1 Gradient and Hamiltonian systems

1.1 Gradient systems

These are quite special systems of ODEs, Hamiltonian ones arising in conservative classical mechanics, and gradient systems, in some ways related to them, arise in a number of applications. They are certainly nongeneric, but in view of their origin, they are common.

A system of the form

$$X' = -\nabla V(X) \tag{1}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is, say, C^∞, is called, for obvious reasons, a gradient system. A critical point of V is a point where $\nabla V = 0$.

These systems have special properties, easy to derive.

Theorem 1. *For the system (1), if V is smooth, we have (i) If c is a regular point of V, then the vector field is perpendicular to the level hypersurface $V^{-1}(c)$ along $V^{-1}(c)$.*

(ii) A point is critical for V iff it is critical for (1).

(iii) At any equilibrium, the eigenvalues of the linearized system are real.

More properties, related to stability, will be discussed in that context.

Proof. □

(i) It is known that the gradient is orthogonal to level surface.

(ii) This is clear essentially by definition.

(iii) The linearization matrix elements are $a_{ij} = -V_{x_i,x_j}$ (the subscript notation of differentiation is used). Since V is smooth, we have $a_{ij} = a_{ji}$, and all eigenvalues are real.

1.2 Hamiltonian systems

If $\mathbf{F}$ is a conservative field, then $\mathbf{F} = -\nabla V$ and the Newtonian equations of motion (the mass is normalized to one) are

$$q' = p \tag{2}$$

$$p' = -\nabla V \tag{3}$$

where $q \in \mathbb{R}^n$ is the position and $p \in \mathbb{R}^n$ is the momentum. That is

$$q' = \frac{\partial H}{\partial p} \tag{4}$$

$$p' = -\frac{\partial H}{\partial q} \tag{5}$$

where

$$H = \frac{p^2}{2} + V(q) \tag{6}$$

is the Hamiltonian. In general, the motion can take place on a manifold, and then, by coordinate changes, H becomes a more general function of q and p. The coordinates q are called generalized positions, and q are the called generalized momenta; they are canonical coordinates on the phase on the cotangent manifold of the given manifold.

An equation of the form (4) is called a Hamiltonian system.

Exercise 1. *Show that a system $x' = F(x)$ is at the same time a Hamiltonian system and a gradient system iff the Hamiltonian H is a harmonic function.*

Proposition 1. *(i) The Hamiltonian is a constant of motion, that is, for any solution $X(t) = (p(t), q(t))$ we have*

$$H(p(t), q(t)) = const \tag{7}$$

where the constant depends on the solution.

(ii) The constant level surfaces of a smooth function $F(p, q)$ are solutions of a Hamiltonian system

$$q' = \frac{\partial F}{\partial p} \tag{8}$$

$$p' = -\frac{\partial F}{\partial x} \tag{9}$$

Proof. (i) We have

$$\frac{dH}{dt} = \nabla_p H \frac{dp}{dt} + \nabla_q \frac{dq}{dt} = -\nabla_p H \nabla_q + \nabla_q H \nabla_p = 0 \tag{10}$$

(ii) This is obtained very similarly. □

1.2.1 Integrability: a few first remarks

Hamiltonian systems (with time-independent Hamiltonian) in one dimension are integrable: the solution can be written in closed form, implicitly, as $H(y(x), x) = c$; in terms of t, once we have $y(x)$ of course we can integrate $x' = G(y(x), x) := f(x)$ in closed form, by separation of variables. Note that for an equation of the form $y' = G(y, x)$, this is *equivalent* to the system having a *constant of motion*. The latter is defined as a function $K(x, y)$ defined globally in the phase space, (perhaps with the exception of some isolated points where it may have "simple" singularities, such as poles), and with the property that $K(y(x), x) = const$ for any given trajectory (the constant can depend on the trajectory, but not on x). Indeed, in this case we have

$$\frac{d}{dx} C(y(x), x) = \frac{\partial C}{\partial y} y' + \frac{\partial C}{\partial x} = 0$$

or

$$y' = -\frac{\partial C}{\partial x} \Big/ \frac{\partial C}{\partial y}$$

which is equivalent to the system

$$\dot{x} = \frac{\partial C}{\partial y}; \quad \dot{y} = -\frac{\partial C}{\partial x} \tag{11}$$

which is a Hamiltonian system.

1.2.2 Local versus global

It is important to mention that a system is actually called Hamiltonian if the function H is defined over a sufficiently large region, preferably the whole phase space.

Indeed, take any smooth first order ODE, $y' = f(y, x)$ and differentiate with respect to the initial condition (we know already that the dependence is smooth; we let $dy/d(y_0) = \dot{y}$):

$$\dot{y}' = \frac{\partial f}{\partial y}\dot{y} \tag{12}$$

with the solution

$$\dot{y} = \exp\left(\int_{x_0}^{x} \frac{\partial f(y(s), s)}{\partial y} ds\right) \tag{13}$$

and thus, in the local solution $y = G(x; x_0)$ we have $G_{x_0}(x; x_0) \neq 0$, *if* G is smooth –i.e. the field is regular–, and the implicit function theorem provides a local function K so that $x_0 = K(y(x), x)$, that is a constant of motion! The big difference between integrable and nonintegrable systems comes from the possibility to extend K globally.

1.3 Example

As an example for both systems, we study the following problem: draw the contour plot (constant level curves) of

$$F(x, y) = y^2 + x^2(x - 1)^2 \tag{14}$$

and draw the lines of steepest descent of F.

For the first part we use Proposition 1 above and we write

$$x' = \frac{\partial F}{\partial y} = 2y \tag{15}$$

$$y' = -\frac{\partial F}{\partial x} = -2x(x - 1)(2x - 1) \tag{16}$$

The critical points are $(0, 0), (1/2, 0), (1, 0)$. It is easier to analyze them using the Hamiltonian. Near $(0, 0)$ H is essentially $x^2 + y^2$, that is the origin is a center, and the trajectories are near-circles. We can also note the symmetry $x \to (1 - x)$ so the same conclusion holds for $x = 1$, and the phase portrait is symmetric about $1/2$.

Near $x = 1/2$ we write $x = 1/2 + s$, $H = y^2 + (1/4 - s^2)^2$ and the leading Taylor approximation gives $H \sim y^2 - 1/2s^2$. Then, $1/2$ is a saddle (check). Now we can draw the phase portrait easily, noting that for large x the curves essentially become $x^4 + y^2 = C$ "flattened circles". Clearly, from the interpretation of the problem and the expression of H we see that *all* trajectories are closed.

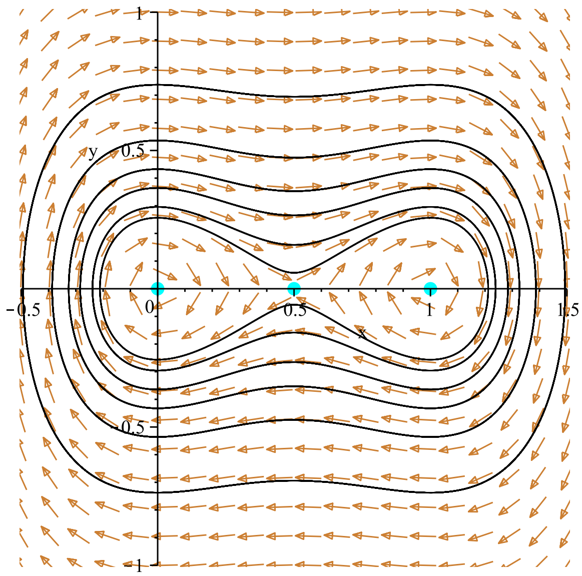

Figure 1:

The perpendicular lines solve the equations

$$x' = -\frac{\partial F}{\partial x} = -2x(x-1)(2x-1) \tag{17}$$

$$y' = -\frac{\partial F}{\partial y} = -2y \tag{18}$$

We note that this equation is separated! In any case, the two equation obviously share the critical points, and the sign diagram can be found immediately from the first figure.

Exercise 2. *Find the phase portrait for this system, and justify rigorously its qualitative features. Find the expression of the trajectories of (17). I found*

$$y = C\left(\frac{1}{(x-1/2)^2} - 4\right)$$

2 Flows, revisited

Often in nonlinear systems, equilibria are of higher order (the linearization has zero eigenvalues). Clearly such points are not hyperbolic and the methods we have seen so far do not apply.

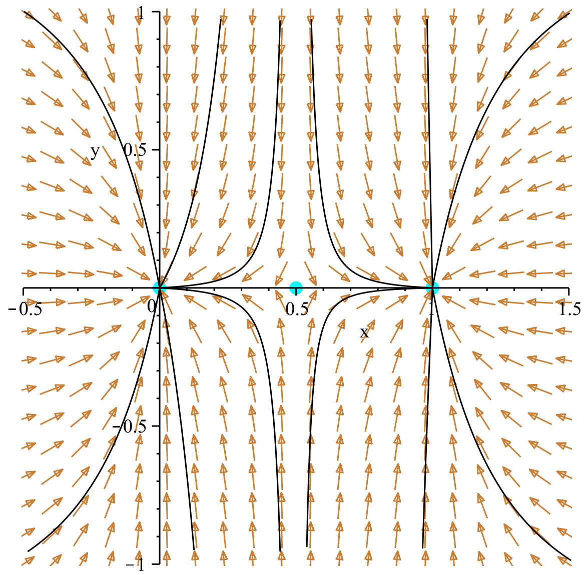

Figure 2:

There are no general methods to deal with all cases, but an important one is based on Lyapunov (or Lyapounov,...) functions.

Definition. A flow is a smooth map

$$(X, t) \to \Phi_t(X)$$

A differential system

$$\dot{x} = F(x) \tag{19}$$

generates a flow

$$(X, t) \to x(t; X)$$

where $x(t; X)$ is the solution at time t with initial condition X.

The derivative of a function G along a vector field F is, as usual,

$$D_F(G) = \nabla G \cdot F$$

If we write the differential equation associated to F, (19), then clearly

$$D_F G = \frac{d}{dt} G(x(t))_{|t=0}$$

2.1 Lyapunov stability

Consider the system (19) and assume $x = 0$ is an equilibrium.

Then

1. $x_e = 0$ is Lyapunov stable (or simply stable) if starting with initial conditions near 0 the flow remains in a neighborhood of zero. More precisely, the condition is: for every $\epsilon > 0$ there is a $\delta > 0$ so that if $|x_0| < \delta$ then $|x(t)| < \epsilon$ for all $t > 0$.

2. $x_e = 0$ is asymptotically stable if furthermore, trajectories that start close to the equilibrium converge to the equilibrium. That is, the equilibrium x_e is asymptotically stable if it is Lyapunov stable and if there exists $\delta > 0$ so that if $|x_0| < \delta$, then $\lim_{t\to\infty} x(t) = 0$.

2.2 Lyapunov functions

Let X^* be a fixed point of (19). A Lyapunov function for (19) is a function defined in a neighborhood $\mathcal{O}$ of X^* with the following properties
(1) L is differentiable in $\mathcal{O}$.
(2) $L(X^*) = 0$ (this can be arranged by subtracting a constant).
(3) $L(x) > 0$ in $\mathcal{O} \setminus \{X^*\}$.
(4) $D_F L \le 0$ in $\mathcal{O}$.
A strict Lyapunov function is a Lyapunov function for which
(4') $D_F L < 0$ in $\mathcal{O}$.

Finding a Lyapunov function is often nontrivial. In systems coming from physics, the energy is a good candidate. In general systems, one may try to find an exactly integrable equation which is a good approximation for the actual one in a neighborhood of X^* and look at the various constants of motion of the approximation as candidates for Lyapunov functions.

Theorem 2 (Lyapunov stability)**.** *Assume X^* is a fixed point for which there exists a Lyapunov function L. Then*
(i) X^* is stable.
(ii) If L is a strict Lyapunov function then X^* is asymptotically stable.

Proof. (i) Consider a small ball $B \ni X^*$ contained in $\mathcal{O}$; we denote the boundary of B (a sphere) by ∂B. Let α be the minimum of L on the ∂B. By the definition of a Lyapunov function, (3), $\alpha > 0$. Consider the following subset:

$$\mathcal{U} = \{x \subset B : L(x) < \alpha\} \tag{20}$$

From the continuity of L, we see that $\mathcal{U}$ is an open set. Clearly, $X^* \subset \mathcal{U}$. Let $X \in \mathcal{U}$. Then $x(t; X)$ is a continuous curve, and it cannot have components outside B without intersecting ∂B. But an intersection is impossible since by monotonicity, $L(x(t)) \le L(X) < \alpha$ for all t. Thus, trajectories starting in $\mathcal{U}$ are confined to $\mathcal{U}$, proving stability.

(ii)

1. Note first that X^* is the only critical point in $\mathcal{O}$ since $\frac{d}{dt}L(x(t; X_1^*)) = 0$ for any fixed point.

2. Note that trajectories $x(t;X)$ with $X \in \mathcal{U}$ are contained in a compact set, and thus they contain limit points. Any limit point x^* is strictly inside $\mathcal{U}$ since $L(x^*) < L(x(t);X) < \alpha$.

3. Let x^* be a limit point of a trajectory $x(t;X)$ where $X \in \mathcal{U}$, i.e. $x(t_n, X) \to x^*$. Then, by 1 and 2, $x^* \in \mathcal{U}$ and x^* is a regular point of the field.

4. We want to show that $x^* = X^*$. We will do so by contradiction. Assuming $x^* \neq X^*$ we have $L(x^*) = \lambda > 0$, again by (3) of the definition of L.

5. By 3 the trajectory $\{x(t;x^*) : t \geq 0\}$ is well defined and is contained in $\mathcal{B}$.

6. We then have $L(x(t;x^*)) < \lambda \forall t > 0$.

7. The set
$$\mathcal{V} = \{X : L(x(t_{n+1} - t_n; X))\} < \lambda \tag{21}$$
is open, so
$$L(x(t_{n+1} - t_n; X_1)) < \lambda \tag{22}$$
for all X_1 close enough to x^*.

8. Let n be large enough so that $x(t_m; X) \in \mathcal{V}$ for all $m \geq n$.

9. Note that, by existence and uniqueness of solutions at regular points we have
$$x(t_{n+1}; X) = x(t_{n+1} - t_n; x(t_n; X)) \tag{23}$$

10. On the one hand $L(x(t_{n+1})) \downarrow \lambda$ and on the other hand we got $L(x(t_{n+1})) < \lambda$. This is a contradiction.

□

2.3 Examples

Hamiltonian systems, in Cartesian coordinates often assume the form
$$H(q,p) = p^2/2 + V(q) \tag{24}$$
where p is the collection of spatial coordinates and p are the momenta. If this ideal system is subject to external dissipative forces, then the energy cannot increase with time. H is thus a Lyapunov function for the system. If the external force is $F(p,q)$, the new system is generally not Hamiltonian anymore, and the equations of motion become
$$\dot{q} = p \tag{25}$$
$$\dot{p} = -\nabla V + F \tag{26}$$
and thus
$$\frac{dH}{dt} = pF(p,q) \tag{27}$$

which, in a dissipative system should be nonpositive, and typically negative. But, as we see, $dH/dt = 0$ along the curve $p = 0$.

For instance, in the ideal pendulum case with Hamiltonian

$$H = \frac{1}{2}\omega^2 + (1 - \cos\theta) \tag{28}$$

The associated Hamiltonian flow is

$$\theta' = \omega \tag{29}$$

$$\omega' = -\sin\theta \tag{30}$$

Then H is a global Lyapunov function at $(0,0)$ for (31) (in fact, this is true for any system with *nonnegative* Hamiltonian). This is clear from the way Hamiltonian systems are defined.

Then $(0,0)$ is a stable equilibrium. But, clearly, it is not asymptotically stable since $H = const > 0$ on any trajectory not starting at $(0,0)$.

If we add air friction to the system (31), then the equations become

$$\theta' = \omega \tag{31}$$

$$\omega' = -\sin\theta - \kappa\omega \tag{32}$$

where $\kappa > 0$ is the drag coefficient. Note that this time, if we take $L = H$, the same H defined in (28), then

$$\frac{dH}{dt} = -\kappa\omega^2 \tag{33}$$

The function H is a Lyapunov function, but it is not strict, since $H' = 0$ if $\omega = 0$. Thus the system is stable. It is however intuitively clear that furthermore the energy still decreases to zero in the limit, since $\omega = 0$ are isolated points on any trajectory and we expect $(0,0)$ to still be asymptotically stable. In fact, we could adjust the proof of Theorem 2 to show this. However, as we see in (27), this degeneracy is typical and then it is worth having a systematic way to deal with it. This is one application of Lasalle's invariance principle that we will prove next.

3 Some important concepts

We start by introducing some important concepts.

Definition 2. *1. An entire solution $x(t;X)$ is a solution which is defined for all $t \in \mathbb{R}$.*

2. A positively invariant set $\mathcal{P}$ is a set such that $x(t,X) \in \mathcal{P}$ for all $t \geq 0$ and $X \in \mathcal{P}$. Solutions that start in $\mathcal{P}$ stay in $\mathcal{P}$. Similarly one defines negatively invariant sets, and invariant sets.

3. *The basin of attraction of a fixed point X^* is the set of all X such that $x(t;X) \to X^*$ when $t \to \infty$.*

4. *Given a solution $x(t;X)$, the set of all points ξ^* such that solution $x(t_n;X) \to \xi^*$ for some sequence $t_n \to \infty$ is called the set of ω-limit points of $x(t;X)$. At the opposite end, the set of all points ξ^* such that solution $x(-t_n;X) \to \xi^*$ for some sequence $t_n \to \infty$ is called the set of α-limit points. These may of course be empty.*

Proposition 3. *Assume $x(t;X)$ belongs to a closed, positively invariant set $\mathcal{P}$ where the field is defined. The ω-limit set $\omega(X)$ is a closed invariant set too; in particular, trajectories through points in $\omega(X)$ are contained in $\omega(X)$. A similar statement holds for the α-set.*

Proof. 1. (Closure) We show the complement is open. Let b be in the complement of the set of ω-limit points. Then $\liminf_{t\to\infty} d(x(t,X),b) > a > 0$ for some a and for all $t > 0$. If b' is close enough to b, then by the triangle inequality, $\liminf_{t\to\infty} d(x(t,X),b') > a/2 > 0$ for all t.

2. (Invariance) Assume $x(t_n,X) \to x^*$. By assumption, the differential equation is well-defined in a neighborhood of any point in $\mathcal{P}$, and since $x^* \in \mathcal{P}$, $x(t;x^*)$ exists for all $t \geq 0$. By continuity with respect to initial conditions and of solutions, we have $x(t;x(t_n)) \to x(t;x^*)$ as $n \to \infty$. Then $x(t;x^*)$ is an omega-point too for any $t \geq 0$ (note that $x(t_n + t, X) \to x(t, x^*)$ by the definition of an ω limit point and continuity .

3. Backward invariance is proved similarly: $x(t_n - t, X) \to x(-t, x^*)$.

□

4 Lasalle's invariance principle

Theorem 3. *Let X^* be an equilibrium point for $X' = F(X)$ and let $L : \mathcal{U} \to \mathbb{R}$ be a Lyapunov function at X^*. Let $X^* \ni \mathcal{P} \subset \mathcal{U}$ be closed, bounded and positively invariant. Assume there is no entire trajectory in $\mathcal{P} - \{X^*\}$ along which L is constant. Then X^* is asymptotically stable, and $\mathcal{P}$ is contained in the basin of attraction of X^*.*

Proof. Since $\mathcal{P}$ is compact and positively invariant, every trajectory in $\mathcal{P}$ has ω-limit points. If X^* is the only limit point, the assumption follows easily (show that all trajectories must tend to X^*). So, we may assume there is an $x^* \neq X^*$ which is also an ω-limit point of some $x(t;X)$. We know that the trajectory $x(t;x^*)$ is entire. Since L is nondecreasing along trajectories, we have $L(x(t;X)) \to \alpha = L(x^*)$ as $t \to \infty$. (This is clear for the subsequence t_n, and the rest follows by inequalities: check!) On the other hand, for any $T \in \mathbb{R}$, positive or negative, $x(T;x^*)$ is arbitrarily close to $x(t_n + T, X)$ if n is large. Since $L(x(T;x^*)) \leq \alpha$ and it is arbitrarily close to $L(x(t,X)) \geq \alpha$, it follows that $L(x(T;x^*)) = \alpha$ for all $t \geq 0$. □

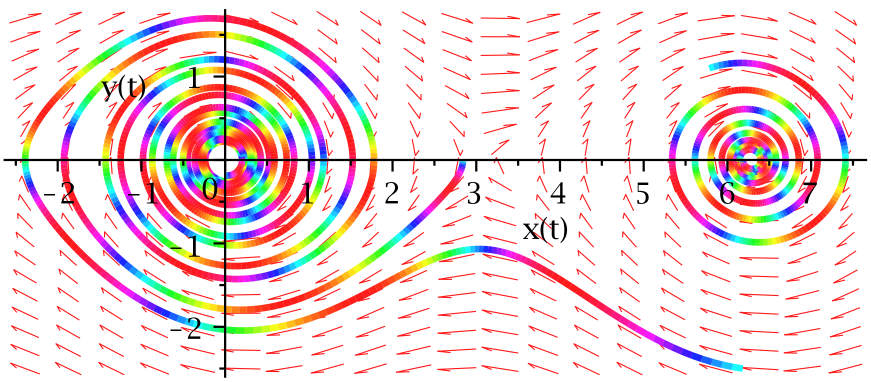

Figure 3:

4.1 Example: analysis of the pendulum with drag

Of course this is a simple example, but the way Lasalle's invariance principle is applied is representative of many other problems.

Intuitively, it is clear that any trajectory that starts with $\omega = 0$ and $\theta \in (-\pi, \pi)$ should asymptotically end up at the equilibrium point $(0, 0)$ (other trajectories, which for the frictionless system would rotate forever, may end up in a different equilibrium, $(2n\pi, 0)$. For zero initial ω, the basin of attraction of $(0, 0)$ should exactly be $(-\pi, \pi)$. In general, the energy should be less than precisely the one in this marginal case, $H = 1 - \cos(\pi) = 2$. Then, the region $\theta_0 \in (-\pi, \pi)$, $H < 1 - \cos(\pi) = 2$ should be the basin of attraction of $(0, 0)$.

So let $c \in (0, 2)$, and let

$$\mathcal{P}_c = \{(\theta, \omega) : H(\theta, \omega) \leq c, \text{ and } |\theta| \leq \arccos(1 - c) \in (-\pi, \pi)\} \tag{34}$$

In H, θ coordinates, this is simply a closed rectangle and since (H, θ) is a continuous map, its preimage in the (p, θ) plane is closed too.

Now we show that $\mathcal{P}_c$ is closed and forward invariant. If a trajectory were to exit $\mathcal{P}_c$, it would mean, by continuity, that for some t we have $H = c + \delta$ for a small $\delta > 0$ (ruled out by $\dot{H} \le 0$ along trajectories) or that $|\theta| = \arccos(1-c)+\epsilon$ for a small $\epsilon > 0$ which implies, from the formula for H the same thing: $H > c$.

Now there is no nontrivial entire solution (that is, other than $X^* = (0, 0)$) along which $H = const$. Indeed, $H = const$ implies, from (33) that $\omega = 0$ identically along the trajectory. But then, from (30) we see that $\sin\theta = 0$ identically, which, within $\mathcal{P}_c$ simply means $\theta = 0$ identically. Lasalle's theorem applies, and all solutions starting in $\mathcal{P}_c$ approach $(0, 0)$ as $t \to \infty$.

Recall that, for a flow, the ω-limit set is defined as

$$\omega(X) := \{x : \lim_{n\to\infty} x(\varphi(t_n) = x \text{ for some sequence } t_n \to +\infty\} \qquad (35)$$

and, similarly, the α-limit set is defined as

$$\alpha(X) := \{x : \lim_{n\to\infty} x(\varphi(t_n) = x \text{ for some sequence } t_n \to -\infty\}. \qquad (36)$$

The phase portrait of the damped pendulum is depicted in Fig. 3

5 Gradient systems and Lyapunov functions

Recall that a gradient system is of the form (1), that is

$$X' = -\nabla V(X) \qquad (37)$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is, say, C^∞ and a critical point of V is a point where $\nabla V = 0$. We have the following result:

Theorem 4. *For the system (1): (i) If c is a regular value of V, then the vector field is orthogonal to the level set of $V^{-1}(c)$.*

(ii) The equilibrium points of the system coincide with the critical points of V.

(iii) If a critical point X^* is an isolated minimum *of V, $V(X) - V(X^*)$ is a strict Lyapunov function at X^*, and then X^* is asymptotically stable.*

(iv) Any $\alpha-$ limit point of a solution of (1), and any $\omega-$ limit point is an equilibrium.

(v) The linearized system at any equilibrium has only real eigenvalues.

Note 1. (a) By (v), any solution of a gradient system tends to a limit point or to infinity.

(b) Thus, descent lines of any smooth manifold have the same property: they link critical points, or they tend to infinity.

(c) We can use some of these properties to determine for instance that a system is not integrable. We write the associated gradient system and determine that it fails one of the properties above, for instance the linearized system at a critical point has an eigenvalue which is not real. Then there cannot exist a smooth H so that $H(x, y(x))$ is constant along trajectories.

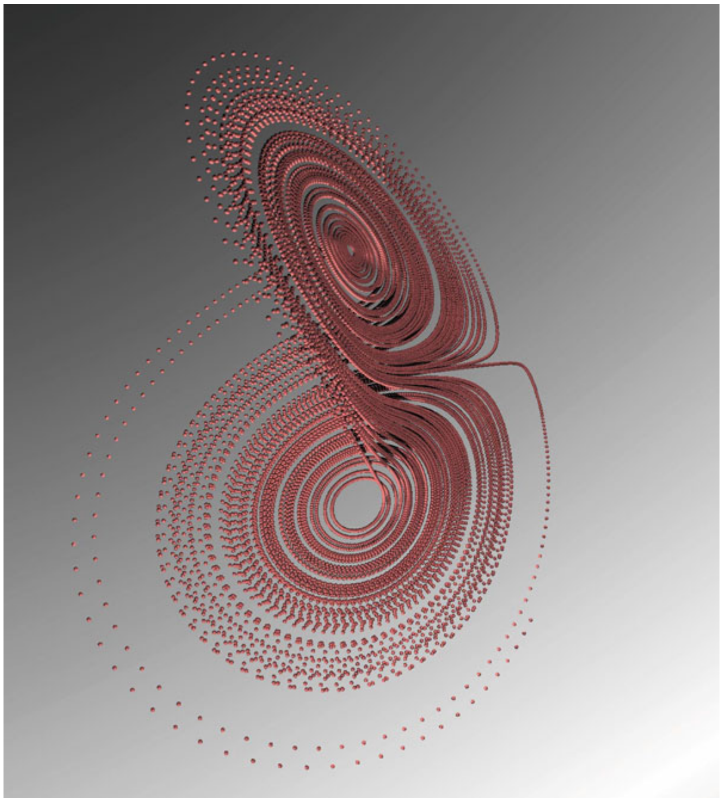

Figure 4: The Lorenz attractor

Proof. We have already shown (i) and (ii), which are in fact straightforward from the definition.

(iii) If an equilibrium point is isolated, then $\nabla V \neq 0$ in a set of the form $|X - X^*| \in (0, a)$. Then $-|\nabla V|^2 < 0$ in this set. Furthermore, $V(X) - V(X^*) > 0$ for all X with $|X - X^*| \in (0, a)$.

(iv) As in the proof of Lasalle's invariance principle, we can check V is constant along any trajectory starting at a limit point. But we see from (iii) that this implies that the trajectory reduces to a point, which is an equilibrium point.

(v) Note that for smooth V, the linearization at X^* is simply the matrix

$$A; \quad A_{ij} = \frac{-\partial^2 V(X^*)}{\partial x_i \partial x_j} \tag{38}$$

which is *symmetric*. □

6 Limit sets, Poincaré maps, the Poincaré Bendixson theorem

In two dimensions, there are typically two types of limit sets: equilibria and periodic orbits (which are thereby limit cycles). Exceptions occur when a limit set contains a number of equilibria, as we will see in examples.

The Poincaré-Bendixson theorem states that if $\omega(X)$ is a nonempty compact limit set of a *planar system of ODEs* containing no equilibria, then Ω is a closed orbit. We will return to this important theorem and prove it.

Beyond two dimensions however, the possibilities are far vaster and limit sets can be quite complicated. Fig. 4 depicts a limit set for the Lorenz system, in three dimensions. Note how the trajectories seem to spiral erratically around two points. The limit set here has a fractal structure.

We begin the analysis with the two dimensional case, which plays an important tole in applications.

We have already studied the system $r' = 1/2(r - r^3)$ in Cartesian coordinates. There the circle of radius one was a periodic orbit, and a limit cycle. All trajectories, except for the trivial one $(0, 0)$ tended to it as $t \to \infty$.

We have also analyzed many cases of nodes, saddle points etc, where trajectories have equilibria as limit sets, or else they go to infinity.

A rather exceptional situation is that where the limit sets contain equilibria. Here is one example

6.1 Example: equilibria on the limit set

Consider the system

$$x' = \sin x(-\cos x - \cos y) \tag{39}$$

$$y' = \sin y(\cos x - \cos y) \tag{40}$$

The phase portrait is depicted in Fig. 5.

Exercise 1. *Justify the qualitative elements in Fig. 5.*

In the example above, we see that the limit set is a collection of fixed points and orbits, none of which periodic.

6.1.1 Closed orbits

A closed orbit is a solution whose trajectory is a closed curve (with no equilibria on it). Let $\mathcal{C}$ be such a trajectory.

Note that the flow is always in the direction of the field, since

$$\dot{x}(t) = f(x(t))$$

and furthermore, the speed is, as we see from the above

$$|\dot{x}(t)| = |f(x(t))|$$

Since trajectories and f are smooth and there are no equilibria along $\mathcal{C}$, $|\dot{x}(t)| = |f(x)|$ is bounded below, and $\mathcal{C}$ is traversed in finite time. That is, starting at a point $x_1 \in \mathcal{C}$, after a (finite) time T, then, the solution returns to x_1. From that time on, the solution must repeat itself identically, by uniqueness of solutions. It then means that the solution is periodic, and there is a smallest τ so that $\Phi_{t+\tau}(x_1) = \Phi(x_1)$. This τ is called the period of the orbit.

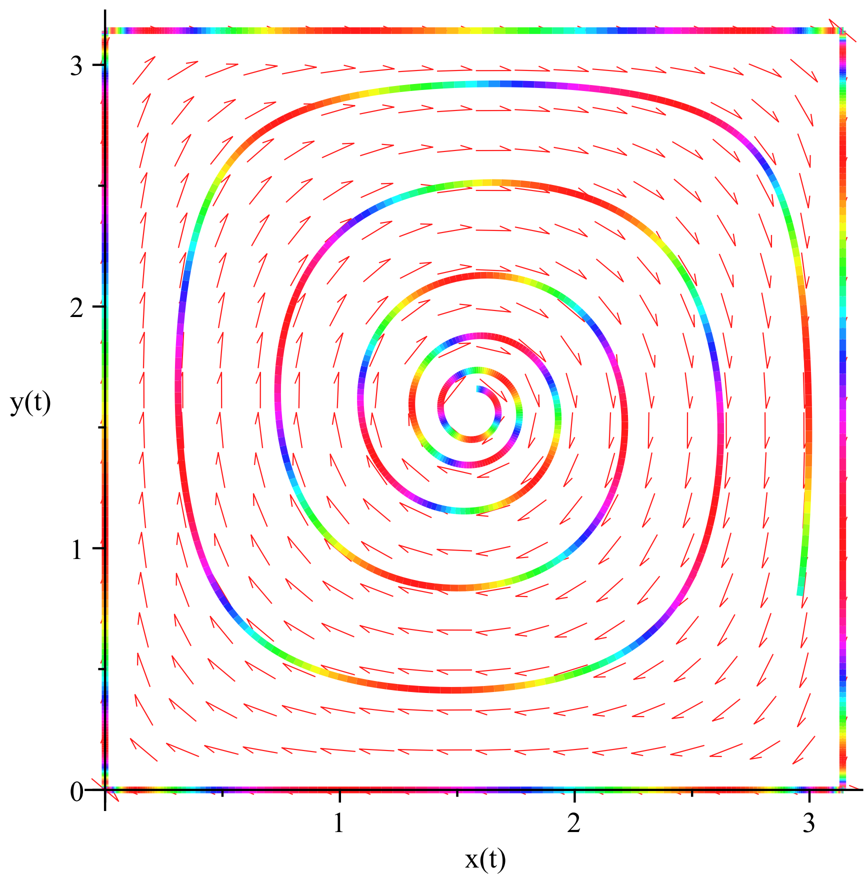

Figure 5: Phase portrait for (39).

Proposition 4. *(i) If x_1 and x_2 lie on the same solution curve, then $\omega(x_1) = \omega(x_2)$ and $\alpha(x_1) = \alpha(x_2)$.*

(ii) If D is a closed, positively invariant set and $x_2 \in D$, then $\omega(x_2) \subset D$; similarly for negatively invariant sets and $\alpha(x_2)$.

(iii) A closed invariant set, and in particular a limit set, contains the $\alpha-$limit and the $\omega-$limit of every point in it.

Proof. Exercise. □

Exercise 2. *Show that τ is the same for any two points x_1, x_2 on $\mathcal{C}$.*

7 Sections; the flowbox theorem

Consider a differential equation $x' = f(x)$ with f smooth, and a point x_0 such that $f(x_0) \neq 0$. A section through x_0 is a curve which is transversal to the flow, and passes through x_0. To be specific, take a unit vector V_0 at x_0 which is orthogonal to $f(x_0)$, say $(-f_2(x_0), f_1(x_0))/|f(x_0)|$. We draw a line segment in the direction of V_0,

$$\mathcal{S} = \{h(u) := x_0 + uV_0 | u \in (-\epsilon, \epsilon)\} \tag{41}$$

Once more, since f is continuous, for small δ there is a small ϵ so that we have $V_0 \cdot (-f_2(h(u)), f_1(h(u)))/|f(h(u))| \geq 1 - \delta$ if $u \in (-\epsilon, \epsilon)$. That is, the field is transversal to the section in a small neighborhood of x_0. By the same estimate, $V_0 \cdot (-f_2(h(u)), f_1(h(u)))/|f(h(u))|$ has constant sign along $\mathcal{S}$, which means that the field, ass well as the flow, cross $\mathcal{S}$ in the same direction throughout $\mathcal{S}$. See the left side of fig. 7.

Definition 5. *The segment $\mathcal{S}$ defined above is called local section at x_0.*

7.0.2 The flowbox theorem

There is a diffeomorphic change of coordinates *in some neighborhood of* x_0, $x \leftrightarrow z$ so that in coordinates z the field is simply $\dot{z} = \mathbf{e}_1 := (1, 0)$.

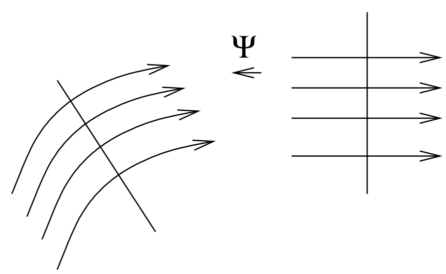

Figure 6: Flowbox and transformation

To straighten the field, we construct the following map, from a neighborhood of x_0 of the form

$$\mathcal{N} = \{\Psi(t, u) := x(t; h(u)) : |t| < \delta, u \in (-\epsilon, \epsilon)\}$$

where ϵ and δ are sufficiently small. Then, $(t, u) \mapsto x(t; h(u)$ is a diffeomorphism since the Jacobian of the transformation at $(0, 0)$ is

$$\det \begin{pmatrix} \dfrac{\partial \Psi_1}{\partial t} & \dfrac{\partial \Psi_1}{\partial u} \\[2ex] \dfrac{\partial \Psi_2}{\partial t} & \dfrac{\partial \Psi_2}{\partial u} \end{pmatrix} = \det \begin{pmatrix} f_1 & V_1 \\ f_2 & V_2 \end{pmatrix} = |f(x_0)| \neq 0 \tag{42}$$

Clearly, the inverse image of trajectories through Ψ are straight lines, (t, u_0), as depicted. The associated flow in the set $\Psi^{-1}(\mathcal{N})$ is

$$\frac{dt}{dt} = 1; \quad \frac{du}{dt} = 0 \tag{43}$$

Figure 7: Time of arrival function

7.1 Time of arrival

We consider all solutions in the domain $\mathcal{O}$ where the field is defined. Some of them intersect $\mathcal{S}$. Since the trajectories are continuous, there is a first time of arrival, the smallest t so that $x(t, z_0) \in \mathcal{S}$.

This time of arrival is continuous in z_0, as shown in the next proposition.

Proposition 6. *Let $\mathcal{S}$ be a local section at x_0 and assume $\phi_{t_0}(z_0) = x_0$. Let $\mathcal{W}$ be a neighborhood of z_0. Then there is an open set $\mathcal{U} \subset \mathcal{W}$ and a differentiable function $\tau : \mathcal{U} \to \mathbb{R}$ such that $\tau(z_0) = t_0$ and*

$$\phi_{\tau(x)}(x) \in \mathcal{S} \tag{44}$$

for each $x \in \mathcal{U}$.

Note 2. *In some sense, a subsegment of the section $\mathcal{S}$ is carried* backwards *smoothly through the field arbitrarily far, assuming that the flow makes sense, and that the subsegment is small enough.*

Proof. A point x_1 belongs to the line ℓ containing $\mathcal{S}$ iff $x_1 = x_0 + uV_0$ for some u. Since V_0 is orthogonal to $f(x_0)$ we see that $x_1 \in \ell$ iff $(x_1 - x_0) \cdot f(x_0) = 0$.

We look now at the more general function

$$G(x, t) = (x(t; x) - x_0) \cdot f(x_0) \tag{45}$$

We have, by construction,

$$G(z_0, t_0) = 0 \tag{46}$$

by construction. We want to see whether we can apply the implicit function theorem to

$$G(x, t) = 0 \tag{47}$$

□

For this we need to check $\frac{\partial}{\partial t}G\big|_{(z_0,t_0)}$. But this equals

$$x'(t;x)\cdot f(x_0) = |f(x_0)|^2 \neq 0 \tag{48}$$

Then, there is a neighborhood of t_0 and a differentiable function $\tau(x)$ so that

$$G(x,\tau(x)) = 0 \tag{49}$$

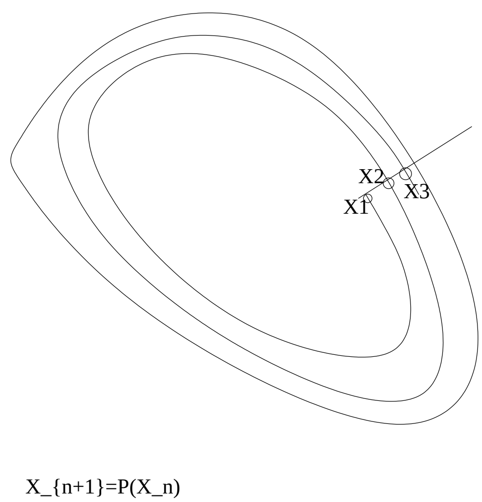

Figure 8: A Poincaré map.

7.2 The Poincaré map

The Poincaré map is a useful tool in determining whether closed trajectories (that is, periodic orbits) are stable or not. This means that taking an initial close enough to the periodic orbit, the trajectory thus obtained would approach the periodic orbit or not.

The basic idea is simple, we look at a section containing a point on the periodic orbit, and then follow the successive re-intersections of the perturbed orbit with the section. Now we are dealing with a discrete map $x_{n+1} = P(x_n)$. If $P(x_n) \to x_0$, the point on the closed orbit, then the orbit is asymptotically stable. See Figure 11.

It is often not easy to calculate the Poincaré map; in general it can't quite be easier than calculating the trajectories, but it is a very useful concept, and it has many theoretical applications; furthermore, we often don't need fully explicit knowledge of P.

Let's define the map P rigorously.

Consider a periodic orbit $\mathcal{C}$ and a point $x_0 \in \mathcal{C}$. We have

$$x(x_0; \tau) = x_0 \tag{50}$$

where τ is the period of the orbit. Consider a section $\mathcal{S}$ through x_0. Then according to Proposition 6, there is a neighborhood of $\mathcal{U}$ of x_0 and a continuous function $\tau(x)$ so that $x(\tau(t), x) \in \mathcal{S}$ for all $x \in \mathcal{U}$. Then certainly $\mathcal{S}_1 = \mathcal{U} \cap \mathcal{S}$ is an open set in $\mathcal{S}$ in the induced topology. The return map is thus defined on $\mathcal{S}_1$. It means that for each point in $x \in \mathcal{S}_1$ there is a point $P(x) \in \mathcal{S}$, so that $x(\tau(x); x) = P(x)$ and $\tau(x)$ is the smallest time with this property. Note that now $\tau(x)$ is not a period, though it is "very close to one": the trajectory does not return to the same point.

This is the Poincaré map associated to $\mathcal{C}$ and to its section $\mathcal{S}$.

This can be defined for planar systems as well as for higher dimensional ones, if we now take as a section a subset of a hyperplane through a point $x_0 \in \mathcal{C}$. The statement and proof of Proposition 6 generalize easily to higher dimensions.

In two dimensions, we can identify the segments $\mathcal{S}$ and $\mathcal{S}_1$ with intervals on the real line, $u \in (-a, a)$, and $u \in (-\epsilon, \epsilon)$ respectively, see also Definition 5. Then P defines an analogous transformation of the interval $(-\epsilon, \epsilon)$, which we still denote by P though this is technically a different function, and we have

$$P(0) = 0$$

$$P(u) \in (-a, a), \quad \forall u \in (-\epsilon, \epsilon)$$

We have the following easy result, the proof of which we leave as an exercise.

Proposition 7. *Assume that $x' = f(x)$ is a planar system with a closed orbit $\mathcal{C}$, let $x_0 \in \mathcal{C}$ and $\mathcal{S}$ a section at x_0. Define the Poincaré map P on an interval $(-\epsilon, \epsilon)$ as above, by identifying the section with a real interval centered at zero. If $|P'(x_0)| < 1$ then the orbit $\mathcal{C}$ is asymptotically stable.*

Example 3. Consider the planar system

$$r' = r(1 - r) \tag{51}$$

$$\theta' = 1 \tag{52}$$

In Cartesian coordinates it has a fixed point, $x = y = 0$ and a closed orbit, $x = \cos t, y = \sin t; x^2 + y^2 = 1$. Any ray originating at $(0, 0)$ is a section of the flow. We choose the positive real axis as $\mathcal{S}$. Let's construct the Poincaré map. Since $\theta' = 1$, for any $x \in \mathbb{R}^+$ we have $x(2\pi; x) = x(0, x)$. We have $P(1) = 1$ since 1 lies on the unit circle. In this case we can calculate explicitly the solutions, thus the Poincaré map and its derivative.

We have

$$\ln r(t) - \ln(r(t) - 1) = t + C \tag{53}$$

and thus

$$r(t) = \frac{Ce^t}{Ce^t - 1} \tag{54}$$

where we determine C by imposing the initial condition $r(0) = x$: $C = x/(x-1)$. Thus,

$$r(t) = \frac{xe^t}{1 - x + xe^t} \tag{55}$$

and therefore we get the Poincaré map by taking $t = 2\pi$,

$$P(x) = \frac{xe^{2\pi}}{1 - x + xe^{2\pi}} \tag{56}$$

Direct calculation shows that $P'(1) = e^{-2\pi}$, and thus the closed orbit is stable. We could have seen this directly from (56) by taking $t \to \infty$.

Note that here we could calculate the orbits explicitly. Thus we don't quite need the Poincaré map anyway, we could just look at (55). When explicit solutions, or at least an explicit formula for the closed orbit is missing, calculating the Poincaré map can be quite a challenge.

8 Monotone sequences in two dimensions

There are two kinds of monotonicity that we can consider. One is monotonicity along a solution: $x_1, ..., x_n$ is monotone along the solution if $x_n = x(t_n, x)$ and t_n is increasing in n. Or, we can consider monotonicity along a segment, or more generally a piece of a curve. On a piece of a smooth curve, or on an interval we also have a natural order (or two rather), by arclength parameterization of the curve: $\gamma_2 > \gamma_1$ if γ_2 is farther from the chosen endpoint. To avoid this rather trivial distinction (dependence on the choice of endpoint) we say that a sequence $\{\gamma_n\}_n$ is monotone along the curve if γ_n is inbetween γ_{n-1} and γ_{n+1} for all n. Or we could say that a sequence is monotone if it is either increasing or else decreasing.

If we deal with a trajectory crossing a curve, then the two types of monotonicity need not coincide, in general. But for sections, they do.

Proposition 8. *Assume $x(t; x); t \in [0, \tau]$ is a solution of a planar system $x' = f(x)$, so that f is regular and nonzero in its neighborhood. Let $\mathcal{S}$ be a local section. Then monotonicity along the solution $x(t; x)$ assumed to intersect $\mathcal{S}$ at $x_1, x_2, ...$ (finitely or infinitely many intersection) and along $\mathcal{S}$ coincide.*

Note that all intersections are taken to be with $\mathcal{S}$, along which, by definition, they are always transversal.

Proof. We assume we have three successive distinct intersections with $\mathcal{S}$, x_1, x_2, x_3 (if two of them coincide, then the trajectory is a closed orbit and there is nothing to prove).

We want to show that x_3 is not inside the interval (x_1, x_2) (on the section, or on its image on $\mathbb{R}$). Consider the curve $\mathcal{C}_1 = \{x(t; x_1) : t \in [0, t_2]\}$ where t_2 is the first time of re-intersection of $x(t; x_1)$ with $\mathcal{S}$. By definition $x(t_2 - t_1; x_1) = x_2$. $\mathcal{C}_1$ is a smooth curve, with no self-intersection (since the field is assumed regular along the curve) thus of finite length. If completed with the line segment $\mathcal{J}$

linking x_1 and x_2, $\mathcal{C}_1 \cup \mathcal{J}$ is a closed continuous curve. By Jordan's lemma, we can define the inside int $\mathcal{C}$ and the outside of the curve, $D =$ext $\mathcal{C}$. Note that the field has a definite direction along $[x_1, x_2]$, by the definition of a section. Note also that it points towards ext $\mathcal{C}$, since $x(t; x_1)$ exits int $\mathcal{C}$ at $t = t_2$. Then,

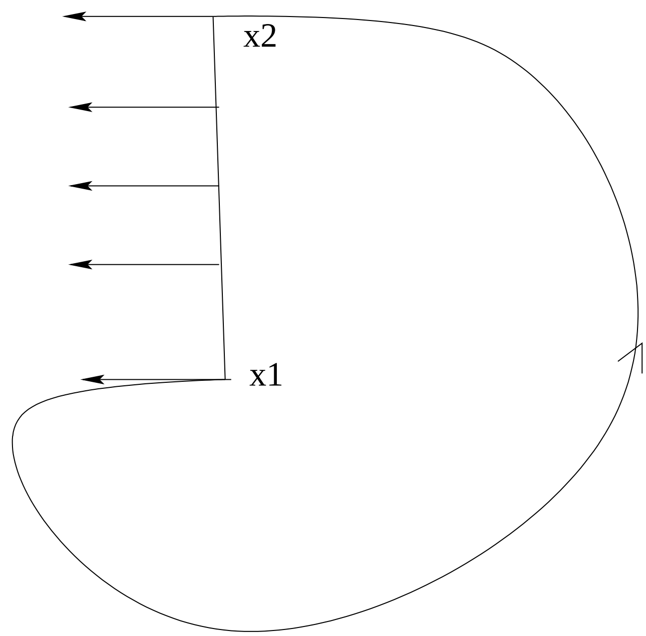

Figure 9: Monotone sequence theorem

no trajectory can enter int $\mathcal{C}$. Indeed, it should intersect either $x(t; x_1)$ or else $[x_1, x_2]$. The first option is impossible by uniqueness of solutions. The second case is ruled out since the field points outwards from $\mathcal{J}$. Thus $x(t_3, x) = x_3$ must lie in ext $\mathcal{C}$, thus outside $[x_1, x_2]$. □

The next result shows points towards limiting points being special: parts of closed curves, or simply infinity.

Proposition 9. *Consider a planar system and $z \in \omega(x)$ (or $z \in \alpha(x)$), assumed a regular point of the field. Consider a local section $\mathcal{S}$ through a regular point $\tilde{z}$. Then the intersection of $\{\Phi_t(z) : t > 0\} \cap \mathcal{S}$ has at most one point (note that we are dealing with $\Phi_t(z)$ and not $\Phi_t(x)$).*

Proof. Assume there are two distinct intersection points $x(t_1, z) = z_1$ and $x(t_2, z) = z_2$ in $\mathcal{S}$. By Proposition 3, $\{\Phi_t(z) : t > 0\} \subset \omega(x)$; in particular, z_1 and z_2 are also in $\omega(x)$. There are then infinitely many points on $\Phi_t(x)$ arbitrarily close to z_1 and infinitely many others arbitrarily close to z_2, by the definition of $\omega(x)$. Consider the first arrival times at $\mathcal{S}$ for the trajectory $\Phi_t(z_1)$: it is clearly zero. Then, by the continuity of τ, if j is large enough, and so that

$\Phi_{t_j}(x)$ is close to z_1, then $\tau(\Phi_{t_j}(x))$ exists by Proposition 6; $\tau(\Phi_{t_j}(x))$ and is arbitrarily small if j is large enough. Thus, by choosing $t_j + \tau(\Phi_{t_j}(x))$ instead of t_j, we can arrange that $\Phi_{t_j}(x) \in \mathcal{S}$. Similarly, we can arrange that the points converging to z_2 are on $\mathcal{S}$.

Also w.l.o.g. (rotating and translating the figure) we can assume that $\mathcal{S} = (-a, b) \in \mathbb{R}$ and $[z_1, z_2] \subset (-a, b)$. We know that $x(t_j, x)$, where t_j are the increasing times when $x(t_j, x) \in \mathcal{S}$, are monotone in $\mathcal{S} = (-a, b)$. Thus they converge. But then, by definition of convergence, they cannot be arbitrarily close to two distinct points.

□

9 The Poincaré-Bendixson theorem

Theorem 5 (Poincaré-Bendixson)**.** *Let $\Omega = \omega(x)$ be a nonempty compact limit set of a planar system of ODEs, containing no equilibria. Then Ω is a closed orbit.*

Proof. First, recall that Ω is invariant. Let $y \in \Omega$. Then $\Phi_t(y)$ is contained Ω, and then $\Phi_t(y)$ has infinitely many accumulation points in Ω. Let z be one of them. By definition, $\Phi_{t_j}(y)$ tend to z as $j \to \infty$ for a sequence $t_j \to \infty$, and, as in the proof of Proposition 9, w.l.o.g. we can assume that they all belong to $\mathcal{S}$, where $\mathcal{S}$ is a section through z, and we can arrange that the sequence t_j contains all intersection times for t large.

Since we are now dealing with $\Phi_t(y)$ and $y \in \omega(x)$, Proposition 9 applies and $\Phi_{t_j}(y)$ must all coincide. Thus $\Phi_{t_j}(y) = \Phi_{t_{j+1}}(y)$. But this means that $\Phi_t(y)$ is periodic, and the period $T = t_{j+1} - t_j$.

Therefore: any point $y \in \omega(x)$ has a periodic orbit, contained in Ω.

It remains to show that $\Omega = \{\Phi_t(y) | t \in [0, T]\}$.

We can now assume that t_j, an increasing sequence, and it contains *all* points of intersection for large t of $\Phi_t(x)$ with $\mathcal{S}_y$, a section through y, y as above. We know that $\tau(x)$ is continuous. If we take a section $\mathcal{S}$ through y and look at $\Phi_\epsilon(y)$, then $\tau(\Phi_\epsilon(y)) = T - o(1)$. Thus, $\tau(\Phi_{t_j}(x)) = T - o(1)$ for large enough j, since $\Phi_{t_j}(x)$ is arbitrarily close to y if j is large enough, and so is $\Phi_\epsilon(y)$ if ϵ is small enough. On the other hand, by continuity with respect to initial conditions, $|\Phi_t(y) - \Phi_{t_j+t}(x)| = o(1)$ if $|t| < 2T$ and if j is large enough. But this means that $d(\{\Phi_t(y) | t \in [0, T]\}, \Phi_t(x)) \to 0$ as $t \to \infty$.

□

Exercise 1. *Where have we used the fact that the system is planar? Think how crucial dimensionality is for this proof.*

10 Applications of the Poincaré-Bendixson theorem

Definition. *A limit cycle is a closed orbit γ which is the ω-set, or an $\alpha-$ set* *of a point $X \notin \gamma$. These are called ω limit cycles or α limit cycles respectively.*

As we see, closed orbits are limit cycles only if *other* trajectories approach them arbitrarily. There are of course closed orbits which are not limit cycles. For instance, the system $x' = -y, y' = x$ with orbits $x^2 + y^2 = C$ for any C clearly has no limit cycles.

But when limit cycles exist, they have at least one-sided stability.

Indeed, if $\gamma = \omega(X)$ ($X \notin \gamma$), then we know that $\text{dist}(x(t, X), \gamma) \to 0$ as $t \to \infty$. If we consider any one-sided neighborhood of γ (on the side of $x(t, X)$), then, because of nonintersection of trajectories and a sandwich argument, all points in that neighborhood are also evolving as $t \to \infty$ towards γ.

Corollary 10. *Assume $\omega(X) = \gamma$, $\gamma \not\ni X$ is a limit cycle. Then there exists a neighborhood $\mathcal{O}$ of X so that $\forall X' \in \mathcal{O}$ we have $\gamma = \omega(X')$.*

Proof. Let t_0 be large enough so that $\Phi_t(X) \in \mathcal{N}$, the one-sided neighborhood of stability of γ, for all $t \geq t_0$. Take any $t_1 > t_0$ and a small enough neighborhood $\mathcal{O}_1$ of $x_1 = \Phi_{t_1}(X)$, so that, in particular, $\mathcal{O}_1 \subset \mathcal{N}$. Clearly, $\Phi_{-t_1}(x_1) = X$. As $\text{diam}(\mathcal{O}_1) \to 0$, we have $\text{diam}(\Phi_{-t_1}(\mathcal{O}_1)) \to 0$ as well, by continuity with respect to initial conditions. Also by continuity of Φ_t, and noting that $\Phi_{-t}(Z) = (\Phi_t)^{-1}(Z)$, we see that $\mathcal{O}_2 := \Phi_{-t_1}(\mathcal{O}_1)$ is an open set, which clearly contains X. By construction, $\omega(X') = \gamma$ for all $X' \in \mathcal{O}_2$. □

Corollary 11. *Hamiltonian systems for which the Hamiltonian H is not constant in any open set have no limit cycles.*

Proof. Indeed, if $\gamma = \omega(X)$ is a limit cycle for some X, then by Corollary 10 there is a neighborhood $\mathcal{O}_X$ so that $\omega(X') = \gamma$ for all $X' \in \mathcal{O}_X$. We know that H is constant along any trajectory. Let $Y_0 \in \gamma$. By continuity, $H(X') = H(Y_0)$ for any $X' \in \mathcal{O}_X$. □

Corollary 12. *Let K be a* compact *positively invariant set. Then K contains at least a limit cycle or an equilibrium.*

Proof. Indeed, K must contain $\omega(X)$ for every $X \in K$, and if K has no equilibrium, clearly neither does $\omega(X)$. But then $\omega(X)$ is a closed orbit, by Poincaré-Bendixson. Assume to get a contradiction that no closed orbit in K is a limit cycle. Take some closed orbit $\gamma \subset K$. The compact set $\gamma \cup \text{int}\gamma$ is (positively and negatively) invariant because of nonintersection of orbits. If $X \in \text{int}\gamma$ then $\omega(X) \neq \gamma$, otherwise γ would be a limit cycle; thus γ_X is strictly contained in γ. We can continue the construction indefinitely by choosing $X_n \in \text{int}\gamma_{X_{n-1}}$. Clearly, the set of $\text{int}\gamma_{X_n}$ thus constructed is a nested sequence of sets. We let ν be the inf of the areas of $\text{int}\gamma_{X_n}$, and we note that a_n=area(γ_n) is a decreasing sequence, thus $a_n \to \inf a_n \leq \nu$. On the other hand, $K_1 = \cap_n(\gamma_n \cup \text{int}\gamma_n)$ is an intersection of compact sets for which no finite intersection is empty; we also clearly have $\text{area}(K_1) \leq \nu$. Let $x \in K_1$. Since $x \in \gamma_n \cup \text{int}\gamma_n$ for all n, we have $\omega(X) \subset (\gamma_n \cup \text{int}\gamma_n)$ for all n. Thus $\omega(X) \subset K_1$ and $\text{area}(\omega(X)) \leq \nu$. If $\nu \neq 0$, then, since $\gamma_X \cup \text{int}\gamma_X$ is invariant, it contains (strictly) a closed orbit, which thus has area $< \nu$, impossible. But $\text{area}(\omega(X)) = 0$ means $\omega(X)$ contains an equilibrium, since otherwise, by Poincaré-Bendixson and the above, it would be a smooth closed curve, which cannot have zero area).

Note 4. *We will see later that in fact K must contain an equilibrium.*

□

Corollary 13. *Let γ be a closed orbit and $\mathcal{U}$ its interior. Then $\mathcal{U}$ contains at least an equilibrium.*

We first prove the following result.

Lemma 14. *Then $\mathcal{U}$ contains either an equilibrium or a limit cycle $\gamma \in int(U)$.*

Proof of the Lemma. The region $\mathcal{U} \cup \gamma = \overline{\mathcal{U}}$ is both positively and negatively invariant, since no trajectory can cross γ. If $X \in \mathcal{U}$ and there are no equilibria or limit cycles in $\mathcal{U}$ then γ itself must be an ω limit cycle, by corollary 12. Reverting the direction of t, the same argument shows that γ is an α limit cycle for $\Phi_t(X)$ as well. But then, the monotonicity along a section of any point on γ would be violated. □

Proof of the Corollary. We first show that if there is no equilibrium in $\mathcal{U}$ then there are infinitely many limit cycles. Indeed, assume to get a contradiction, that there are finitely many limit cycles inside $\mathcal{U}$ (and no equilibrium). Then there is one of minimum area, γ_1. But by Lemma 14, there should be a limit cycle strictly inside it (impossible by assumption) or else an equilibrium.

Thus, that there are infinitely many limit cycles γ_n in U. We can furthermore assume they are contained in each other, since each limit cycle contains an equilibrium or yet another limit cycle (strict inclusion). Now we can repeat the last part of the proof of Corollary 12, since a limit cycle is, in particular, a closed orbit. (At the end of that proof, $\omega(X)$ cannot be a closed orbit, otherwise, once more, it would contain an even smaller one.) □

Corollary 15. *If K is positively (or negatively) invariant, then it contains an equilibrium.*

Proof. Indeed, we know by Corollary 12 that K contains an equilibrium or a limit cycle. The rest follows from Corollary 13. □

11 The Painlevé property

As mentioned on p.2, Hamiltonian systems (with time-independent Hamiltonian) in one dimension are integrable: the solution can be written in closed form, implicitly, as $H(y(x), x) = c$; in terms of t, once we have $y(x)$ of course we can integrate $x' = G(y(x), x) := f(x)$ in closed form, by separation of variables. The classification of equations into integrable and nonintegrable, and in the latter case finding out whether the behavior is chaotic plays a major role in the study of dynamical systems.

As usual, for an $n-$th order differential equation $x' = f(x)$, a constant of motion is a function $K(u_1, ..., u_n, t)$ with a predefined degree of smoothness (analytic, meromorphic, C^n etc.) and with the property that for any solution $y(t)$ we have

$$\frac{d}{dt}K(y(t), y'(t), ..., y^{(n-1)}(t), t) = 0$$

There are multiple precise definitions of integrability, and no one perhaps is comprehensive enough to be widely accepted. For us, let us think of a system as being integrable, relative to a certain regularity class of first integrals, if there are sufficiently many global constants of motion so that a particular solution can be found by knowledge of the values of the constants of motion.

If f is analytic, it is usually required that K is analytic too, except perhaps for *isolated* singularities (in particular, single-valued; e.g., the log does not have an isolated singularity at zero, whereas $e^{1/x}$ does).

We note once more that an integral of motion needs to be defined in a wide region. The existence of local constants along trajectories follows immediately either from the flowbox theorem, or from the implicit function theorem: indeed, if $x' = f(x)$ is a system of equations near a regular point, x_0, then evidently there exists a local solution $x(t; x_0) = \phi_t(x_0)$. It is easy to check that $D_{x_0}x_{|t=0} = I$, so we can write, near $x_0, t = 0$, $x_0 = K(x, t) = \Phi_{-t}(x)$. Clearly K is constant along trajectories. Not a very explicit function, admittedly, but smooth, at least locally. Given $x(t)$ it asks, where did it start, when t was zero. K is thus obtained by integrating the equation backwards in time.

Is this an integral of motion?

Not really. This cannot be defined for t which is not small enough, in general since we cannot integrate backwards from any t to zero, without running into singularities.

Assume now that f is an analytic function, so that it makes sense to extend the equation to $\mathbb{C}^n$.

If we think of t in the complex domain, we may think of circumventing possible singularities, and define K by analytic continuation around singularities. But what does that mean? If the singularities are always isolated, and in particular solutions are single valued, it does not matter which way we go. But if these are, say, square root branch points, if we avoid the singularity on one side we get $+\surd$ and on the other $-\surd$. There is no consistency.

But we see, if we impose the condition that the equation have only isolated singularities (at least, those depending on the initial condition, or *movable*, then we have a single valued global constant of motion, take away some lower dimensional singular manifolds in $\mathbb{C}^2$.

Such equations are said to have the Painlevé property (PP) and are integrable, at least in the sense above. But it turns out, in those considered so far in applications, that more is true: they were all ultimately re-derived from linear equations.

11.1 The Painlevé equations

11.2 Spontaneous singularities: The Painlevé's equation P_I

Let us analyze local singularities of the Painlevé equation P_I,

$$y'' = y^2 + x \tag{57}$$

In a neighborhood of a point where y is large, keeping only the largest terms in the equation (*dominant balance*) we get $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we may search for a power-like behavior

$$y \sim A(x - x_0)^p$$

where $p < 0$ obtaining, to leading order, the equation $Ap(p-1)x^{p-2} = A^2(x - x_0)^2$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). Let's look for a power series solution, starting with $6(x - x_0)^{-2}$: $y = 6(x-x_0)^{-2} + c_{-1}(x-x_0)^{-1} + c_0 + \cdots$. We get: $c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = -x_0/10, c_3 = -1/6$ and c_4 is undetermined, thus free. Choosing a c_4, all others are uniquely determined. To show that there indeed is a convergent such power series solution we substitute $y(x) = 6(x-x_0)^{-2} + \delta(x)$ where for consistency we should have $\delta(x) = o((x - x_0)^{-2})$ and taking $x = x_0 + z$ we get the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{58}$$

Note now that our assumption $\delta = o(z^{-2})$ makes $\delta^2/(\delta/z^2) = z^2\delta = o(1)$ and thus the nonlinear term in (58) is *relatively* small. Thus, *to leading order*, the new equation is linear. This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and the equation is better and better approximately by a linear equation. It is then natural to separate out the large terms from the small terms and write a fixed point equation for the solution based on this separation. We write (58) in the form

$$\delta'' - \frac{12}{z^2}\delta = z + x_0 + \delta^2 \tag{59}$$

and integrate as if the right side was known. This leads to an equivalent integral equation. Since all unknown terms on the right side are chosen to be *relatively smaller*, by construction this integral equation is expected to be contractive.

Click here for Maple file of the formal calculation ($y'' = y^2 + x$)

The indicial equation for the Euler equation corresponding to the left side of (59) is $r^2 - r - 12 = 0$ with solutions $4, -3$. By the method of variation of parameters we thus get

$$\delta = \frac{D}{z^3} - \frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 - \frac{1}{7z^3}\int_0^z s^4\delta^2(s)ds + \frac{z^4}{7}\int_0^z s^{-3}\delta^2(s)ds$$
$$= -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \tag{60}$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$; C is arbitrary. To find δ formally, we would simply iterate (60) in the following way: We take $r := \delta^2 = 0$ first and obtain $\delta_0 = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4$. Then we take $r = \delta_0^2$ and compute δ_1 from (60) and so on. This yields:

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + ... \tag{61}$$

This series is actually convergent. To see that, we scale out the leading power of z in δ, z^2 and write $\delta = z^2 u$. The equation for u is

$$\begin{aligned} u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z su^2(s)ds \\ = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \end{aligned} \tag{62}$$

It is straightforward to check that, given C_1 large enough (compared to $x_0/10$ etc.) there is an ϵ such that this is a contractive equation for u in the ball $\|u\|_\infty < C_1$ in the space of analytic functions in the disk $|z| < \epsilon$. We conclude that δ is analytic and that y is meromorphic near $x = x_0$.

$$y'' = 6y^2 + x \quad (1)$$

$$y'' = 2y^3 + xy + \alpha \quad (2)$$

$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{\alpha y^2 + \beta}{x} + \gamma y^3 + \frac{\delta}{y} \quad (3)$$

$$y'' = \frac{y'^2}{2y} + \frac{3}{2} y^3 + 4 x y^2 + 2\left(x^2 - \alpha\right) y + \frac{\beta}{y} \quad (4)$$

$$y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right) y'^2 - \frac{y'}{x} + \frac{(y-1)^2}{x^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma y}{x} + \frac{\delta y (y+1)}{y-1} \quad (5)$$

$$y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right) y'^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right) y' + \frac{y(y-1)(y-x)}{x^2 (x-1)^2}\left[\alpha + \frac{\beta x}{y^2} + \frac{\gamma (x-1)}{(y-1)^2} + \frac{\delta x (x-1)}{(y-x)^2}\right] \quad (6)$$

Figure 10: The six Painlevé equations, all equations of the form $y'' = R(y', y, x)$, with R rational, having the PP.

The equation $y'' = y^2 + x^2$ does not have the Painlevé property.
Click here for Maple file of the formal calculation, for $y'' = y^2 + x^2$

12 Asymptotics of ODEs: first examples

Asymptotic behavior typically refers to behavior near an irregular singular point.

Remember from Frobenius theory that regular singular points (say $z = 0$) of an nth order ODE are characterized by the order of the poles relative to the order of differentiation. Homogeneous equations with regular singularities are of the form

$$y^{(n)} + \frac{A_1(z)}{z} y^{(n-1)} + \cdots \frac{A_j(z)}{z^j} y^{(n-j)} + \cdots \frac{A_n(z)}{z^n} y = 0 \tag{63}$$

where $A_j(z)$ are analytic at zero (note once more $n - j + j = n$; note that removing the first j terms of the equation still leaves you with a Frobenius type equation).

There is a fundamental system of solutions in the form of a finite combination of terms of the form $z^a, \mathcal{A}(z), \ln^j z$ where a may be complex, $j \le n-1$. Written in terms of (generalized) series, these series are convergent.

Two new things happen near irregular singular points.

- Solutions can have exponential behavior.
- Series can be divergent.

Consider first the very simple ODE

$$y' = y/z^p; \quad p > 1 \tag{64}$$

near $z = 0$. The general solution is

$$y = C \exp(-z^{-p+1}/(p-1)) \tag{65}$$

Note that this function has no power series at $z = 0$, and the behavior is exponential.

Most often, irregular singularities are placed at infinity (to characterize a singularity at infinity, make the substitution $z = 1/x$). Then, in first order equations a singularity is irregular at infinity if the equation is of the form $y' = Ax^q(1 + o(1))y,\ q > -1$.

For the second new phenomenon, consider the equation

$$y' = -y + 1/x; \quad y \to \infty \tag{66}$$

We can make it homogeneous by multiplying by x and differentiating once more. By taking $z = 1/x$ you convince yourself that the resulting equation is second order with a fourth order pole at zero.

Eq. (66) has a power series solution. Indeed, inserting

$$y = \sum_{k=0}^{\infty} c_k/x^k \tag{67}$$

in (66) we get $c_k = (k-1)c_{k-1};\ \ c_1 = 1\ \Rightarrow c_k = (k-1)!$ and thus

$$y = \sum_{k=0}^{\infty} k!/x^{k+1} \tag{68}$$

The domain of convergence of this expansion in *empty*.

Many equations for special functions have an irregular singularity at infinity.

Typical equations

1. Bessel:
$$y'' + x^{-1}y' + (1 - \alpha^2/x^2)y = 0 \tag{69}$$

2. Parabolic cylinder functions
$$y'' + (\nu + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0 \tag{70}$$

3. Airy functions
$$y'' = xy \tag{71}$$
as well as many nonlinear ones

4. Elliptic functions
$$y'' = y^2 + 1 \tag{72}$$

5. Painlevé P1
$$y'' = 6y^2 + x \tag{73}$$
etc.

It is important to understand the behavior of irregular singularities. Start again from the example (64). It is clear that the singularity remains irregular if z^{-p} is replaced by $z^{-p} + \cdots$ where $\cdots$ are terms with *higher* powers of z.

Given the exponential behavior $\exp(Az^{-b})$, it is natural to make an *exponential substitution* $y = e^w$. Of course, at the end, we re-obtain the solution we had before. Only, the equation for w' will admit power-like, instead of exponential behavior.

This substitution works in much more generality, and it is behind what is known as the WKB method.

12.1 A brief discussion of the WKB method

This method works for *homogeneous, linear equations with meromorphic coefficients* but it can be adapted no non-homogeneous, or even nonlinear ones. But more about this later.

It turns out (and it is not extremely hard to show, but it is beyond our scope now) that solutions of equations of the form $\sum_{k=0}^{n} y^{(k)} P_k(x)$ where P_k are polynomials have exponential behavior at infinity ($e^{Ax_1^p+\cdots}$) where $p_1 > 0$ and "$\cdots$" are lower powers of x, and the whole expansion (the "$\cdots$") might have zero domain of convergence.

Note that if we write $y = e^w$, then $w''/w'^2 \to 0$ as $x \to \infty$. Indeed, to leading order, $w''/{w'}^2 \sim Cx^{2p-1}/x^{2p-2}$. A a WKB substitution leads to approximate reduction of the order of the equation. In fact, to leading order, the new equation is always first order.

Let's take as an example (71). Substituting $y = e^w$ we get

$$w'' + {w'}^2 = x \tag{74}$$

Here we expect that ${w'}^2 \gg w''$ and thus ${w'}^2 \sim x$ which also means

$$x \gg w'' \tag{75}$$

$$w' = \pm\sqrt{x - w''} \tag{76}$$

Let's take one of the signs,

$$w' = \sqrt{x - w''} \tag{77}$$

By (75) we have

$$w' = \sqrt{x - w''} = \sqrt{x} - \frac{w''}{2\sqrt{x}} - \frac{{w''}^2}{8x^{3/2}} + \cdots \tag{78}$$

It is convenient o write $w' = f$; then, (78) becomes

$$f = \sqrt{x - f'} = \sqrt{x} - \frac{f'}{2\sqrt{x}} - \frac{{f'}^2}{8x^{3/2}} + \cdots \tag{79}$$

where, at a formal level for now, we iterate the equality a la Picard, by first discarding f' on the right side,

$$f^{[0]} = \sqrt{x} \tag{80}$$

then write

$$f^{[1]} = \sqrt{x} - \frac{f^{[0]\prime}}{2\sqrt{x}} \tag{81}$$

etc.

This gives the following sequence of approximations:

$$f^{[0]} = \sqrt{x} \tag{82}$$

$$f^{[1]} = \sqrt{x} - \frac{1}{4x} \tag{83}$$

$$f^{[2]}\sqrt{x} - \frac{1}{4x} - \frac{5}{32x^{5/2}} \tag{84}$$

etc. In terms of w, we get

$$w = C_1 + \frac{2}{3}x^{3/2} - \tfrac{1}{4}\ln x + \tfrac{5}{48x^{3/2}} \tag{85}$$

and thus

$$y \sim Ce^{\frac{2}{3}x^{\frac{3}{2}}} x^{-1/4}(1 + \tfrac{5}{48x^{3/2}} + \cdots) \tag{86}$$

We will justify this procedure in the next subsection.

12.2 WKB: rigorous justification of the asymptotics

Theorem 6. *There exist two linearly independent solutions of* (71) *with the (two) asymptotic behaviors (corresponding to different choices of sign)*

$$y_{\pm} \sim e^{\pm\frac{2}{3}x^{\frac{3}{2}}} x^{-1/4}(1 + o(1)) \text{ as } x \to +\infty \tag{87}$$

A similar analysis can be performed for $x \to -\infty$.

Note 5. *The notation above simply means that*

$$\frac{y_{\pm}}{e^{\pm\frac{2}{3}x^{\frac{3}{2}}} x^{-1/4}} \to 1 \text{ as } x \to +\infty \tag{88}$$

Proof. It is enough to show that $w_{\pm} - [\pm\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\ln x] \to 0$ as $x \to \infty$. We choose the sign $+$, for which the analysis is slightly more involved. Define $w = \sqrt{x} + g$ and consider the equation for g,

$$g' + 2\sqrt{x}g = -\frac{1}{2\sqrt{x}} - g^2 \tag{89}$$

or, more generally

$$g' + 2\sqrt{x}g = H(x) \tag{90}$$

The differential equation (90) with initial condition $g(x_0) = 0$ (chosen for simplicity) where $x_0 > 0$ will be chosen large, is equivalent to

$$g(x) = \mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} H(s)\, \mathrm{e}^{4/3\, s^{3/2}} ds \tag{91}$$

In our specific case, we have

$$H(x) := -\frac{1}{2\sqrt{x}} - g(x)^2 \tag{92}$$

□

and thus

$$g(x) = -\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds - \mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} g^2(s) \mathrm{e}^{4/3\, s^{3/2}} ds \tag{93}$$

What is the expected behavior of the first integral? We can see this by L'Hospital (which, you can check, applies). We have

$$\frac{\left(\displaystyle\int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds \right)'}{\left(\dfrac{\mathrm{e}^{4/3\, x^{3/2}}}{4x} \right)'} = \frac{1}{1 - 2x^{-3/2}} \to 1 \ (x \to +\infty) \tag{94}$$

and thus

$$= -\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} \frac{1}{2\sqrt{s}} \mathrm{e}^{4/3\, s^{3/2}} ds \sim -\frac{1}{4x} \quad x \to +\infty \tag{95}$$

Let's more generally, look at the behavior of

$$\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} s^{-a} \mathrm{e}^{4/3\, s^{3/2}} ds \tag{96}$$

We can apply the same method. We look at the value of b for which, by L'Hospital, we get

$$\frac{\left(\displaystyle\int_{x_0}^{x} s^{-a} \mathrm{e}^{4/3\, s^{3/2}} ds \right)'}{\left(s^{-b} \mathrm{e}^{4/3\, x^{3/2}} \right)'} = \frac{x^{b+a-1}}{2x^{3/2} - (a - 1/2)x^{-3/2}} \to C \tag{97}$$

where $C \neq 0$ is some constant. Clearly, for that, we have to choose $b = a + 1/2$ which gives

$$\mathrm{e}^{-4/3\, x^{3/2}} \int_{x_0}^{x} s^{-a} \mathrm{e}^{4/3\, s^{3/2}} ds \sim \frac{1}{2x^{a+1/2}}, \ x \to +\infty \tag{98}$$

Since the behavior of the first term of (91) is $-1/(4x)$, consistent with our formal WKB analysis and thus g should be $O(1/x)$, this suggests we write $g = u/x$. We get

$$u(x) = -x\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} \frac{1}{2\sqrt{s}}\mathrm{e}^{4/3\,s^{3/2}} ds - x\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} u^2(s)s^{-2}\mathrm{e}^{4/3\,s^{3/2}} ds =: \mathcal{N}u \tag{99}$$

We analyze this equation in $L^\infty[x_0,\infty)$. We first need bounds on the main ingredients of (99), that is on integrals of the form

$$\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} s^{-a}\mathrm{e}^{4/3\,s^{3/2}} ds \tag{100}$$

which are valid on $[x_0,\infty)$ and not merely as $x \to \infty$. The asymptotic information (98) is still useful, but we have to use it wisely. For instance, we expect that for any $A > 1$, if x_0 is large enough, we should have

$$\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} s^{-a}\mathrm{e}^{4/3\,s^{3/2}} ds < A\frac{1}{2x^{a+1/2}}, \tag{101}$$

We can check this by monotonicity. We look at

$$f(x) = \int_{x_0}^{x} s^{-a}\mathrm{e}^{4/3\,s^{3/2}} ds - A\mathrm{e}^{4/3\,x^{3/2}}\frac{1}{2x^{a+1/2}} \tag{102}$$

We note that $f(x_0) < 0$. Calculating f', we get

$$f'(x) = x^{-a}\mathrm{e}^{4/3\,x^{3/2}} - Ax^{-a}\mathrm{e}^{4/3\,x^{3/2}}\left(1 - \frac{a+1/2}{2x^{3/2}}\right) = -x^{-a}\mathrm{e}^{4/3\,x^{3/2}}\left[A - 1 - \frac{A(a+1/2)}{2x^{3/2}}\right] \tag{103}$$

It is clear that $f' < 0$ if $x > x_0$ where

$$x_0^{3/2} = \frac{A(1+2a)}{4(A-1)} \tag{104}$$

Thus we proved

Lemma 16. *If $A > 1$ and x_0 is given in* (104)*, then*

$$\int_{x_0}^{x} s^{-a}\mathrm{e}^{4/3\,s^{3/2}} ds < A\mathrm{e}^{4/3\,x^{3/2}}\frac{1}{2x^{a+1/2}} \tag{105}$$

for all $x > x_0$.

We will now write (99) in contractive form in a suitable ball in $L^\infty[x_0,\infty)$. We will make some choices of A, x_0 etc, to write down something specific. The proof of the theorem is completed by the following result.

Lemma 17. *Let $A = 2$ and $x_0 \geq \max\{(10A(1+4)/4)^{2/3}, 2\}$. Consider the ball*

$$B = \{u : \sup_{x\geq x_0} |u(x)| \leq 1\} \tag{106}$$

Then $\mathcal{N}$ is contractive in B, and thus (99) has a unique solution u_0 there.

Proof of the lemma. It is straightforward to check that $\mathcal{N}B \subset B$. We have

$$\begin{aligned} |\mathcal{N}(u_2-u_1)| &= \left| x\mathrm{e}^{-4/3\,x^{3/2}} \int_{x_0}^{x} (u_2-u_1)(u_2+u_1)s^{-2}\mathrm{e}^{4/3\,s^{3/2}}ds \right| \\ &\leq \|u_2-u_1\| \frac{2|x|}{|x|^{5/2}} \leq \frac{2}{|x_0|^{3/2}}\|u_2-u_1\| \leq 2^{-1/2}\|u_2-u_1\| \end{aligned} \tag{107}$$

□

On the other hand, as $x\to\infty$, using (98) and the fact that $\|u_0\| < 1$, we have

$$g = -\frac{1}{4x} + o(1/x) \quad \text{as} \quad x\to\infty \tag{108}$$

13 Elements of eigenfunction theory–material complementary to Coddington-Levinson

13.1 Properties of the Wronskian of a system

Lemma 18. *Let A be a matrix on $\mathbb{C}^n$. We have*

$$\det(I+\varepsilon A) = 1 + \varepsilon \mathrm{Tr}\ A + O(\epsilon^2) \quad \textit{as} \quad \varepsilon\to 0 \tag{109}$$

Proof 1. The property is obvious for

$$\begin{pmatrix} 1+\varepsilon a_{11} & \varepsilon a_{12} \\ \varepsilon a_{21} & 1+\varepsilon a_{22} \end{pmatrix} \tag{110}$$

For the general case, use induction and row expansion. □

Proof 2. Note that $\det B = \prod_j (1+b_j)$, where b_j are the eigenvalues of B (repeated if the multiplicity is not one). If $(I+\varepsilon A)v = \mu v$ then $\varepsilon A v = (\mu-1)v$ that is, $v = v_j$ is an eigenvector of A: $Av_j = a_j v$. Thus $(1+\varepsilon a_j)v_j = (I+\varepsilon A)v_j = \mu v_j \Rightarrow \mu = (1+\varepsilon a_j)$. The property now follows. □

13.1.1 The Wronskian

The definition is

$$W[f_1, ..., f_n] = \begin{vmatrix} f_1 & ... & f_n \\ f_1' & ... & f_n' \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)} & ... & f_n^{(n-1)} \end{vmatrix} \tag{111}$$

Lemma 19. *Let*

$$M' = AM \tag{112}$$

be a matrix equation in $\mathbb{C}^n$. *We have*

$$\det M(t) = \det M(0) \exp\left(\int_0^t \mathrm{Tr}\, A(s) ds\right) \tag{113}$$

Proof. We have (just by differentiability)

$$M(t+\varepsilon) - M(t) = A(t)M(t)\varepsilon + o(\varepsilon) \tag{114}$$

and thus

$$\begin{aligned} M^{-1}(t)M(t+\varepsilon) &= I + M^{-1}(t)A(t)M(t)\varepsilon + o(\varepsilon) \\ \Rightarrow \det\left(M^{-1}(t)M(t+\varepsilon)\right) &= \det\left(I + M^{-1}(t)A(t)M(t)\varepsilon + o(\varepsilon)\right) \\ &= 1 + \mathrm{Tr}\,(A)\varepsilon + o(\varepsilon) \end{aligned} \tag{115}$$

and thus

$$\begin{aligned} \frac{\det M(t+\varepsilon)}{\det M(t)} = 1 + \mathrm{Tr}\,(A)\varepsilon + o(\varepsilon) \Rightarrow \frac{\det M(t+\varepsilon) - \det M(t)}{\varepsilon} \\ = \det M(t)\mathrm{Tr}\,(A(t)) + o(1) \Rightarrow (\det M(t))' = \det M(t)\mathrm{Tr}\,(A(t)) \end{aligned} \tag{116}$$

and the result follows by integration. ☐

Note that an equation of the kind we are considering,

$$Lf = p_0(t)f^{(n)} + p_1(t)f^{(n-1)} + \cdots + p_n(t)f = \lambda f \tag{117}$$

has the matrix equation counterpart

$$M' = AM \tag{118}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \cdots & \cdots & \cdots & \cdots & \\ -\frac{p_n}{p_0} & -\frac{p_{n-1}}{p_0} & -\frac{p_{n-2}}{p_0} & ... & -\frac{p_1}{p_0} \end{pmatrix} \tag{119}$$

and

$$M = \begin{pmatrix} f_1 & \dots & f_n \\ f_1' & \dots & f_n' \\ \cdots & \cdots & \cdots \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{pmatrix} \tag{120}$$

Clearly, $\mathrm{Tr}\, A = -p_1/p_0$. Thus we have

Corollary 20. *The Wronskian W of a fundamental system for* (117) *satisfies*

$$W(t) = W(0) \exp\left(- \int_0^t \frac{p_1(s)}{p_0(s)} ds \right) \tag{121}$$

14 Discrete dynamical systems

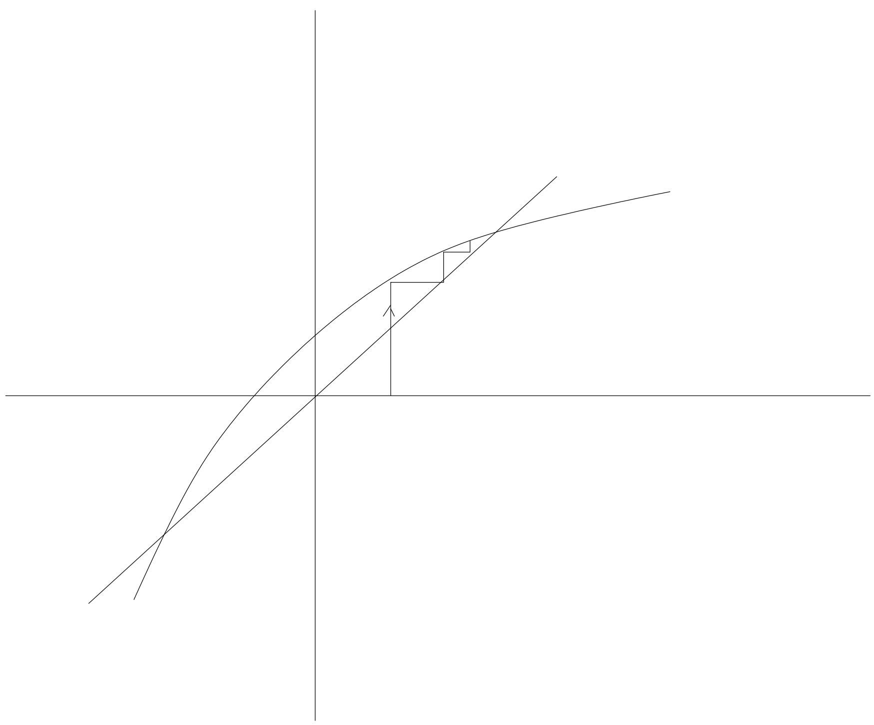

Figure 11:

The study of the Poincaré map leads naturally to the study of discrete dynamics. In this case we have closed trajectory, x_0 a point on it, $\mathcal{S}$ a section through x_0 and we take a point x_1 near x_0, on the section. If x_1 is sufficiently close to x, it must cross again the section, at x_1', still close to x_1, after the *return time* which is then close to the period of the orbit. The application $x_1 \to x_1'$ defines the Poincaré map, which is smooth on the manifold near x_0.

The study of the behavior of differential systems is near closed orbits is often more easily understood by looking at the properties of the Poincaré map.

In one dimension first, we are dealing with a smooth function f, where the iterates of f are what we want to understand.

We write $f^n(x) = f(f(...(f(x))))$ n times. The *orbit* of a point x_0 is the sequence $\{f^n(x_0)\}_{n\in\mathbb{N}}$, assuming that $f^n(x_0)$ is defined for all n. In particular, we may assume that $f : J \to J$, where $J \subset \mathbb{R}$ is an interval, possibly the whole line.

The effects of the iteration are often easy to see on the graph of the iteration, in which we use the bisector $y = x$ to conveniently determine the new point. We have $(x_0, 0) \to (x_0, f(x_0)) \to (f(x_0), f(x_0)) \to (f(x_0), f(f(x_0))$, where the two-dimensionality and the "intermediate" step helps in fact drawing the iteration faster: we go from x_0 up to the graph, horizontally to the bisector, vertically back to the graph, and repeat this sequence.

There are simple iterations, for which the result is simple to understand globally, such as

$$f(x) = x^2$$

where it is clear that $x = 1$ is a fixed point, if $|x_0| < 1$ the iteration goes to zero, and it goes to infinity if $|x_0| > 1$.

Local behavior near a fixed point is also, usually, not difficult to understand, analytically and geometrically.

Theorem 7. *(a) Assume f is smooth, $f(x_0) = x_0$ and $|f'(x_0)| < 1$. Then x_0 is a sink, that is, for x_1 in a neighborhood of x_0 we have $f^n(x_1) \to x_0$.*

(b) If instead we have $|f'(x_0)| > 1$, then x_0 is a source, that is, for x_1 in a small neighborhood $\mathcal{O}$ of x_0 we have $f^n(x_1) \notin \mathcal{O}$ for some n (this does not mean that $f^m(x_1)$ cannot return "later" to $\mathcal{O}$, it just means that points very nearby are repelled, in the short run.)

Proof. We show (a), (b) being very similar. Without loss of generality, we take $x_0 = 0$. There is a $\lambda < 1$ and ϵ small enough so that $|f'(x)| < \lambda$ for $|x| < \epsilon$. If we take x_1 with $|x_1| < \epsilon$, we have $|f(x_1)| = |f'(c)||x_1| < \lambda|x_1|(< \epsilon)$, so the inequality remains true for $f(x_1) : |f(f(x_1))| < \lambda|f(x_1)| < \lambda^2|x_1|$ and in general $f^n(x_1) = O(\lambda^n) \to 0$ as $n \to \infty$.

In fact, it is not hard to show that, for smooth f, the evolution is essentially geometric decay.

□

When the derivative is one, in absolute value, the fixed point is called neutral or indifferent. It does not mean that it can't still be a sink or a source, just that we cannot resort to an argument based on the derivative, as above.

Example 6. *We can examine the following three cases:*

(a) $f(x) = x + x^3$.

(b) $f(x) = x - x^3$

(c) $f(x) = x + x^2$.

It is clear that in the first case, any positive initial condition is driven to $+\infty$. Indeed, the sequence $f^n(x_1)$ is increasing, and it either goes to infinity or else it has a limit. But the latter case cannot happen, because the limit should satisfy $l = l + l^3$, that is $l = 0$, whereas the sequence was increasing.

The other cases are analyzed similarly: in (a), if $x_0 < 0$ then the sequence still diverges. Case (c) is more interesting, since the sequence converges to zero if $x_1 < 0$ is small enough and to ∞ for all $x_1 > 0$. We leave the details to the reader.

It is useful to see what the behavior of such sequences is, in more detail.

Let's take the case (c), where $x_1 < 0$. We have

$$x_{n+1} = x_n + x_n^2$$

where we expect the evolution to be slow, since the relative change is vanishingly small. We then approximate the true evolution by a differential equation

$$(d/dn)x = x^2$$

giving

$$x_n = (C - n)^{-1}$$

We can show rigorously that this is the behavior, by taking $x_n = -1/(n+c_0)+\delta$, $\delta_{n_0} = 0$ and we get

$$\delta_{n+1} - \delta_n = \frac{1}{n^2(n+1)} - \frac{2}{n}\delta_n + \delta_n^2 \tag{122}$$

and thus

$$\delta_n = \sum_{j=n_0}^{n} \left(\frac{1}{j^2(j+1)} - \frac{2}{j}\delta_j + \delta_j^2 \right) \tag{123}$$

Exercise 1. *Show that (123) defines a contraction in the space of sequences with the property $|\delta_n| < C/n^2$, where you choose C carefully.*

Exercise 2. *Find the behavior for small positive x_1 in (b), and then prove rigorously what you found.*

14.1 Bifurcations

The local number of fixed points can only change when $f'(x_0) = 1$. As before, we can assume without loss of generality that $x_0 = 0$.

We have

Theorem 8. *Assume $f(x, \lambda)$ is a smooth family of maps, that $f(0, 0) = 0$ and that $f_x(0, 0) \neq 1$. Then, for small enough λ there exists a smooth function $\varphi(\lambda)$, also small, so that $f(\varphi(\lambda), \lambda) = \varphi(\lambda)$, and the character of the fixed point (source or sink) is the same as that for $\lambda = 0$.*

Exercise 3. *Prove the theorem, using the implicit function theorem.*

References

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