point $z = 0$ is a regular singular point of (62) and in fact we can check that $g(z) = C_1 z^{-2} B(z)$ with C_1 an arbitrary constant and $B(z)$ analytic at $z = 0$. Thus $\int g(s)ds = C_1(a/z + b\ln(z) + A_1(z)) + C_2$ where $A_1(z)$ is analytic at $z = 0$. Undoing the substitutions we see that we have a fundamental set of solutions in the form $\{y_0(z), B_1(z)\ln z + B_2(z)\}$ where B_1 and B_2 are analytic.

3.2.2 Singularities at infinity

An equation has a singularity of first kind at infinity, if after the change of variables $z = 1/\zeta$, the equation in ζ has a singularity of first kind at zero.

For instance, (60) changes into

$$y'' + \frac{2}{\zeta}y + \frac{y}{\zeta^2(1-\zeta)} = 0 \tag{63}$$

As a result, we see that (60) only has singularities of the first kind on the Riemann sphere, $\mathbb{C}_\infty$.

Exercise 3. (i) Show that *any* nonzero solution of (60) has at least one branch point in $\mathbb{C}$. (Hint: Examine the indicial equations at: $0, 1$ and ∞. Alternatively, you can use the indicial equation at ∞ and (61).)

(ii) Use the substitution (50) to bring the equation to a system form. What is the matrix B_0', the matrix B' in the notation of Theorem 2 at $z = 0$? What is its Jordan normal form?

(iii) If we write the B_1' corresponding to the singular point $z = 1$, can B_0' and B_1' commute?

4 General isolated singularities

We now take a system of the form

$$y' = By \tag{64}$$

Interpreted as as a matrix equation, we write

$$Y' = BY \tag{65}$$

where, for some $\rho > 0$ the matrix $B(z)$ is analytic in $\mathbb{D}_\rho \setminus \{0\}$. We do not assume anymore that the singularity is a pole. It is clear that (65) has, at any point $z_0 \in \mathbb{D}_\rho \setminus \{0\}$, a fundamental matrix solution Y_0, and that the general matrix solution of (65) is $Y_0 K$ where K is an invertible constant matrix. Indeed, Y_0 is invertible, and if Y is any solution we can thus always write $Y = Y_0 K$, clearly, for $K = Y_0^{-1} Y$. Then, we can check that $Y_0 K' = 0$, or $K' = 0$ which is what we claimed. By our general arguments, Y_0 is analytic (at least) in a disk of radius $|z_0|$. If we take a point $z_1 = z_0 e^{i\phi}$, with ϕ small enough, then the disk $\mathbb{D}_{|z_0|}(z_0)$ and the disk $\mathbb{D}_{|z_0|}(z_1)$ overlap nontrivially, and then $Y_0 = Y_1 K_1$ for some constant matrix K. We see that Y_0 is analytic in $\mathbb{D}_{|z_0|}(z_1)$. It follows that Y_0 is analytic on the Riemann surface of the log at zero, that is, it can be continued along any curve in $\mathbb{D}$ not crossing zero: $Y_0 \to Y_1 K_1 \to Y_2 K_1 K_2 \cdots$. Does this mean that

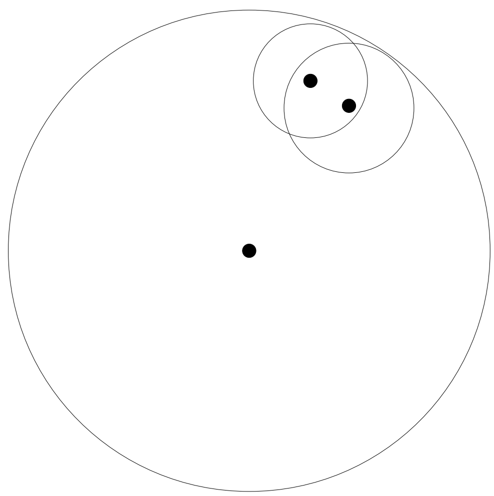

Figure 1:

Y_0 is analytic in $\mathbb{D} \setminus \{0\}$? Absolutely not. This is because after one full loop, we may return at z_0 with $Y_n K_n \cdots K_1 = Y_0 K$ for some nontrivial K. To see that, we can just recall the general solution $M(z) z^{B'}$ valid when zB is analytic, or simply look at the solution of the equation $y' = \sqrt{2} y/z,\ y = z^{\sqrt{2}}$. However, note that K is invertible, and thus it can be written in the form e^C. Indeed, the Jordan form J of K is of the type $D + N$, where D is diagonal with all elements on the diagonal nonzero, and N is a nilpotent commuting with D. We then write $D + N = D(1 + D^{-1} N)$ and note that $N_1 := D^{-1} N$ is also nilpotent. We define $\log(D + N) = \log D + \sum_{j=1}^{m} (-1)^{j-1} N_1 j/j$ where m is the size of the largest Jordan block in J. We define $\ln K = U^{-1} \ln J U$ where U is the matrix that brings K to its Jordan form. We can check that $e^{\ln J} = J$ and consequently $e^{\ln K} = K$. If we write $K = e^{2\pi i P}$ we note that the matrix $Y_0 z^{-P}$ is single-valued, thus analytic in $\mathbb{D}_\rho \setminus \{0\}$. Thus we have proved,

Theorem 4. *The matrix equation (65), under the assumptions there, has a fundamental solution of the form* $A(z)z^P$ *where* A *is analytic in* $\mathbb{D}_\rho \setminus \{0\}$.

4.1 Converse of Frobenius' theorem

For the proof, we note that we can always change coordinates so that $P = J$ is in Jordan normal form. Then, the equation (38) has the general solution in the form $A(x)x^J K$ where K is a constant matrix. Then, check that there is a solution of the form $x^\lambda y_1(x)$ where y_1 is analytic in $\mathbb{D}_\rho \setminus \{0\}$. By performing a reduction of order on the associated nth order equation and rewriting that as a system, check that we get a system of order $n-1$, otherwise of the same form (38).

Exercise 1. *Use induction on* n *to complete the proof.*

5 Nonlinear systems

A point, say $z = 0$ is a singular point of the first kind of a nonlinear system if the system can be written in the form

$$y' = z^{-1}h(z, y) = z^{-1}(L(z)y + f(z, y)) \tag{66}$$

where h is analytic in z, y in a neighborhood of $(0, 0)$. We will not analyze these systems in detail, but much is known about them, [3] [2]. The problem, in general, is nontrivial and the most general analysis to date for one singular point is in [3], and utilizes techniques beyond the scope of our course now. We present, without proofs, some results in [2], which are more accessible. They apply to several singular points, but we will restrict our attention to just one, in the setting of (66). In the nonlinear case, a "nonlinear nonresonance" condition is needed, namely: if λ_i are the eigenvalues of $L(0)$, we need a *diophantine condition*: for some $\nu > 0$ we have

$$\inf\left\{(|\mathbf{m}| + k)^\nu |k + \mathbf{m}\cdot\boldsymbol{\lambda} - \lambda_i| \,\middle|\, \mathbf{m} \in \mathbb{N}^n, |\mathbf{m}| > 1, k \in \mathbb{N}\cup\{0\}; i \le n\right\} > 0 \tag{67}$$

Furthermore, $L(0)$ is assumed to be diagonalizable. (In [3] a weaker nonresonance condition is imposed, known as the Brjuno condition, which is known to be optimal.)

Proposition 3. *Under these assumptions, There is a change of coordinates* $y = \Phi(z)u(z)$ *where* Φ *is analytic with analytic inverse, so that the system becomes*

$$u' = z^{-1}h(z, u) = z^{-1}(Bu + f(z, u)) \tag{68}$$

where B *is a constant matrix.*

Proposition 4. *The system (68) is analytically equivalent* in a neighborhood of $(0, 0)$*, that is for small* u *as well as small* z*, to its linear part, namely to the system*

$$w' = z^{-1}Bw \tag{69}$$

In terms of solutions, it means that the general *small* solution of (66) can be written as

$$y = H(z, \Phi(z)z^B C) \tag{70}$$

where $H(u,v)$ is analytic as a function of two variables, C is an arbitrary constant vector. The diophantine, and more generally, Brjuno condition is generically satisfied. If the Brjuno condition fails, equivalence is still possible, but unlikely. The structure of y in (70) is

$$y_j(z) = \sum_{m,k} c_{k,\mathbf{m}} z^k z^{\mathbf{m}\cdot\boldsymbol{\lambda}} \tag{71}$$

i.e., a convergent multiseries in powers of $z, z^{\lambda_1}, ..., z^{\lambda_n}$.

6 Variation of parameters

As we discussed, a linear nonhomogeneous equation can be brought to a linear homogeneous one, of higher order. While this is useful in a theoretical quest, in practice, it is easier to solve the associated homogeneous system and obtain the solution to the nonhomogeneous one by integration. Indeed, if the matrix equation

$$Y' = B(z)Y \tag{72}$$

has the solution $Y = M(z)$, then in the equation

$$Y' = B(z)Y + C(z) \tag{73}$$

we seek solutions of the form $Y = M(z)W(z)$. We get

$$M'W + MW' = B(z)MW + C(z) \quad \text{or} \quad M(z)W' = C(z) \tag{74}$$

giving

$$Y = M(z)\int_a^z M^{-1}(s)C(s)ds \tag{75}$$

7 Equilibria

We start with the simple example of the harmonic oscillator. It is helpful in a number of ways, since we have a good intuitive understanding of the system. Yet, the ideal (frictionless) oscillator has nongeneric features.

We can use conservation of energy to write

$$\frac{1}{2}mv^2 + mgl(1 - \cos x) = const \tag{76}$$

where x is the angle and $v = dx/dt$, so with $l = 1$ we get

$$x'' = -\sin x \tag{77}$$

7.1 Exact solutions

This equation can be solved exactly, in terms of Weierstrass elliptic functions. Integration could be based on (78), and also by multiplication by x' and integration, which leads to the same.

$$\frac{1}{2}{x'}^2 - \cos x = C \tag{78}$$

$$\int_0^x \frac{ds}{\sqrt{C + 2\cos s}} = t + t_0 \tag{79}$$

With the substitution $\tan(x/2) = u$ we get

$$\int_0^{\tan(x/2)} \frac{du}{\sqrt{1+u^2}\sqrt{C+1+(C-1)u^2}} = t + t_0 \tag{80}$$

On the other hand, by definition the elliptic integral of the first kind, $F(z,k)$ is defined as

$$F(z,k) = \int_0^z \frac{ds}{\sqrt{1-s^2}\sqrt{1-k^2s^2}} \tag{81}$$

and we get, with $K = \sqrt{2}/\sqrt{1+C}$,

$$iKF\left(\cos(z/2), K\right)\Big|_0^x = t + t_0 \tag{82}$$

At this point, we should study elliptic functions to proceed. They are in fact very interesting and worthwhile studying, but we'll leave that for later. For now, it is easier to gain insight on the system from the equation than from the properties of elliptic functions.

7.2 Discussion and qualitative analysis

Written as a system, we have

$$x' = v \tag{83}$$

$$v' = -\sin x \tag{84}$$

The point $(0,0)$ is an equilibrium, and $x = 0, v = 0$ is a solution. So are the points $x = n\pi, y = 0$, $n \in \mathbb{N}$.

Note that (83) is a *Hamiltonian system*, i.e., it is of the form

$$x' = \frac{\partial H(x,v)}{\partial y} \tag{85}$$

$$v' = -\frac{\partial H(x,v)}{\partial x} \tag{86}$$

where $H(x,v) = \frac{1}{2}v^2 + 1 - \cos x$. In all such cases, we see that is a *conserved quantity*, that is $H(x(t), y(t)) = const$ along a given *trajectory* $\{(x(t), y(t)) : t \in \mathbb{R}\}$. The trajectories are thus the level lines of H, that is

$$H(x,y) = \frac{1}{2}y^2 + 1 - \cos x = C \tag{87}$$

the *trajectories* (we artificially added 1, since H is defined up to an additive constant, to make $H \geq 0$.

We now see the importance of critical points: If H is analytic (in our case, it is entire), at all points where the right side of (85) is nonzero, either $x(y)$ or $y(x)$ are locally analytic, by the implicit function theorem, whereas otherwise, in general, the curves are nonuniquely defined and possibly singular.

We have $H(0,0) = 0$ and we see that $H(x,y) = h$ for $0 < h < 2$ are closed curves.

Indeed, we have in this case,

$$|v| \leq \sqrt{2h} \tag{88}$$

$$1 - \cos x < h \tag{89}$$

and thus both x and v are bounded, $(x,v) \in K$, in particular $x \in (-\pi/2, \pi/2)$. Then, $H(x,y) \leq h$ is compact, and since, if $C < 2$ we have $\nabla H = 0$ only at the origin, where H is zero, and H is positive otherwise, its maximum occurs on the boundary of $\{(x,y) : H(x,y) \leq h\}$. Furthermore, $H(x,y) = h$ is an analytic curve, in the sense above, since $\nabla H \neq 0$ in this region.

Physically, for initial conditions close to zero, the pendulum would periodically swing around the origin, with amplitude limited by the total energy.

Fig. 8 represents a numerical contour plot of $y^2/2 - \cos x$. If we zoom in, we see that the program had difficulties at the critical points $\pm\pi$, showing once more that there is something singular there.

7.3 Linearization of the phase portrait

Take

$$(1 - u^2/2) = \cos x; \quad u \in [-2, 2] \tag{90}$$

We can write this as

$$u^2 = 4\sin(x/2)^2 \tag{91}$$

which defines two holomorphic changes of coordinates

$$u = \pm 2\sin(x/2) \tag{92}$$

These are indeed biholomorphic changes of variables until $\sin(x/2)' = 0$ that is, $x = \pm\pi$. With any of these changes of coordinates we get

$$\frac{u}{\sin x} u' = v \tag{93}$$

$$v' = -\sin x \tag{94}$$

or

$$uu' = v \sin x \tag{95}$$

$$v' = -\sin x \tag{96}$$

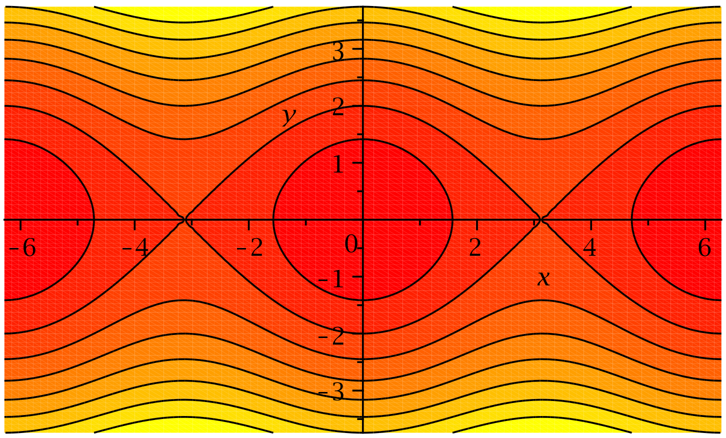

Figure 2: Contour plot of $y^2/2 - \cos x$

which would give the same trajectories family as

$$u' = v \tag{97}$$
$$v' = -u \tag{98}$$

for which the exact solution, $A \sin t, A \cos t$ gives rise to circles. The same could have been seen easily seen by making the same substitution, (101) in (87). We note again that in (101) we have $u^2 \in [0, 4]$, so the equivalence does not hold beyond $u = \pm 2$.

What about the other equilibria, $x = (2k+1)\pi$? It is clear, by periodicity and symmetry that it suffices to look at $x = \pi$. If we make the change of variable $x = \pi + s$ we get

$$s' \quad = v \tag{99}$$
$$v' \quad = \sin s \tag{100}$$

In this case, the same change of variable, $u = 2\sin(s/2)$ gives

$$u' = v \tag{101}$$
$$v' = u \tag{102}$$

implying $v^2 - u^2 = C$ as long as the change of variable is meaningful, that is, for $u < 2$, or $|s| < \pi$. So the curves associated to (99) are analytically conjugated to the hyperbolas $v^2 - u^2 = C$. The equilibrium is unstable, points starting nearby necessarily moving far away. The point $\pi, 0$ is a saddle point.

The trajectories starting at π are *heteroclinic*: they link different saddles of the system. In general, they do not necessarily exist.

In our case, these trajectories correspond to $H = 2$ and this gives

$$v^2 = 2(1 + \cos(x)) \tag{103}$$

or

$$v^2 = 4\cos(x/2)^2 \tag{104}$$

that is, the trajectories are given explicitly by

$$v = \pm 2\cos(x/2) \tag{105}$$

This is a case where the elliptic function solution reduces to elementary functions: The equation

$$\frac{dx}{dt} = 2\cos(x/2) \tag{106}$$

has the solution

$$x = 2\arctan(\sinh(t + C)) \tag{107}$$

We see that the time needed to move from one saddle point to the next one is infinite.

7.4 Connection to regularly perturbed equations

Note that at the equilibrium point $(\pi, 0)$ the system of equations is analytically equivalent, insofar as trajectories go, to the system

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \tag{108}$$

The eigenvalues of the matrix are ± 1 with (unnormalized) eigenvectors $(1, 1)$ and $(-1, 1)$. Thus, the change of variables to bring the system to a diagonal form is $x = \xi + \eta, v = \xi - \eta$. We get

$$\xi' + \eta' = \xi - \eta \tag{109}$$

$$\xi' - \eta' = \xi + \eta \tag{110}$$

By adding and subtracting these equations we get the diagonal form

$$\xi' = \xi \tag{111}$$

$$\eta' = -\eta \tag{112}$$

or

$$\frac{d\xi}{d\eta} = -\frac{\xi}{\eta}; \ \textit{or} \ \xi_\eta + \frac{1}{\eta}\xi = 0 \tag{113}$$

a standard regularly perturbed equation. Clearly the solutions of (113) are $\xi = C/\eta$ with $C \in (-\infty, \infty)$, and insofar as the phase portrait goes, we could have written $\eta_\xi + \frac{1}{\xi}\eta = 0$, which means that the trajectories are the curves $\xi = C/\eta$ with $C \in [-\infty, \infty]$, hyperbolas and the coordinate axes. In the original variables, the whole picture is rotated by 45°.

7.5 Completing the phase portrait

We see that, for $H > 2$ we have

$$v = \pm\sqrt{2h + 2\cos(x)} \tag{114}$$

where now $h > 2$. With one choice of branch of the square root (the solutions are analytic, after all), we see that $|v|$ is bounded, and it is an open curve, defined on the whole of $\mathbb{R}$. Note that the explicit form of the trajectories, given by (87) does not, in general, mean that we can solve the second order differential equation. The way the pendulum position depends on time, or the way the point moves along these trajectories, is still transcendental.

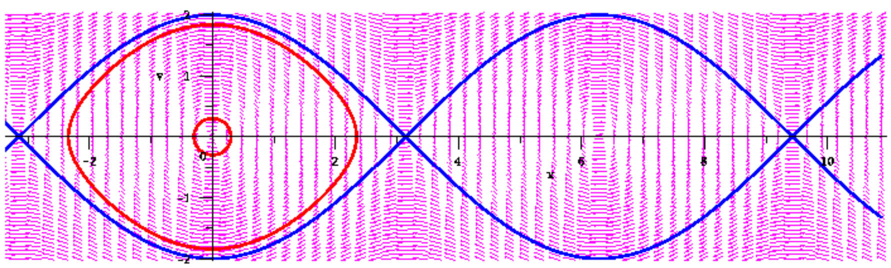

Figure 3: Contour plot of $y^2/2 - \cos x$

7.6 Local and asymptotic analysis

Near the origin, for $C = a^2$ small, we have

$$x' = v \tag{115}$$
$$v' = x - x^3/6 + ... \tag{116}$$

implying

$$x' = v \tag{117}$$
$$v' \approx x \tag{118}$$

which means

$$x \approx a \sin t \tag{119}$$
$$v \approx a \cos t \tag{120}$$

For C very large, we have

$$\frac{dx}{\sqrt{C+\cos x}} = dx(C+\cos x)^{-1/2} = dxC^{-1/2}(1+\cos x/C)^{-1/2}$$
$$= dx(C^{-1/2} - \frac{1}{2}\cos x/C^{-3/2} + \cdots) \tag{121}$$

which means

$$C^{-1/2}x + \frac{1}{2}\sin x/C^{-3/2} + \cdots = t + t_0 \tag{122}$$

or

$$x = C^{1/2}(t+t_0) - \frac{1}{2}\sin(C^{1/2}t)/C^{-3/2} + \cdots = \tag{123}$$

The solutions near the critical point $(\pi, 0)$ can be analyzed similarly.

Local and asymptotic analysis often give sufficient qualitative, and sometimes quantitative information about all solutions of the equation.

8 Equilibria, more examples and results

8.1 Flows

Consider the system

$$\frac{dx}{dt} = X(x) \tag{124}$$

where X is smooth enough. Such equations can be considered in $\mathbb{R}^n$ or, more generally, in Banach spaces.

The initial condition x_0 is mapped, by the solution of the differential equation (124) into $x(t)$ where $t \in (-a, b)$.

The map $x(0) \to x(t)$ written as $f^t(x_0)$ is the flow associated to X.

For $t \geq 0$ we note the semigroup property $f^0 = I, f^{s+t} = f^s f^t$.

Fixed points, hyperbolic fixed points in $\mathbb{R}^n$. Example. If $X(x) = Bx$ where B **does not depend on** x, then the general solution is

$$x = e^{Bt}x_0 \tag{125}$$

where x_0 is the initial condition at $t = 0$. (Note again that a simple exponential formula does not exist, in general, if M depended on t.)

In this case, the flow f is given by the linear map

$$f^t(x_0) = e^{Bt}x_0 \tag{126}$$

Note that $(D_x f)(0) = e^{Bt}$. This is the case in general, as we will see.

Note 1. Check also (for instance from the power series of the exponential) that the eigenvalues of $e^{B\alpha}$ are $e^{\lambda_i \alpha}$ where λ_i are the eigenvalues of B.

Definition 5. *The point x_0 is a fixed point of f if $f^t(x_0) = x_0$ for all t.*

Proposition 6. *If f is associated to X, then x_0 is a fixed point of f iff $X(x_0) = 0$.*

Proof. Indeed, we have $x(t + \Delta t) = x(t) + X(x_0)\Delta t + O((\Delta t)^2)$ for small Δt. Then $x(t+\Delta t) = x(t)$ implies $X(x_0)+O(\Delta t) = 0$, or, $X(x_0) = O(\Delta t)$, that is, $X(x_0) = 0$. Conversely, it is obvious that $X(x_0) = 0$ implies that $x(t) = x_0$ is a solution of (124). □

Proposition 7. *If f is associated to X, and 0 is a fixed point of f, then $D_x f^t|_{x=0} = e^{DX(0)t}$.*

Proof. Let t be fixed and take x_0 small enough. Let $DX(0) = B$. We have $x' = Bx + O(x^2)$, and thus, by taking $x = e^{Bt}u$ we get

$$e^{Bt}u' + Be^{Bt}u = Be^{Bt}u + g(e^{Bt}u) \tag{127}$$

where $g(s) = O(s^2)$. Thus

$$u = x_0 + e^{-Bt}\int_0^t g(e^{Bs}u(s))ds \tag{128}$$

We can check that for given t and x_0 small enough, this equation is contractive in the sup norm, in the ball $|u| < 2x_0$. Then, we see that

$$u = u_0 + O(C(t)u_0^2) \tag{129}$$

where we emphasized that the correction depends on t too. Then,

$$x = e^{Bt}x_0 + O(C(t)x_0^2) \tag{130}$$

proving the statement.

Definition 8. • *The fixed point $x = 0$ is hyperbolic if the matrix $D_x f|_{x=0}$ has no eigenvalue on the unit circle.*

• *Equivalently, if f is associated with X, the fixed point 0 is hyperbolic if the matrix $DX(0)$ has no purely imaginary eigenvalues.*

□

8.2 The Hartman-Grobman theorem

The following result generalizes to Banach space settings.

Let U and V be open subsets of $\mathbb{R}^n$. Let f be a diffeomorphism between U and V with a **hyperbolic** fixed point, that is there is $x \in U \cap V$ so that $f(x) = x$. Without loss of generality, we may assume that $x = 0$.

Theorem 5 (Hartman-Grobman for maps)**.** *Under these assumptions, f and $Df(0)$ are topologically conjugate, that is, there are neighborhoods U_1, V_1 of zero, and a homeomorphism h from U_1 to V_1 so that $h^{-1} \circ f \circ h = Df(0)$.*

The proof is not very difficult, but it is preferable to leave it for later.

Theorem 6 (Hartman-Grobman for flows, [5])**.** *Let consider $x' = X(t)$ over a Banach space, where X is a C^1 vector field defined in a neighborhood of the origin 0 of E. Suppose that 0 is a hyperbolic fixed point of the flow described by X. Then there is a homeomorphism between the flows of X and $DX(0)$, that is a homeomorphism between a neighborhood of zero into itself so that*

$$f^t = h \circ e^{tDX(0)} \circ h^{-1} \tag{131}$$

See also [4].

The more regularity is needed, the more conditions are required.

Differentiable linearizations

Theorem 7 (Sternberg-Siegel, see [5])**.** *Assume f is differentiable, with a hyperbolic fixed point at zero, and the derivative Df is Hölder continuous near zero. Assume further that $DX(0)$ is such that its eigenvalues satisfy*

$$\mathrm{Re}\lambda_i \neq \mathrm{Re}\lambda_j + \mathrm{Re}\lambda_k \tag{132}$$

when $\mathrm{Re}\lambda_j < 0 < \mathrm{Re}\lambda_k$. Then the functions h in Theorems 5 and 6 can be taken to be diffeomorphisms.

Smooth linearizations

Theorem 8 (Sternberg-Siegel, see [5])**.** *Assume $f \in C^\infty$ and the eigenvalues of $Df(0)$ are nonresonant, that is*

$$\lambda_i - \mathbf{k}\boldsymbol{\lambda} \neq 0 \tag{133}$$

for any $\mathbf{k}$ with $|\mathbf{k}| > 1$. Then the functions h in Theorems 5 and 6 can be taken to be C^∞ diffeomorphisms.

We will prove, in simpler settings, the Hartman-Grobman theorem for flows.

For the analytic case, see Proposition 3.

8.3 Bifurcations

Bifurcations occur in systems depending on a parameter (or more), call it s. Thus, the system is

$$\frac{d}{dt}x(t; s) = X(x; s) \tag{134}$$

A *local* bifurcation at an equilibrium, say $x = 0, X(0) = 0$, may occur when at least one of the eigenvalues of $DX(0)$ becomes purely imaginary. (Otherwise, the linearization theorem shows that the phase portrait is locally similar to that of the linearized system. In this case, the topology does not change unless we indeed go through purely imaginary eigenvalues.) We will explore bifurcation types and prove theorems about some of them, but before that let's see what types of equilibria are possible in linear systems. Those that are associated to hyperbolic fields represent, again by the linearization theorem, the local behavior of general hyperbolic systems.

9 Types of equilibria of linear systems with constant coefficients in 2d

The equation is now

$$x' = Bx \tag{135}$$

where B is a 2×2 matrix with constant coefficients.

9.1 Distinct eigenvalues

In this case, the system can be diagonalized, and it is equivalent to a pair of trivial first order ODEs

$$x' = \lambda_1 x \tag{136}$$

$$y' = \lambda_2 y \tag{137}$$

9.1.1 Real eigenvalues

The change of variables that diagonalizes the system has the effect of rotating and rescaling the phase portrait of (136). The phase portrait of (136) can be fully described, since we can solve the system in closed form, in terms of simple functions:

$$x = x_0 e^{\lambda_1 t} \tag{138}$$

$$y = y_0 e^{\lambda_2 t} \tag{139}$$

On the other hand, we have

$$\frac{dy}{dx} = \frac{\lambda_2}{\lambda_1}\frac{y}{x} = a\frac{y}{x} \Rightarrow y = C|x|^a \tag{140}$$

where we also have as trajectories the coordinate axes: $y = 0$ ($C = 0$) and $x = 0$ ("$C = \infty$"). These trajectories are generalized parabolas. If $a > 0$ then the system is either (i) **a sink**, when both λ's are negative, in which case, clearly, the solutions converge to zero. See Fig. 4, or (ii) **a source**, when both λ's are positive, in which case, the solutions go to infinity.

The other case is that when $a < 0$; then the eigenvalues have opposite sign. Then, we are dealing with a **saddle**. The trajectories are generalized hyperbolas,

$$y = C|x|^{-|a|} \tag{141}$$

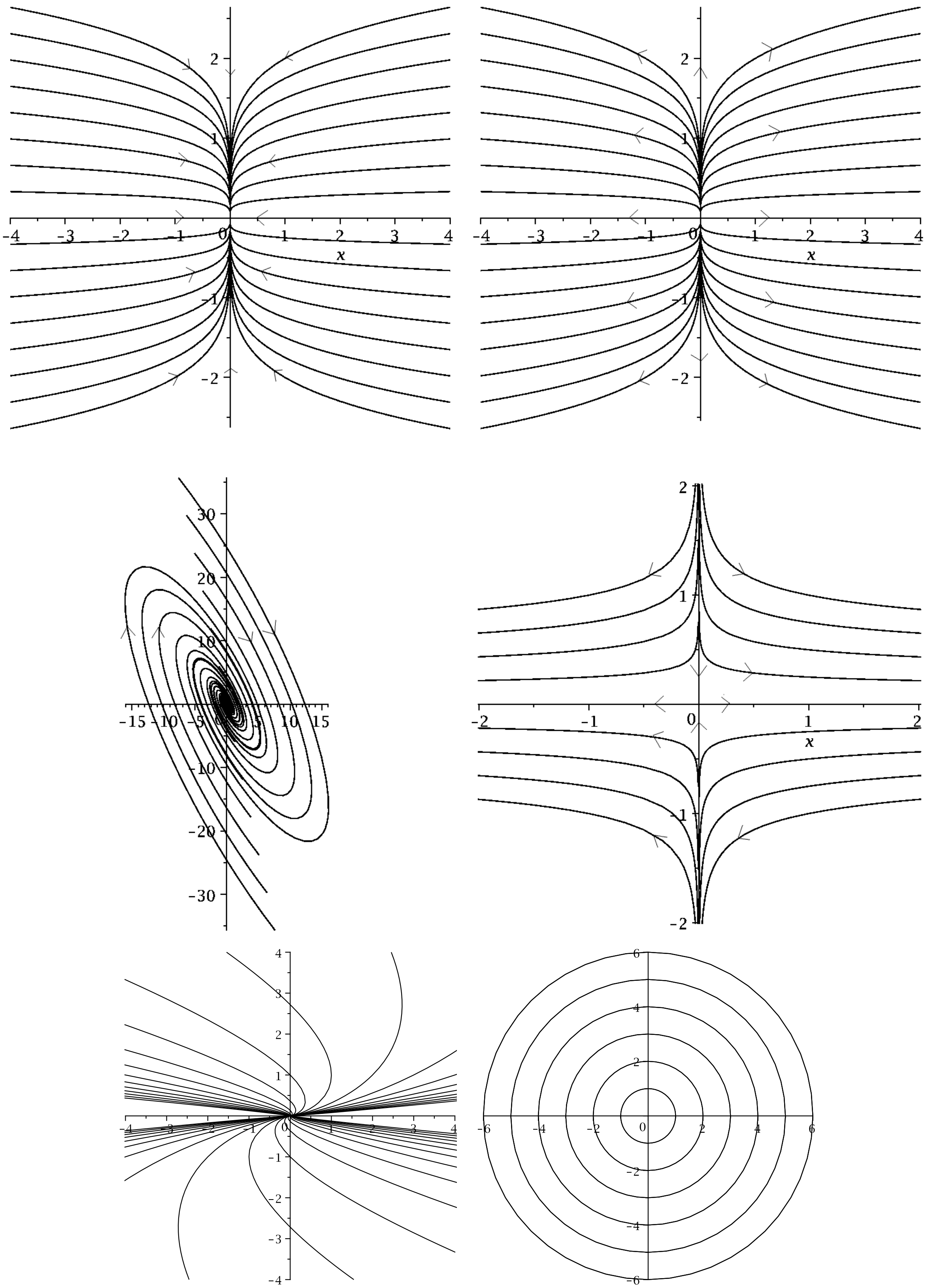

Figure 4: All types of linear equilibria in 2d, modulo euclidian transformations and rescalings: sink, source, spiral sink, saddle, nontrivial Jordan form, center resp. In the last two cases, the arrows point according to the sign of λ or ω, resp.

Say $\lambda_1 > 0$. In this case there is a **stable manifold** the y axis, along which solutions converge to zero, and an **unstable manifold** in which trajectories go to zero as $t \to -\infty$. Other trajectories go to infinity both forward and backward in time. In the other case, $\lambda_1 < 0$, the figure is essentially rotated by $\pi/2$.

9.1.2 Complex eigenvalues

In this case we just keep the system as is,

$$x' = ax + by \tag{142}$$
$$y' = cx + dy \tag{143}$$

We solve for y, assuming $b \neq 0$ (check the case $b = 0$!), introduce in the second equation and we obtain a second order, constant coefficient, differential equation for x:

$$x'' - (a + d)x' + (ad - bc)x = 0 \quad \textit{or} \tag{144}$$
$$x'' - \mathrm{tr}(B)x' + \det(B)x = 0 \tag{145}$$

If we substitute $x = e^{\lambda t}$ in (144) we obtain

$$\lambda^2 - \mathrm{tr}(B)\lambda + \det(B) = 0 \tag{146}$$

and, evidently, since $\lambda_1 + \lambda_2 = \mathrm{tr}(B)$ and $\lambda_1\lambda_2 = \det(B)$, this is the same equation as the one for the eigenvalues of B. The eigenvalues of B have been assumed complex, and since the coefficients we are working with are real, the roots are complex conjugate:

$$\lambda_i = \alpha \pm i\omega \tag{147}$$

The real valued solutions are

$$x = Ae^{\alpha t}\sin(\omega t + \varphi) \tag{148}$$

where A and φ are free constants. Substituting in

$$y = b^{-1}x' - ab^{-1}x \tag{149}$$

we get

$$y(t) = Ae^{\alpha t}b^{-1}[(\alpha - 1)\cos(\omega t + \varphi) - \omega\sin(\omega t + \varphi) \tag{150}$$

which can be written, as usual,

$$y(t) = A_1 e^{\alpha t}\sin(\omega t + \varphi_1) \tag{151}$$

If $\lambda < 0$, then we get the **spiral sink**. If $\alpha > 0$ then we get a spiral source, where the arrows are reverted.

A special case is that when $\alpha = 0$. This is the only non-hyperbolic fixed point with distinct eigenvalues. In this case, show that for some c we have $x^2 + cy^2 = A^2$, and thus the trajectories are ellipses. In this case, we are dealing with a **center**.

9.2 Repeated eigenvalues

In 2d this case there is exactly one eigenvalue, and it must be real, since it coincides with its complex conjugate. Then the system can be brought to a Jordan normal form; this is either a diagonal matrix, in which case it is easy to see that we are dealing with a sink or a source, or else we have

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{152}$$

In this case, we obtain

$$\frac{dx}{dy} = \frac{x}{y} + \frac{1}{\lambda} \tag{153}$$

with solution

$$x = ay + \lambda^{-1} y \ln |y| \tag{154}$$

As a function of time, we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} = e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t} = e^{\lambda t} \left[I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \right] \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{155}$$

$$x(t) = (At + B)e^{\lambda t} \tag{156}$$

$$y(t) = Ae^{\lambda t} \tag{157}$$

We see that, in this case, only the x axis is a special solution (the y axis is not), and thus, all solutions approach (as $t \to \infty$ or $t \to -\infty$ for $\lambda < 0$ or $\lambda > 0$ respectively) the x axis.

Note 2. The eigenvalues of a matrix depend continuously on the coefficients of the matrix. In two dimensions you can see this by directly solving $\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$. Thus, if a linear or nonlinear system depends on a parameter α (scalar or not) and the equilibrium is hyperbolic when $\alpha = \alpha_0$, then the real part of the eigenvalues will preserve their sign in a neighborhood of $\alpha = \alpha_0$. The type of equilibrium is the same and of local phase portrait changes smoothly unless the real part of an eigenvalue goes through zero.

Note 3. When conditions are met for a diffeomorphic local linearization at an equilibrium, then we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi \begin{pmatrix} u \\ v \end{pmatrix} \tag{158}$$

where the equation in (u, v) is linear and the matrix φ is a diffeomorphism. We then have

$$\begin{pmatrix} x \\ y \end{pmatrix} = (D\varphi) \begin{pmatrix} u \\ v \end{pmatrix} + o(u, v) \tag{159}$$

which implies, in particular that the phase portrait very near the equilibrium is changed through a linear transformation.

9.3 Further examples, [4]

Consider the system

$$x' = x + y^2 \tag{160}$$

$$y' = -y \tag{161}$$

The linear part of this system is

$$x' = x \tag{162}$$

$$y' = -y \tag{163}$$

The associated matrix is simply

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{164}$$

with eigenvalues 1 and -1, and the conditions of a differentiable homeomorphism are satisfied.

Locally, near zero, the phase portrait of the system (164) is thus the prototypical saddle.

We will see that, again insofar as the field lines are concerned, this system can be *globally* linearized too.

How about the global behavior? In this case, we can completely solve the system. First, insofar as the field lines go, we have

$$\frac{dx}{dy} = -\frac{x}{y} + y \tag{165}$$

a linear inhomogeneous equation that can be solved by variation of parameters, or more easily noting that, by homogeneity, $x = ay^2$ must be a particular solution for some a, and we check that $a = -1/3$. The general solution of the homogeneous equation is clearly $xy = C$. It is interesting to make it into a homogeneous second order equation by the usual method. We write

$$\frac{1}{y}\frac{dx}{dy} = -\frac{x}{y^2} + 1 \tag{166}$$

and differentiate once more to get

$$\frac{d^2x}{dy^2} = -2\frac{x}{y^2} \tag{167}$$

which is an Euler equation, with indicial equation $(\lambda - 2)(\lambda + 1) = 0$, and thus the general solution is

$$x(y) = ay^2 + \frac{b}{y} \tag{168}$$

where the constants are not arbitrary yet, since we have to solve the more stringent equation (165). Inserting (168) into (165) we get $a = -1/3$. Thus, the general solution of (166) is

$$3xy + y^3 = C \tag{169}$$

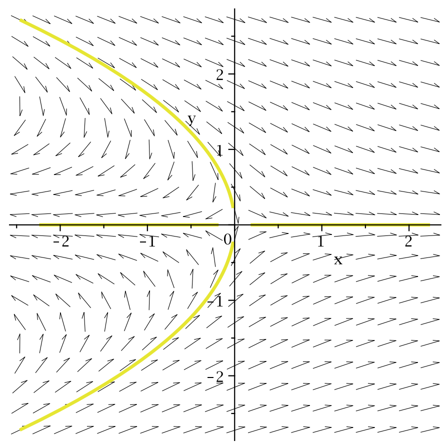

Figure 5: Phase portrait of (160)

which can be, of course, solved for x. The phase can be obtained in the following way: we note that near the origin, the system is diffeomorphic to the linear part, thus we have a saddle there. There is a particular solution with $x = -1/3y^2$ and the field can be completed by analyzing the field for large x and y. This separates the initial conditions for which the solution ends up in the right half plane from those confined to the left half plane.

Global linearization. This is another case of "accidental" analytic linearizability since we can write the *conserved quantity* $3xy(1+y^2/3) = C$, which or $(x + y^2/3)y = C$ and thus passing to the variables $u = x + y^2/3$, $v = y$ the system (160) becomes linear, of the form (162) (check!)

Note 4. The change of coordinates is thus

$$\begin{pmatrix} u \\ v \end{pmatrix} = \left(I + \begin{pmatrix} 0 & y^2/3 \\ 0 & 0 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} \tag{170}$$

and in particular we see that the transformation is, to leading order, the identity.

Exact solution of the time dependent system. Due to the very special nature of the equation, an exact solution is possible too: we note that the second equation contains y alone, and it gives immediately

$$y = y(0)e^{-t}$$

while x can be either solved from the first equation or, more simply, from (176):

$$x(t) = \frac{c}{y(0)}e^t - \frac{1}{3}y(0)e^{-2t} \tag{171}$$

In the nonlinear system, $y = 0$ is still a solution, but $x = 0$ is not; $x = 0$ is "deformed" into the parabola $x = (-1/3)y^2$.

9.4 Stable and unstable manifolds in 2d

Assume that φ is differentiable, and that the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \varphi \begin{pmatrix} x \\ y \end{pmatrix} \tag{172}$$

has an equilibrium at zero, which is a saddle, that is, the eigenvalues of $(D\varphi)(0)$ are $-\mu$ and λ, where λ and μ are positive. We can make a linear change of variables so that $(D\varphi)(0) = \text{diag}(-\mu, \lambda)$. Consider the linearization tangent to the identity, that is, with $D\varphi(0) = I$. We call the linearized variables (u, v).

Theorem 9. *Under these assumptions, in a disk of radius $\epsilon > 0$ near the origin there exist two functions $y = f_+(x)$ and $x = f_-(y)$ passing through the origin, tangent to the axes at the origin and so that all solutions with initial conditions $(x_0, f_+(x_0))$ converge to zero as $t \to \infty$, while the initial conditions $(f_-(y_0), y_0)$ converge to zero as $t \to -\infty$. The graphs of these functions are called the* **stable and unstable manifolds, resp**. *All other initial conditions necessarily leave this disk as time increases, or decreases.*

Proof. We show the existence of the curve f_+, the proof for f_- being the same, by reverting the signs. We have

$$\begin{aligned} x(t) &= \varphi_1(u(t), v(t)) \\ y(t) &= \varphi_2(u(t), v(t)) \end{aligned} \tag{173}$$

Consider a point $(\varphi_1(u_0, 0), \varphi_2(u_0, 0))$. There is a unique solution passing through this point, namely $(\varphi_1(u_+(t), 0), \varphi_2(0_+(t), 0))$ where $u_+(0) = u_0, v_+(0) = 0$. Since $u_+(t) \to 0$ as $t \to \infty$ and φ is continuous, we have

$$(\varphi_1(u_+(t), 0), \varphi_2(u_+(t), 0)) \to 0$$

as $t \to \infty$. Since $\varphi = I + o(1)$, we have $\partial\varphi_2/\partial y = 1$ at $(0, 0)$, and the implicit function theorem shows that $\varphi_2(x, y) = 0$ defines a differentiable function $y = f(x)$ near zero, and $y'(0) = 0$ (check). For other solutions we have, from (173), that x, y exits any small enough disk (check). □

9.5 A limit cycle

We follow again [4], but with a different starting point. Let's look at the simple system

$$r' = r(1 - r^2)/2 \tag{174}$$

$$\theta' = 1 \tag{175}$$

Obviously, we can solve this in closed form. The flow clearly has no fixed point, since the field never vanishes. To solve the first equation, note that if we multiply by $2r$ we get

$$2rr' = r^2(1 - r^2) \tag{176}$$

or, with $u = r^2$,

$$u' = u(1-u) \tag{177}$$

The exact solution is

$$r = \pm(1 + Ce^{-t})^{-1/2};\ \text{or } r = 0;\ \pm 1 \text{ are special constant solutions} \tag{178}$$

$$\theta = t + t_0 \tag{179}$$

We see that all solutions that start away from zero converge to one as $t \to \infty$. What if interpret r and θ as polar coordinates and write the equations for x

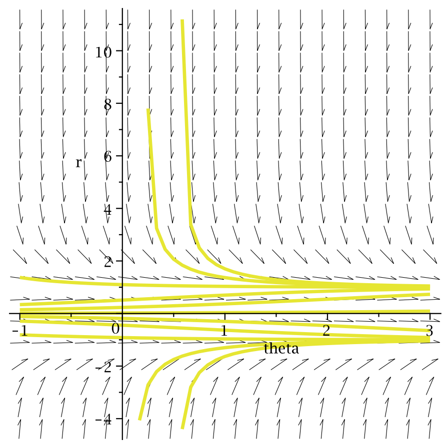

Figure 6: Phase portrait of (174)

and y? We get

$$x' = r'\cos\theta - r\sin\theta\theta' = \frac{1}{2}r(1-r^2)\cos\theta - r\sin\theta$$
$$= \frac{1}{2}x - y - \frac{1}{2}(x^3 + xy^2) \tag{180}$$

$$y' = r'\sin\theta + r\cos\theta\theta' = \frac{1}{2}r(1-r^2)\sin\theta + r\cos\theta$$
$$= x + \frac{1}{2}y - \frac{1}{2}(x^2y + y^3) \tag{181}$$

thus the system

$$x' = \frac{1}{2}x - y - \frac{1}{2}(x^3 + xy^2) \tag{182}$$

$$y' = x + \frac{1}{2}y - \frac{1}{2}(x^2y + y^3) \tag{183}$$

which looks rather hopeless, but we know that it can be solved in closed form.

To analyze this system, we see first that at the origin the matrix is

$$\begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \tag{184}$$

with eigenvalues $1/2 \pm i$. Thus the origin is a spiral source. There are no other equilibria (why?)

Now we know the solution globally, by looking at the solution of (174) and/or its phase portrait. We note that $r = 1$ is a solution of (174), thus the unit circle

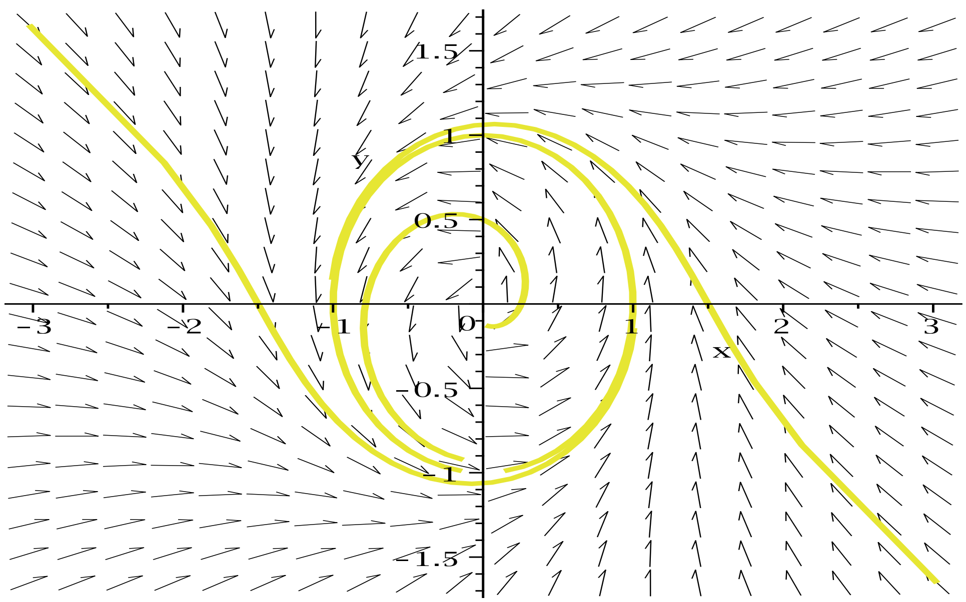

Figure 7: Phase portrait of (174)

is a trajectory of the system (182). It is a closed curve, all trajectories tend to it asymptotically. This is a limit cycle.

9.6 Application: constant real part, imaginary part of analytic functions

Assume for simplicity that f is entire. The transformation $z \to f(z)$ is associated with the planar transformation $(x, y) \to (u(x, y), v(x, y))$ where $f = u + iv$. The grid $x = const, y = const$ is transformed into the grid $u = const, v = const$. We can first look at what this latter grid is transformed back into, by the transformation. The analysis is more general though, nothing below requires $u + iv$ to be analytic. We use only use this information to shortcut through some calculations.

We take first $u(x(t), y(t)) = const$. We have

$$\frac{\partial u}{\partial x}x'(t) + \frac{\partial u}{\partial y}y'(t) = 0 \tag{185}$$

which we can write, for instance, as the system

$$x' = \frac{\partial u}{\partial y} \tag{186}$$

$$y' = -\frac{\partial u}{\partial x} \tag{187}$$

which, in particular, is a Hamiltonian system. We have a similar system for v. We can draw the curves $u = const, v = const$ either by solving this implicit equation, or by analyzing (186), or even better, by combining the information from both. Let's take, for example $f(z) = z^3 - 3z^2$. Then, $v = 3x^2y - y^3 - 6xy$. It would be rather awkward to solve $v = c$ for either x or y. The system of equations reads

$$x' = -6x + 3x^2 - 3y^2 \tag{188}$$

$$y' = 6y - 6xy \tag{189}$$

Note that $\nabla u = 0$ or $\nabla v = 0$ are equivalent to $z' = 0$. For equilibria, we thus solve $3z^2 - 6z = 0$ which gives $z = 0; z = 2$. Near $z = 0$ we have

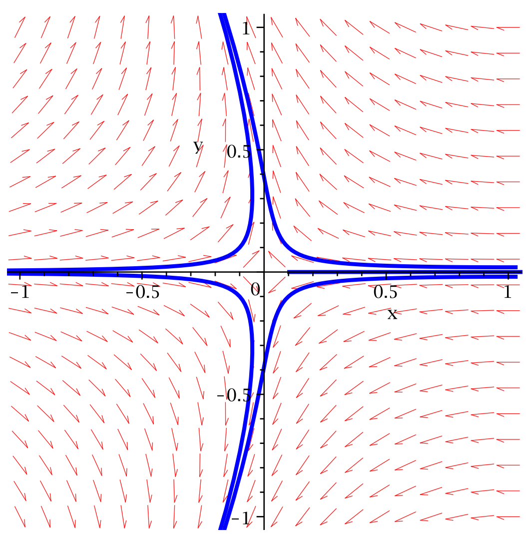

Figure 8: Phase portrait of (188) near $(0, 0)$.

$$x' = -6x + o(x, y) \tag{190}$$

$$y' = 6y + o(x, y) \tag{191}$$

which is clearly a saddle point, with x the stable direction and y the unstable one. At $x = 2, y = 0$ we have, denoting $x = 2 + s$,

$$s' = 6s + o(s, y) \tag{192}$$
$$y' = -6y + o(s, y) \tag{193}$$

another saddle, where now $y = 0$ is the unstable direction. We note that $y = 0$ is,

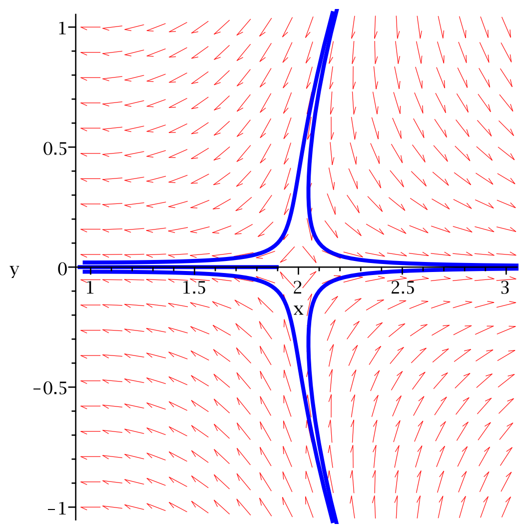

Figure 9: Phase portrait of (188) near $(0, 2)$.

in fact, a special trajectory, and it is in the nonlinear unstable/stable manifold at the equilibrium points. Note also that a nonlinear stable manifold exists locally. In this case it changes character as it happens to pass through another equilibrium.

We draw the phase portraits near $x = 0$, near $x = 2$, mark the special trajectory, and look at the behavior of the phase portrait at infinity. Then we "link" smoothly the phase portraits at the special points, and this should suffice for having the phase portrait of the whole system.

For the behavior at infinity, we note that if we write

$$\frac{dy}{dx} = \frac{y(1 - 6x)}{-6x + 3x^2 - 3y^2} \tag{194}$$

we have the special solution $y = 0$, and otherwise the nonlinear terms dominate and we have

$$\frac{dy}{dx} \approx \frac{-6yx}{3x^2 - 3y^2} \tag{195}$$

By homogeneity, we look for special solutions of the form $y = ax$ (which would be asymptotes for the various branches of $y(x)$. We get, to leading order,

$$a = \frac{-6a}{3 - 3a^2} \tag{196}$$

We obtain

$$a = 0, a = \pm\sqrt{3} \tag{197}$$

We also see that, if $x = o(y)$, then $y' = o(1)$ as well. This would give us information about the whole phase portrait, at least qualitatively.

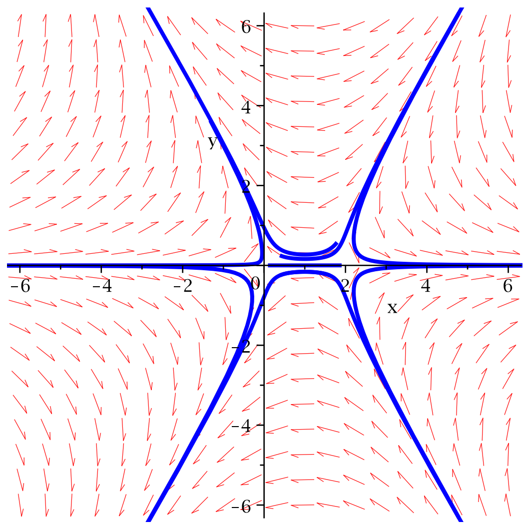

Figure 10: Phase portrait of (188), $v = const$.

Exercise 1. Analyze the phase portrait of $u(x, y) = const$.

The two phase portraits, plotted together give Note how the fields intersect at right angles, except at the saddle points. The reason, of course, is that $f(z)$ is a conformal mapping wherever $f' \neq 0$.

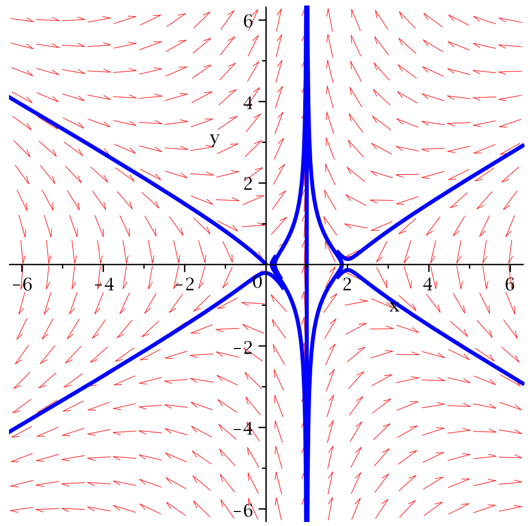

Figure 11: Phase portrait of $u = const$

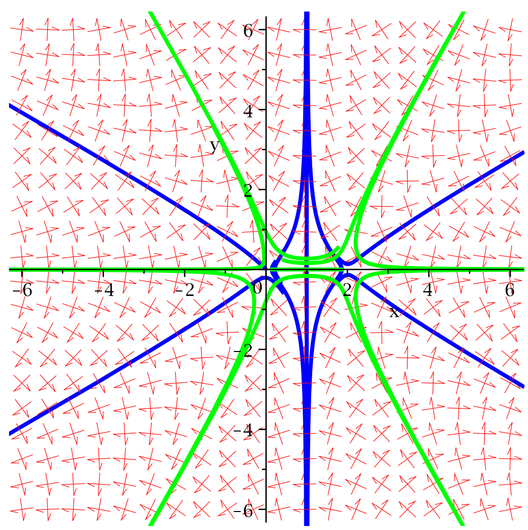

Figure 12: Phase portrait of $u = const$, and $v = const$.

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