Transseries and Écalle-Borel summability

June 5, 2007

Contents

Chapter 1

Transseries

Transseries are studied carefully in Chapter 3.

Informally, they are finitely generated asymptotic combinations of powers, exponentials and logs and are defined inductively. In the case of a power series, finite generation means that the series is an integer multiseries in $\mu_1, ..., \mu_n$ where $\mu_j = x^{-\alpha_j}, \Re(\alpha_j) > 0$. An example is

$$\log\log x + \sum_{k=0}^{\infty} e^{-k\exp(\sum_{k=0}^{\infty} k! x^{-k})}$$

A single term in a transseries is a transmonomial.

1. A term of the form $m = x^{-\alpha_1 k_1 - ... - \alpha_n k_n}$ with $\alpha_i > 0$ is a level zero **(trans)monomial**. We denote $-\alpha_1 k_1 - ... - \alpha_n k_n = \boldsymbol{\alpha} \cdot \mathbf{k}$.

   For the purpose of having a unified construction at all levels, we denote $x^{-\alpha_1 k_1 - ... - \alpha_n k_n}$ by $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^n$.

2. Real transseries of level zero are simply finitely generated *asymptotic* power series. That is, given $\alpha_1, ..., \alpha_n$ with $\alpha_i > 0$ a level zero transseries is a sum of the form

$$S = \sum_{k_i \geq M_i} c_{k_1,...,k_n} x^{-\alpha_1 k_1 - ... - \alpha_n k_n} := \sum_{\mathbf{k} \geq \mathbf{M}} c_{\mathbf{k}} \mu_{\mathbf{k}} \tag{1.1}$$

   where $M_1, ..., M_n$ are *integers*, positive or negative; the terms of S are therefore nonincreasing in k_i and bounded *above* by $O(x^{-\alpha_1 M_1 - ... - \alpha_n M_n})$.

3. We note that $\mathbf{k} \mapsto \boldsymbol{\mu}_{\mathbf{k}}$ defines a morphism between $\mathbb{Z}^n$ and the abelian multiplicative group with generators $\mu_1, ..., \mu_n$.

4. A transseries of level zero can be written (uniquely) in *collected* form,

$$\sum_{\beta \in B} d_\beta \mu_\beta \tag{1.2}$$

where $\beta_1 > \beta_2 > ... > \beta_j > ...$ and d_β are real (positive or negative) numbers in the following way:

Note that there can only be finitely many $k_1, ..., k_n$ so that

$$\boldsymbol{\alpha} \cdot \mathbf{k} = \beta \tag{1.3}$$

Indeed, let α_1 be the least of the α_j, $\|\mathbf{x}\| = \max\{|x_1|, ..., |x_n|\}$, and Q be the least N so that condition $\alpha_1 N_1 > \|\beta\| + \|\boldsymbol{\alpha}\| \ |\mathbf{M}|$ holds. Then, there are finitely many k_i such that $M_i \le k_i \le Q$ and thus no solutions of (1.3) exist if $k_i > Q$. We then define the corresponding d_β by

$$d_\beta = \sum_{\boldsymbol{\alpha}\cdot\mathbf{k}=\beta} c_{k_1}\mu_{k_1} + ... + c_{k_n}\mu_{\alpha_n k_n} \tag{1.4}$$

where therefore the sum contains finitely many terms.

Exercise. Show that the numbers β_n can have no accumulation point. In particular, $\beta_n \to \infty$ as $n \to \infty$.

5. We assume $c_{\beta_1} \neq 0$, unless all $c_{\beta_n} = 0$.

6. Terminology: The magnitude of a transseries S of level zero is $\mathrm{mag}(S) = \mu_{\beta_1}$ where β_1 is the smallest of the β_j. The dominance $\mathrm{dom}(S) = d_{\beta_1}\mu_{\beta_1}$. We convene to write $\mathrm{dom}(0) = 0$. For $\mathbb{R}^n \ni \boldsymbol{\alpha} > 0$ we write $\mathcal{T}_{\boldsymbol{\alpha}}^{[0]}$ the space of multiseries generated by $x^{-\alpha_i}, i = 1, ..., n$.

7. The operations are defined in a natural way. If n_1 and n_2 are in $\mathbb{N}$ then we take the larger between the two, say n_1. We embed the second transseries in the set of transseries indexed by multiindices in $\mathbb{Z}^n$ by setting in the second transseries $c_{k_1,k_2,...,k_j,...k_n} = 0$ if $j > n$. Then

$$S_1 + S_2 := \sum_{\mathbf{k}\ge\mathbf{M}} (c_{1;\mathbf{k_1}} + c_{2;\mathbf{k_2}})\mu_{\mathbf{k}} \tag{1.5}$$

Multiplication is defined, after a similar embedding, as

$$S_1 S_2 := \sum_{\mathbf{k}\ge\mathbf{M_1}+\mathbf{M_2}} \mu^{\mathbf{k}} \sum_{\mathbf{k_1}+\mathbf{k_2}=\mathbf{k}} c_{1;\mathbf{k}} c_{2;\mathbf{k}} \tag{1.6}$$

Exercise: show that for every $\mathbf{k}$ there are finitely many $\mathbf{k}_1 \ge \mathbf{M}_1$ and $\mathbf{k}_2 \ge \mathbf{M}_2$ so that $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}$. Therefore the sum in (1.6) is well defined.

8. A transseries can be uniquely decomposed as

$$S = L + c + s = \sum_{\beta<0} d_\beta \mu_\beta + d_0 + \sum_{\beta>0} d_\beta \mu_\beta \tag{1.7}$$

where L is a *purely large transseries*, c is a constant and s is a *small transseries*. Of course, any of the L, c, s could be zero. The decomposition can be written either in collected form or with multiindices, whichever is more convenient, whenever S originates in a multiindexed transseries of level zero.

9. We note that by 4 the large part of at transseries only contains finitely many terms.

10. More general operations. For any small transseries s and coefficients $\{c_k\}_{k\geq M}$, the sum

$$s_1 = \sum_{k\geq M} c_k s^k = \sum_{\gamma_i;i\in\mathbb{N}} d_{\gamma_i}\mu_{\gamma_i} \tag{1.8}$$

where

$$d_{\gamma_i}\mu_{\gamma_i} = \mu_{\beta_1}\cdots\mu_{\beta_r} \sum_{r\in\mathbb{N},\mathbf{k}\in\mathbb{Z}^r,\beta_1+...+\beta_r=\gamma} c_{\beta_1+...+\beta_r} \tag{1.9}$$

is well defined: there are finitely many r and β_r so that $\beta_1 + ... + \beta_r = \gamma$ since β_j are positive for all sufficiently large j.

Exercise: Show that s_1 is a transseries, that its generators are the same as the generators of s and that the γ's have no accumulation point.

11. **Product form.** We can write uniquely

$$S = c_{\beta_1}\mu_{\beta_1}\left(1 + \sum_{j>1} c_{\beta_j} c_{\beta_1}^{-1}\right)$$

$$e^s = 1 + s + s^2/2 + ... \tag{1.10}$$

It is easy to check that $(e^s)' = e^s$ and all other properties of the usual exponential hold.

12. A simple instance of transcomposition. If $S = \sum_{\beta\in B} d_\beta x^\beta$ is a transseries of level zero in collected form, and $s_{\beta_1} \gg s_{\beta_2} \gg \cdots$ are transseries with *the same generators* then the formal sum $\sum_{\beta\in B} d_\beta x^\beta s_\beta$ defines unambiguously a transseries of level zero, *the same generators*.

Exercise. Show that this follows from the fact that, since $\beta_1 + \beta_2 = \beta$ has finitely many solutions, so does $\beta_1 + \beta_2 = \beta_3 + \beta_4$.

13. Ordering. There are two types of ordering: they are induced by induced we write $S \gg 1$ if $\text{mag}(S_1) > 1$ and $S > 0$ if $c_{\beta_1} > 0$. It follows from (5) $S = 0$ iff all $c_\beta = 0$.

14. It follows that $S' = 0$ iff $S = c$.

Exercise. Show that if $S_1 \gg S_2$ are constant-free transseries of level zero, then $S_1' \gg S_2'$ and that if S is constant free and positive, then so is S'.

15. In the sequel, whenever operating with transseries, all small transseries in exponentials are expanded out as above, to obtain in the end a canonical form of the transseries.

16. The space of transseries of level zero is

$$\mathcal{T}^{[0]} = \bigcup_{r\in\mathbb{Z},\boldsymbol{\alpha}>0\in\mathbb{Z}^r} \mathcal{T}_{\boldsymbol{\alpha}}^{[0]} \tag{1.11}$$

with the embeddings mentioned before.

17. Note that although there is a continuum of generators in $\mathcal{T}^{[0]}$, every particular transseries has a finite number of generators.

18. With the operations defined thus far, transseries of level zero form a differential field.

1.1 Abstracting from this construction

19. Let $(\mathcal{G}, \cdot, \ll)$ be a finitely generated, totally ordered (any two elements are comparable) abelian group, with generators $\mu_1, \mu_2, ...\mu_n$, such that $\ll$ is compatible with the group operations, that is, $g_1 \ll g_2$ and $g_3 \ll g_4$ implies $g_1 g_3 \ll g_2 g_4$, and such that $1 \gg \mu_1 \gg \cdots \gg \mu_n$. This is the case when μ_i are transmonomials of level zero.

20. We write $\mu_{\mathbf{k}} = \mu^{\mathbf{k}} := \mu_1^{k_1} \cdots \mu_n^{k_n}$.

Lemma 1.12 *Consider the partial order relation that we introduced before on $\mathbb{Z}^n$, $\mathbf{k} > \mathbf{m}$ iff $k_i \geq m_i$ for all $i = 1, 2, ..., n$ and at least for some j we have $k_j > m_j$. If $B \subset A = \{\mathbf{k} \in \mathbb{Z}^n : \mathbf{k} \geq \mathbf{m}\}$, then there is no infinite nonascending chain in B. That, is there is no infinite sequence in $B, b_n \neq b_m$ for $n \neq m$, and $b_{n+1} \not> b_n$ for all n.*

Proof. Assume there is an infinite nonascending sequence, $\{\mathbf{k}(m)\}_{m\in\mathbb{N}}$. Then at least for some $i \in \{1, 2, ..., n\}$ the sequence $\{k_i(m)\}_{m\in\mathbb{N}}$ must have infinitely many distinct elements. Since the $k_i(m)$ are bounded below, then the set $\{k_i(m)\}_{m\in\mathbb{N}}$ is unbounded above, and we can extract a strictly increasing subsequence $\{k_i(m_l)\}_{l\in\mathbb{N}}$. We now take the sequence $\{\mathbf{k}(m_l)\}_{l\in\mathbb{N}}$. At least for some $j \neq i$ the set $k_j(m_l)$ needs to have infinitely many elements too. Indeed if the sets $\{k_j(m_l); j \neq i\}$ are finite, we can split $\{\mathbf{k}(m_l)\}_{l\in\mathbb{N}}$ into a finite set of subsequences, in each of which all $k_j(m_l)$, $j \neq i$, are constant while k_i is strictly increasing. But every such subsequence would be strictly decreasing, which is impossible. By finite induction we can extract a subsequence $\{\mathbf{k}(m_t)\}_{t\in\mathbb{N}}$ of $\mathbf{k}(m)\}_{m\in\mathbb{N}}$ in which all $k_l(m_t)$ are increasing, a contradiction.

Remark. This is a particular, much easier result of Kruskal's tree theorem. which we briefly mention here. A relation is well-founded if and only if it contains no countable infinite descending sequence $\{x_j\}_{\in\mathbb{N}}$ of elements of X such that $x_{n+1}Rx_n$ for every $n\in\mathbb{N}$. The relation R is a quasiorder if it is reflexive and transitive. Well-quasi-ordering is a well-founded quasi-ordering such that that there is no sequence $\{x_j\}_{\in\mathbb{N}}$ with $x_i \not\leq x_j \forall i<j$. A tree is a collection of vertices in which any two vertices are connected by exactly one line. *J. Kruskal's tree theorem states that the set of finite trees over a well-quasi-ordered set is well-quasi-ordered.*

21. *Exercises. (1) Show that the equation* $\mathbf{k}_1+\mathbf{k}_2=\mathbf{l}$ *has only finitely many solutions in the set* $\{\mathbf{k}:\mathbf{k}\geq\mathbf{m}\}$.

    (2) Show that for any $\mathbf{l}\in\mathbb{R}^n$ *there can only be finitely many* $p\in\mathbb{N}$ *and* $\mathbf{k}_j\in\mathbb{R}^n, j=1,...,p$ *such that* $\mathbf{k}_1+\mathbf{k}_2+\cdots\mathbf{k}_p=\mathbf{l}$.

Corollary 1.13 *For any set* $B\subset A=\{\mathbf{k}\in\mathbb{Z}^n:\mathbf{k}\geq\mathbf{m}\}$ *there is a set* $B_1=\mathrm{mag}(B)$ *with* **finitely many elements***, such that* $\forall b\in B\setminus B_1$ *there exists* $b_1\in B_1$ *such that* $b_1<b$.

Consider the set of all elements which not greater than other elements of B, $B_1=\{b_1\in B|b\neq b_1\Rightarrow b\not> b_1\}$. In particular, no two elements of B_1 can be compared with each-other. But then, by Lemma 1.12 this set cannot be infinite since it would contain an infinite non-ascending chain.

Now, if $b\in B\setminus B_1$, then by definition there is a $b'>b$ in B. If $b'\in B_1$ there is nothing to prove. Otherwise there is a $b''>b'$ in B. Eventually some $b^{(k)}$ must belong to B_1, finishing the proof, otherwise $b<b'<...$ would form an infinite nonascending chain.

Corollary 1.14 *For any set* $B\subset A=\{\mathbf{k}\in\mathbb{Z}^n:\mathbf{k}\geq\mathbf{m}\}$ *there is a set* $\mathrm{Mag}(B)$ *with* finitely many elements*, such that* $\forall b\in B\setminus B_1$ *there exists* $b_1\in B_1$ *such that* $b_1<b$.

22. For any $\mathbf{m}\in\mathbb{Z}^n$ and any set $B\subset\{\mathbf{k}|\mathbf{k}\geq\mathbf{m}\}$, the set $A=\{\mu_\mathbf{k}|\mathbf{k}\in B\}$ has a largest element with respect to $>$. Indeed, if such was not the case, then we would be able to construct an infinitely ascending sequence.

Lemma 1.15 *No set of elements of* $\mu_\mathbf{k}\in\mathcal{G}$ *such that* $\mathbf{k}\geq\mathbf{m}$ *can contain an infinitely* ascending *chain, that is a sequence of the form*

$$g_1\ll g_2\ll\cdots$$

Proof. For such a sequence, the corresponding $\mathbf{k}$ would be strictly nonascending, in contradiction with Lemma 1.12.

23. It follows that for any $\mathbf{m}$ every $B \subset A_{\mathbf{m}} = \{g \in \mathcal{G} | g = \mu_{\mathbf{k}}; \mathbf{k} \geq \mathbf{m}\}$ is well ordered (every subset has a largest element) and thus B can be indexed by ordinals. By this we mean that there exists a set of ordinals Ω (or, which is the same, an ordinal) which is in one-to-one correspondence with B and $g_\beta \ll g_{\beta'}$ if $\beta > \beta'$.

24. If A is as in 22, and *if* $g \in \mathcal{G}$ has a *successor* in A, that is, there is a $\tilde{g} \in A$, $g \gg \tilde{g}$ then it has an *immediate successor*, the largest element in the subset of A consisting of all elements less than g. There may not be an immediate *predecessor* though, as is the case of e^{-x} in $A_1 = \{x^{-n}, n \in \mathbb{N}\} \cup \{e^{-x}\}$. Note also that, although e^{-x} has infinitely many predecessors, there is no infinite ascending chain in A_1.

    Lemma 1.16 *For any $g \in \mathcal{G}$, and $\mathbf{m} \in \mathbb{Z}^n$, there exist finitely many (distinct) $\mathbf{k} \geq \mathbf{m}$ such that $\mu_{\mathbf{k}} = g$.*

    Proof. Assume the contrary. Then for at least one i, say $i = 1$ there are infinitely many k_i in the set of $(\mathbf{k})_i$ such that $\mu_{\mathbf{k}} = g$. As in Lemma 1.23, we can extract a strictly increasing subsequence. But then, along it, $\mu_1^{k_2} \cdots \mu_n^{k_n}$ would form an infinite strictly ascending sequence, a contradiction.

    Proof: Exercise.

25. For any coefficients $c_{\mathbf{k}} \in \mathbb{R}$, consider the formal multiseries, which we shall call *transseries* over $\mathcal{G}$,

$$T = \sum_{\mathbf{k} \in \mathbb{Z}^n; \mathbf{k} \geq \mathbf{M}} c_{\mathbf{k}} \mu_{\mathbf{k}} \tag{1.17}$$

    Transseries actually needed in analysis are constructed in the sequel, with a particular inductive definition of generators μ_k.

26. *More generally a* **transseries over** *G is a sum which can be written in the form (1.17) for some (fixed) $n \in \mathbb{N}$ and for some* **some choice of generators** *$\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^n$.*

27. The fact that a transseries s is small does *not* mean that the corresponding $\mu_{\mathbf{k}}$ have positive $\mathbf{k}$; s could contain terms such as xe^{-x} of $x^{\sqrt{2}}x^{-2}$ etc.). But positiveness can be *arranged* by a suitable choice of generators as follows from the next result.

28. **Note** It is important that a transseries is defined over a set of the form $A_{\mathbf{m}}$. For instance, in the group $\mathcal{G}$ with two generators x^{-1} and $x^{-\sqrt{2}}$ an expression of the form

$$\sum_{\{(m,n) \in \mathbb{Z}^2 | m\sqrt{2}+n>0\}} x^{-m\sqrt{2}-n} \tag{1.18}$$

is not acceptable. The behavior of a function whose "asymptotic expansion" is given by (1.18) is not at all manifest.

Exercise 1.19 *Consider the numbers the form* $m\sqrt{2}+n$*, where* $m, n \in \mathbb{Z}$*. It can be shown, for instance using continued fractions, that one can choose a subsequence from this set such that* $s_n \uparrow 1$*. Show that* $\sum_n x^{-s_n}$ *is not a transseries over any group of monomials of order zero.*

Expressions similar to the one in the exercise do appear in some problems in discrete dynamics. The very fact that transseries are closed under many operations, including solutions of ODEs, shows that such functions are "highly transcendental".

29. Given $\mathbf{m} \in \mathbb{Z}^n$ and $g \in \mathcal{G}$, the set $S_g = \{\mathbf{k}|\mu_{\mathbf{k}} = g\}$ contains, by Lemma 1.16 finitely many elements (possibly none). Thus the constant $d(g) = \sum_{\mathbf{k}\in S_g} c_{\mathbf{k}}$ is well defined. By 22 there is a largest $g = g_1$ in the set $\{\mu_{\mathbf{k}}|d(g) \neq 0\}$, unless all coefficients are zero. We call this g_1 the magnitude of T, $g_1 = \mathrm{mag}(T)$, and we write $\mathrm{dom}(T) = d(g_1)g_1 = d_1 g_1$.

30. By 23, the set $\{g = \mu_{\mathbf{k}}|\mathbf{k} \geq \mathbf{m}\}$ can be indexed by ordinals, and we write

$$T = \sum_{\beta\in\Omega} d_\beta g_\beta \tag{1.20}$$

where $g_\beta \ll g_{\beta'}$ if $\beta > \beta'$. By convention, the first element in (1.20), $d_1 g_1 \neq 0$.

Convention. To simplify the notation and terminology, we will say, with some abuse of language, that a group element g_β appearing in (1.20) belongs to T.

Whenever convenient, we can also select the elements of $d_\beta g_\beta$ in T with nonzero coefficients. As a subset of a well ordered set, it is well ordered too, by a set of ordinals $\tilde{\Omega} \subset \Omega$ and write

$$T = \sum_{\beta\in\tilde{\Omega}} d_\beta g_\beta \tag{1.21}$$

where all d_β are nonzero.

31. **Notation** To simplify the exposition we will denote by $A_{\mathbf{m}}$ the set $\{\mu_{\mathbf{k}}|\mathbf{k} \geq \mathbf{m}\}$, $\mathbf{K}_{\mathbf{m}} = \{\mathbf{k}|\mathbf{k} \geq \mathbf{m}\}$ and $\mathcal{T}_{A_{\mathbf{m}}}$ the set of transseries over $A_{\mathbf{m}}$.

32. Any transseries can be written in the form

$$T = L + c + s = \sum_{\beta\in\Omega;\, g_\beta \gg 1} d_\beta g_\beta + c + \sum_{\beta\in\Omega;\, g_\beta \ll 1} d_\beta g_\beta \tag{1.22}$$

where L is called a purely large transseries, c is a constant and s is called a small transseries.

Note that L, c and s are transseries since, for instance, the set $\{\beta \in \Omega; g_\beta \ll 1\}$ is a subset of ordinals, thus an ordinal itself.

Lemma 1.23 *If $\mathcal{G}$ is finitely generated, if $A_{\mathbf{m}} \subset \mathcal{G}$ and s is a small transseries over $A_{\mathbf{m}}$ we can always assume, for an $n \geq n'$ that the generators $\nu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^{n'}$ are such that for all $\nu_{\mathbf{k}'} \in s$ we have $\mathbf{k}' > 0$.*

$$s = \sum_{\mathbf{k} \geq \mathbf{m}} \mu_{\mathbf{k}} c_{\mathbf{k}} = \sum_{\beta \in \tilde{\Omega}} d_\beta g_\beta = \sum_{\mathbf{k}' > 0} \nu_{\mathbf{k}'} c'_{\mathbf{k}'} \tag{1.24}$$

Proof. In the first sum on the left side we can retain only the set of indices I such that $\mathbf{k} \in I \Rightarrow \mu_{\mathbf{k}} = g_\beta$ has nonzero coefficient d_β. In particular, since all $g_\beta \ll 1$, we have $\mu_{\mathbf{k}} \ll 1 \ \forall \mathbf{k} \in I$. Let $I_1 = \mathrm{Mag}(I)$. We adjoin to the generators of $\mathcal{G}$ all the $\nu_{\mathbf{k}'} = \mu_{\mathbf{k}}$ with $\mathbf{k}' \in I_1$. The new set of generators is still finite and for all $\mathbf{k} \in I$ there is a $\mathbf{k}' \in \mathrm{Mag}(I)$ such that $\mathbf{k} \geq \mathbf{k}'$ and $\mu_{\mathbf{k}}$ can be written in the form $\nu^1_{\mathbf{k}'} \mu_{\mathbf{l}}$ where all $\mathbf{l} \geq 0$.

Remark. After the construction, generally, there will be nontrivial relations between the generators. But nowhere do we assume that generators are relation-free, so this creates no difficulty.

33. An algebra over $\mathcal{G}$ can be defined as follows. Let A and $\tilde{A}$ be well ordered sets in Ω. The set of pairs $(\beta, \tilde{\beta}) \in A \times \tilde{A}$ is well ordered (check!). For every g, the equation $g_\beta \cdot g_{\tilde{\beta}} = g$ has finitely many solutions. Indeed, otherwise there would be an infinite sequence of g_β which cannot be ascending, thus there is a subsequence of them which is strictly descending. But then, along that sequence, $g_{\tilde{\beta}}$ would be strictly ascending; then the set of corresponding ordinals $\tilde{\beta}$ would form an infinite strictly descending chain, which is impossible. Thus, in

$$T \cdot \tilde{T} := \sum_{\gamma \in A \times \tilde{A}} g_\gamma \sum_{g_\beta \cdot g_{\tilde{\beta}} = g_\gamma} d_\beta d_{\tilde{\beta}} \tag{1.25}$$

the inner sum contains finitely many terms.

34. We denote be $\mathcal{T}_{\mathcal{G}}$ the algebra of transseries over $\mathcal{G}$. $\mathcal{T}_{\mathcal{G}}$ is a commutative algebra with respect to $(+, \cdot)$. We will see in the sequel that $\mathcal{T}_{\mathcal{G}}$ is in fact a field. We make it an ordered algebra by writing

$$T_1 \ll T_2 \Leftrightarrow \mathrm{mag}(T_1) \ll \mathrm{mag}(T_2) \tag{1.26}$$

and writing

$$T > 0 \Leftrightarrow \mathrm{dom}(T) > 0 \tag{1.27}$$

35. **Product form**. With the convention $\operatorname{dom}(0) = 0$, any transseries can be written in the form

$$T = \operatorname{dom}(T)(1+s) \tag{1.28}$$

where s is small (check).

36. Embeddings. If $\mathcal{G}_1 \subset \mathcal{G}$, we write that $\mathcal{T}_{\mathcal{G}_1} \subset \mathcal{T}_{\mathcal{G}}$ in the natural way.

37. **Topology** on $\mathcal{T}_{\mathcal{G}}$. We consider a sequence of transseries over a *common* set $A_{\mathbf{m}}$ of elements of $\mathcal{G}$, indexed by the ordinal Ω.

$$\{T^{[j]}\}_{j\in\mathbb{N}};\ T^{[j]} = \sum_{\beta\in\beta} d_\beta^{[j]} g_\beta^{[j]}$$

Definition. We say that $T^{[j]} \to 0$ as $j \to \infty$ if for any $\beta \in \Omega$ there is a $j(\beta)$ such that the coefficient $d_\beta^{[j]} = 0$ for all if $j > j(\beta)$.

Thus the transseries $T^{[j]}$ must be *eventually depleted of all coefficients*. This aspect is very important. The mere fact that $\operatorname{dom}(S) \to 0$ does not suffice. Indeed the sequence $\sum_{k>j} x^{-k} + je^{-x}$, though "rapidly decreasing" is not convergent according to the definition, and probably should not be considered convergent in any reasonable topology.

38. Equivalently, the sequence $T^{[j]} \to 0$ is convergent if there is a representation such that

$$T^{[j]} = \sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}}^{[j]} \mu_{\mathbf{k}} \tag{1.29}$$

and in the sum $\mu_{\mathbf{k}} = g$ has only one solution (we know that such a choice is possible), and $\min\{|k_1| + \cdots + |k_n| : c_{\mathbf{k}}^{[j]} \neq 0\} \to 0$ as $j \to \infty$.

39. Let $\mu_1, ..., \mu_n$ be any generators for $\mathcal{G}$, $\mathbf{m} \in \mathbb{Z}^d$, as in 23 and $T_j \in \mathcal{T}_{A_{\mathbf{m}}}$ a sequence of transseries. Let $N_j := \min\{k_1 + ... + k_n | \mu_1^{p_1} \cdots \mu_n^{p_n} \in T_j\}$. Note that we can write min since, by Lemma 1.12, the minimum value is attained (check this!). If $N_j \to \infty$ then $T_j \to 0$. Indeed, if this was not the case, then there would exist a g_β such that $g_\beta \in T_j$ with $d_\beta \neq 0$ for infinitely many j. Since $N_j \to \infty$ there is a sequence $\mu_{\mathbf{k}} \in A_{\mathbf{m}}$ such that $k_1 + ... + k_n \to \infty$ and $\mu_{\mathbf{k}} = g_\beta$. This would yield an infinite set of solutions of $\mu_{\mathbf{k}} = g_\beta$ in $A_{\mathbf{m}}$, which is not possible. The function $\max\{e^{-|k_1|+\cdots+|k_n|} : \sum_{\mu_k=g} c_{\mathbf{k}} \neq 0\}$ is a semimetric (it satisfies all properties of the metric except the triangle inequality) which induces the same topology.

    More generally, transseries are a subset of functions f defined on $\mathcal{G}$ with real values and for which there exists a $\mathbf{k}_0(f) = \mathbf{k}_0$ such that $f(g_{\mathbf{k}}) = 0$ for all $\mathbf{k} < \mathbf{k}_0$. On these functions we can define a topology by writing $f^{[j]} \to 0$ if there exists $\mathbf{k}_0(f^{[j]})$ does not depend on j and for any g_β there is an N we have $f^{[n]}(g_\beta) = 0$ for all $n > N$ and such . The first restriction is imposed to disallow, say, the convergence of x^n to zero, which would not be compatible with a good structure of transseries.

40. This topology is metrizable[1]. For example we can proceed as follows. Let $A_{\mathbf{m}}$ be the common set over which the transseries are defined. The elements of $\mathcal{G}$ are countable. We choose any counting on $A_{\mathbf{m}}$. We then identify transseries over $A_{\mathbf{m}}$ with the space $\mathcal{F}$ of real-valued functions defined on the natural numbers. We define $d(f,g) = 1/n$ where n is the least integer such that $f(n) \neq g(n)$ and $d(f,f) = 0$. The only property that needs to be checked is the triangle inequality. Let $h \in \mathcal{F}$. If $d(g,h) \geq 1/n$, then clearly $d(f,g) \leq d(f,h)+d(h,g)$. If $d(g,h) < 1/n$ then $d(f,h) = 1/n$ and the inequality holds too.

41. The topology cannot come from a norm, since in general $a_n\mu \not\to 0$ as $a_n \to 0$.

42. We also note that the topology is *not* compatible with the order relation. For example $s_n = x^{-n} + e^{-x} \to e^{-x}$ as $n \to \infty$, $s_n \gg e^{-\sqrt{x}}$ for all n while $e^{-x} \not\gg e^{-\sqrt{x}}$. The same argument shows that there is no distance compatible with the order relation.

43. In some sense, there is no "good" topology compatible with the order relation $\ll$. Indeed, if there was one, then the sequences $s_n = x^{-n}$ and $t_n = x^{-n} + e^{-x}$ which are interlaced in the order relation should have the same limit, but then addition would be discontinuous[2].

44. Giving up compatibility with asymptotic order allows us to ensure continuity of most operations of interest.

    Exercise. Show that a Cauchy sequence in $\mathcal{T}_{A_{\mathbf{m}}}$, is convergent, and $\mathcal{T}_{A_{\mathbf{m}}}$ is a topological algebra.

45. If $\mathcal{G}$ is finitely generated, then for any small transseries

$$s = \sum_{\beta\in\Omega:g_\beta\ll 1} d_\beta g_\beta \tag{1.30}$$

    we have $s^j \to 0$ as $j \to \infty$.

    Proof. Indeed, by Lemma 1.23 we may assume that the generators of $\mathcal{G}$, $\mu_1, ..., \mu_n$, are chosen such that all $\mathbf{k} > 0$ in s. Let $g \in \mathcal{G}$. The terms occurring in the formal sum of s^j are of the form $const.\mu_1^{l_1^1+...+l_1^j} \cdots \mu_n^{l_n^1+\cdots l_n^j}$ where $l_m^s \geq 0$ and at least one $l_j^s > 0$. Therefore $l_1^1 + ... + l_1^j \to \infty$ and $\sum_{l=1..M} s^l \to 0$ by 39 for any $j, M \to \infty$.

    As a side remark, finite generation is not needed at this point. More generally, let $A \subset \mathcal{G}$ be well ordered. It follows from J. Kruskal's theorem that the set $\tilde{\mathcal{A}} \supset A$ of all products of elements of A is also well quasi-ordered.

[1]Zhi pointed out difficulties in defining a topology. A similar construction was suggested by M. Tychonievich.

[2]This example was pointed out by G. Edgar.

Note 1.31 The sum $\sum_{k=0}^{\infty} c_k s^k$ might belong to a space of transseries defined over a larger, but still finite, number of generators. For instance, if

$$\frac{1}{xe^x+1} = \frac{1}{xe^x(1+xe^{-x})} = \frac{e^{-x}}{x}\sum_{j=0}^{\infty}(-1)^j x^j e^{-jx} \tag{1.32}$$

then the generators of (1.32) can be taken to be x^{-1}, e^{-x}, xe^{-x} but certainly cannot stay e^{-x}, x^{-1} since then the power of x^{-1} would be unbounded below.

46. In particular if $f(\mu) := \sum_{k=0}^{\infty} c_k \mu^k$ is a formal series and s is a small transseries, then

$$f(s) := \sum_{k=0}^{\infty} c_k s^k \tag{1.33}$$

is well defined.

Exercise 1.34 *Show that f is continuous, in the sense that* $s^{[n]} \to 0$ *implies* $f(s) \to c_0$.

47. If $T_1 \gg T_2$, $T_3 \ll T_1$ and $T_4 \ll T_2$ then $T_1 + T_3 \gg T_2 + T_4$. Indeed, mag(T_1+T_3)=mag(T_1) and mag(T_2+T_4)=mag(T_2).

48. It is easily checked that $(1+s)\cdot 1/(1+s) = 1$, where

$$\frac{1}{1+s} := \sum_{j\geq 0}(-1)^j s^j \tag{1.35}$$

More generally we define

$$(1+s)^a = 1 + a\,s + \frac{a(a-1)}{2} + \cdots$$

49. Writing $S = \mathrm{dom}(S)(1+s)$ we define $S^{-1} = \mathrm{dom}(S)^{-1}(1+s)^{-1}$.

50. if μ^r is defined for a real r (this will be the case for the power-exponential transseries), then we then adjoin μ^r to $\mathcal{G}$ and define

$$T^r := d_1^r g_1^r (1+s)^r$$

51. If $\mu_j \mapsto \mu_j'$ is a "derivation" defined from the generators μ_j into $\mathcal{T}_{\mathcal{G}}$, where we assume that derivation is compatible with the relations between the generators, we can extend it by $(g_1g_2)' = g_1'g_2 + g_1g_2'$, $1' = 0$ to the whole of $\mathcal{G}$ and by linearity to $\mathcal{T}_{\mathcal{G}}$,

$$\left(\sum_{\mathbf{k}\in\mathbb{Z}^n} c_{\mathbf{k}}\mu_{\mathbf{k}}\right)' = \sum_{j=1}^{n}\mu_j' \sum_{\mathbf{k}\in\mathbb{Z}^n} k_j c_{\mathbf{k}} \mu_1^{k_1}\cdots\mu_j^{k_j-1} \tag{1.36}$$

and the latter sum is a well defined finite sum of transseries.

Exercise. Show that with these operations, $\mathcal{T}_{\mathcal{G}}$ *is a differential field.*

52. If s is a small series, we define

$$e^s = \sum_{k\geq 0} \frac{s^k}{k!} \tag{1.37}$$

Exercise. Show that e^s has the usual properties with respect to multiplication and differentiation.

53. **Transseries are limits of finite sums**. We let $\mathbf{m} \in \mathbb{Z}^n$ and $\mathbf{M}_p = (p, p, ..., p) \in \mathbb{N}^n$. Note that

$$T_p := \sum_{g_\beta=\mu_{\mathbf{k}};\mathbf{m}\leq\mathbf{k}\leq\mathbf{M}_p;\beta\in\Omega} d_\beta g_\beta \underset{p\to\infty}{\to} \sum_{\beta\in\Omega} d_\beta g_\beta$$

Indeed, it can be checked that $d(T_p, T) \to 0$ as $p \to \infty$.

54. More generally, let $\mathcal{G}$ be finitely generated and $\mathbf{k}_0 \in \mathbb{Z}$. Assume $s_{\mathbf{k}} \to 0$ as $\mathbf{k} \to \infty$. Then, for any sequence of real numbers $c_{\mathbf{k}}$, the sequence

$$\sum_{\mathbf{k}_0\leq\mathbf{k}\leq\mathbf{M}_p} c_{\mathbf{k}} s_{\mathbf{k}} \tag{1.38}$$

where $\mathbf{M}_p = (p, ..., p), p \in \mathbb{N}$ is Cauchy and the limit

$$\lim_{p\to\infty} \sum_{\mathbf{k}_0\leq\mathbf{k}\leq\mathbf{M}_p} c_{\mathbf{k}} s_{\mathbf{k}} \tag{1.39}$$

is well defined. In particular, for a given transseries

$$T \triangleleft \mathbf{s} = \sum d_{\mathbf{k}} s_{\mathbf{k}} \tag{1.40}$$

we define the **transcomposition**

$$T \triangleleft \mathbf{s} = \sum_{\mathbf{k}\geq\mathbf{k}_0} d_{\mathbf{k}} s_{\mathbf{k}} \tag{1.41}$$

55. As an example of transcomposition, we see that transseries are closed under right pseudo-composition with *large* (not necessarily purely large) transseries $\mathbf{T} = T_i; i = 1, 2, ..., n$ by

$$T_1(1/\mathbf{T}) = \sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}} \mathbf{T}^{-\mathbf{k}} \tag{1.42}$$

if

$$T_1 = \sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}} \mu^{\mathbf{k}}$$

(cf. 45) We should mention that at this level of abstractness pseudo-composition may not behave as a composition, for instance it may not be compatible with chain rule in differentiation.

56. **Contractive operators** Contractivity is usually defined in relation to a metric, but given a topology, contractivity depends on the metric while convergence does not. There is apparently no natural metric on transseries.

Definition 1.43 *Let first J be a linear operator from $\mathcal{T}_{A_{\mathbf{m}}}$ or from one of its subspaces, to $A_{\mathbf{k}}$,*

$$JT = J\sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}}\mu_{\mathbf{k}} = \sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}} J\mu_{\mathbf{k}} \tag{1.44}$$

Then J is called asymptotically contractive on $\tilde{\mathcal{A}}_{\mathbf{m}}$ if

$$J\mu_{\mathbf{j}} = \sum_{\mathbf{p}>0} c_{\mathbf{p}}\mu_{\mathbf{j}+\mathbf{p}} \tag{1.45}$$

Remark 1.46 *Contractivity depends on the set of generators.*

Remark 1.47 *It can be checked that contractivity holds if*

$$J\mu_{\mathbf{j}} = \sum_{\mathbf{p}>0} c_{\mathbf{p}}\mu_{\mathbf{j}+\mathbf{p}}(1+s_{\mathbf{j}}) \tag{1.48}$$

where $s_{\mathbf{j}}$ are small transseries.

Exercise 1.49 *Check that for any $\mu_{\mathbf{j}}$ we have*

$$\sup_{p>0}\sum_{k=n}^{n+p} J^k\mu_{\mathbf{j}} \to 0$$

as $n\to\infty$.

We then have

$$JT = \sum_{\mathbf{k}\geq\mathbf{m}} J\mu_{\mathbf{k}} \tag{1.50}$$

Definition 1.51 *The linear or nonlinear operator J is (asymptotically) contractive in the set $A\subset A_{\mathbf{m}}$ if $J: A\mapsto A$ and the following condition holds. Let T_1 and T_2 in A be arbitrary and let*

$$T_1 - T_2 = \sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}}\mu_{\mathbf{k}} \tag{1.52}$$

Then

$$J(T_1) - J(T_2) = \sum_{\mathbf{k}\geq\mathbf{m}} c'_{\mathbf{k}}\mu_{\mathbf{k}+\mathbf{p}_{\mathbf{k}}}(1+s_{\mathbf{k}}) \tag{1.53}$$

where $\mathbf{p}_{\mathbf{k}}>0$ and $s_{\mathbf{k}}$ are small.

Remark 1.54 *The sum of asymptotically contractive operators is contractive; the composition of contractive operators, whenever defined, is contractive.*

Theorem 1.55 *(i) If J is linear and contractive on $\mathcal{T}_{A_{\mathbf{m}}}$ then for any $T_0 \in \mathcal{T}_{A_{\mathbf{m}}}$ the fixed point equation $T = JT + T_0$ has a unique solution $T \in \mathcal{T}_{A_{\mathbf{m}}}$.*

(ii) In general, if $A \subset A_{\mathbf{m}}$ is closed and $J : A \mapsto A$ is a (linear or nonlinear) contractive operator on A, then $T = J(T)$ has a unique solution is A.

Proof. For (ii) we define the sequence $T_{n+1} = J(T_n)$ is convergent since for some coefficients $c_{j,\mathbf{k}}$ we have

$$J^q(T) - J(T) = \sum_{\mathbf{k}\geq m} c_{j,\mathbf{k}} \mu_{\mathbf{k}+q\mathbf{p}_{\mathbf{k}}} \to 0$$

as $q \to \infty$. Uniqueness is immediate. □

57. When working with transseries we often encounter this fixed point problem in the form $X = Y + \mathcal{N}(X)$, where Y is given, X is the unknown Y is given, and $\mathcal{N}$ is "small".

    Exercise. Show the existence of a unique inverse of $(1 + s)$ where s is a small transseries, by showing that the equation $T = 1 - sT$ is contractive.

58. For example ∂ is contractive on transseries of level zero. This is clear since in every monomial the power of x decreases by one. But note that ∂ is not contractive anymore if we add "terms beyond all orders", e.g., $(e^{-x^2})' = -2xe^{-x^2} \gg e^{-x^2}$.

    We cannot expect any contractivity of ∂ in general, since if y_1 is the level zero solution of $T = 1/x - T'$ then $T + Ce^{-x}$ is a solution for any C so uniqueness fails.

    This is one reason the WKB method works near irregular singularities, where exponential behavior is likely, and naive approximations don't.

59. We take the union

$$\mathcal{T} = \bigcup_{\mathcal{G}} \mathcal{T}_{\mathcal{G}}$$

with the natural embeddings. It can be easily checked that $\mathcal{T}$ is a differential field too. The topology is that of inductive limit, namely a sequence of transseries converges if they all belong to some $\mathcal{T}_{\mathcal{G}}$ and they converge there.

60. One can check that algebraic operations, exponentiation, composition with functions for which composition is defined, are continuous wherever the functions are "C^∞".

Exercise 1.56 *Let $T \in A_{\mathbf{m}}$. Show that the set $\{T_1 \in A_{\mathbf{m}} | T_1 \ll T\}$ is closed.*

1.2 General logarithmic-free transseries

1.2a Assumption on the inductive step

1. We have already constructed transseries of level zero. Transseries of any level are constructed inductively, level by level.

   Since we have already studied the properties of abstract multiseries, the construction is relatively simple, all we have to do is essentially watch for consistency of the definitions at each level.

2. Assume finitely generated transseries of level at most n have already been constructed. We assume a number of properties, and then build level $n+1$ transseries and show that these properties are conserved.

   (a) Transmonomials μ_j of order at most N are totally ordered, with respect to two order relations, $\ll$ and $<$. Multiplication is defined on the transmonomials, it is commutative and compatible with the order relations.

   (b) For a set of n small transmonomials, a transseries of level at most N is defined as expression of the form (1.17).

   It follows that the set $\{g = \mu_{\mathbf{k}} | \mathbf{k} \geq \mathbf{m}\}$ can be indexed by ordinals, and we can write the transseries in the form (1.20). The decomposition 1.22 then applies.

   It also follows that two transseries are equal iff their corresponding d_β coincide.

   The ordering relation on transseries of level N is defined as before, $T \gg 1$ if, by definition $g_1 \gg 1$ and $T > 0$ iff $d_1 > 0$.

   Transseries of level at most N are defined as the union of all $\mathcal{T}_{\mathcal{A}_{\mathbf{m}}}$ where $A_{\mathbf{m}}$ is as before.

   (c) A transmonomial or order at most N is of the form $x^a e^L$ where L is a purely large or null transseries of level $N-1$, and e^L is defined recursively. There are no transseries of level -1, so for $N = 1$ we take $L = 0$.

   Exercise. Show that any transmonomial is of the form $x^a e^{L_1} e^{L_2} \cdots e^{L_j}$ where L_j are of order exactly j meaning that they are of order j but not of lower order.

   (d) For any transmonomial, $(x^a e^L)^r$ is defined as $x^{ar} e^{rL}$ where the ingredients have already been defined. It may be a adjoined to the generators of $\mathcal{G}$ and then, as in the previous section, T^r is well defined.

(e) By definition, $x^a e^L = e^L x^a$ and $x^{a_1} e^{L_1} x^{a_2} e^{L_2} = x^{a_1+a_2} e^{L_1+L_2}$. Furthermore $e^{L_1} \gg x^a e^{L_2}$ for any a if $L_1 > 0$ is a purely large transseries of level strictly higher than the level of L_2.

(f) There is a differentiation with the usual properties on the generators, compatible with the group structure and equivalences. We have $(x^a e^L)' = a x^{a-1} x^L + x^a L' e^L$ where L' is a (finitely generated) transseries of level at most $N-1$.

We define

$$T' = \sum_{\mathbf{k}\in\mathbb{Z}^n;\mathbf{k}\geq\mathbf{M}} c_{\mathbf{k}} \left[(x^{-\mathbf{k}\cdot\boldsymbol{\alpha}})' e^{-\mathbf{L}\cdot\boldsymbol{\beta}} + x^{-\mathbf{k}\cdot\boldsymbol{\alpha}} (e^{-\mathbf{L}\cdot\boldsymbol{\beta}})' \right] \tag{1.57}$$

where, according to the definition of differentiation, (1.57) is a finite sum of products of transseries of level at most N.

We have $T' = 0$ iff $T = const$. If $\mathrm{dom}(T_{1,2}) \neq const., then\ T_1 \ll T_2$ implies $T_1' \ll T_2'$.

3. It can be checked by induction that $T > 0, T \gg 1$ implies $T' > 0$. In this sense, differentiation is compatible with the order relations.

4. It can then be checked that differentiation has the usual properties.

5. if c is a constant, then e^c is a constant, the usual exponential of c, and if $L + c + s$ is the decomposition of a transseries of level $N-1$ we write $e^{L+c+s} = e^L e^c e^s$ where e^s is *reexpanded* according to formula (1.37) and the result is a *transseries* of level N.

   We convene to write e^T, for any T transseries of level at most N *only* in this reexpanded form.

   Then it is always the case that $e^T = T_1 e^{L_2}$ where T_1 and L_2 are transseries of level $N-1$ and L_2 is purely large or zero. The transseries e^T is finitely generated, with generators e^{-L_1} , if $L_1 > 0$ or e^{L_1} otherwise, together with all the generators of L_1.

   Sometimes it is convenient to adjoin to the generators of T all the generators in the exponents of the transmonomials in T, and then the generators in exponents in the exponents of the transmonomials in T etc. Of course, this process is finite, and we end up with a finite number of generators, which we will call *the complete set of generators* of T.

6. This **defines** the exponential of any transseries of level at most $N-1$ if $L \neq 0$ and the exponential of any transseries of level at most N if $L = 0$. We can check that $e^{T_1} = e^{T_2}$ iff $T_1 = T_2$.

7. If all transseries of level N are written in the canonical form (1.20) then $T_1 = T_2$ iff all g_β at all levels have exactly the same coefficients. Transseries, in this way, have a unique representation in a strong sense.

8. The space of transseries of level N, $\mathcal{T}^{[N]}$, is defined as the union of all spaces of transseries over finitely generated groups of transmonomials of level N.

$$\mathcal{T}^{[N]} = \bigcup_{\mathcal{G}_N} \mathcal{T}_{\mathcal{G}_N}$$

with the inductive limit topology.

9. The abstract theory of transseries we have developed in the previous section applies. In particular the definition $1/(1-s) = \sum_j s^j$ $1/T = 1/\operatorname{dom}(T)(1+s)^{-1}$ and transseries of level N form a differential field closed under the contractive mappings.

10. Note that transseries of order N are closed under the contractive mapping principle.

1.2b Passing from step N to step $N+1$

1. We now proceed in defining transseries of level at most $N+1$. We have to check that the construction preserves the properties in §1.2a .

2. For any purely large transseries of level N we define $x^a e^L$ to equal the already defined transmonomial of order N. If L is a (finitely generated) purely large transseries of level exactly N we define a new primitive object, $x^a e^L$, a transmonomial of order $N+1$, with the properties

   (a) $e^0 = 1$.

   (b) $x^a e^L = e^L x^a$.

   (c) $x^{a_1} e^{L_1} x^{a_2} e^{L_2} = x^{a_1+a_2} e^{L_1+L_2}$.

   (d) If $L > 0$ is a purely large transseries of level exactly N then we have $e^L \gg x^a$ for any a.

   Exercise. Show that if L_1 and L_1 are purely large transseries and the level of L_1 strictly exceeds the level of L_2, then $e^{L_1} \gg x^a e^{L_2}$ for any a.

   Note that $L_1 \pm L_2$ may be of lower level but it is either purely large or else zero; $L_1 L_2$ is purely large.

   Note 1.58 At this stage, no meaning is given to e^L, or even to e^x; they are treated as primitives. There are possibly many models of this construction. We will interpret many of them later by finding an extended isomorphism between a family of transseries and a set of functions. Then e^x would correspond to the usual exponential, convergent multiseries will correspond to their sums etc. Finite generation would play a role throughout that process, and "good" transseries come as solutions of well defined classes of problems, with "coefficients" which are themselves "good" transseries. We will have $(1-1/x)^{-1} = \sum_j x^{-j}$ but also selected divergent series will have a meaning, e.g. $e^x \sum_{k=0}^{\infty} n!/x^{n+1} = PV \int_{-\infty}^{x} t^{-1}e^t dt$ The latter transseries, and its associated sum solve $f' + f = 1/x$. But it is not to be expected to have a summation process that applies to all series.

3. If $\alpha > 0$ and L is a *positive* transseries of level N we define a generator of order N to be $\mu = x^{-\alpha}e^{-L}$. We choose a number of generators $\mu_1, ..., \mu_n$, and define the abelian multiplicative group generated by them, with the multiplication rule just defined. We can check that $\mathcal{G}$ is a totally ordered, of course finitely generated, abelian group, and that the order relation is compatible with the group structure.

4. We can now define transseries over $\mathcal{G} = \mathcal{G}^{[N+1]}$ as in §1.1.

5. We define transseries of order $N+1$ to be the union over all $\mathcal{T}_{\mathcal{G}^{[N+1]}}$, with the natural embeddings. We denote these transseries by $\mathcal{T}^{[N+1]}$.

6. *Compatibility of differentiation with the order relation.* We have already assumed that this is the case for transseries of level at most N. (i) We first show that it holds for transmonomials of level $N+1$. If $L_1 - L_2$ is a positive transseries, then $(x^a e^{L_1})' \gg (x^b e^{L_2})'$ follows directly from the formula of differentiation, the fact that $e^{L_1 - L_2}$ is large and the induction hypothesis. If $L_1 = L_2$ then $a > b$ and the property follows from the fact that L_1 is either zero, or else $L \gg x^{\beta}$ for some $\beta > 0$ for some positive β (check!).

   (ii) For the general case we note that

$$\left(\sum_{\beta} d_{\beta} \mu_{\beta} \right)' = \sum_{\beta} d_{\beta} \mu'_{\beta}$$

   and $\mu'_{\beta_1} \ll \mu'_{\beta_2}$ if $\beta_1 > \beta_2$. Then $\operatorname{dom}(T)' = (\operatorname{dom}(T))'$ and the property follows.

7. Differentiation is continuous. Indeed, if $T^{[m]} \to 0$,

$$T^{[m]} = \sum_{\mathbf{k} \geq \mathbf{m}} c_{\mathbf{k}}^{[m]} x^{\mathbf{k} \cdot \mathbf{a}} e^{-\mathbf{k} \cdot \mathbf{L}} \to 0 \quad \text{as} \quad m \to \infty$$

   where the transseries $L_1, ..., L_n$ are purely large, then

$$(T^{[m]})' = \frac{1}{x} \sum_{\mathbf{k} \geq \mathbf{m}} (\mathbf{k} \cdot \mathbf{a}\, c_{\mathbf{k}}^{[m]}) x^{\mathbf{k} \cdot \mathbf{a}} e^{-\mathbf{k} \cdot \mathbf{L}} - \mathbf{L}' \cdot \sum_{\mathbf{k} \geq \mathbf{m}} (\mathbf{k} c_{\mathbf{k}}^{[m]}) x^{\mathbf{k} \cdot \mathbf{a}} e^{-\mathbf{k} \cdot \mathbf{L}}$$

   and the rest follows from continuity of multiplication and the definition of convergence.

8. Therefore, if a property of differentiation holds for finite sums of transmonomials, then it holds for transseries.

9. By direct calculation, if μ_1, μ_2 are transmonomials of order $N+1$ then $(\mu_1\mu_2)' = \mu_1'\mu_2 + \mu_1\mu_2'$. Then, one can check by usual induction, the product rule holds for finite sums of transmonomials. Using 8 the product rule follows for general transseries.

Composition

10. Composition *to the right* with a *large* (not necessarily purely large) transseries T of level m is defined as follows.

    The power of a transseries $T = x^a e^L(1+s)$ is defined by $T^p = x^{ap}e^{pL}(1+s)^p$, where the last expression is well defined and $(T^p)' = pT'T^{p-1}$ (check).

    The exponential of a transseries is defined, inductively, in the following way.

$$T = L + c + s \Rightarrow e^T = e^L e^c e^s = S e^L e^c \tag{1.59}$$

    where S is given in (1.37).

    A general exponential-free transseries of level zero has the form

$$T_0 = \sum_{\mathbf{k}\geq\mathbf{m}} c_{\mathbf{k}} x^{-\mathbf{k}\cdot\boldsymbol{\alpha}} \tag{1.60}$$

    where $(\alpha_1, ..., \alpha_n) \in \mathbb{R}^{+n}$ for some n.

    Then we take $\mathbf{T} = (T^{\alpha_1}, ..., T^{\alpha_n})$ and define $T_0(1/T)$ by (1.42); $T_0(1/T)$ has level m. If the sum (1.60) contains finitely many terms, it is clear that $[T_0(1/T)]' = T_0'(1/T)T'$. By continuity, this is true for a general T_0 of level zero.

11. Assume that composition with T has been defined for all transseries of level N. It is assumed that this composition is a transseries of level $N+m$. Then $L(T) = L_1 + c_1 + s_1$ (it is easily seen that $L(T)$ is not necessarily purely large). Then

$$(x^a e^L) \circ (T) := T^a e^{L(T)} = x^b(1 + s_1(T))e^{L_1(T)} \tag{1.61}$$

    where $L_1(T)$ is purely large. Since L_1 has level $N+m$, then $(x^a e^L) \circ (T)$ has level $N+m+1$. We have $(e^{L_1})' = L_1' e^{L_1}$ and the chain rule follows by induction and from the sum and product rules.

Exercise 1.62 *If $T^{[n]}$ is a sequence of transseries, then $e^{T^{[n]}}$ is a not necessarily a valid sequence of transseries. But if it is, then there is an L_0 such that $L^{[n]} = L_0$ for all large n. If $e^{T^{[n]}}$ is a sequence of transseries and $T^{[n]} \to 0$, then $e^{T^{[n]}} \to 1$.*

12. The exponential is continuous. This follows from the Exercise 1.62 and Exercise 1.34.

13. Take now a general transseries of level $N+1$ and write $T = x^a e^L(1+s)$

$$t = \sum_{\mathbf{k}\geq\mathbf{m}} x^{-\mathbf{k}\cdot\boldsymbol{\alpha}} e^{-\mathbf{k}\cdot\mathbf{l}} \tag{1.63}$$

Then $t(T)$ is well defined as the limit of the following finite sum with generators $x^{-|a\alpha_j|}, x^{-\alpha_j} e^{-l_j(T)}, e^{-l_j(T)}$; $, j = 1, ..., n$:

$$t(T) = \sum_{\mathbf{M}_p\geq\mathbf{k}\geq\mathbf{m}} x^{-a(\mathbf{k}\cdot\boldsymbol{\alpha})} e^{-\mathbf{k}\cdot\mathbf{l_1}(T)}(1+\mathbf{s}(T)) \tag{1.64}$$

14. The chain rule holds by continuity.

15. The general theory we developed in §1.1 applies and guarantees that the properties listed in §1.2a hold (check!).

Small transseries as infinitesimals; expansions beyond all orders

16. Let T be a transseries of level N over $\mathcal{G}$ and dx a small transseries with dominance e^{-L} where L is a positive large transseries of level $N+p$, $p>0$. Then $(T(x+dx)-T(x))/dx = T'(x)+s(T)$ where $s(T)$ is a small transseries of level $N+p$.

The proof is by induction on the level. By linearity and continuity it is enough to prove the statement for transmonomials. We have

$$(x+dx)^a e^{-L_1(x+dx)} = x^a(1+dx/x)^a e^{L_1(x)+L_1'(x)dx+s(L)}$$

where $L_1'dx$ is a small transseries (since $L_1 e^{-L}$ is small) and $s(L_1)$ is of level $N+p$. The claim follows after reexpansion of the two terms in the product. Note that dx must be far less than all terms in T; $dx \ll 1$ is not enough.

Exercise 1.65 *Show that, under the same assumptions that*

$$T(x+dx) = \sum_{j=0}^{\infty} T^{(n)}(x)\frac{dx^n}{n!} \tag{1.66}$$

In this sense, transseries behave like analytic functions.

An inequality helpful in WKB analysis.

Proposition 1.67 *If $L \gg 1$ then $L'' \ll (L')^2$ (or, which is the same, $L' \ll L^2$).*

Proof. If $L = x^a e^{L_1}$ where $L_1 \neq 0$ then L_1 is purely large, then the dominance of L' is of the form $x^b e^{L_1}$, whereas the dominance of L is of the form $x^a e^{2L_1}$ and the property is obvious. If $L_1 = 0$ the property is obvious as well. □ In WKB analysis this result is mostly used in the form (1.69 below.

Exercise 1.68 *Show that if $T \gg 1$, T positive or negative, we have*

$$\mathrm{dom}[(e^T)^{(n)}] = \mathrm{dom}[(T')^n e^T] \tag{1.69}$$

1.2c General logarithmic-free transseries

These are simply defined as

$$\mathcal{T}_e = \bigcup_{N \in \mathbb{N}} \mathcal{T}^{[N]} \tag{1.70}$$

with the natural embeddings.

The general theory we developed in §1.1 applies to $\mathcal{T}_e$ as well. Since any transseries belongs to some level, any finite number of them share some level. There are no operations defined which involve infinitely many levels, because they would involve infinitely many generators. Then, the properties listed in §1.2a hold in $\mathcal{T}_e$ (check!).

1.2d Écalle's notation

- $\sqcup$ —small transmonomial.
- $\sqcap$ —large transmonomial.
- $\square$ —any transmonomial, large or small.
- $\sqcup\!\sqcup$ —small transseries.
- $\sqcap\!\sqcap$ —large transseries.
- $\square\!\square$ —any transseries, small or large.

Further properties of transseries

Definition. The level $l(T)$ of T is n if $T \in \mathcal{T}^{[n]}$ and $T \notin \mathcal{T}^{[n-1]}$.

Further properties of differentiation

We denote $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}x}$

Corollary 1.71 *We have* $\mathcal{D}T = 0 \iff T = \text{Const.}$

Proof. We have to show that if $T = L + s \neq 0$ then $T' \neq 0$. If $L \neq 0$ then for some $\beta > 0$ we have $L + s \gg x^\beta + s$ and then $L' + s' \gg x^{\beta-1} \neq 0$. If instead $L = 0$ then $(1/T) = L_1 + s_1 + c$ and we see that $(L_1 + s_1)' = 0$ which, by the above, implies $L_1 = 0$ which gives $1/s = s_1$, a contradiction. □

Proposition 1.72 *Assume* $T = L$ *or* $T = s$. *Then:*
(i) If $l(\operatorname{mag}(T)) \geq 1$ *then* $l(\operatorname{mag}(T^{-1}T')) < l(\operatorname{mag}(T))$.
(ii) $\operatorname{dom}(T') = \operatorname{dom}(T)'(1 + s)$.

Proof. Straightforward induction. □

Transseries with complex coefficients

Complex transseries $\mathcal{T}_{\mathbb{C}}$ are constructed in a similar way as real transseries, replacing everywhere $L_1 > L_2$ by $\Re L_1 > \Re L_2$. Thus there is only one order relation in $\mathcal{T}_{\mathbb{C}}$, $\gg$. Difficulties arise when exponentiating transseries whose dominant term is imaginary. Operations with complex transseries are then limited. We will only use complex transseries in contexts that will prevent these difficulties.

Differential systems in $\mathcal{T}_e$

The theory of differential equations in $\mathcal{T}_e$ is similar in many ways to the corresponding theory for functions.

Example. The general solution of the differential equation

$$f' + f = 1/x \tag{1.73}$$

in $\mathcal{T}_e$ (for $x \to +\infty$) is $T(x;C) = \sum_{k=0}^{\infty} k! x^{-k} + Ce^{-x} = T(x;0) + Ce^{-x}$.

The particular solution $T(x;0)$ is the unique solution of the equation $f = 1/x - \mathcal{D}f$ which is manifestly contractive in the space of level zero transseries.

Indeed, the fact that $T(x;C)$ is a solution follows immediately from the definition of the operations in $\mathcal{T}_e$ and the fact that e^{-x} is a solution of the homogeneous equation.

To show uniqueness, assume T_1 satisfies (1.73). Then $T_2 = T_1 - T(x;0)$ is a solution of $\mathcal{D}T + T = 0$. Then $T_2 = e^x T$ satisfies $\mathcal{D}T_2 = 0$ i.e., $T_2 = \text{Const.}$

1.2e The space $\mathcal{T}$ of general transseries

We define

$$\log_n(x) = \underbrace{\log\log ... \log(x)}_{n\ times} \tag{1.74}$$

$$\exp_n(x) = \underbrace{\exp\exp ... \exp(x)}_{n\ times} \tag{1.75}$$

$$\tag{1.76}$$

with the convention $\exp_0(x) = \log_0(x) = x$.
We write $\exp(\log x) = x$ and then any log-free transseries can be written as $T(x) = T \circ \exp_n(\log_n(x))$. This defines right composition with $\log_n$ in this trivial case, as $T_1 \circ \log_n(x)) = (T \circ \exp_n) \circ \log_n(x) := T(x)$.

More generally, we define $\mathcal{T}$, the space of general transseries , as a set of formal compositions

$$\mathcal{T} = \{T \circ \log_n : T \in \mathcal{T}_e\}$$

with the algebraic operations and inequalities (symbolized below by $\odot$) inherited from $\tilde{\tilde{\mathcal{T}}}$ by

$$(T_1 \circ \log_n) \odot (T_2 \circ \log_{n+k}) = [(T_1 \circ \exp_k) \odot T_2] \circ \log_{n+k} \tag{1.77}$$

and using (1.77), differentiation is defined by

$$\mathcal{D}(T \circ \log_n) = x^{-1} \left[(\prod_{k=1}^{n-1} \log_k)^{-1} \right] (\mathcal{D}T) \circ \log_n$$

Proposition 1.78 *$\mathcal{T}$ is an ordered differential field, closed under restricted composition.*

Proof. Exercise. □

The logarithm of a transseries. This is defined by first considering the case when $T \in \mathcal{T}_e$ and then taking right composition with iterated logs.

If $T = c\,\mathrm{mag}(T)(1 + s) = cx^a e^L(1 + s)$ then we define

$$\log(T) = \log(\,\mathrm{mag}(T)) + \log c + \log(1+s) = a \log x + L + \log c + \log(1+s) \tag{1.79}$$

where $\log c$ is the usual log, $\log(1 + s)$ is defined by expansion which we know is well defined on small transseries.

1. If $L \gg 1$ is large, then $\log L \gg 1$ and if $s \ll 1$, then $\log s \gg 1$.

Restricted composition

Proposition 1.80 *$\mathcal{T}$ is closed under integration.*

Proof. The idea behind the construction of $\mathcal{D}^{-1}$ is the following: we first find an invertible operator J which is *to leading order* $\mathcal{D}^{-1}$; then the equation for the correction will be contractive. Let $T = \sum_{\mathbf{k}\ge\mathbf{k}_0} \mu^{\mathbf{k}} \circ \log_n$. To unify the treatment, it is convenient to use the identity

$$\int_x T(s)ds = \int_{\log_{n+2}(x)} \left(T \circ \exp_{n+2}\right)(t) \prod_{j\le n+1} \exp_j(t)\mathrm{d}t \; = \int_{\log_{n+2}(x)} T_1(t)\mathrm{d}t$$

where the last integrand, $T_1(t)$ is a log-free transseries and moreover

$$T_1(t) = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\mu_1^{k_1}\cdots\mu_M^{k_M} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}e^{-k_1L_1-\ldots-k_ML_M}$$

The case $\mathbf{k}=0$ is trivial and it thus suffices to find $\partial^{-1}e^{\pm L}$, where $n = l(L)\ge 1$ where $L>0$. We analyse the case $\partial^{-1}e^{\pm L}$, the other one being similar. Then $L \gg x^m$ for any m and thus also $\partial L \gg x^m$ for all m. Therefore, since $\partial e^{-L} = -(\partial L)e^{-L}$ we expect that $\operatorname{dom}(\partial^{-1}e^{-L}) = -(\partial L)^{-1}e^{-L}$ and we look for a Δ so that

$$\partial^{-1}e^{-L} = -\frac{e^{-L}}{\partial L}(1+\Delta) \tag{1.81}$$

Then Δ should satisfy the equation

$$\Delta = -\frac{\partial^2 L}{(\partial L)^2} - \frac{\partial^2 L}{(\partial L)^2}\Delta + (\partial L)^{-1}\partial\Delta \tag{1.82}$$

Since $s_1 = 1/L'$ and $s_2 = L''/(L')^2$ are small, by Lemma 1.23, there is a set of generators in which all the magnitudes of $s_{1,2}$ are of the form $\boldsymbol{\mu}^{\mathbf{k}}$ with $\mathbf{k}>0$. By Proposition 1.67 and Exercise 1.56, (1.82) is contractive and has a unique solution in the space of transseries with the complete set of generators of L and x^{-1} and $\Delta \ll L$ and the generators constructed above. For the last term, note that if $\Delta = \sum c_\omega e^{-L_\omega}$ and $L = e^{L_1}$, then $\Delta'/L' = \sum c_\omega L'_\omega e^{-L_\omega}e^{-L_1}$ and $L'_\omega e^{-L} = \mu_\omega \ll 1$.
□

1. Since the equation is contractive, it follows that $\operatorname{mag}(\Delta) = \operatorname{mag}(L''/L'^2)$.

In the following we also use the notation $\partial T = T'$ and we write $\mathcal{P}$ for the antiderivative ∂^{-1} constructed above.

Proposition 1.83 *$\mathcal{P}$ is an antiderivative without constant terms, i.e,*

$$\mathcal{P}T = L + s$$

Proof. This follows from the fact that $\mathcal{P}e^{-L} \ll 1$ while $P(e^L)$ is purely large, since all small terms are of lower level. Check! □

Proposition 1.84 *We have*

$$\begin{aligned}\mathcal{P}(T_1+T_2) &= \mathcal{P}T_1+\mathcal{P}T_2\\ (\mathcal{P}T)' = T;\ \mathcal{P}T' &= T_{\bar{0}}\\ \mathcal{P}(T_1T_2') &= (T_1T_2)_{\bar{0}} - \mathcal{P}(T_1'T_2)\\ T_1 \gg T_2 \implies \mathcal{P}T_1 &\gg \mathcal{P}T_2\\ T>0 \text{ and } T\gg 1 \implies \mathcal{P}T &> 0\end{aligned} \tag{1.85}$$

where

$$T=\sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}}\mu^{\mathbf{k}} \implies T_{\bar{0}} = \sum_{\mathbf{k}\geq\mathbf{k}_0;\mathbf{k}\neq 0} c_{\mathbf{k}}\mu^{\mathbf{k}}$$

Proof. Exercise. □

There exists only one $\mathcal{P}$ with the properties (1.85), for any two would differ by a constant.

Remark 1.86 *Let $s_0 \in \mathcal{T}$. The operators defined by*

$$J_1(T) = \mathcal{P}(e^{-x}(\text{Const.}+s_0)T(x)) \tag{1.87}$$

$$J_2(T) = e^{\pm x}x^{\sigma}\mathcal{P}(x^{-2}x^{-\sigma}e^{\mp x}(\text{Const.}+s_0)T(x)) \tag{1.88}$$

are contractive on $\mathcal{T}$.

Proof. For (1.87) it is enough to show contractivity of $\mathcal{P}(e^{-x}\cdot)$. If we assume the contrary, that $T' \not\ll Te^{-x}$ it follows that $\log T \not\gg 1$. We know that if $\log T$ is small then $\operatorname{mag}(T)=c$, c constant. But if $\operatorname{mag}(T)=c$ then the property is immediate. The proof of (1.87) is very similar.

□

1.3 Equations in $\mathcal{T}$: examples

Remark 1.89 The general contractivity principle stated in Theorem 1.55, which we have used in proving closure of transseries with respect to a number of operations can be used to show closure under more general equations. Our main focus is on differential systems.

Nonlinear ODEs in $\mathcal{T}$

We start with an example, a first order equation:

$$f'+f=\frac{1}{x}+\frac{1}{x}f+f^3 \tag{1.90}$$

The analysis however can be easily carried out for higher order systems of the form

$$\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n \tag{1.91}$$

see §3.1c . We now address the issue of what is the general solution in transseries of (1.90).

Proposition 1.92 *Eq. (1.90) has exactly a one parameter family of transseries solutions. They are of the form*

$$\sum_{k=0}^{\infty} C^k (xe^{-x})^k \tilde{y}_{k;0} \tag{1.93}$$

and two zero parameter solutions of the form

$$\pm 1 + \tilde{y}_{k;\pm} \tag{1.94}$$

where $\tilde{y}_{k;0,\pm}$ *are power series which only depend on the equation while* C *is a free parameter.*

Proof. At a formal level, we could think of the proof as rigorous asymptotics.

First of all, we show there are no large transseries solutions. Indeed, if such was the case, then $f^3 \gg f$ and we are left with the dominant balance, or balance of dominances,

$$A(x^b e^L)' = A^3 x^{3B} e^{3L} \tag{1.95}$$

L may contain logs, in which case $L = L_1(\log_k(x))$ where L_1 is log-free. We can make it of level at least one, of the form e^L, by writing $L = L_2(\log_{k+1}(x))$. Then,

$$L_2' = \prod_{j=0}^{k+1} \exp_n(x) e^{2L} \tag{1.96}$$

Since the product on the right side is much larger than one, we would have $L' \gg e^{2L}$, a contradiction.

We then see that the dominant part is either of the order of a constant, $A_j \in \{-1, 0, 1\}$. We look at the case $A = 0$ the others cases being very similar, after substituting $f = A_j + y_j$. For $A = 0$, f is small. Then, $f/x, f^3 \ll f$ and the dominance is decided by $f, 1/x$ and possibly f'.

We have

$$\mathrm{dom}(f) = \mathrm{dom}(1/x) - f'$$

There three possibilities which we analyze separately: $f' \ll 1/x$, $f' \sim 1/x$ or $f' \gg 1/x$. The last two entail, by applying $\mathcal{P}$, that f is larger or of the order of $\ln x$. But this is not possible since it gives $\log x = O(1/x)$.

Thus $f' \ll 1/x$. Within level zero transseries, we find immediately a solution f_0, using the contractive equation

$$f = \frac{1}{x} - f' - \frac{1}{x} f - f^3 \tag{1.97}$$

We write $f = f_0 + \delta$ and get

$$\delta' + \delta = o(\delta) \tag{1.98}$$

and thus, as before, $\mathrm{dom}(\delta) = Ce^{-x}$ for some C. We thus write $\delta = \Delta e^{-x}$ and get

$$\Delta' = \frac{1}{x}\Delta + 3f_0^2\Delta + 3f_0\Delta^2 e^{-x} + \Delta^3 e^{-2x} \tag{1.99}$$

where the dominant balance, by (1.98), is between Δ' and Δ/x. This gives $\mathrm{dom}\Delta = Cx$, still large. If $\Delta = xT$ then $T' = 3f_0^2 T + 3f_0 x T^2 e^{-x} + x^3 e^{-2x} T^3$. In any contractive formulation, we need to account for the free constant, since otherwise we would have no uniqueness, preventing contractivity. We write the equation in integral form:

$$T = C + \mathcal{P}\left(3f_0^2 T + 3f_0 x T^2 e^{-x} + x^3 e^{-2x} T\right) \tag{1.100}$$

This equation is contractive in the space of small transseries whose generators include $e^{-x}, x^{-1}, 1/xe^{-x}$ (we can always adjoin them). The rest of the proof is straightforward. $\Box$

Formal linearization

Let $z = Cx^{\beta}e^{-x}$. We have

$$C(x,\delta) = x^{-\beta}e^{x}\sum_{k\geq 1}\delta^k \tilde{g}_k(x)$$

A direct calculation shows that $C' = C_x + C_\delta \delta' = 0$. The transformation $(x \mapsto x; y \mapsto C(x, y - f_0))$ formally linearizes scalar equations of the form (1.91).

Chapter 2

Correspondence with functions

Analyzable functions are constructed in such a way that closure under most operations is preserved. The only way we know how to proceed is by summing transseries, which are already closed, in a way that commutes with all operations defined on transseries. Compatibility with the topolgy is unknown, but at least a weak form of compatibility is expected to hold.

Not all transseries are summable, but only the minimal space of transseries that originate in "natural" problems such as ODEs, difference equations etc, contained in the closure of, say, polynomials at infinity under all operations. This limitation is probably unavoidable. Some formal series are provably non-summable by any procedure which preserves elementary properties. Such is the case of the formal series

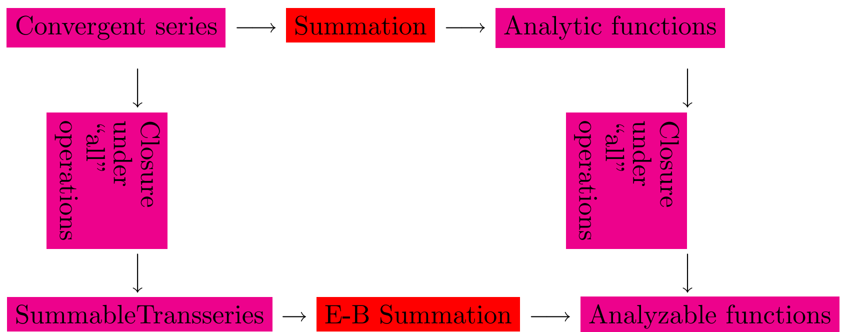
$$\tilde{S} = \sum_{q\in\mathbb{Q}} \frac{1}{(x-q)^2 + (x-q) + 1} \tag{2.1}$$

If Σ was a summation procedure compatible with the usual operations, then $S = \Sigma\tilde{S}$ would be a function with arbitrarily small oeriods. Since it is constructively defined it is measurable, thus constant. It is easily seen that this is inconsistent too since differentiating formally and using the symmetry of the problem, we have $S' < 0$ on all rationals. Furthermore, thus (2.1) cannot arise in any natural problem as *the* formal solution.

The limitation is not too serious, since the main purpose of studying transseries is in solving problems, in a constructive way. Even with this limitation though some restrictions must be imposed since the equation $x_{n+1} = 2^{x_n}$ has no transseries solution. The topology on transseries is too weak, since it allows for any growth of coefficients.

2.1 Construction of analyzable functions

Analyzable functions are constructed using the following diagram:



A few notes are in order, to understand why Borel summation is natural.

1. If a problem has analytic coefficients and is nonsingular, or regularly perturbed, the series expansions are convergent. In differential systems, a problem is singularly perturbed if the highest derivative is formally small, for instance in problems like $f' + f = 1/x$ or, exiting the realm of one variable, $\epsilon f'' + h(x)f = g$.

2. In singularly perturbed problems the highest derivative belongs formally to the right side. One then iterates upon the highest derivative. For generic analytic functions, by Cauchy's formula, $f^{(n)}$ grows roughly like $const^n n!$

3. It is then natural to diagonalize d/dx.

4. Then, by repeated iteration of d/dx we get geometric rather that factorial divergence. This is much easier to resolve.

5. The operator d/dx is diagonalized by the Fourier transform. Since it is often the case that we deal with asymptotic problems, for say a large variable x, we would like to perform it while keeping x large. A Fourier transform on a vertical contour in the complex domain is the inverse Laplace transform,

$$\mathcal{L}^{-1}f := \frac{1}{2\pi i}\int_{-i\infty+x_0}^{i\infty+x_0} f(t)e^{pt}dt \tag{2.2}$$

6. $\mathcal{L}^{-1}f' = pf$ thus repeated differentiation means repeated multiplication by p.Factorial growth is replaced by geometric growth, much easier to control.

7. The formal inverse Laplace transform (Borel transform, $\mathcal{B}$) of a small zero level transseries, that is of a small multiseries, is defined, roughly, as the

term-by-term inverse Laplace transform of the series. It is still a level zero transseries,

$$\mathcal{B}\sum_{\mathbf{k}>0} c_{\mathbf{k}} x^{-\mathbf{k}\cdot\mathbf{a}} = \sum_{\mathbf{k}>0} c_{\mathbf{k}} p^{\mathbf{k}\cdot\mathbf{a}-1}/(\mathbf{k}\cdot\mathbf{a}-1)! \tag{2.3}$$

where the factorial is understood in terms of the Gamma function.

The result of summing a formal series is still a formal series, convergent or not.

8. One difference between $\mathcal{L}^{-1}$ and $\mathcal{B}$ is that $\mathcal{L}^{-1}x^{-b-1} = p^b/b!$ for all small p, not only for $\Re p > 0$.

9. A series is classically Borel summable if (a) the series in p in (2.3) is convergent (as a Puiseux series) for small p, (b) the sum admits analytic continuation along $\mathbb{R}^+$ and (c) the sum is a function f analytic in a neighborhood of the real line, along which f does not grow faster than exponentially. The norm can be taken the sup norm with weight $e^{-\nu p}$ for some ν, or $L^1(\mathbb{R}^+, e^{-\nu p}dp)$ etc.

10. The Borel sum is then, by definition the Laplace transform of f. Since, in some sense, we applied Laplace transform to an inverse Laplace transform, formally the identity, this summation should preserve all properties. Some rigorous results follow, preceded by a heuristic argument.

11. We will shortly extend this summation to transseries. In practice however, rarely does one need to Borel sum several levels of a transseries: once the lowest level has been summed, usually the remaining object is convergent.

    Furthermore, in practice the conditions of Borel summability are not satisfied, and a more general summation replaces it. Even the prototypical example $\sum_{k=0}^{\infty} k!x^{-k-1}$ is not summable since its Borel transform $(1-p)^{-1}$ is not real-analytic. As we shall see, generic formal solutions which allow for small real valued exponential corrections are not Borel summable.

12. Higher powers of the factorial can often be easily dealt with by changes of the independent variable. For instance, in $\sum_{k=0}^{\infty}(k!)^2x^{-k+1}$ we achieve that by taking $x = y^2$ to get $\sum_{k=0}^{\infty}(k!)^2y^{-2k+2}$. Note that $k!^2$ roughly behaves like $(2k)!$. In some special cases however, no single change of variable suffices, and that is dealt with by *multisummability.*

13. As a rule of thumb, we pass to the variable in which divergence is factorial. It will turn out that this is intimately linked to the form of small exponential corrections. If these are of the form e^{-x^q} then divergence is usually like $(n!)^{1/q}$. The variable should then be chosen to be $t = x^q$. This t is the *critical time.* The Laplace transform leaves room for exponentially small corrections where the exponent is linear in x. For instance we can take the Laplace transform of $(1-p)^{-1}$ along any ray other than $\mathbb{R}^+$. An upper and lower transform differ precisely by a small exponential. Sub or

super-exponential corrections cannot originate in Borel sums. This will be proved shortly.

14. For instance, to find the antiderivative of e^{x^2}, $g' = e^{x^2}$ we can write $g = ue^{x^2}$ and then

$$2xu' + u = 1 \tag{2.4}$$

The freedom is that of an additive constant, $g = \mathcal{P}e^{x^2} + C$ and thus the correction is roughly e^{-x^2} times the dominant term. The critical time is $t = x^2$. We then take it is convenient to take $g = h(x^2)e^{x^2}$. The equation for h is

$$h' + h = \frac{1}{2\sqrt{t}} \tag{2.5}$$

15. If the transform of the solution of an equation is summable, then it is expected that the transformed equation should be more regular. In this sense, Borel summation is a regularizing transformation.

    In the case of (2.5) it becomes

$$-pH + H = p^{-1/2}\pi^{-1/2} \tag{2.6}$$

an algebraic equation, with algebraic singularities. The irregular singularity has been removed.

16. We see though again that the transformed function is not Laplace transformable, since it has a singularity on the real line. This is a situation we will deal with frequently, and is dealt with by an appropriate Écalle *medianization*. This is a suitable universal linear combination of analytic continuations, chosen in such a way that averaging commutes with Laplace convolution

$$f * g = \int_0^p f(s)g(p-s)ds \tag{2.7}$$

and the Borel sum of a product is the product of Borel transforms. We will return to this. For (2.6) it all amounts to taking the half-sum of the Laplace transforms along contours from 0 to $(1 \pm i\epsilon)\infty$.

17. It is crucial to perform Borel summation in the adequate variable. If the divergence is not fully compensated, then obviously we are still left with a divergent series. "Oversummation", the result of overcompensating divergence usually leads to superexponential growth of the transformed function. The presence of singularities in Borel plane is in fact a good sign.

    For equation (2.4), one can check that the divergence is like $\sqrt{n!}$. The equation is oversummed if we inverse Laplace transform it in x; what we get is

$$2H' - pH = 0; \quad H(0) = 1/2 \tag{2.8}$$

and thus $H = \frac{1}{2}e^{p^2/4}$. There are no singularities anymore but we have superexponential growth; this combination is a sign of oversummation. After oversummation there is no obvious way of taking the Laplace transform close to the real line. In some cases, a simple change of variable, as we have seen, can cure the problem. Of course, one can mix together series with different rates of divergence. Then multisummation becomes needed.

18. Commutation of Borel summation with all operations of natural origin can be understood heuristically as follows. We can think of $\sum_{n!}(-x)^{n+1}$ as the analytic continuation in λ of the series $\sum(\lambda n)!(-x)^{-n-1}$, which is convergent if $\lambda = i$ and x is large. This analytic continuation is nothing else but the Borel sum! For analytic continuation, commutation with all operations of natural origin comes under the umbrella of the vaguely stated "principle of permanence of relations" which cannot be formulated rigorously in any obvious way without giving up some legitimate "relations".

2.1a Laplace transform, Inverse Laplace transform

For convenience we provide some standard results on Laplace transforms.

Lemma 2.9 *Assume that $c \geqslant 0$ and $f(z)$ is analytic in $H_c := \{z : \Re z \geqslant c\}$. Assume further that $g(t) := \sup_{c' \geqslant c} |f(c'+it)| \in L^1(\mathbb{R}, dt)$. Let*

$$F(p) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px} f(x)dx =: (\mathcal{L}^{-1}f)(p) \tag{2.10}$$

Then for any $x \in \{z : \Re z > c\}$ we have $\mathcal{L}F = \int_0^\infty e^{-px}F(p)dp = f(x)$

Note that for any $x' = x_1' + iy_1' \in \{z : \Re z > c\}$

$$\int_0^\infty dp \int_{c-i\infty}^{c+i\infty} \left| e^{p(s-x')} f(s) \right| d|s| \leqslant \int_0^\infty dp e^{p(c-x_1')} \|g\|_1 \leqslant \frac{\|g\|_1}{x_1' - c} \tag{2.11}$$

and thus, by Fubini we can interchange the orders of integration:

$$\begin{aligned} U(x') &= \int_0^\infty e^{-px'} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px} f(x)dx \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dx f(x) \int_0^\infty dp e^{-px'+px} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{f(x)}{x'-x}dx \end{aligned} \tag{2.12}$$

Since $g \in L^1$ there exist subsequences $\{\tau_n\}, \{-\tau_n'\}$ tending to infinity such that $|g(\tau_n)| \to 0$. Let $x' > \Re x = x_1$ and consider the box $B_n = \{z : \Re z \in [x_1, x'], \Im z \in [-\tau_n', \tau_n]\}$ with positive orientation.

$$\int_{B_n} \frac{f(s)}{x'-s} ds = -f(x') \tag{2.13}$$

while, by construction,

$$\lim_{n\to\infty}\int_{B_n}\frac{f(s)}{x'-s}ds=\int_{x'-i\infty}^{x'+i\infty}\frac{f(s)}{x'-s}ds-\int_{c-i\infty}^{c+i\infty}\frac{f(s)}{x'-s}dx \tag{2.14}$$

On the other hand, by dominated convergence, we have

$$\int_{x'-i\infty}^{x'+i\infty}\frac{f(s)}{x'-s}ds\to 0 \quad\text{as}\quad x'\to\infty \tag{2.15}$$

Analytic behavior in x versus properties in p

Proposition 2.16 *If $F\in L^1(\mathbb{R}^+)$ then $\mathcal{L}F$ is analytic in the right half plane H and continuous on the imaginary axis ∂H, and $\mathcal{L}\{F\}(x)\to 0$ as $x\to\infty$ in H.*

Proof. Continuity and analyticity are preserved by integration against a finite measure ($F(p)dp$). Equivalently, these properties follow by dominated convergence, as $\epsilon\to 0$, of $\int_0^\infty e^{-isp}(e^{-ip\epsilon}-1)F(p)dp$ and of $\int_0^\infty e^{-xp}(e^{-p\epsilon}-1)\epsilon^{-1}F(p)dp$ respectively, the last integral for $\Re(x)>0$. The stated limit also follows easily from dominated convergence, if $|\arg(x)\pm\pi/2|>\delta$; the general case follows from the case $|\arg(x)|=\pi/2$ which is a consequence of the Riemann-Lebesgue lemma. □

Proposition 2.17 *(i) Assume f is analytic in an open sector $H_\delta:=\{x:|\arg(x)|<\pi+\delta\}$, $\delta\ge 0$ and is continuous on ∂H_δ, and that for some $K>0$ and any $x\in\overline{H_\delta}$ we have*

$$|f(x)|\le K(|x|^2+1)^{-1} \tag{2.18}$$

Then $\mathcal{L}^{-1}f$ is well defined by

$$F=\mathcal{L}^{-1}f=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\mathrm{d}t\,e^{pt}f(t) \tag{2.19}$$

and

$$\int_0^\infty \mathrm{d}p\,e^{-px}F(p)=\mathcal{L}\mathcal{L}^{-1}f=f(x)$$

and in addition $\|\mathcal{L}^{-1}\{f\}\|_\infty\le K\pi$ and $\mathcal{L}^{-1}\{f\}\to 0$ as $p\to\infty$.

(ii) If $\delta>0$ then $F=\mathcal{L}^{-1}f$ is analytic in the sector $S=\{p\ne 0:|\arg(p)|<\delta\}$. In addition, F is continuous in $\overline{S}$ and $\max_S|F|\le K\pi$, $F(p)\to 0$ as $p\to\infty$ in S.

This also means that growth of F at infinity indicates singularities or change of asymptotic behavior of f past the right half plane. Of course the half plane can be shifted right or left by a constant and a similar statement holds.
Proof.

(i) We have

$$\int_0^\infty \mathrm{d}p\, e^{-px} \int_{-\infty}^{\infty} \mathrm{d}s\, e^{ips} i f(is) = \int_{-\infty}^{\infty} \mathrm{d}t\, f(is) \int_0^\infty \mathrm{d}p\, e^{-px} e^{ips} \tag{2.20}$$

$$= \int_{-i\infty}^{i\infty} f(z)(x-z)^{-1}\mathrm{d}z = 2\pi i f(x) \tag{2.21}$$

where we applied Fubini's theorem and then pushed the contour of integration past x to infinity. The norm is obtained by majorizing $|fe^{ips}|$ by $K(|x^2|+1)^{-1}$.

(ii) We have for any $\delta' < \delta$, by (2.18),

$$\int_{-i\infty}^{i\infty} \mathrm{d}s\, e^{ps} f(s) = \left(\int_{-i\infty}^{0} + \int_0^{i\infty}\right) \mathrm{d}s\, e^{ps} f(s)$$

$$= \left(\int_{-i\infty e^{-i\delta'}}^{0} + \int_0^{i\infty e^{i\delta'}}\right) \mathrm{d}s\, e^{ps} f(s) \tag{2.22}$$

and analyticity is clear in (2.22).

For (ii) we note that (i) applies in $\bigcup\limits_{|\delta'|<\delta} e^{i\delta'} H_0$. Continuity follows by dominated convergence. □

Many cases can be reduced to this one after transformations. For instance if $g = \sum_{j=1}^{N} a_j x^{-k_j} + f(x)$, with $k_j > 0$ and f satisfying the assumptions above, then g is inverse Laplace transformable since the finite sum in its definition is explicitly transformable.

As we shall see, nonanaliticities in p plane, when they are infinitely many, translate usually in singularities in the "physical" domain x. Likewise, these singularities, or their absence can be used to show properties fo the inverse Laplace transform.

Proposition 2.23 *Let F be analytic in the open sector $S_p = e^{i\phi}\mathbb{R}^+$ with $\phi \in (-\delta, \delta)$ be such that $|F(|x|e^{i\phi})| \le g(|x|)$ for some $g \in L^1[0, \epsilon)$ bounded as $x \to \infty$. Then $f = \mathcal{L}F$ is analytic in the sector $S_x = \{x : |\arg(x)| < \pi/2 + \delta\}$ and $f(x) \to 0$ as $|x| \to \infty, \arg(x) = \theta \in (-\pi/2 - \delta, \pi/2 + \delta)$.*

Proof. Because of the analyticity of F and the decay conditions for large p, the path of Laplace integration can be rotated by any angle $\phi \in (-\delta, \delta)$ without changing $(\mathcal{L}F)(x)$ (see also the next example). This means Proposition 2.16 applies in $\cup_{|\phi|<\delta} e^{i\phi} H$.

Note that without further assumptions on $\mathcal{L}F$, F is *not* necessarily analytic at $p = 0$.

Uniqueness

Remark 2.24 (Uniqueness) *Assume $F \in L^1(\mathbb{R}^+)$ and $\mathcal{L}F = 0$ for a set of x with an accumulation point. Then $F = 0$ a.e.*

Proof. By analyticity, $\mathcal{L}F = 0$ in the open right half plane and by continuity, for $s \in \mathbb{R}$, $\mathcal{L}F(is) = 0 = \hat{\mathcal{F}}F$ where $\hat{\mathcal{F}}F$ is the Fourier transform of F (extended by zero for negative values of p). Since $F \in L^1$ and $0 = \hat{\mathcal{F}}F \in L^1$, by the known Fourier inversion formula [15], $F = 0$. □

2.1b Asymptotic properties. Watson's Lemma

Since solutions of a vast class of solutions are analyzable, therefore representable as combinations of Laplace transforms, the asymptotic behavior of these transforms is important to us.

Lemma 2.25 *Let $F \in L^1(\mathbb{R}^+)$, $x = \rho e^{i\phi}$, $\rho > 0$, $\phi \in (-\pi/2, \pi/2)$ and assume*

$$F(p) \sim p^\beta \quad \text{as } p \to 0^+$$

with $\Re(\beta) > -1$. Then

$$\int_0^\infty F(p)e^{-px}dp \sim \Gamma(\beta+1)x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. If $U(p) = p^{-\beta}F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let χ_A be the characteristic function of the set A and $\phi = \arg(x)$. We choose C and a positive so that $|F(p)| < C|p^\beta|$ on $[0, a]$. Since

$$\left| \int_a^\infty F(p)e^{-px}\mathrm{d}p \right| \le e^{-xa}\|F\|_1 \tag{2.26}$$

we have by dominated convergence, and after the change of variable $s = p/|x|$,

$$x^{\beta+1}\int_0^\infty F(p)e^{-px}\mathrm{d}p = e^{i\phi(\beta+1)}\int_0^\infty s^\beta U(s/|x|)\chi_{[0,a]}(s/|x|)e^{-se^{i\phi}}\mathrm{d}s \\ + O(|x|^{\beta+1}e^{-xa}) \to \Gamma(\beta+1) \quad (|x| \to \infty) \tag{2.27}$$

Watson's Lemma

This important tool states that the asymptotic series at infinity of $(\mathcal{L}F)(x)$ is obtained by formal term-by-term integration of the asymptotic series of $F(p)$ for small p, provided F has such a series.

Lemma 2.28 *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^\infty c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants β_i with $\Re(\beta_i) > 0$, $i = 1, 2$. Then*

$$\mathcal{L}F \sim \sum_{k=0}^\infty c_k \Gamma(k\beta_1 + \beta_2)x^{-k\beta_1-\beta_2}$$

along any ray ρ in the open right half plane H.

Proof. Induction, using Lemma 2.25. □

2.1c Uniqueness results

Proposition 2.29 *Assume that F and G are real-analytic and Laplace transformable (for instance, they are in $L^1(\mathbb{R}^+)$) and that $\mathcal{L}F$ and $\mathcal{L}G$ have the same asymptotic series. Then $F \equiv G$.*

Proof. This follows from Watson's lemma, which implies that F and G have the same (convergent) series at the origin, thus they coincide everywhere. □

Corollary 2.30 *If F is analytic and $\mathcal{L}F(x_n) = 0$ for x_n in the right half plane with an accumulation point, possibly $\infty e^{i\phi}$, $\phi \in (-\pi/2, \pi/2)$, then $F \equiv 0$.*

Lemma 2.31 *Assume $F \in L^1(\mathbb{R}^+)$ and for some $\epsilon > 0$ we have*

$$\mathcal{L}F(x) = O(e^{-\epsilon x}) \quad \text{as} \quad x \to +\infty \tag{2.32}$$

Then $F = 0$ a.e. on $[0, \epsilon]$. (The result is sharp as discussed after the proof.)

Corollary 2.33 *Assume $F \in L^1$ and $\mathcal{L}F = O(e^{-ax})$ as $x \to +\infty$ for all $a > 0$. Then $F = 0$ a.e. on $\mathbb{R}^+$.*

This shows that representability by Laplace transforms of (mostly) analytic functions is a good way to take into account exponentially small terms. No other freedom is possible except for exponentially small terms which are visible as nonanalyticities of the integrand. All this, as we see, assuming we deal with analytic functions of exponential order one at $+\infty$.

Proof of Lemma 2.31

We write

$$\int_0^\infty e^{-px} F(p)dp = \int_0^\epsilon e^{-px} F(p)dp + \int_\epsilon^\infty e^{-px} F(p)dp \tag{2.34}$$

we note that

$$\left| \int_\epsilon^\infty e^{-px} F(p)dp \right| \leqslant e^{-\epsilon x} \int_\epsilon^\infty |F(p)|dp \leqslant e^{-p\epsilon} \|F\|_1 = O(e^{-\epsilon x}) \tag{2.35}$$

Therefore

$$g(x) = \int_0^\epsilon e^{-px} F(p)dp = O(e^{-\epsilon x}) \quad \text{as} \quad x \to +\infty \tag{2.36}$$

The function g is manifestly entire. Let $h(x) = e^{\epsilon x} g(x)$. Then by assumption h is entire and uniformly bounded for $x \in \mathbb{R}$ (since by assumption, for some x_0 and all $x > x_0$ we have $|h| \leqslant C$ and by continuity $\max |h| < \infty$ on $[0, x_0]$). The function is bounded by $\|F\|_1$ for $x \in i\mathbb{R}$. By Phragmén-Lindelöf's theorem (first applied in the first quadrant and then in the fourth quadrant, with $\beta = 1, \alpha = 2$) h is bounded in the closed right half plane. Now, for $x = -s < 0$ we have

$$e^{-s\epsilon} \int_0^\epsilon e^{sp} F(p)dp \leqslant \int_0^\epsilon |F(p)| \leqslant \|F\|_1 \tag{2.37}$$

Again by Phragmén-Lindelöf (and again applied twice) h is bounded in the closed left half plane thus bounded in C, and it is therefore a constant. But, by the Riemann-Lebesgue lemma, $h \to 0$ for $x = is$ when $s \to +\infty$. Thus $h \equiv 0$. Therefore, with χ_A the characteristic function of A,

$$\int_0^{\epsilon} F(p)e^{-isp}dp = \hat{\mathcal{F}}(\chi_{[0,\epsilon]}F) = 0 \tag{2.38}$$

for all $s \in \mathbb{R}$ entailing the conclusion.

Note 2.39 In the opposite direction, by Laplace's method it is easy to check that for any small $\epsilon > 0$ we have $\mathcal{L}e^{-p^{-\frac{1-\epsilon}{\epsilon}}} = o\left(e^{-x^{1-\epsilon}}\right)$ and for any n $\mathcal{L}\left(e^{-E_{n+1}(1/p)}\right) = o(e^{-x/L_n(x)})$ where E_n is the n-th composition of the exponential with itself and L_n is the n-th composition of the log with itself. □.

2.1d Definition of Borel summation and basic properties

Series of the form $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-\beta_1 k_1 - \ldots - \beta_m k_m - r}$ with $\Re(\beta_j) > 0$ frequently arise as formal solutions of differential systems. We will first analyze the case $m = 1, r = 1, \beta = 1$ but the theory extends without difficulty to more general series.

Borel summation is relative to a direction, see Remark 2.57. The same formal series $\tilde{f}$ may yield different functions by Borel summation in different directions.

Borel summation along $\mathbb{R}^+$ consists in three operations, assuming they are possible:

1. Borel Transform, $\tilde{f} \mapsto \mathcal{B}\{\tilde{f}\}$.
2. Summation of the series $\mathcal{B}\{\tilde{f}\}$ and analytic continuation along $\mathbb{R}^+$; denote this function by F.
3. Laplace Transform, $F \mapsto \int_0^{\infty} F(p)e^{-px}\mathrm{d}p =: \mathcal{LB}\{\tilde{f}\}$, which requires exponential bounds on F, defined in some half plane $\Re(x) > x_0$.

The *domain* of Borel summation is the subspace $S_{\mathcal{B}}$ of series for which the conditions for the steps 1-3 above are met. For 3 we can require that for some constants C_F, ν_F we have $|F(p)| \le C_F e^{\nu_F p}$. Or we can require that $\|F\|_\nu < \infty$ where, for $\nu > 0$ we define

$$\|F\|_\nu := \int_0^{\infty} e^{-\nu p}|F(p)|dp \tag{2.40}$$

Exercise 2.41 *Show, using dominated convergence, Morera's and Fubini's theorems that if $F \in L^1_\nu$ then $\mathcal{L}F$ is analytic in x in the half plane $\Re(x) \ge \nu$.*

Note 2.42 *Equivalently we can say that the series $\tilde{f}$ is Borel summable if it is the asymptotic series as $x \to +\infty$ of $\mathcal{L}F$ with F analytic in a neighborhood $\mathcal{D}_{\mathbb{R}^+}$ of $\mathbb{R}^+$ (in particular, we say such a function is* real-analytic *on $[0, +\infty)$) and* *exponentially bounded at infinity. The domain $\mathcal{D}_{\mathbb{R}^+}$ as well as the bounds may depend on F. The definition is unambiguous since on the one hand the asymptotic series of a function is unique, and, by Watson's Lemma, if the asymptotic series of $\mathcal{L}F$ is zero, then the Taylor series of F at $p = 0$ is zero as well, and then $F \equiv 0$.*

Definition 2.43 (Inverse Laplace space convolution) *If $f, g \in L^1_{loc}$ then*

$$(f * g)(p) := \int_0^p f(s)g(p-s)ds \tag{2.44}$$

Lemma 2.45 *The space of functions which are in $L^1[0,\epsilon)$ for some $\epsilon > 0$ and real-analytic on $(0,\infty)$ is closed under convolution. If F and G are exponentially bounded then so is $F * G$. If $F, G \in L^1_\nu$ then $F * G \in L^1_\nu$.*

Proof. The statement about L^1 follows easily from Fubini's theorem. Analyticity follows by writing

$$\int_0^p f_1(s)f_2(p-s)ds = p\int_0^1 f_1(pt)f_2(p(1-t))dt \tag{2.46}$$

which is manifestly analytic in p. Clearly, if $|F_1| \le C_1 e^{\nu_1 p}$ and $|F_2| \le C_2 e^{\nu_2 p}$, then

$$|F_1 * F_2| \le C_1 C_2 p\, e^{(\nu_1+\nu_2)p} \le C_1 C_2\, e^{(\nu_1+\nu_2+1)p}$$

Finally, we note that

$$\int_0^\infty e^{-\nu p}\left|\int_0^p F(s)G(p-s)ds\right| dp \le \int_0^\infty e^{-\nu s}e^{-\nu(p-s)}\int_0^p |F(s)||G(p-s)|dsdp$$
$$= \int_0^\infty \int_0^\infty e^{-\nu s}|F(s)|e^{-\nu\tau}|G(\tau)|d\tau = \|F\|_\nu \|G\|_\nu \tag{2.47}$$

by Fubini.

Remark 2.48 *The results above can be rephrased for more general series of the form $\sum_{k=0}^\infty c_k x^{-k-r}$ by noting that for $\Re(\rho) > -1$ we have*

$$\mathcal{L}p^\rho = x^{-\rho-1}\Gamma(\rho+1)$$

and thus

$$\mathcal{B}\left(\sum_{k=0}^\infty c_k x^{-k-r}\right) = c_0\frac{p^{r-1}}{\Gamma(r)} + \frac{p^{r-1}}{\Gamma(r)} * \mathcal{B}\left(\sum_{k=1}^\infty c_k x^{-k}\right)$$

Furthermore, Borel summation naturally extends to series of the form

$$\sum_{k=-M}^\infty c_k x^{-k-r}$$

where $M \in \mathbb{N}$ by defining

$$\mathcal{LB}\left(\sum_{k=-M}^{\infty} c_k x^{-k-r}\right) = \sum_{k=-M}^{0} c_k x^{-k-r} + \mathcal{LB}\left(\sum_{k=0}^{\infty} c_k x^{-k-r}\right) \tag{2.49}$$

and more general powers can be allowed, replacing analyticity in p with analyticity in $p^{\beta_1}, ..., p^{\beta_m}$.

Proposition 2.50 *(i) $S_{\mathcal{B}}$ is a differential field,[1] and $\mathcal{LB} : S_{\mathcal{B}} \mapsto \mathcal{LB}S_{\mathcal{B}}$ is a differential algebra isomorphism.*

(ii) If $S_c \subset S_{\mathcal{B}}$ denotes the differential algebra of convergent power series, and we identify a convergent power series with its sum, then $\mathcal{LB}$ is the identity on S_c.

(iii) In addition, for $\tilde{f} \in S_B$, $\mathcal{LB}\{\tilde{f}\} \sim \tilde{f}$ as $|x| \to \infty$, $\Re(x) > 0$.

Proof. (i) Clearly $S_{\mathcal{B}}$ is a linear space; furthermore, $\tilde{f} = 0 \iff \mathcal{B}\tilde{f} = 0 \iff \mathcal{LB}\{\tilde{f}\} = 0$ (the last step follows from the injectivity of $\mathcal{L}$ which, in our case also follows from Watson's Lemma as in Note 2.42 above.)

Exercise 2.51 *Show that if $\mathcal{B}\tilde{f} = F$ and $\mathcal{B}\tilde{g} = G$ then $\mathcal{B}\tilde{f}\tilde{g}$ is the power series in p of $F * G$.*

To show multiplicativity, we use Note 2.42. Analyticity and exponential bounds of $|F * G|$ follow from Lemma 2.45. Consequently, $F * G$ is Laplace transformable, and by elementary properties of Laplace transforms (or by performing a simple change of variables in a double integral) we see that

$$\mathcal{L}(F * G) = \mathcal{L}F\,\mathcal{L}G$$

We have to show that if $\tilde{f}$ is a Borel summable series, so is $1/\tilde{f}$. We have $f = Cx^m(1+s)$ for some m where s is a small series.

We want to show that

$$1 - s + s^2 - s^3 + \cdots \tag{2.52}$$

is Borel summable, or that

$$-s + s^2 - s^3 + \cdots \tag{2.53}$$

is Borel summable. Let $\mathcal{B}s = H$. We examine the series

$$S = -H + H * H - H^{*3} + \cdots \tag{2.54}$$

where H^{*n} is the self convolution of H n times. Each term of the series is analytic, by Lemma 2.45. If $\max_{p \in \mathcal{D}} |H(p)| = m$, then it is easy to see that

$$|H^{*n}| \le m^n 1^{*n} = m^n \frac{p^{n-1}}{(n-1)!} \tag{2.55}$$

[1] with respect to formal addition, multiplication, and differentiation of power series.

Thus the function series in (2.54) is absolutely and uniformly convergent in $\mathcal{D}$ and the limit is analytic. Let now ν be large enough so that $\|H\|_\nu < 1$. (The fact that this is possible follows from dominated convergence.) Then the series in (2.54) is norm convergent, thus an element of L^1_ν.

Exercise 2.56 *Check that* $(1 + \mathcal{L}H)(1 + \mathcal{L}S) = 1$.

It remains to show that the asymptotic expansion of $\mathcal{L}(F * G)$ is indeed the product of the asymptotic series of $\mathcal{L}F$ and $\mathcal{L}G$, which is a consequence of the more general fact that the asymptotic series of a product is the product of the corresponding asymptotic series.

(ii) Since $\tilde{f}_1 = \tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k-1}$ is convergent, then $|c_k| \le CR^k$ for some C, R and $F(p) = \sum_{k=0}^{\infty} c_k p^k/k!$ is entire, $|F(p)| \le \sum_{k=0}^{\infty} CR^k p^k/k! = Ce^{Rp}$ and thus F is Laplace transformable for $|x| > R$. By dominated convergence we have for $|x| > R$,

$$\mathcal{L}\Big\{\sum_{k=0}^{\infty} c_k p^k/k!\Big\} = \lim_{N\to\infty} \mathcal{L}\Big\{\sum_{k=0}^{N} c_k p^k/k!\Big\} = \sum_{k=0}^{\infty} c_k x^{-k-1} = f(x)$$

(iii) This part follows simply from Watson's lemma, cf. § 2.1b . □

Remark 2.57 We note that in the last step in Borel summation we may take the integral in p along a different half-line in $\mathbb{C}$, as long as $\Re(xp) > 0$, and the algebraic properties are preserved. But it is also easy to check that the path *matters*, in general. For instance, if $x \in \mathbb{R}^+$ and $\mathcal{B}\tilde{f} = (1-p)^{-1}$, the half line can be any ray in the open right half plane, other than $\mathbb{R}^+$. But

$$\int_0^{\infty e^{i0+}} \frac{e^{-xp}}{1-p} dp - \int_0^{\infty e^{i0-}} \frac{e^{-xp}}{1-p} dp = 2\pi i e^{-x}$$

thus a convention for a choice of ray is needed.

Definition. The Borel sum of a series in the direction ϕ ($\arg x = \phi$), $(\mathcal{L}\mathcal{B})_\phi \tilde{f}$ is by convention, the Laplace Transform of $\mathcal{B}\tilde{f}$ in the direction that ensures $xp \in \mathbb{R}^+$,

$$(\mathcal{L}\mathcal{B})_\phi \tilde{f} = \int_0^{\infty e^{-i\phi}} e^{-px} F(p) dp = \mathcal{L}_{-\phi} F = \mathcal{L}F(\cdot e^{-i\phi}) \tag{2.58}$$

We can also say that Borel summation of $\tilde{f}$ along the ray $\arg(x) = \phi$ is defined as the (real) Borel summation of $\tilde{f}(xe^{i\phi})$.

Since in most cases of interest $\mathcal{B}\tilde{f}$ has singularities in the complex plane, different functions $\mathcal{L}\mathcal{B}_\phi \tilde{f}$ are obtained for different ϕ. For example, we have

$$\mathcal{L}\mathcal{B}_\phi \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} = \mathcal{L}_{-\phi}\{(1-p)^{-1}\} = \begin{cases} \mathrm{e}^{-x}(\mathrm{Ei}(x) - \pi i) & \text{for } \phi \in (-\pi, 0) \\ \mathrm{e}^{-x}(\mathrm{Ei}(x) + \pi i) & \text{for } \phi \in (0, \pi) \end{cases} \tag{2.59}$$

while the series is *not* classically Borel summable along $\mathbb{R}^+$, because of the pole at $p = 1$.

(iv) On the other hand it can be seen by deforming the contour in $\mathcal{L}$ that if $\mathcal{B}\tilde{f}$ is analytic and has uniform exponential bounds at infinity for $\arg(p) \in (-\delta_1, \delta_2)$, then the function $\mathcal{L}\mathcal{B}_\phi \tilde{f}$ is the same for all $\arg(x) \in (-\delta_2, \delta_1)$, in contrast to (2.59).

Note 2.60 (Connection between rate of divergence and exponential freedom) If f is a solution of some linear problem and $f = \mathcal{L}F$ where F is analytic except for nonaccumulating singularities and exponentially bounded then, by the shown isomorphism, $\mathcal{L}F$ is a solution of the problem (not necessarily satisfying the initial conditions) for any direction of integration. The integrals differ if the function is not entire. It means that there is freedom in the general solution of the problem. This freedom is visible as singularities in p, and it is exponentially small: if the singularity of F closest to the origin is at p_0, then the difference between the integrals is roughly $e^{-p_0 x}$. Since F is analytic at zero, assuming it is not entire, the asymptotic series of f diverges exactly factorially (up to geometric corrections). Factorial divergence is linked to the presence of exponentially small corrections.

Recovering exact solutions from formal series.

If a differential equation has a formal solution $\tilde{f} \in S_{\mathcal{B}}$ then $\mathcal{L}\mathcal{B}\tilde{f}$ is an actual solution of the same equation. For example

$$f' - f = x^{-1} \tag{2.61}$$

for $x \to \infty$ has the series solution $\tilde{f} = \sum_{k=0}^{\infty} (-x)^{-k-1} k!$ and $\mathcal{B}\{\tilde{f}\} = \sum_{k=0}^{\infty} (-p)^k$ sums to the Laplace transformable function $(1+p)^{-1}$. Now, for any $\tilde{f} \in S_{\mathcal{B}}$ and $f \in \mathcal{L}\mathcal{B}(S_{\mathcal{B}})$ we have

$$\tilde{f}' - \tilde{f} - x^{-1} = 0 \iff \mathcal{L}\mathcal{B}\big(\tilde{f}' - \tilde{f} - x^{-1}\big) = 0 \tag{2.62}$$

$$\iff \big(\mathcal{L}\mathcal{B}\{\tilde{f}\}\big)' - \mathcal{L}\mathcal{B}\{\tilde{f}\} - x^{-1} = 0 \tag{2.63}$$

In particular,

$$\mathcal{L}\mathcal{B}\{\tilde{f}\} = \int_0^\infty \frac{e^{-px}\,\mathrm{d}p}{1+p} = f \tag{2.64}$$

is an actual solution of (2.61). Solving the analytic problem (2.61) in $\mathcal{L}\mathcal{B}(S_{\mathcal{B}})$ has reduced thus to an essentially *algebraic* question, that of finding $\tilde{f}$.

Stokes phenomena: first examples

Dependence of Borel summation on the angle is reflected in changes of behavior of the summed function.

We illustrate this on a simple case:

$$y(x) := \int_0^\infty \frac{e^{-px}}{1+p} dp \tag{2.65}$$

and the question is to find the asymptotic behavior in the x complex plane (or, in fact, on a Riemann surface) of y. A simple estimate of the integral over an arc of radius R shows that for $x \in \mathbb{R}^+$ $y(x)$ also equals

$$y(x) = \int_0^{\infty e^{-i\pi/4}} \frac{e^{-px}}{1+p} dp \tag{2.66}$$

Then the functions given in (2.65) and (2.66) agree in $\mathbb{R}^+$ thus they agree everywhere they are analytic. Furthermore, the expression (2.66) is analytic for $\arg x \in (-\pi/4, 3\pi/4)$ and by the very definition of analytic continuation f admits analytic continuation in a sector $\arg(x) \in (-\pi/2, 3\pi/4)$. Now we take x with $\arg x = \pi/4$ and note that along this ray, by the same argument as before, the integral equals

$$y(x) = \int_0^{\infty e^{-\pi i/2}} \frac{e^{-px}}{1+p} dp \tag{2.67}$$

we can continue this rotation process until $\arg(x) = \pi - \epsilon$ in which case we have

$$y(x) = \int_0^{\infty e^{-\pi i + i\epsilon}} \frac{e^{-px}}{1+p} dp \tag{2.68}$$

which is now manifestly analytic for $\arg(x) \in (\pi/2 - \epsilon, 3\pi/2 - \epsilon)$. To proceed further, we relate the integral below the pole to the integral above the pole, noting that their difference is simply calculated in terms of the at the pole:

$$\int_0^{\infty e^{-\pi i - i\epsilon}} \frac{e^{-px}}{1+p} dp - \int_0^{\infty e^{-\pi i + i\epsilon}} \frac{e^{-px}}{1+p} dp = 2\pi i e^x \tag{2.69}$$

and thus

$$f(x) = \int_0^{\infty e^{-\pi i - i\epsilon}} \frac{e^{-px}}{1+p} dp - 2\pi i e^x \tag{2.70}$$

which is manifestly analytic for $\arg(x) \in (\pi/2 + \epsilon, 3\pi/2 + \epsilon)$. We can now freely proceed with the analytic continuation in similar steps until $\arg(x) = 2\pi$ and get

$$f(xe^{2\pi i}) = f(x) - 2\pi i e^x \tag{2.71}$$

The function has *nontrivial monodromy at infinity.* We also note that by Watson's Lemma, as long as f can be written as a pure Laplace-like integral, f has an asymptotic series in a half-plane. The relation (2.70) shows that this ceases to be the case when $\arg(x) = \pi$. This line is called a **Stokes line**. The exponential, "born" there is smaller than the terms of the series until $\arg(x) = 3\pi/2$ when it becomes the dominant term of the expansion. This line is called an **Antistokes line**. The fact that the function itself is not single-valued in a neighborhood of infinity is also seen from the calculation (take first $x \in \mathbb{R}^+$)

$$\begin{aligned} f(x) &= e^{-x}\int_1^\infty \frac{e^{-xt}}{t}dt = e^{-x}\int_x^\infty \frac{e^{-s}}{s}ds = e^{-x}\left(\int_x^1 \frac{e^{-s}}{s}ds + \int_1^\infty \frac{e^{-s}}{s}ds\right) \\ &= e^{-x}\left(C_1 + \int_x^1 \frac{e^{-s}}{s}ds\right) = e^{-x}\left(C_1 + \int_x^1 \frac{e^{-s}-1}{s}ds - \ln x\right) \\ &= e^{-x}\left(\text{entire} - \ln x\right) \end{aligned} \tag{2.72}$$

The Stokes phenomenon however is *not* due to the multivaluedness of the function but to the divergence of the asymptotic series, as seen from the following simple remark.

Remark 2.73 *Assume f is analytic outside a compact set and is asymptotic to $\tilde{f}$ as $|x| \to \infty$ (in all directions). Then $\tilde{f}$ is convergent.*

Proof. By the change of variable $x = 1/z$ we move the analysis at zero. The existence of an asymptotic series as $z \to 0$ implies in particular that f is bounded at zero. Since it is analytic in $\mathbb{C} \setminus \{0\}$ then zero is a removable siungularity of f, and thus the asymptotic series, which as we know is unique, must coincide with the Taylor series of f at zero, a convergent series. $\Box$ The exercise below also shows that the Stokes phenomenon is not due to multivaluedness.

Exercise 2.74 ** (1) Show that the function $f(x) = \int_x^\infty e^{-s^2}ds$ is entire.*
(2) Note that

$$\int_x^\infty e^{-s^2}ds = \frac{1}{2}\int_{x^2}^\infty \frac{e^{-t}}{\sqrt{t}}dt = \frac{1}{2x}\int_1^\infty \frac{e^{-x^2u}}{\sqrt{u}}du = \frac{e^{-x^2}}{2x}\int_0^\infty \frac{e^{-x^2p}}{\sqrt{1+p}}dp \tag{2.75}$$

Do a similar analysis to the one in the text and identify the Stokes and antistokes lines for f. Note that the critical time now is x^2.

2.2 Gevrey classes, least term truncation, and Borel summation

A function is asymptotic to $\tilde{f}$ if the difference between it and the truncates of the series is smaller than any prescribed power of x^{-1}. Such information cannot see exponentially small corrections, which are natural to factorially divergent series. We can attempt to obtain more information from the series by optimizing the truncation index in a way which depends on x, to minimize the errors. The errors cannot be expected to be smaller than the smallest term of the series, but they can be of the same order of magnitude. If this is possible for a series in a wide enough sector, then, perhaps surprisingly, for that series there is a unique function with this property.

The formal series in the polynomial

$$\tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k}, \quad x \to \infty$$

is by definition Gevrey of order $1/m$, or Gevrey-$(1/m)$ if $|c_k| \le C_1 C_2^k (k!)^m$ for some C_1, C_2. Taking $x = y^m$ and $\tilde{g}(y) = \tilde{f}(x)$, then $\tilde{g}$ is Gevrey-1 (albeit not necessarily an integer power series, the generalization to noninteger power series is immediate) and we will focus on this case. Also, the corresponding classification for series in z, $z \to 0$ is obtained by taking $z = 1/x$. A very useful approach is due to *Gevrey* (see e.g. [18]).

Remarks 2.76 (a) The Gevrey order of the series $\sum_k (k!)^r x^{-k}$ $r > 0$, is the same as that of $\sum_k (rk)! x^{-k}$. Indeed, if $\epsilon > 0$ we have, by Stirling's formula,

$$Const\ (1+\epsilon)^{-k} \le (rk)!/(k!)^r \sim Const\ k^{\frac{1}{2}-r} \le Const\ (1+\epsilon)^k$$

(b) There is a simple connection between the Gevrey order of formal power series solutions of a differential equation at an irregular singular point and the type of exponentials of the associated homogeneous equation. For illustration consider the example of the equation $z^{q+1}y' - ay = 1$ in a neighborhood of zero, with $q \in \mathbb{N}$. The coefficients c_k of a formal power series solution $\tilde{y} = \sum_{k \ge 0} c_k z^k$ satisfy the recurrence $a_0 = 0$ and $(k-q)c_{k-q} + ac_k = 0$ if $k - q > 0$. If $q \ge 1$ we get $c_{jq+q} = a^j j!$, the series diverges and $x = 0$ is an irregular singularity. Using part (a) above we see that the series is Gevrey-q. On the other hand, the solution of the homogeneous equation $z^{q+1}y' - ay = 0$ is $C \exp\left(-\frac{a}{q} z^{-q}\right)$.

Exercise. Formulate and prove a more general result in the spirit of Remark 2.76 (b) for n-th order linear differential equations.

*

Let $\tilde{f}$ be Gevrey-1. A function f is *Gevrey*-1 *asymptotic* to $\tilde{f}$ as $x \to \infty$ in a sector S if for some C_3, C_4, C_5, and all $x \in S$ with $|x| > C_5$ and all N we have

$$|f(x) - \tilde{f}^{[N]}| \le C_1 C_2^{N+1} |x|^{-N-1} (N+1)! \tag{2.77}$$

i.e. the error $f - \tilde{f}^{[N]}$ is of the same form as the first omitted term in $\tilde{f}$.

Remark 2.78 *If $\tilde{f}$ is Gevrey-*1 *and f is Gevrey-*1 *asymptotic to $\tilde{f}$ then f can be approximated by $\tilde{f}$ with exponential precision in the following sense. Let $N = \lfloor |x/C_2| \rfloor$ ($\lfloor \cdot \rfloor$ is the integer part) then for any $C > C_2$ we have*

$$f(x) - \tilde{f}^{[N]}(x) = O(|x|^{-1/2} e^{-|x|/C})\ |x| \text{ large}) \tag{2.79}$$

Indeed, letting $|x| = NC_2 + \epsilon$ with $\epsilon \in ([0, 1)$ and applying Stirling's formula we have

$$N!(N+1)C_2^N|NC_2 + \epsilon|^{-N-1} = O(|x|^{1/2}e^{-|x|/C_2})$$

□

It is also interesting that when there is (a unique) f in $S_{\pi+}$ with the property (2.79), then $\tilde{f}$ is Borel summable, and f is *precisely the Borel sum of* $\tilde{f}$ (Theorem 2.80 below).[2]

(c) However the same theorem suggests that unless the series $\tilde{f}$ is trivial, there must exist *some* $S_{\pi+}$ in which *no* f is Gevrey−1-asymptotic to $\tilde{f}$ and where this method of associating an f to $\tilde{f}$ fails.

(d) *Summation to the least term* as will be detailed in the Chapter 4, is in a sense a refined version of Gevrey asymptotics. It requires *optimal constants* in addition to an improved form of Rel. (2.77). In this way the imprecision of approximation of f by $\tilde{f}$ turns out to be smaller than the largest exponentially small "possible" term beyond all orders, and thus the cases in which uniqueness is ensured are more numerous.

Connection between Gevrey asymptotics and Borel summation

Theorem 2.80 *Let* $\tilde{f} = \sum_{k=2}^{\infty} c_k x^{-k}$ *be a Gevrey-*1 *series and assume the function* f *is analytic for large* x *in* $S_{\pi+} = \{x : |\arg(x)| < \pi/2 + \delta\}$ *for some* $\delta > 0$ *and Gevrey-*1 *asymptotic to* $\tilde{f}$ *in* $S_{\pi+}$*. Then*

(i) f *is unique.*

(ii) $\tilde{f}$ *is Borel summable in any direction* $e^{i\theta}\mathbb{R}^+$ *with* $|\theta| < \delta$ *and* $f = \mathcal{L}\mathcal{B}_\theta \tilde{f}$.

(iii) $\mathcal{B}(\tilde{f})$ *is analytic (at* $p = 0$ *and) in the sector* $S_\delta = \{p : \arg(p) \in (-\delta, \delta)\}$, *and Laplace transformable in any closed subsector.*

(iv) Conversely, if $\tilde{f}$ *is Borel summable along any ray in the sector* $S_{\delta;\epsilon} = |\arg(p)| < \delta$ *or* $p : |p| < \epsilon\}$, *and if* $\mathcal{B}\tilde{f}$ *is uniformly bounded in any closed subsector of* $S_{\delta;\epsilon}$, *then* f *is Gevrey-1 with respect to its asymptotic series* $\tilde{f}$ *in the sector* $|\arg(x)| \le \pi/2 + \delta$..

Notes. (i) In particular, when the assumptions of the theorem are met, Borel summability follows using only *asymptotic estimates.*

(ii) We also see that the cases described in Theorem 2.80 in which Gevrey estimates ensure uniqueness of the association between $\tilde{f}$ and f are weaker than those in which $\tilde{f}$ is Borel summable, since Borel summability requires analyticity in some neighborhood of $\mathbb{R}^+$ and not in a sector.

Proof of Theorem 2.80. Let us note first a possible pitfall. Inverse Laplace transformability of f follows immediately from the assumptions. What doesn't follow is analyticity of the transform at zero. On the other hand, the formal inverse Laplace of $\tilde{f}$ trivially converges to an analytic function. But there is no guarantee that this analytic function has anything to do with the inverse Laplace transform of f! This is where Gevrey estimates enter.

---

[2]Borel summability is clearly not ensured by the Gevrey character of $\tilde{f}$ alone, since such estimates give no information about $\sum \mathcal{B}\tilde{f}$ beyond the implied disk of convergence.

(iii) By Proposition 2.17, the function $F = \mathcal{L}^{-1}f$ is analytic for $|p| > 0, |\arg(p)| < \delta$, and $F(p)$ is analytic and uniformly bounded if $|\arg(p)| < \delta_1 < \delta$.

(iv) Let $|\phi| < \delta$ and $a'' < a' < a < 1$ We have

$$
\begin{aligned}
|f(x) - \tilde{f}^{N+1}| = \Big| \int_0^{a'} (F(p) - F^N(p)) e^{-xp} dp \\
+ \int_{a'}^{\infty} (F(p) - F^{[N]}(p)) e^{-xp} dp \Big| \le C_1 (a')^{-N+1} \frac{N!}{|x|^{N+1}} + O(e^{-a'\Re(px)}) \\
\overset{W}{\le} C_2 (a'')^{-N+1} \frac{N!}{|x|^{N+1}} \quad (2.81)
\end{aligned}
$$

where ^{W} indicates usage of Watson's lemma.

(i)–(ii) By (iii) it remains to show that that F is analytic for amall $|p|$. Indeed, if the inverse Laplace transform of any such function is analytic in a neighborhood of $\mathbb{R}^+ \cup \{0\}$, this proves both existence and uniqueness, by Theorems 2.29 and 2.28, and uniqueness of the asymptotic series of a function. By a simple change of variables we arrange $C_1 = C_2 = 1$. We let $\tilde{f}^{[N]}$ denote the truncate of the power series $\tilde{f}$ to $o(x^{-N})$. Let x be real, $N = \lfloor x \rfloor$, and F_1 the convergent Borel transform of $\tilde{f}$, and $f_1 = \mathcal{L}(F_1 \chi[0, a''])$. As in (2.81) we have, for real x, using Remark 2.78

$$
|f(x) - f_1(x)| \le |\tilde{f}^{N+1}(x) - f(x)| + |\tilde{f}^{N+1}(x) - f_1(x)| \le C x^{1/2} e^{-|\frac{x}{a''}|}
$$

By Corollary 2.30 and Proposition 2.17, $F = F_1$ on $[0, a'']$ thus F is analytic on $[0, a'']$. □

Regularizing the heat equation

When the solutions of a problem are Borel summable, then factorial divergence is replaced by geometric divergence and it follows that the Borel transform of the problem is more regular, and thus the trnsformed problem must be more regular. Often, it is then convenient to Borel transform the equation even before finding formal solutions, solve the transformed problem and Laplace transform its solutions. This is particularly useful in PDEs, where finding solutions, and even more, a sufficient family of solutions, formal or not, may be quite challenging. We analyze now a rather trivial example, the heat equation.

$$
f_{xx} - f_t = 0 \qquad (2.82)
$$

Since (2.82) is parabolic, power series solutions

$$
f = \sum_{k=0}^{\infty} t^k F_k(x) = \sum_{k=0}^{\infty} \frac{F_0^{(2k)}}{k!} t^k \qquad (2.83)
$$

are divergent even if F_0 is analytic (but not entire). Nevertheless, under suitable assumptions, Borel summability results of such formal solutions have been shown by Lutz, Miyake, and Schäfke [14] and more general results of multi-summability of linear PDEs have been obtained by Balser [2].

Since the expansion is in one variable only, we may think of the other as being a parameter, so we are not departing too much from the theory that we developed.

The heat equation can be regularized by a suitable Borel summation. The divergence implied, under analyticity assumptions, by (2.83) is $F_k = O(k!)$ which indicates that the critical time is t^{-1}. Indeed, the substitution

$$t = 1/\tau; \quad f(t,x) = t^{-1/2} g(\tau, x) \tag{2.84}$$

yields

$$g_{xx} + \tau^2 g_\tau + \frac{1}{2}\tau g = 0$$

which becomes after formal inverse Laplace transform (Borel transform) in τ,

$$p\hat{g}_{pp} + \frac{3}{2}\hat{g}_p + \hat{g}_{xx} = 0 \tag{2.85}$$

which is brought, by the substitution $\hat{g}(p,x) = p^{-\frac{1}{2}} u(x, 2p^{\frac{1}{2}})$; $y = 2p^{\frac{1}{2}}$, to the wave equation, which is hyperbolic, thus *regular*

$$u_{xx} - u_{yy} = 0. \tag{2.86}$$

Existence and uniqueness of solutions to regular equations is guaranteed by Cauchy-Kowalevsky theory. For this simple equation the general solution is certainly available in explicit form: $u = f_1(x-y) + f_2(x+y)$ with f_1, f_2 arbitrary twice differentiable functions. Since the solution of (2.86) is related to a solution of (2.82) through (2.84), to ensure that we do get a solution it is easy to check that we need to choose $f_1 = f_2 =: u$ (up to an irrelevant additive constant which can be absorbed into u) which yields,

$$f(t,x) = t^{-\frac{1}{2}} \int_0^\infty y^{-\frac{1}{2}} \left[u\left(x + 2\,y^{\frac{1}{2}}\right) + u\left(x - 2\,y^{\frac{1}{2}}\right)\right] \exp\left(-\frac{y}{t}\right) dy \tag{2.87}$$

which, after splitting the integral and making the substitutions $x \pm 2\,y^{\frac{1}{2}} = s$ is transformed into the usual Heat kernel solution,

$$f(t,x) = t^{-\frac{1}{2}} \int_{-\infty}^{\infty} u(s) \exp\left(-\frac{(x-s)^2}{4t}\right) ds \tag{2.88}$$

*

Chapter 3

Borel summability of nonlinear systems of ODEs

3.1a Convolutions: elementary properties

The spaces below are well suited for the study of convolution algebras.

(1) Let $\nu \in \mathbb{R}^+$ and define $L^1_\nu := \{f : \mathbb{R}^+ : f(p)e^{-\nu p} \in L^1(\mathbb{R}^+)\}$; then the norm $\|f\|_\nu$ is defined as $\|f(p)e^{-\nu p}\|_1$ where $\|\cdot\|_1$ denotes the L^1 norm.

Proposition 3.1 *L^1_ν is a Banach algebra with respect to convolution and $\mathcal{L}$ is a Banach algebra isomorphism onto $\mathcal{L}(L^1_\nu)$ endowed with $(+,\cdot)$ and the induced toplogy.*

Note 3.2 *It follows in particular that $*$ is not distinguished, algebraically or topologically, from usual multiplication.*

Proof. Note first that if $f \in L^1_\nu$ then the Laplace transform of f exists for $\Re(x) \geq \nu$. If f and g are positive, then $\mathcal{L}(f * g) = \mathcal{L}f\mathcal{L}g$ (since convolution is the inverse Laplace image of multiplication or by direct calculation). In general, it is easy to check that $|f * g| \leq |f| * |g|$.

Commutativity, associativity, distributivity of $*$ follow from the fact that the kernel of $\mathcal{L}$ is $\{0\}$ and the fact that multiplication has all these properties. For instance,

$$\mathcal{L}(f * (g + h)) = \mathcal{L}f(\mathcal{L}g + \mathcal{L}h) = \mathcal{L}f\mathcal{L}g + \mathcal{L}f\mathcal{L}h = \mathcal{L}(f * g + f * h)$$

□

(2) A generalization is to allow p to be complex. We say that $f \in L^1_\nu(\mathbb{R}^+e^{i\phi})$ (along the ray $\{p = te^{i\phi} : t \in \mathbb{R}^+\}$) if $f(p) := f(te^{i\phi}) \in L^1_\nu$. Convolution is defined as

$$(f * g)(p) = \int_0^p f(s)g(p-s)ds = e^{i\phi}(f_\phi * g_\phi)(|p|e^{i\phi}) \tag{3.3}$$

and it is clear that $L^1_\nu(\mathbb{R}^+e^{i\phi})$ is also a Banach algebra with respect to convolution.

Similarly, we define $f \in L^1_\nu(S)$ for a domain $S \in \mathbb{C}$ with the norm

$$\|f\|^\nu_{\nu;S} := \sup_S \|f\chi(S)\|^1_\nu$$

$L^1_\nu(S)$ is also a Banach algebra.

Proposition 3.4 *The space of analytic functions in $f \in L^1_\nu(S)$ which are analytic in $A \subset S$ is a closed subspace of $L^1_\nu(S)$.*

Proof. Convergence of $\{f_n\}_{n\in\mathbb{N}}$ in the norm $\|\cdot\|^\nu_{\nu;S}$ entails uniform convergence of $\int_a^p f_n$ on compact sets, if $a, p \in S$, which by standard complex analytic arguments, entails the uniform convergence of f_n, and thus the limit is analytic. □

Spaces of sequences of functions

In Borel summing not simply series but transseries it is convenient to look at sequences of functions belonging to one ore more of the spaces introduced before. We let

$$L^1_{\nu\mu} = \{\mathbb{Y} \in (L^1_\nu)^{\mathbb{N}^d} : \sum_{\mathbf{k}\geq 0}^{\infty} \mu^{-|\mathbf{k}|}\|\mathbf{Y_k}\|_\nu < \infty\} \tag{3.5}$$

We introduce the following convolution on $L^1_{\nu\mu}$

$$(\mathbb{Y} \circledast \mathbb{G})_k = \sum_{j=1}^{n-1} F_j * G_{k-j} \tag{3.6}$$

Exercise 3.7 ** Show that*

$$\|\mathbb{F} \circledast \mathbb{G}\|_{\nu,\mu} \leq \|\mathbb{F}\|_{\nu,\mu}\|\mathbb{G}\|_{\nu,\mu} \tag{3.8}$$

$(L^1_{\nu,\mu}, +, \circledast, \|\,\|_{\nu,\mu})$ where $\|\,\|_{\nu,\mu}$ is the norm introduced in (3.5) is a Banach algebra.

Focal (contractive) spaces and algebras

An important property of the norms introduced, on the spaces L^1_ν and $\mathcal{A}_{K,\nu;0}$ is that for any f in these spaces $\|f\| \to 0$ as $\nu \to \infty$. In the case L^1_ν this is an immediate consequence of dominated convergence.

More generally, we say that a family of norms $\|\|_\nu$ depending on a parameter $\nu \in \mathbb{R}^+$ is **contractive** if for any f with $\|f\|_{\nu_0} < \infty$

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{3.9}$$

Let $\mathcal{E}$ be a linear space and $\{\|\|_\nu\}$ a family of norms satisfying (3.9). For each ν we define a Banach space $\mathcal{B}_\nu$ as the completion of $\{f \in \mathcal{E} : \|f\|_\nu < \infty\}$. Enlarging $\mathcal{E}$ if needed, we may assume that $\mathcal{B}_\nu \subset \mathcal{E}$. For $\alpha < \beta$, (3.9) shows that the identity is an embedding of $\mathcal{B}_\alpha$ in $\mathcal{B}_\beta$. Let $\mathcal{F} \subset \mathcal{E}$ be the inductive limit of the $\mathcal{B}_\nu$. That is to say

$$\mathcal{F} := \varinjlim \mathcal{B}_\nu \tag{3.10}$$

is endowed with the topology in which a sequence is convergent if it converges in *some* $\mathcal{B}_\nu$. We call $\mathcal{F}$ a **focal space**.

Consider now the case when $(\mathcal{B}_\nu, +, *, \|\|_\nu)$ are commutative Banach algebras. Then $\mathcal{F}$ inherits a structure of a commutative algebra, in which $*$ ("convolution") is continuous. We say that $(\mathcal{F}, *, \|\|_\nu)$ is a **focal algebra**.

Examples.

$$L^1_{\underset{\to}{\nu}} := \varinjlim_{\nu>0} L^1_\nu; \quad L^1_{\underset{\to}{\mu\nu}} := \varinjlim_{\mu,\nu>0} L^1_{\nu\mu}; \tag{3.11}$$

The last space is focal as $\nu \to \infty$ and/or $\mu \to \infty$.

Remark 3.12 *The following result is immediate. Let A, B be any sets and assume that the equation $f(x) = 0$ is well defined and has a unique solution x_1 in A, a unique solution x_2 in B and a unique solution x_3 in $A \cap B$. Then $x_1 = x_2 = x_3 = x$. In particular, if $A \subset B$ then $x \in A \cap B$.*

3.1b Convolution calculus

To simplify the notation, from now on, unless otherwise indicated, a function or series in the original space is denoted with a small letter; when capitalized, it denotes its inverse Laplace or Borel transform.

We have seen that there is an isomorphism between the Banach algebra with convolution L^1_ν and a Banach algebra of analytic functions with usual multiplication. The isomorphism can be pushed further. Let

$$g(\mathbf{z}) = \sum_{k=0}^{\infty} g_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} \tag{3.13}$$

be an analytic function in a polydisk of radia r near the origin. We endow this space with the norm $\sum_{\mathbf{k}} |g_{\mathbf{k}}| r^{-\mathbf{k}}$. This is a slightly stronger norm than the usual sup norm, it defines a Banach space (since it is a weighted L^1) and it contains as a closed subspace analytic functions continuous to the boundary. Let $\mathbf{Y} \in L^1_{\underset{\to}{\nu}}$. Then

$$g(*\mathbf{Y}) = \sum_{k=0}^{\infty} g_{\mathbf{k}} \mathbf{Y}^{*\mathbf{k}} \tag{3.14}$$

is well defined in $L^1_{\underset{\to}{\nu}}$ since it converges in all L^1_ν if ν is large enough. It is $*$*analytic function* of $\mathbf{Y}$.

More generally, if we have $g(1/x, \mathbf{z})$ an analytic function of $1/x$ and $\mathbf{z}$ we write, by Taylor expanding in $1/x$,

$$g(1/x, \mathbf{z}) = g_0(\mathbf{z}) + x^{-1} g_1(\mathbf{z}) + g_2(x^{-1}, \mathbf{z})$$

where $g_2 = O(x^{-2})$; in this case we write

$$G(p, *Y) = \sum_{\mathbf{k}\geq 0} g_{\mathbf{k}} Y^{*\mathbf{k}} + \sum_{\mathbf{k}\geq 0} g_{\mathbf{k}} Y^{*\mathbf{k}} * 1 + G_2(p)(*Y) \tag{3.15}$$

where the last term is defined as in (3.14).

We have

$$g(p, *(\mathbf{Y} + d\mathbf{Y})) = g(p, *\mathbf{Y}) + \boldsymbol{\nabla}_{\mathbf{z}} g(p, *\mathbf{Y}) \ * d\mathbf{Y} + o(d\mathbf{Y}) \tag{3.16}$$

where $\boldsymbol{\nabla}$ is the usual gradient.

Lemma 3.17 *Let $\mathbf{Y} \in L^1_{\underset{\to}{\nu}}$ and $g = O(\mathbf{Y}^{*2})$ be $*$-analytic. Then $\boldsymbol{\nabla}_{\mathbf{z}} g(p, *\mathbf{Y}) = O(\mathbf{Y}) \to 0$ as $\nu \to \infty$.*

Proof. Straightforward. □

Convergent series composed with Borel summable series

Proposition 3.18 *Assume A is an analytic function in the disk of radius ρ centered at the origin, $a_k = A^{(k)}(0)/k!$, and $\tilde{s} = \sum s_k x^{-k}$ is a small series which is Borel summable along $\mathbb{R}^+$. Then the formal power series obtained by reexpanding*

$$\sum a_k s^k$$

in powers of x is Borel summable along $\mathbb{R}^+$.

Proof. Let $S = \mathcal{B}s$ and choose ν be large enough so that $\|S\|_\nu < \rho^{-1}$ in L^1_ν. Then

$$\|F\|_\nu := \|A(*S)\|_\nu := \|\sum_{k=0}^{\infty} a_k S^{*k}\|_\nu \leq \sum_{k=0}^{\infty} a_k \|S\|_\nu^k \leq \sum_{k=0}^{\infty} a_k \rho^k < \infty \tag{3.19}$$

thus $A(*S) \in L^1_\nu$. Similarly, $A(*S)$ is in $L^1_\nu([0, a))$, in $\mathcal{A}_{K,\nu}\nu([0, a))$ for any a. □

3.1c Analytic systems of ODEs at rank one singularities

Consider the differential system

$$\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n \tag{3.20}$$

We look at solutions $\mathbf{y}$ such that $\mathbf{y}(x) \to 0$ as $x \to \infty$ along some direction $d = \{x \in \mathbb{C} : \arg(x) = \phi\}$. The following conditions are assumed

(a1) The function $\mathbf{f}$ is analytic at $(0,0)$.
(a2) Nonresonance: the eigenvalues λ_i of the linearization

$$\hat{\Lambda} := -\left(\frac{\partial f_i}{\partial y_j}(0,0)\right)_{i,j=1,2,\ldots n} \tag{3.21}$$

are linearly independent over $\mathbb{Z}$ (in particular nonzero) and such that $\arg\lambda_i$ are different from each other (i.e., the Stokes lines are distinct; we will require somewhat less restrictive conditions, see § 3.1).

Many systems can be brought to this form by chamges of variables, or to a ramified form tractable by the same approach. The cases when this is not possible, notably those in which the linearized system has a resonant matrix, require other tools, for instance Écalle multisummation.

By elementary changes of variables, the system (3.20) can be brought to the *normalized form* [8],

$$\mathbf{y}' = -\hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x^{-1},\mathbf{y}) \tag{3.22}$$

where $\hat{\Lambda} = \operatorname{diag}\{\lambda_j\}$, $\hat{A} = \operatorname{diag}\{\alpha_j\}$ are constant matrices, $\mathbf{g}$ is analytic at $(0,\mathbf{0})$ and $\mathbf{g}(x^{-1},\mathbf{y}) = O(x^{-2}) + O(|\mathbf{y}|^2)$ as $x \to \infty$ and $\mathbf{y} \to 0$. Performing a further transformation of the type $\mathbf{y} \mapsto \mathbf{y} - \sum_{k=1}^{M} \mathbf{a}_k x^{-k}$ (which takes out M terms of the formal asymptotic series solutions of the equation), makes

$$\mathbf{g}(|x|^{-1},\mathbf{y}) = O(x^{-M-1}; |\mathbf{y}|^2; |x^{-2}\mathbf{y}|) \quad (x \to \infty;\ \mathbf{y} \to 0) \tag{3.23}$$

where

$$M \geq \max_j \Re(\alpha_j)$$

and $O(a;b;c)$ means (at most) of the order of the largest among a,b,c.

Our analysis applies to solutions $\mathbf{y}(x)$ such that $\mathbf{y}(x) \to 0$ as $x \to \infty$ along some arbitrarily chosen direction $d = \{x \in \mathbb{C} : \arg(x) = \phi\}$. We shall exemplify some of these transformations in the sequel.

Given a direction d in the complex x-plane the *transseries* (on d), are, in our context, those exponential series (3.25) which are formally *asymptotic* on d, i.e. the terms $\mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} x^{-r}$ (with $\mathbf{k} \in (\mathbb{N}\cup\{0\})^n$, $r \in \mathbb{N}\cup\{0\}$) form a well ordered set with respect to $\gg$ on d (see also [8]). In other words, indices i for which the corresponding term $e^{-\lambda_i x}$ is not formally small in d do **not** appear, that is, they must be associated with $C_i = 0$.

Exercise 3.24 Justify the following description (most steps require no more than straightforward calculation). Let d be a ray in $\mathbb{C}$. There is $m \leq n$-parameter of level one transseries solutions (1.91) (under the assumptions mentioned)

$$\tilde{\mathbf{y}}(x) = \sum_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \tilde{\mathbf{y}}_{\mathbf{k}}(x) \tag{3.25}$$

where indices i for which the corresponding term $e^{-\lambda_i x}$ is not formally small in d must be absent, that is, they must be associated with $C_i = 0$. There is no other formal transseries solution, of any level.

3.1d How do we normalize a system?

A first order nonintegrable Abel equation

Consider the equation

$$u' = u^3 - z \tag{3.26}$$

Formal solutions provide a good guide in finding the normalization transformations. A transformation bringing the equation to its normal form also brings its transseries solutions to the form (3.25). It is simpler to look for substitutions with this latter property, and then the first step is to find the transseries solutions of (3.26).

Power series solutions. Since at this stage we are merely looking for useful transformation hints, rigor is naturally not required. Substituting of $u \sim Az^p$ in (3.26) and looking for maximal balance [3] give $p = 1/3$, $A^3 = 1$. Then $u \sim Az^{1/3} + Bz^q$ with $q < 1/3$ determines $B = \frac{1}{9}A^2$, $q = -4/3$. Inductively, one obtains a power series formal solution $\tilde{u}_0 = Az^{1/3}(1 + \sum_{k=1}^{\infty} \tilde{u}_{0,k} z^{-5k/3})$.

General transseries solutions of (3.26). In order to determine the form of the exponentials in the transseries of u, the method is to look for transcendentally small corrections beyond $\tilde{u}_0$, by linear perturbation theory. Substituting $u = \tilde{u}_0 + \delta$ in (3.26) yields to leading order in δ, the equation

$$\delta' = \left(3A^2 z^{2/3} + \frac{2}{3z}\right)\delta \tag{3.27}$$

whence $\delta \propto z^{2/3} \exp\left(\frac{9}{5}A^2 z^{5/3}\right)$. In (3.25) the exponentials have *linear* exponent, with negative real part. The independent variable should thus be $x = -(9/5)A^2 z^{5/3}$ and $\Re(x) > 0$. Then $\tilde{u}_0 = x^{1/5}\sum_{k=0}^{\infty} u_{0;k} x^{-k}$, which suggests the change of dependent variable $u(z) = Kx^{1/5}h(x)$. Choosing for convenience $K = A^{3/5}(-135)^{1/5}$ yields

$$h' + \frac{1}{5x}h + 3h^3 - \frac{1}{9} = 0 \tag{3.28}$$

The next step is to achieve leading behavior $O(x^{-2})$. This is easily done by subtracting out the leading behavior of h (which can be found by maximal balance, as above). With $h = y + 1/3 - x^{-1}/15$ we get the normal form

$$y' = -y + \frac{1}{5x}\, y + g(x^{-1}, y) \tag{3.29}$$

where

$$g(x^{-1}, y) = -3(y^2 + y^3) + \frac{3y^2}{5x} - \frac{1}{15x^2} - \frac{y}{25x^2} + \frac{1}{3^2 5^3 x^3} \tag{3.30}$$

The Painlevé equation $\mathrm{P_I}$

The Painlevé functions were studied asymptotically in terms of doubly periodic functions by Boutroux (see, for example, [11]). Solutions of the $\mathrm{P_I}$ equation turn out to have arrays of poles and they can be asymptotically represented by elliptic functions whose parameters change with the direction in the complex plane. We consider solutions of the Painlevé $\mathrm{P_I}$ equation (in the form of [6], which by rescaling gives the form in [11])

$$\frac{d^2y}{dz^2} = 6y^2 + z \tag{3.31}$$

in a region centered on a Stokes line, say $d = \{z : \arg z = \pi\}$.

To bring (3.31) to a normal form the transformations are suggested by the general methodology explained in §3.1d . There is a one parameter family of solutions for each of the behaviors $y \sim \pm\sqrt{\frac{-z}{6}}$ for large z along d. We will study the family with $y \sim +\sqrt{\frac{-z}{6}}$, since the other can be treated similarly. Its transseries can be obtained as in the previous example, namely determining first the asymptotic series $\tilde{y}_0$, then by linear perturbation theory around it one finds the form of the small exponential, and notices the exponential is determined up to one multiplicative parameter. We get the transseries solution

$$\tilde{y} = \sqrt{\frac{-z}{6}} \sum_{k=0}^{\infty} \xi^k \tilde{y}_k \tag{3.32}$$

where

$$\xi = \xi(z) = Cx^{-1/2}\mathrm{e}^{-x}; \quad \text{with} \quad x = x(z) = \frac{(-24z)^{5/4}}{30} \tag{3.33}$$

and $\tilde{y}_k$ are power series, in particular

$$\tilde{y}_0 = 1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} - \frac{7^2}{2^8 \cdot 3}\frac{1}{z^5} - ... - \frac{\tilde{y}_{0;k}}{(-z)^{5k/2}} - ...$$

We note that in the sector $|\arg(z) - \pi| < \frac{2}{5}\pi$ the constant C of a particular solution y (see (3.123)) changes only once, on the Stokes line $\arg(z) = \pi$ [8].

As in Example 1, the form of the transseries solution (3.32), (3.33) suggests the transformation

$$x = \frac{(-24z)^{5/4}}{30}; \ y(z) = \sqrt{\frac{-z}{6}}\, Y(x)$$

which, in fact, coincides with Boutroux's (cf. [11]); P_I becomes

$$Y''(x) - \frac{1}{2}Y^2(x) + \frac{1}{2} = -\frac{1}{x}Y'(x) + \frac{4}{25}\frac{1}{x^2}Y(x) \tag{3.34}$$

To fully normalize the equation, we subtract the $O(1)$ and $O(x^{-1})$ terms of the asymptotic behavior of $Y(x)$ for large x. It is convenient to subtract also the $O(x^{-2})$ term (since the resulting equation becomes simpler). Then the substitution

$$Y(x) = 1 - \frac{4}{25x^2} + h(x)$$

transforms P_I to

$$h'' + \frac{1}{x}h' - h - \frac{1}{2}h^2 - \frac{392}{625x^4} = 0 \tag{3.35}$$

Written as a system, with $\mathbf{y} = (h, h')$ this equation is now in normal form.

3.1e Summary of summability results

The results proven for this type of equations may be, informally, summarized as follows.

Proposition 3.36 All $\tilde{\mathbf{y}}_{\mathbf{k}}$ are generalized Borel summable at the same time.

i) The Borel sum $\mathcal{LB}\tilde{\mathbf{y}}_{\mathbf{k}} = \mathbf{y}_{\mathbf{k}}$[1] exist if $\Re x > |x_0|$ and x is outside a compact set in polar complements of any sector [2] contained outside a neighborhood of two successive Stokes lines in p plane. The minimal $x_0 = \Re(x)$ does not depend on $\mathbf{k}$. Furthermore,

$$\mathbf{y}_{\mathbf{k}}(x) \sim \tilde{\mathbf{y}}_{\mathbf{k}}(x) \tag{3.37}$$

ii) There exists a constant $\mathbf{c}$ independent of $\mathbf{k}$ so that $\sup_{x \in H} |\mathbf{y}_{\mathbf{k}}| \leq \mathbf{c}^{\mathbf{k}}$. Thus, the new series,

$$\mathbf{y} = \sum_{\mathbf{k} \in (\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathbf{y}_{\mathbf{k}}(x) \equiv \mathcal{LB} \sum_{\mathbf{k} \in (\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \tilde{\mathbf{y}}_{\mathbf{k}}(x) \tag{3.38}$$

is convergent for any $\mathbf{C}$ for which the corresponding expansion (3.25) is a transseries, in a region given by the condition $|C_i e^{-\lambda_i x} x^{\alpha_i}| < c_i^{-1}$ (remember that C_i is zero if $|e^{-\lambda_i x}|$ is not small).

[1]Here and in the following we allow $\tilde{\mathbf{y}_{\mathbf{k}}}$ to start with a constant, understanding summability in the more general sense as in (2.49).

[2]The polar is the larger sector obtained by drawing perpendiculars to the edges of a given sector.

iii) The function $\mathbf{y}$ obtained in this way is a solution of the differential equation (3.20).

iv) Any solution of the differential equation (3.20) which tends to zero in some direction d can be written in the form (3.38) for a unique $\mathbf{C}$, this constant depending usually on the sector where d is. This dependence is a manifestation of the Stokes phenomenon. Two different representations may yield the same solution, and this happens iff the $\mathcal{LB}$ summation direction for the two reprentations are on different sides of a Stokes line.

v) The Borel summation operator $\mathcal{LB}$ is the usual Borel summation in any direction d of x which is not a Stokes direction. However $\mathcal{LB}$ is still an isomorphism, whether d is a Stokes direction or not.

Characterization of solutions in the "x" plane

Proposition 3.39 *i) Let $\mathbf{y}$ be a solution of (3.20) which goes to zero as $x \to \infty$ in some direction d. Then $\mathbf{y} \sim \tilde{\mathbf{y}}_0$.*

(ii) Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be solutions of (3.20) so that $\mathbf{y}_{1,2} \sim \tilde{\mathbf{y}}_0$ for large x in some direction d; then

$$\mathbf{y}_1 - \mathbf{y}_2 = \sum_j C_j \mathrm{e}^{-\lambda_{i_j} x} x^{-\beta_{i_j}} (\mathbf{e}_{i_j} + o(1)) \tag{3.40}$$

for some constants C_j, where the indices run over the eigenvalues λ_{i_j} with the property $\Re(\lambda_{i_j} x) > 0$ in S (or d).

(iii) If $\mathbf{y}_1 - \mathbf{y}_2 = o(\mathrm{e}^{-\lambda_{i_j} x} x^{-\beta_{i_j}})$ for all j, then $\mathbf{y}_1 = \mathbf{y}_2$.

iv) Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be solutions of (3.20) and assume that $\mathbf{y}_1 - \mathbf{y}_2$ has differentiable asymptotics of the form $\mathbf{K} a \exp(-ax) x^b (1 + o(1))$ with $\Re(ax) > 0$ and $\mathbf{K} \neq 0$, for large x. Then $a = \lambda_i$ for some i.

v) Let $\mathbf{U}_\mathbf{k} \in \mathcal{T}_{\{\cdot\}}$ for all $\mathbf{k}$, $|\mathbf{k}| > 1$. Assume in addition that for large ν there is a function $\delta(\nu)$ vanishing as $\nu \to \infty$ such that

$$\sup_{\mathbf{k}} \delta^{-|\mathbf{k}|} \int_d \left| \mathbf{U}_\mathbf{k}(p) \mathrm{e}^{-\nu p} \right| \mathrm{d}|p| < K < \infty \tag{3.41}$$

Then, if $\mathbf{y}_1, \mathbf{y}_2$ are solutions of (3.20) in S where in addition

$$\mathbf{y}_1 - \mathbf{y}_2 = \sum_{|\mathbf{k}|>1} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}+1} \int_d \mathbf{U}_\mathbf{k}(p) \exp(-xp) \mathrm{d}p \tag{3.42}$$

[3] *where $\boldsymbol{\lambda}, x$ are as in (c1), then $\mathbf{y}_1 = \mathbf{y}_2$, and $\mathbf{U}_\mathbf{k} = 0$ for all $\mathbf{k}$, $|\mathbf{k}| > 1$.*

[3] We have added 1 to the power of x since by Watson's lemma, the Laplace transform of an analytic function if $O(1/x)$. See the footnote in Proposition 3.36, 1.

Corollary 3.43 *Any solution that goes to zero in a direction d can be written uniquely in the form (3.38) if the direction of summation is* **the same***.*

Proof. (of the corollary). If we take any solution $\mathbf{y}_s$ which vanishes as $x \to \infty$, and a solution $\mathbf{y}_{s1}$ of the form (3.38) then their difference is necessarily of the form (3.40). But it is easy to check that for any right-hand side of the form (3.40) we can find yet another solution $\mathbf{y}_{s2}$ of the form (3.38) for which the difference is also given by (3.40). Then, by Proposition 3.39 (iii), we have $\mathbf{y}_s = \mathbf{y}_{s2}$.
□

Proof. (of the proposition) (i), (ii) and (iii) are classical results (see [7] for the general treatment and [23] for a brief presentation of special cases and further references). However, what is actually needed for our purposes can be reduced to the more familiar *linear* asymptotic theory in the following way. Let d be a direction in the complex plane.

Let $\mathbf{y}_s \to 0$. Then

$$\mathbf{y}_s' = \mathbf{v}(x) + \hat{\Lambda}\mathbf{y}_s + \frac{1}{x}\hat{A}\mathbf{y}_s + \frac{1}{x^2}\mathbf{g}_1(x^{-1}, \mathbf{y}_s) + \mathbf{y}_s\mathbf{g}_2(x^{-1}, \mathbf{y}_s) \qquad (3.44)$$

where $\mathbf{g}_1$ is bounded and $\mathbf{g}_2 \to 0$ as $x \to \infty, \mathbf{y} \to 0$ is also a small solution of the equation

$$\mathbf{y}' = \mathbf{v}(x) + \hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \frac{1}{x^2}\mathbf{g}_1(x^{-1}, \mathbf{y}_s) + \mathbf{y}\mathbf{g}_2(x^{-1}, \mathbf{y}_s)$$
$$= \mathbf{v}(x) + \hat{\Lambda}\mathbf{y} + \mathbf{g}_3(x)\mathbf{y} \qquad (3.45)$$

where $\mathbf{g}_3 \to 0$ as $x \to \infty$. Of course, this new equation may have unintended solutions, but our solution is *one of them*. Everything now follows from linear asymptotic theory.

The proof of (ii) is very similar. Let $\mathbf{y}_0$, $\mathbf{y}_1$ be solutions of (3.20) such that $\mathbf{y}_{0,1} \sim \tilde{\mathbf{y}}_0$ for large x along d. Then, by (3.23), $\mathbf{y}_{0,1}(x) = O(x^{-M})$ and for any j, $\mathbf{g}^{(\mathbf{e}_j)}(x, \mathbf{y}_{0,1}(x)) = O(x^{-M})$. If $\boldsymbol{\delta} = \mathbf{y}_1 - \mathbf{y}_0$ then by hypothesis $\boldsymbol{\delta}(x) = o(x^{-l})$ along d, for all l. The function $\boldsymbol{\delta}$ is locally analytic and satisfies the equation

$$\boldsymbol{\delta}' = -\hat{\Lambda}\boldsymbol{\delta} - \frac{1}{x}\hat{B}\boldsymbol{\delta} + \sum_{|\mathbf{k}|=1}\mathbf{g}^{(\mathbf{k})}(x, \mathbf{y}_0)\boldsymbol{\delta}^{\mathbf{k}} + \sum_{|\mathbf{k}|>1}\mathbf{g}^{(\mathbf{k})}(x, \mathbf{y}_0)\boldsymbol{\delta}^{\mathbf{k}} =$$
$$-\hat{\Lambda}\boldsymbol{\delta} - \frac{1}{x}\hat{B}\boldsymbol{\delta} + \frac{1}{x^M}\sum_{j=1}^{n}(\boldsymbol{\delta})_j\mathbf{h}_{\mathbf{e}_j}(x) \qquad (3.46)$$

where $\mathbf{h}_{\mathbf{k}}(x)$ are bounded along d. Obviously, because of the link between $\boldsymbol{\delta}$ and $\mathbf{h}_{\mathbf{k}}$, the $\boldsymbol{\delta}$ we started with might be the only solution of (3.46) which is also a difference of solutions of (3.20). The asymptotic characterization we need holds nevertheless for *all* decaying solutions solutions of (3.46): since no two eigenvalues are equal, there exists by the well-known linear asymptotic theory [23] a fundamental set $\{\boldsymbol{\delta}_i\}_{1\le i\le n}$ of solutions of (3.46) such that $\boldsymbol{\delta}_i \sim e^{-\lambda_i x}x^{-\beta_i}(\mathbf{e}_i + o(1))$. Thus $\boldsymbol{\delta} = \sum_{i=1}^n C_i\boldsymbol{\delta}_i = \sum_{i=1}^n C_i e^{-\lambda_i x}x^{-\beta_i}(\mathbf{e}_i + o(1))$. Since $\Re(-\beta_i) > 0$ and the λ_i are distinct we must have $C_i = 0$ for all i for which $\Re(-\lambda_i x) \ge 0$, otherwise $|\boldsymbol{\delta}(x)|$ would be unbounded for large x; the first part of (i) is proven. If on the other hand $\boldsymbol{\delta} = o(e^{-\lambda_{i_j}x}x^{-\beta_{i_j}})$ for all j, again because the λ_i are independent, it follows that $C_i = 0$ for all $i = 1, 2, \dots, n$, thus $\boldsymbol{\delta} = 0$.

(iv) is now obvious.

For (v), note first that by (3.41) and (c1) the right-hand side of (3.42) converges uniformly for large x in some open sector. In addition, by an arbitrarily small change in $\xi = \arg(x)$, we can make the set $\{\Re(x\lambda_i)\}_i$ $\mathbb{Z}$-independent (the existence of $\mathbf{k}(\xi) \ne 0$ s.t. $\Re(e^{i\xi}\mathbf{k}\cdot\boldsymbol{\lambda}) = 0$ for ξ in an interval of would imply the existence of a *common* $\mathbf{k}$ for a set of ξ with an accumulation point, giving $\mathbf{k}\boldsymbol{\lambda} = 0$). We choose such a ξ. Assume now there exist $\mathbf{k}$ so that $\mathbf{U}_\mathbf{k} \ne 0$; among them let $\mathbf{k}_0$ have the least $\Re(x\mathbf{k}\cdot\boldsymbol{\lambda})$. By (3.41) for large x, $\mathbf{y}_1 - \mathbf{y}_2 \sim e^{-\boldsymbol{\lambda}\cdot\mathbf{k}_0 x}x^{\boldsymbol{\alpha}\cdot\mathbf{k}_0}\mathcal{L}_\phi\mathbf{U}_{\mathbf{k}_0}(1 + o(1))$. Because $\mathbf{U}_{\mathbf{k}_0} \in \mathcal{T}_{\{\cdot\}}$, and by (3.41), $\mathcal{L}_\phi\mathbf{U}_{\mathbf{k}_0}$ has a differentiable power series asymptotics which is the term-by-term Laplace transform of the Puiseux series at the origin of $\mathbf{U}_{\mathbf{k}_0}$, and thus nonzero. This contradicts (i) because with $|\mathbf{k}_0| > 1$ we have $\boldsymbol{\lambda}\cdot\mathbf{k}_0 \ne \lambda_j$ for all j ($\mathbb{Z}-$independence). Thus $\mathbf{U}_\mathbf{k} = 0$ for all $\mathbf{k}$.

□

Borel plane equations

We write $|\mathbf{f}| := \max_i\{|f_i|\}$.

The logic of the approach is approximately as follows. We write the formal inverse Laplace transform of the equation for $\mathbf{y}_0$ and of the equations of the $\mathbf{y}_\mathbf{k}$ (formal in the sense that no assumptions are yet made on t the transform to be justified). We then solve the transformed equations, and show that these "p" plane solutions are Laplace transformable to actual solutions. We then use Proposition 3.39 to show that these are in fact all solutions. We then go on to explore the analytic structure in p space.

The inverse Laplace transform of (3.20) is the convolution equation:

$$-p\mathbf{Y} = \mathbf{F}_0 - \hat{\Lambda}\mathbf{Y} + \hat{A}(1 * \mathbf{Y}) + \mathbf{G}(p, *\mathbf{Y}) \tag{3.47}$$

Let $\mathbf{d}_\mathbf{j} := \sum_{\mathbf{l}\ge\mathbf{j}}\binom{\mathbf{l}}{\mathbf{j}}\mathbf{g}_\mathbf{l}(x)\mathbf{y}_0^{\mathbf{l}-\mathbf{j}}$. Straightforward calculation shows that the components $\tilde{\mathbf{y}}_\mathbf{k}$ of the transseries satisfy the hierarchy of differential equations

$$\mathbf{y}_\mathbf{k}' + \left(\hat{\Lambda} + \frac{1}{x}\left(-\hat{A} + \mathbf{k}\cdot\boldsymbol{\alpha}\right) - \mathbf{k}\cdot\boldsymbol{\lambda}\right)\mathbf{y}_k + \sum_{|\mathbf{j}|=1}\mathbf{d}_\mathbf{j}(\mathbf{y}_\mathbf{k})^\mathbf{j} = \mathbf{t}_{\mathbf{k}^-} \tag{3.48}$$

where $\mathbf{t}_{\mathbf{k}^-} = \mathbf{t}_{\mathbf{k}^-}\left(\mathbf{y}_0, \{\mathbf{y}_{\mathbf{k}'}\}_{0<\mathbf{k}'<\mathbf{k}}\right)$ is a *polynomial* in $\{\mathbf{y}_{\mathbf{k}'}\}_{0<\mathbf{k}'<\mathbf{k}}$ and in $\{\mathbf{d}_\mathbf{j}\}_{\mathbf{j}\le\mathbf{k}}$ (compare with (3.140)), with $\mathbf{t}(\mathbf{y}_0, \emptyset) = 0$; $\mathbf{t}_{\mathbf{k}^-}$ satisfies the homogeneity relation

$$\mathbf{t}_{\mathbf{k}^-}\left(\mathbf{y}_0, \left\{C^{\mathbf{k}'}\mathbf{y}_{\mathbf{k}'}\right\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right) = C^{\mathbf{k}}\mathbf{t}_{\mathbf{k}^-}\left(\mathbf{y}_0, \{\mathbf{y}_{\mathbf{k}'}\}_{0\prec\mathbf{k}'\prec\mathbf{k}}\right) \tag{3.49}$$

It is convenient to note the following. Consider the generators $\mu_j = x^{a_j}e^{-x\lambda_j}$.As in 20 Page 4 we write $\mu_{\mathbf{k}} = x^{\mathbf{k}\cdot\mathbf{a}}e^{-\mathbf{k}\boldsymbol{\lambda}}$. We let $\boldsymbol{\mu}^{\mathbf{k}} = \mathbf{C}^{\mathbf{k}}\mu_{\mathbf{k}}$. We see that

$$\mathbf{g}\Big(x^{-1}, \sum_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}}\mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x}x^{\boldsymbol{\alpha}\cdot\mathbf{k}}\mathbf{y}_{\mathbf{k}}(x)\Big) = \mathbf{g}\Big(x^{-1}, \mathbf{y}_0 + \mathbf{y}_c(x^{-1}, \boldsymbol{\mu})\Big) \tag{3.50}$$

where

$$\mathbf{y}_c(x^{-1}, \boldsymbol{\mu}) = \sum_{\mathbf{k}>0}\mathbf{y}_{\mathbf{k}}(x)\boldsymbol{\mu}^{\mathbf{k}} \tag{3.51}$$

Then, noting that

$$\frac{d\boldsymbol{\mu}^{\mathbf{k}}}{dx} = -\left(\mathbf{k}\cdot\boldsymbol{\lambda} + \frac{1}{x}\mathbf{k}\cdot\mathbf{a}\right)\boldsymbol{\mu}^{\mathbf{k}} \tag{3.52}$$

and expanding in powers of $\boldsymbol{\mu}$, treated as independent (since the monomials are additively independent) we see that for the the coefficient of $\boldsymbol{\mu}^{\mathbf{k}}$ to be zero we must have, using $\hat{\mathbf{D}}^{\mathbf{k}}_{\boldsymbol{\mu}}$ as in the usual multinomial expansion in $\boldsymbol{\mu}$,

$$\mathbf{y}_{\mathbf{k}}' + \left(-\mathbf{k}\cdot\boldsymbol{\lambda} + \frac{1}{x}\mathbf{k}\cdot\mathbf{a} + \hat{\Lambda} - \frac{1}{x}\hat{A} - \nabla_{\mathbf{y}_0}\mathbf{g}(x^{-1}, \mathbf{y}_0)\right)\mathbf{y}_{\mathbf{k}}$$
$$= \hat{\mathbf{D}}^{\mathbf{k}}_{\boldsymbol{\mu}}\mathbf{g}\Big(x^{-1}, \mathbf{y}_0 + \mathbf{y}_c(x^{-1}, \boldsymbol{\mu})\Big)\Big|_{x^{-1},\mathbf{y}_0,\boldsymbol{\mu}=\mathbf{1}} = \mathbf{T}_{\mathbf{k}^-}(*\mathbf{Y}) \tag{3.53}$$

We also note that $\mathbf{T}_{\mathbf{k}}$ is a convolution polynomial in $\mathbf{Y} = \mathbf{y}_{\mathbf{m}}$, ${}^-\mathbf{k}$ indicates that all $\mathbf{m} < \mathbf{k}$. This polynomial is $\mathbf{k}$-homogeneous, that is, replacing $\mathbf{y}_{\mathbf{j}}$ by $\mathbf{c}\mathbf{y}_{\mathbf{j}}$, the new polynomial gets simply multiplied by $\mathbf{c}\mathbf{y}_{\mathbf{k}}$ (since this amounts to replacing $\boldsymbol{\mu}$ by $\mathbf{c}\boldsymbol{\mu}$).

For a more explicit form of these equations, see Appendix 3.2e We also note an important feature: all equations, except the one for $\mathbf{y}_0$, which will be treated in §3.1f , are *linear*.

3.1f Borel summation of $\tilde{\mathbf{y}}_0$ away from Stokes directions

Stokes directions are the p directions along which $\hat{\Lambda} - p$ is not invertible. There are of course n of them, the directions of the eigenvalues of $\hat{\Lambda}$. We write (3.47) in the form

Proposition 3.54 *The Borel transform $\mathbf{Y}_0$ is analytic and belongs to L^1_ν for large enough ν in any connected region which contains a neighborhood of the origin and is at a distance $\epsilon > 0$ from the rays originating at the λ_j, in the direction of λ_j. The value of ν may depend on ϵ.*

Proof. We have

$$\mathbf{Y_0} = (p - \hat{\Lambda})^{-1}\Big(\mathbf{F}_0 - \hat{\Lambda}\mathbf{Y_0} - \hat{B}\mathbf{Y}_0 * 1 + \mathbf{G}(p, *\mathbf{Y}_0)\Big) \tag{3.55}$$

By Lemma 3.17, Eq. (3.55) is contractive in $L^1_{\underset{\rightarrow}{\nu}}$ and thus it has a unique solution there. It is also contractive, for the same reason, in any $L^1_{\underset{\rightarrow}{\nu}}(S)$ if S contains no eigenvalue of $\hat{\Lambda}$, ensuring analyticity of $\mathbf{Y}_0$ in $\mathbb{C} \setminus \{\lambda_i t : t \in \mathbb{R}^+\}$ where we made (arbitrary) cuts at the eigenvalues.

□

3.1g The equations with $|\mathbf{k}| = 1$

The equations with $|\mathbf{k}| = 1$ are also special, in that the right side is zero (and also many terms on the left side cancel out). Thus, if there are acceptable solutions, and indeed this is the case, there is necessarily more than one. This degree of freedom has to be eliminated (it is only then that a contractive mapping approach can succeed), for instance by writing the equations in integral form with specific constants of integration. We first write the equations. We let $\mathbf{e}_j$ be the unit vector in the direction j.

$$y_j' = [\nabla_{\mathbf{y}_0}\mathbf{g}(x^{-1}, \mathbf{y}_0)\mathbf{y_k}]_j =: \mathbf{a}_j \cdot \mathbf{y_k}, \quad |\mathbf{k}| = 1 \tag{3.56}$$

or

$$y_j = c_j + \int_x^\infty \mathbf{a}_j \cdot \mathbf{y_k} dx, \quad |\mathbf{k}| = 1 \tag{3.57}$$

At this stage there is one more step before taking the inverse Laplace transform (ILT), namely to make all terms small; the constant (and thus y) is not. There are several ways to achieve this, and they lead to the same result. The question is really one of simplicity of the transformation. Probably the easiest one here is to subtract out the constant out of y. We write $y_j = c_j + u_j$. The equations for the $\mathbf{u}$s are

$$\begin{aligned} u_j = \int_x^\infty \mathbf{a}_j \cdot \mathbf{y_k c} dx + \int_x^\infty \mathbf{a}_j \cdot \mathbf{u_k} dx, \quad |\mathbf{k}| = 1 \\ = C_j f_j + \int_x^\infty \mathbf{a}_j \cdot \mathbf{u_k} dx, \quad |\mathbf{k}| = 1 \end{aligned} \tag{3.58}$$

We note that by our assumptions the matrix $\hat{\mathbf{h}}$ is $O(x^{-M+1}, x^{-2}\mathbf{y}_0, \mathbf{y}_0^2)$ thus $O(x^{-M+1})$ since $\mathbf{y}_0$ is also $O(x^{-M+1})$, where M is chosen by us since it is of the order of magnitude

After ILT the equation becomes is

$$U_j = C_j F_j(p) + p^{-1}\hat{\mathbf{H}} * \mathbf{u_k}, \quad |\mathbf{k}| = 1 \tag{3.59}$$

where $F_j = p^{M-1}\check{F}_j$ and $\mathbf{H} = p^{M-1}\check{\mathbf{H}}$ where the "$\vee$" functions are bounded. In any compact set, f is bounded and in L^1_ν, we can write

$$p^{-1}\int_0^p s^M f(s) g(p-s) ds = p^M \int_0^1 t^M f(t) g(p(1-t) ds$$

which manifestly preserves analyticity and is contractive for small p. It is also contractive in L^1_ν for large enough ν.

Thus $\mathbf{u_k}$ is Borel summable. We see that the solution depends on $\mathbf{C}$. With slight abuse of notation we write the ILT of this solution as $\mathbf{CU}$.

3.1h The set of equations with $|\mathbf{k}| > 1$

Now the sysyem of equations in ILT space, with

$$\mathbb{K} = (\mathbf{C^k})_{\mathbf{k>1}}, \mathbb{L}^{\mathbf{-1}} = \mathbb{K}(-\mathbf{k}\cdot\boldsymbol{\lambda} + \hat{\mathbf{\Lambda}})^{\mathbf{-1}}_{\mathbf{k}}, \mathbb{Y} = (\mathbf{Y_k})_{\mathbf{k}},\ \mathbb{T} = (\mathbf{T_k})_{\mathbf{k}} \tag{3.60}$$

where $\mathbf{k} \in \mathbb{Z}^n$; $\|\mathbf{k}| > 1$, can be written as

$$\mathbb{Y} = \mathbb{K}\mathbb{L}^{-1}\mathbb{T}(\mathbb{Y}) \tag{3.61}$$

Lemma 3.62 *(i) Equation (3.61) is contractive in $L^1_{\mu\nu}$ for large enough μ, ν in the same regions as $\mathbf{Y}_0$.*

(ii) Thus (3.60) is contractive, and all the $\mathbf{y_k}$ are summable, in a common domain $\Re x > \nu$ with sup norm $|\mathbf{y_k}| < \nu^{|\mathbf{k}|}$. The whole summed transseries converges, as a function series, provided

$$\prod_j \left|\mathbf{x}^{\mathbf{a_j}}\mathbf{C}\mathrm{e}^{-\lambda_j \mathrm{x}}\right| < 1/\mu \tag{3.63}$$

Again, μ and ν may depend on the region.

Proof. This is straightforward and left as an exercise. □

We also note that $\mathbf{y}_0$ is analytic in any direction which is not a Stokes direction. However, the condition (3.63) can fail before reaching a Stokes line! Then the function series diverges geometrically, and, as we shall see, this means formation of singularities in the x plane.

3.1i Analytic structure along $\mathbb{R}^+$

Two lemmas on analytic structure

Lemma 3.64 *Let f be analytic in the unit disc cut along the positive axis and let $0 < g(x) \in C^1[0,1]$. Assume that $\lim_{\epsilon\downarrow 0} f(x \pm \mathrm{i}\epsilon g(x)) = f^{\pm}(x)$ in $L^1[0,1]$ and*

$$f^+(x) - f^-(x) = f_\delta(x) = x^r A(x) \tag{3.65}$$

with $\Re(r) > -1$, where $A(\xi)$ extends to an analytic function for $|\xi| < a \le 1$. Then there exists a function B analytic in $|\xi| < a$ so that

$$f(\xi) = \frac{1}{1 - \mathrm{e}^{2\pi i r}}\xi^r A(\xi) + B(\xi) \quad (r \notin \mathbb{N})$$

$$f(\xi) = \frac{i}{2\pi}\ln(\xi)\xi^r A(\xi) + B(\xi) \quad (r \in \mathbb{N}) \tag{3.66}$$

If $f^+(x) - f^-(x)$ is a linear combination $\sum_{i=1}^N x^{r_i} A_i(x)$ (under the same assumptions on r_i and A_i), then f is given by the corresponding superposition of terms of the form (3.66).

Note 3.67 *This is a simple setting similar to a Riemann-Hilbert problem; it has an elementary proof.*

Proof. Take first $r \notin \mathbb{Z}$. Choose a_1, a_2 so that $0 < a_1 < a_2 < a$ and consider the closed contour C going along the upper cut from $\xi = 0$ to $\xi = a_2$, continuing towards the lower cut anticlockwise along the circle $C(a_2)$ of radius a_2 centered at origin, and finally coming from $\xi = a_2$ back to $\xi = 0$ along the lower cut. For $|\xi| < a_1$ we have, by the assumptions of the lemma,

$$2\pi i f(\xi) = \oint_C \frac{f(s)}{s-\xi}\mathrm{d}s = \oint_{C(a_2)} \frac{f(s)}{s-\xi}\mathrm{d}s + \int_0^{a_2} \frac{s^r A(s)}{s-\xi}\mathrm{d}s \tag{3.68}$$

On the other hand, defining $z^r A(z)$ in the interior of $C(a)$ cut along the positive axis (with the usual convention $\arg(z) = 0$ on the upper cut), we have, for the same contour as above and $\xi \in \mathcal{V}_{a_1}$

$$2\pi\, \mathrm{i}\xi^r A(\xi) = \oint_{C(a_2)} \frac{A(s)}{s-\xi}\mathrm{d}s + \left(1 - \mathrm{e}^{2\pi i r}\right)\int_0^{a_2} \frac{s^r A(s)}{s-\xi}\mathrm{d}s \tag{3.69}$$

Comparing (3.68) to (3.69) we get:

$$\begin{aligned} f(\xi) &= \frac{1}{1-\mathrm{e}^{2\pi i r}}\xi^r A(\xi) \\ &- \frac{1}{2\pi i(1-\mathrm{e}^{2\pi i r})}\oint_{C(a_2)} \frac{A(s)}{s-\xi}\mathrm{d}s + \frac{1}{2\pi i}\oint_{C(a_2)} \frac{f(s)}{s-\xi}\mathrm{d}s \end{aligned} \tag{3.70}$$

As integrals of analytic functions with respect to complex absolutely continuous measures ($A(s)\mathrm{d}s$ and $f(s)\mathrm{d}s$), the last two terms in (3.70) are analytic in ξ for $|\xi| < a_1$. Since a_1 can be chosen arbitrarily close to a, the case $r \notin \mathbb{Z}$ is proven. For $r \in \mathbb{Z}$ the argument is essentially the same, in terms of $A(\xi)\xi^r \ln \xi$ instead of $\xi^r A(\xi)$. The proof generalizes immediately to linear combinations of $\xi^r A(\xi)$. $\square$

Study of the analytic structure

We let $\mathbf{C} \in (\mathbb{C}\backslash\{0\})^{n_1}$ be an arbitrary constant vector. As before, we can assume that by a rescaling of the independent variable we arranged that the Stokes line in $\mathbb{R}^+$ and that $\lambda_1 = 1$.
For x large enough, we let $\mathbf{y}^+$, a solution of (3.20), be defined by taking the Laplace transform slightly above (below,resp.) the real line. We now use Lemma 3.62 to write a same solution $\mathbf{y}$ (3.38) in terms of functions analytic in p in the first and fourth quadrant. We denote these *representations*[4] of the same solution $\mathbf{y}$ by

$$\mathbf{y}^+ = \mathcal{L}_-\mathbf{Y}_0 + \sum_{|\mathbf{k}|=1} \mathbf{C}_-^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}+1} \mathcal{L}_-\mathbf{Y}_{\mathbf{k}} + \sum_{|\mathbf{k}|>1} \mathbf{C}_-^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}+1} \mathcal{L}_-\mathbf{Y}_{\mathbf{k}} \quad (3.71)$$

$$\mathbf{y}^- = \mathcal{L}_+\mathbf{Y}_0 + \sum_{|\mathbf{k}|=1} \mathbf{C}_+^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}+1} \mathcal{L}_+\mathbf{Y}_{\mathbf{k}} + \sum_{|\mathbf{k}|>1} \mathbf{C}_+^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}+1} \mathcal{L}_+\mathbf{Y}_{\mathbf{k}} \quad (3.72)$$

[5]

We have thus

$$\mathcal{L}_-\mathbf{Y}_0 + \sum_{|\mathbf{k}|=1} \mathbf{C}_-^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathcal{L}_-\mathbf{Y}_{\mathbf{k}}$$
$$= \mathcal{L}_+\mathbf{Y}_0 + \sum_{|\mathbf{k}|=1} \mathbf{C}_+^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathcal{L}_+\mathbf{Y}_{\mathbf{k}} + o(x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x}) \quad (3.73)$$

We note however that $\mathbf{Y}_{\mathbf{k}}$ are analytic *at the origin*. Thus in fact,

$$(\mathcal{L}_- - \mathcal{L}_+)\,\mathbf{Y}_0 = \sum_{|\mathbf{k}|=1} (\mathbf{C}_-^{\mathbf{k}} - \mathbf{C}_+^{\mathbf{k}}) \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathcal{L}_t\mathbf{Y}_{\mathbf{k}} + o(x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x}) \quad (3.74)$$

where we denoted by $\mathcal{L}_t$ the truncated Laplace transform $\int_0^t$ for some small t. Here we can drop the $\pm$ from $\mathbf{Y}_{\mathbf{k}}$, $|\mathbf{k}| = 1$, since the functions are analytic in this region. The contribution of the integrals on the right-hand side from t to infinity is exponentially smaller that $\mathcal{L}_t$.

We now claim that only $(\mathbf{C}_-)_1$ and $(\mathbf{C}_+)_1$ may be different; for all the other indices the constants are the same. Indeed, by assumption, the directions of the λ's are different. On the other hand, by Proposition 3.36 (i), all $\mathbf{Y}_{\mathbf{k}}$ are analytic, say, above the real line and until the first eigenvalue direction, say that of λ_2. We now look at the eigenvalue which is *is represented in the transseries, with nonzero constant* and is the *farthest* in angle as measured clockwise, down from

[4] According to the definition, xp must be real and positive. Therefore, y^- corresponds to summation in the *first quadrant* etc.

[5] We do have $\mathcal{L}\mathcal{B}\tilde{\mathbf{y}}_k = \mathbf{y}_k$, but with the convention $\mathcal{L}\mathcal{B}1 = 1$. If we want to write the solutions in terms of classical Laplace transforms, we had to multiply by another power of x, to allow for the fact that, by Watson's Lemma, the transform of an analytic function is $O(x^{-1}$.

the real line, say λ_q. This λ_q has to be in the fourth quadrant to generate a small exponential in the transseries along $\mathbb{R}^+$.

The direction perpendicular to it in the upper half plane is an antistokes direction for it and the transseries representation remains valid until we reach a direction, in the upper half plane, where Condition 3.63 breaks. It clearly breaks because of λ_q, which is, by construction the first one to give rise to oscillatory exponentials in the upper half plane. But for large enough x and close enough to the breakdown of (3.63), the terms containing C_q in (3.73) become *dominant*, larger than even x^{-M}, the leading term of $\mathbf{Y}_0$, and the classical asymptotics of $\mathbf{y}$ will have, as a *leading* term,

$$(C_-)_q \mathrm{e}^{-\lambda_q x} x^{\alpha_q} \mathcal{L} Y_q (1+o(1)) = (C_+)_q \mathrm{e}^{-\lambda_q x} x^{\alpha_q} \mathcal{L} Y_q (1+o(1)) \tag{3.75}$$

This equality holds because we are dealing with the *same* solution $\mathbf{y}$. But this, as it is easy to see, can only happen if

$$(C_-)_q = (C_+)q \tag{3.76}$$

We can proceed (finite)-inductively to show that

$$(C_-)_l = (C_+)_l; \quad l \neq 1. \tag{3.77}$$

We denote

$$(C_-)_1 - (C_+)_1 = S_1 \tag{3.78}$$

Note 3.79 *The* Stokes constant S_1 *(and similarily, all others, do not depend on the solution but only on the equation, since* $\mathbf{Y}_0$ *is unique.*

Now we take $m > \Re\alpha - 1$, say m is even, and write

$$\frac{\partial}{\partial x^m} (\mathcal{L}_- - \mathcal{L}_+)_t \left[\partial^{-m}\mathbf{Y}_0 \; - \; S_1 x^{\alpha_1+m} \mathbf{Y}_{\mathbf{e}_1}(p-1)\chi(p>1)\right] \\ = o(e^{-x-t}x^{\alpha_1}) \tag{3.80}$$

Therefore, in a neighborhood of the real line it is easy to see that we have

$$(\mathcal{L}_- - \mathcal{L}_+)_t \mathbf{Y}_0 = S_1 e^{-x} x^{\alpha_1} \mathcal{L}_t \mathbf{Y}_{\mathbf{e}_1} + o(e^{-x-t}x^{\alpha_1}) \tag{3.81}$$

At this stage we make use of Lemma 2.31 to conclude that

$$\Delta \mathbf{Y}_0 = S_1 \mathbf{Y}_{\mathbf{e}_1}^{(m)} \tag{3.82}$$

where Δ denoted the difference between the analytic continuations of $\mathbf{Y}_0$ above and below $p = 1$.

This is a **resurgence** relation. It shows that $\mathbf{Y}_0$ contains all the information about $\mathbf{Y}_{\mathbf{e}_1}$! In fact, as it is clear, the index "1" is not important here, and it follwows that $\mathbf{Y}_0$ contains the information about all $\mathbf{Y}_{\mathbf{e}_k}$.

Now Lemma 3.64 gives the complete information about the singularity nature of $\mathbf{Y}_0$ at $p = 1$. But it can be seen by the same type of arguments, inductivrly, by straightforward though perhaps at places tedious calculations, that all $\mathbf{Y}'s$ are linked to each other in a similar fashion. The net result is:

Theorem 3.83 *(i)* $\mathbf{Y}_0 = \mathcal{B}\tilde{\mathbf{y}}_0$ *is analytic in* $\mathcal{R}\cup\{0\}$.

The singularities of $\mathbf{Y}_0$ *(which are contained in the set* $\{l\lambda_j : l\in\mathbb{N}^+, j = 1,2,\ldots,n\}$*) are described as follows. For* $l\in\mathbb{N}^+$ *and small* z

$$\mathbf{Y}_0^{\pm}(z+l\lambda_j) = \pm\Big[(\pm S_j)^l(\ln z)^{0,1}\mathbf{Y}_{l\mathbf{e}_j}(z)\Big]^{(lm_j)} + \mathbf{B}_{lj}(z) = \\ \Big[z^{l\beta'_j-1}(\ln z)^{0,1}\,\mathbf{A}_{lj}(z)\Big]^{(lm_j)} + \mathbf{B}_{lj}(z)\ (l=1,2,\ldots) \quad (3.84)$$

where the power of $\ln z$ *is one iff* $l\beta_j\in\mathbb{Z}$, *and* $\mathbf{A}_{lj},\mathbf{B}_{lj}$ *are analytic for small* z. *The functions* $\mathbf{Y}_{\mathbf{k}}$ *are, exceptionally, analytic at* $p = l\lambda_j$, $l\in\mathbb{N}^+$, *iff,*

$$S_j = r_j\Gamma(\beta'_j)\left(\mathbf{A}_{1,j}\right)_j(0) = 0 \quad (3.85)$$

where $r_j = 1-\mathrm{e}^{2\pi i(\beta'_j-1)}$ *if* $l\beta_j\notin\mathbb{Z}$ *and* $r_j = -2\pi i$ *otherwise. The* S_j *are Stokes constants.*

(ii) $\mathbf{Y}_{\mathbf{k}} = \mathcal{B}\tilde{\mathbf{y}}_{\mathbf{k}}$, $|\mathbf{k}|>1$, *are analytic in* $\mathcal{R}\backslash\{-\mathbf{k}'\cdot\boldsymbol{\lambda}+\lambda_i : \mathbf{k}'\le\mathbf{k}, 1\le i\le n\}$. *For* $l\in\mathbb{N}$ *and* p *near* $l\lambda_j$, $j=1,2,\ldots,n$ *there exist* $\mathbf{A}=\mathbf{A}_{\mathbf{k}jl}$ *and* $\mathbf{B}=\mathbf{B}_{\mathbf{k}jl}$ *analytic at zero so that (z is as above)*

$$\mathbf{Y}_{\mathbf{k}}^{\pm}(z+l\lambda_j) = \pm\left[(\pm S_j)^l\binom{k_j+l}{l}(\ln z)^{0,1}\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(z)\right]^{(lm_j)} + \mathbf{B}_{\mathbf{k}lj}(z) = \\ \Big[z^{\mathbf{k}\cdot\boldsymbol{\beta}'+l\beta'_j-1}(\ln z)^{0,1}\,\mathbf{A}_{\mathbf{k}lj}(z)\Big]^{(lm_j)} + \mathbf{B}_{\mathbf{k}lj}(z)\ (l=0,1,2,\ldots) \quad (3.86)$$

where the power of $\ln z$ *is* 0 *iff* $l=0$ *or* $\mathbf{k}\cdot\boldsymbol{\beta}+l\beta_j-1\notin\mathbb{Z}$ *and* $\mathbf{A}_{\mathbf{k}0j} = \mathbf{e}_j/\Gamma(\beta'_j)$. *Near* $p\in\{\lambda_i-\mathbf{k}'\cdot\boldsymbol{\lambda} : 0<\mathbf{k}'\le\mathbf{k}\}$, *(where* $\mathbf{Y}_0$ *is analytic)* $\mathbf{Y}_{\mathbf{k}}$, $\mathbf{k}\neq 0$ *have convergent Puiseux series.*

REMARK: The fact that the singular part of $\mathbf{Y}_{\mathbf{k}}(p+l\lambda_j)$ in (3.84) and (3.86) is a *multiple* of $\mathbf{Y}_{\mathbf{k}+l\mathbf{e}_j}(p)$ is the effect of *resurgence* and provides a way of determining the $\mathbf{Y}_{\mathbf{k}}$ given $\mathbf{Y}_0$ provided the S_j are nonzero. Since, generically, the S_j are nonzero this is a surprising upshot: given one formal solution, (generically) an n parameter family of solutions can be constructed out of it, without using (3.20) in the process; the differential equation itself is then recoverable.

3.1j Heuristic discussion of transasymptotic matching

There is a sharp distinction between linear and nonlinear systems with respect to the behavior beyond S_{trans}, the region where the formal or summed transseries is valid, namely

$$S_{trans} = \left\{x\in\mathbb{C}\,;\ \text{if } C_j\neq 0 \text{ then } x^{a_j}e^{\lambda_j} = o(1), j=1,...,n\right\} \quad (3.87)$$

This sector might be the whole $\mathbb{C}$ if all $C_j = 0$; otherwise it lies between two antistokes lines, and has opening at most π.

If we have normalized the equation in such a way that $\lambda_1 = 1$, and λ_m is the eigenvalue in the fourth quadrant (if there is such an eigenvalue) with the most *negative* angle, then in the upper half plane, S_{trans} will be controlled, roughly, by the condition $\Re(\lambda_m x) > 0$. If there is no such eigenvalue, then the rigion in the first quadrant will be determined by $\lambda_1 = 1$, namely $x^{a_1}e^{-x} = o(1)$. If we examine the first quadrant, it is now convenient to rotate again the independent variable so that $\lambda_m = 1$, since this eigenvalue is the determining one. Since originally no exponentials associated with λ_j belonging to the second or third quadrant were allowed, then after this new rotation there will be no eigenvalue in the fourth quadrant, and the region of validity *in the first quadrant* would be, roughly, up to the imaginary line.

In the linear case there are only *finitely many* (at most n) nonzero $\mathbf{y_k}$ in (3.38) and thus the function series is valid in a full (possibly ramified) neighborhood of infinity, except for jumps in the components of $\mathbf{C}$, one at each Stokes line (see [20], [8], [22] and the references therein). The map $\mathcal{LB}$ is continuous at the antistokes lines, and thus the transseries $\tilde{\mathbf{y}}$ of $\mathbf{y}$ is the same on both sides of an antistokes line. The phenomenon that does take place in these directions is an interchange of dominance between the components of the transseries, and since classical asymptotics only retains the dominant series in $\tilde{\mathbf{y}}$, the *classical* asymptotic expansion of $\mathbf{y}$ changes. But from the point of view of exponential asymptotics, when the whole transseries is considered, the behavior of solutions of linear equations at antistokes lines is relatively simple.

Now we note that if we decide to perform Écalle-Borel summation of the transseries in a direction above but not too far from the real line, no $\mathbf{Y_k}$ will be singular, thus, in the summed transseries, the corresponding $\mathbf{y_k}$ will be well defined, bounded and analytic in a sector whose upper edge *has an angle strictly greater than* $\pi/2$. Thus, the transseries, summed or not, becomes invalid (the terms become rapidly growing due to the simpler ingredient $x^a e^{-x}$.

The divergence of (3.38) turns out to mark an actual change in the behavior of $\mathbf{y}(x)$, which usually develops singularities in this region. The information about the singularities is contained in (3.38).

The key to understanding the behavior of $\mathbf{y}(x)$ for x beyond the sector of analyticity is to look carefully at the borderline region where (3.38) converges but barely so. Because of nonresonance, for $\arg(x) = \pi/2$ we have $\Re(\lambda_j x) > 0, j = 2, ..., n_1$.[6] By (3.37) all terms in (3.25) with $\mathbf{k}$ not a multiple of $\mathbf{e}_1 = (1, 0, ..., 0)$ are subdominant (small). Thus, for x near $i\mathbb{R}^+$ we only need to look at

$$\mathbf{y}^{[1]}(x) = \sum_{k \geq 0} C_1^k \mathrm{e}^{-kx} x^{kM_1} \mathbf{y}_{k\mathbf{e}_1}(x) \tag{3.88}$$

The region of convergence of (3.88) (thus of (3.38)) is then determined by the effective variable $\xi = C_1 \mathrm{e}^{-x} x^{\alpha_1}$ (since $\mathbf{y}_{k\mathbf{e}_1} \sim \tilde{\mathbf{y}}_{k\mathbf{e}_1} = \mathbf{e}_1 + o(1)$). Convergence

[6] We have $C_j = 0$ for $j > n_1$.

is marginal along curves such that ξ is small enough but, as $|x| \to \infty$, is nevertheless larger than all *negative* powers of x. In this case, any term of the form $C_1^k e^{-kx} x^{kM_1} \mathbf{y}_{k\mathbf{e}_1;0}$ is much larger than the terms $C_1^l e^{-lx} x^{l\alpha_1} \mathbf{y}_{l\mathbf{e}_1}$ if $k, l \geq 0$ and $r > 0$. Hence the leading behavior of $\mathbf{y}^{[1]}$ is expected to be

$$\mathbf{y}^{[1]}(x) \sim \sum_{k \geq 0} (C_1 \mathrm{e}^{-x} x^{\alpha_1})^k \tilde{\mathbf{s}}_{k\mathbf{e}_1;0} \equiv F_0(\xi) \tag{3.89}$$

moreover, taking into account all terms in $\tilde{\mathbf{s}}_{k\mathbf{e}_1}$ we get

$$\mathbf{y}^{[1]}(x) \sim \sum_{r=0}^{\infty} x^{-r} \sum_{k=0}^{\infty} \xi^k \tilde{\mathbf{y}}_{k\mathbf{e}_1;r} \equiv \sum_{j=0}^{\infty} \frac{\mathbf{F}_j(\xi)}{x^j} \tag{3.90}$$

Expansion (3.90) has a two-scale structure, with the scales ξ and x.

It may come as a surprise that each $\mathbf{F}_j$ is a **convergent** series in ξ (though the whole expansion (3.90) is still divergent).

It turns out that the reshuffling (3.90) is meaningful and yields the correct asymptotic representation of $\mathbf{y}^{[1]}$, and therefore of $\mathbf{y}$, beyond the upper edge of S_{transs}. In fact, (3.90) extends $\mathbf{y}(x) \sim \tilde{\mathbf{y}}_{\mathbf{0}}$ right into the regions in $\mathbb{C}$ where $\mathbf{y}$ is singular. Once these two scales are known and once the validity of (3.90) is proved for our class of systems (Theorems 1 and 3 below), it is easier to calculate the $\mathbf{F}_j$ by direct substitution of (3.90) in (1.91) and identification of the powers of x. The exact form of the second scale ξ is decisive for the domain of validity of the expansion, see §3.1d .

3.1k The recursive system for $\mathbf{F}_m$

The functions are $\mathbf{F}_m$ recursively, from their differential equation. Formally the calculation is the following.

The series $\tilde{\mathbf{F}} = \sum_{m \geq 0} x^{-m} \mathbf{F}_m(\xi)$ is a formal solution of (1.91); substitution in the equation and identification of coefficients of x^{-m} yields the recursive system

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \mathbf{F}_0 = \xi^{-1} \left(\hat{\Lambda} \mathbf{F}_0 - \mathbf{g}(0, \mathbf{F}_0) \right) \tag{3.91}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \mathbf{F}_m + \hat{N} \mathbf{F}_m = \alpha_1 \frac{\mathrm{d}}{\mathrm{d}\xi} \mathbf{F}_{m-1} + \mathbf{R}_{m-1} \qquad \text{for } m \geq 1 \tag{3.92}$$

where $\hat{N}$ is the matrix

$$\xi^{-1} (\partial_{\mathbf{y}} \mathbf{g}(0, \mathbf{F}_0) - \hat{\Lambda}) \tag{3.93}$$

and the function $\mathbf{R}_{m-1}(\xi)$ depends only on the $\mathbf{F}_k$ with $k < m$:

$$\xi \mathbf{R}_{m-1} = -\left[(m-1)I + \hat{A}\right]\mathbf{F}_{m-1} - \frac{1}{m!}\frac{\mathrm{d}^m}{\mathrm{d}z^m}\mathbf{g}\left(z; \sum_{j=0}^{m-1} z^j \mathbf{F}_j\right)\Bigg|_{z=0} \tag{3.94}$$

For more detail see [4] Section 4.3.

To leading order we have $\mathbf{y} \sim \mathbf{F}_0$ (see also (3.89)) where $\mathbf{F}_0$ satisfies the autonomous (after a substitution $\xi = e^{\zeta}$) equation

$$\mathbf{F}_0' = \hat{\Lambda}\mathbf{F}_0 - \mathbf{g}(0, \mathbf{F}_0)$$

which can be solved in closed form for first order equations ($n = 1$) (the equation for F_0 is separable, and for $k \geq 1$ the equations are linear), as well as in other interesting cases (see e.g. §3.1d , §3.2c).

(3.91), (3.92). To determine the $\mathbf{F}_m$'s associated to $\mathbf{y}$ we first note that these functions are analytic at $\xi = 0$ (cf. Theorem 3.101). Denoting by $F_{m,j}$, $j = 1, .., n$ the components of $\mathbf{F}_m$, a simple calculation shows that (3.91) has a unique analytic solution satisfying $F_{0,1}(\xi) = \xi + O(\xi^2)$ and $F_{0,j}(\xi) = O(\xi^2)$ for $j = 2, ..., n$. For $m = 1$, there is a one parameter family of solutions of (3.92) having a Taylor series at $\xi = 0$, and they have the form $F_{1,1}(\xi) = c_1\xi + O(\xi^2)$ and $F_{1,j}(\xi) = O(\xi^2)$ for $j = 2, ..., n$. The parameter c_1 is determined from the condition that (3.92) has an analytic solution for $m = 2$. For this value of c_1 there is a one parameter family of solutions $\mathbf{F}_2$ analytic at $\xi = 0$ and this new parameter is determined by analyzing the equation of $\mathbf{F}_3$. The procedure can be continued to any order in m, in the same way; in particular, the constant c_m is only determined at step $m + 1$ from the condition of analyticity of $\mathbf{F}_{m+1}$.

3.1l Notation

Let d be a direction in the x-plane which is not not an antistokes line. Consider a solution $\mathbf{y}(x)$ of (1.91) satisfying the assumptions in §3.1c . We define

$$S_{an} = S_{an}(\mathbf{y}(x); \epsilon) = S_\epsilon^+ \cup S_\epsilon^- \tag{3.95}$$

where

$$S_\epsilon^{\pm} = \Big\{x\,;\, |x| > R\ ,\ \arg(x) \in [-\frac{\pi}{2} \mp \epsilon, \frac{\pi}{2} \mp \epsilon] \text{ and } \\ \left|C_j^- e^{-\lambda_j x} x^{-\beta_j}\right| < \delta^{-1} \text{ for } j = 1, ..., n\Big\} \tag{3.96}$$

We use be the representation of $\mathbf{y}$ as summation of its transseries $\tilde{\mathbf{y}}(x)$ (3.38) in the direction d. Let

$$p_{j;\mathbf{k}} = \lambda_j - \mathbf{k} \cdot \boldsymbol{\lambda}\ ,\quad j = 1, ..., n_1\ ,\quad \mathbf{k} \in \mathbb{Z}_+^{n_1} \tag{3.97}$$

For simplicity we *assume*, what is generically the case, that no $\overline{p_{j;\mathbf{k}}}$ lies on the antistokes lines bounding S_{trans}.

We *assume* that not all parameters C_j are zero, say $C_1 \neq 0$. Then S_{trans} is bounded by two antistokes lines and its opening is at most π.

We arrange that

(a) $\arg(\lambda_1) < \arg(\lambda_2) < ... < \arg(\lambda_{n_1})$

and, by construction,

(b) $\Im\lambda_k \geq 0$.

The solution $\mathbf{y}(x)$ is then analytic in a region S_{an}.

The locations of singularities of $\mathbf{y}(x)$ depend on the constant C_1 (constant which may change when we cross the Stokes line $\mathbb{R}^+$). We need its value in the sector between $\mathbb{R}^+$ and $i\mathbb{R}_+$, the next Stokes line.

Fix some small, positive δ and c. Denote

$$\xi = \xi(x) = C_1 \mathrm{e}^{-x} x^{\alpha_1} \tag{3.98}$$

and

$$\mathcal{E} = \Big\{ x\,;\, \arg(x) \in \Big[-\frac{\pi}{2} + \delta, \frac{\pi}{2} + \delta\Big] \text{ and } \Re(\lambda_j x/|x|) > c \text{ for all } j \text{ with } 2 \leq j \leq n_1 \Big\} \tag{3.99}$$

Also let

$$\mathcal{S}_{\delta_1} = \{x \in \mathcal{E}\,;\, |\xi(x)| < \delta_1\} \tag{3.100}$$

The sector $\mathcal{E}$ contains S_{trans}, except for a thin sector at the lower edge of S_{trans} (excluded by the conditions $\Re(\lambda_j x/|x|) > c$ for $2 \leq j \leq n_1$, or, if $n_1 = 1$, by the condition $\arg(x) \geq -\frac{\pi}{2} + \delta$), and may extend beyond $i\mathbb{R}_+$ since there is no condition on $\Re(\lambda_1 x)$—hence $\Re(\lambda_1 x) = \Re(x)$ may change sign in $\mathcal{E}$ and $\mathcal{S}_{\delta_1}$.

Figure 1 is drawn for $n_1 = 1$; $\mathcal{E}$ contains the gray regions and extends beyond the curved boundary.

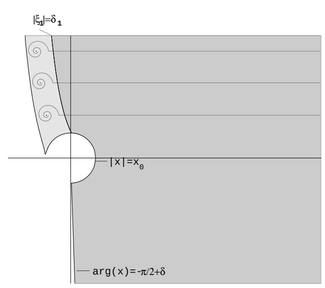


Fig. 1 Region $\mathcal{D}_x$ where (3.103) holds, when $n_1 = 1$. The dark gray subregion is S_{δ_1}. Curves like the spiraling gray curves surround points in X

(close to singularities of $\mathbf{y}$) generate the region $\mathcal{D}_x$. The picture is drawn with $n_1 = 1, \lambda = \frac{1}{10}, \alpha = -\frac{1}{2}, \delta_1 = 3 \cdot 10^6, x_0 = 40$. In this case S_{trans} is a sector where $|\arg(x)| < \frac{\pi}{2} - 0$.

Theorem 3.101 *(i) The functions $\mathbf{F}_m(\xi)$; $m \geq 1$, are analytic in $\mathcal{D}$ (note that by construction $\mathbf{F}_0$ is analytic in $\mathcal{D}$) and for some positive B, K we have*

$$|F_m(\xi)| \leq Km!B^m, \quad \xi \in \mathcal{D} \tag{3.102}$$

(ii) For large enough R, the solution $\mathbf{y}(x)$ is analytic in $\mathcal{D}_x$ and has the asymptotic representation

$$\mathbf{y}(x) \sim \sum_{m=0}^{\infty} x^{-m}\mathbf{F}_m(\xi(x)) \quad (x \in \mathcal{D}_x,\ |x| \to \infty) \tag{3.103}$$

In fact, the following Gevrey-like estimates hold

$$\left| \mathbf{y}(x) - \sum_{j=0}^{m-1} x^{-j}\mathbf{F}_j(\xi(x)) \right| \leq K_2 m! B_2^m |x|^{-m} \quad (m \in \mathbb{N}^+,\ x \in \mathcal{D}_x) \tag{3.104}$$

(iii) Assume $\mathbf{F}_0$ has an isolated singularity at $\xi_s \in \Xi$ and that the projection of $\mathcal{D}$ on $\mathbb{C}$ contains a punctured neighborhood of (or an annulus of inner radius r around) ξ_s.

Then, if $C_1 \neq 0$, $\mathbf{y}(x)$ is singular at a distance at most $o(1)$ $(r + o(1)$, respectively) of $x_n \in \xi^{-1}(\{\xi_s\}) \cap \mathcal{D}_x$, as $x_n \to \infty$.
The collection $\{x_n\}_{n\in\mathbb{N}}$ forms a nearly periodic array

$$x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C_1 - \ln \xi_s + o(1) \tag{3.105}$$

as $n \to \infty$.

Some of the conclusions of the theorem hold with $\mathcal{D}$ noncompact, under some natural restrictions, see Proposition 3.106.

Comments. 1. The singularities x_n satisfy $C_1 e^{-x_n} x_n^{\alpha_1} = \xi_s(1 + o(1))$ (for $n \to \infty$). Therefore, the singularity array lies slightly to the left of the antistokes line $i\mathbb{R}_+$ if $\Re(\alpha_1) < 0$ (this case is depicted in Figure 1) and slightly to the right of $i\mathbb{R}_+$ if $\Re(\alpha_1) > 0$.

2. In practice it is useful to normalize the system (1.91) so that α_1 is as small as possible (see the Comment 1. in § 3.1d and § 3.1d).

3. By (3.104) a truncation of the two-scale series (3.103) at an m dependent on x ($m \sim |x|/B$) is seen to produce exponential accuracy $o(\mathrm{e}^{-|x/B|})$, see e.g. [18].

4. Theorem 3.101 can also be used to determine precisely the nature of the singularities of $\mathbf{y}(x)$. In effect, for any n, the representation (3.103) provides

$o(\mathrm{e}^{-K|x_n|})$ estimates on $\mathbf{y}$ down to an $o(\mathrm{e}^{-K|x_n|})$ distance of an actual singularity x_n. In most instances this is more than sufficient to match to a suitable local integral equation, contractive in a tiny neighborhood of x_n, providing rigorous control of the singularity. See also §3.2.

General comments. 1. The expansion scales, x and $x^{-1/2}\mathrm{e}^{-x}$ are crucial. Only for this choice one obtains an expansion which is valid both in S_{trans} and near poles of (3.35). For instance, the more general second scale $x^a\mathrm{e}^{-x}$ introduces logarithmic singularities in H_j, except when $a \in -\frac{1}{2} + \mathbb{Z}$. With these logarithmic terms, the two scale expansion would only be valid in an $O(1)$ region in x, what is sometimes called a "patch at infinity", instead of more than a sector. Also, $a \in -\frac{1}{2} - \mathbb{N}$ introduces obligatory singularities at $\xi = 0$ precluding the validity of the expansion in S_{trans}. The case $a \in -\frac{1}{2} + \mathbb{N}$ produces instead an expansion valid in S_{trans} but not near poles. Indeed, the
so that

$$\lambda_1 = 1, \alpha_1 = -\frac{1}{2} - \frac{3}{2}\frac{B\alpha}{A}$$

implying

$$\xi^2 F_0'' + \xi F_0' - F_0 = 3BF_0^2 - F_0^3$$

and, with the same initial condition as above, we now have

$$F_0 = \frac{2\xi(1+B\xi)}{\xi^2+2}$$

The first normalization applies for the manifold of solutions such that $y \sim -\frac{\alpha}{x}$ (for $\alpha = 0$ y is exponentially small and behaves like an Airy function) while the second one corresponds to $y \sim -B - \frac{\alpha}{2}x^{-3/2}$.

The following is an extension, in some respects, of Theorem 3.101 (ii).

Proposition 3.106 *Assume $\mathcal{D}$ is not necessarily compact, Γ is a curve of possibly infinite length in $\mathcal{D}$ with the following properties:*
(a) For some $\epsilon > 0$, $\mathbf{T}_{1,2}(z, \boldsymbol{\delta})$ and $\hat{N}(z)$ are analytic for z in an ϵ neighborhood of Γ and for $|\boldsymbol{\delta}| < \epsilon$ and in addition $\mathbf{T}_{1,2}(z, \boldsymbol{\delta}) = O(z\boldsymbol{\delta}, \boldsymbol{\delta}^2)$
(b) $\hat{M}(\xi, \xi_{1,0})$ is bounded in an ϵ neighborhood of Γ and for some K and all $\xi \in \Gamma$ we have $\int_{\xi_{1,0}}^{\xi} \left|\hat{M}(\xi, \xi_{1,0})\right| \mathrm{d}|s| < K$ (where $|\hat{M}|$ is some Euclidian norm of the matrix $\hat{M}(\xi, \xi_{1,0})$).

Then the conclusions of Theorem 3.101 (ii) hold in the x domain $\mathcal{D}_x$ corresponding to $\mathcal{D}$.

3.1m Proof of Theorem 3.101 (iii)

We need the following result which is in some sense a converse of Morera's theorem.

To show Theorem 3.101 (iii), assume ξ_s is an isolated singularity of $\mathbf{F}_0$ (thus $\xi_s \neq 0$) and $X = \{x : \xi(x) = \xi_s\}$. By lemma 1.15 there is a circle $\mathcal{C}$ around ξ_s

and a function $g(\xi)$ analytic in $B_r(\xi - \xi_s)$ such that $\oint_{\mathcal{C}} \mathbf{F}_0(\xi) g(\xi) d\xi = 1$. In a neighborhood of $x_n \in X$ the function $f(x) = \mathrm{e}^{-x} x^{\alpha_1}$ is conformal and for large x_n

$$\oint_{f^{-1}(\mathcal{C})} \mathbf{y}(x) \frac{g(f(x))}{f(x)} dx \\ = -\oint_{\mathcal{C}} (1 + O(x_n^{-1}))(\mathbf{F}_0(\xi) + O(x_n^{-1})) g(\xi) d\xi = 1 + O(x_n^{-1}) \neq 0 \quad (3.107)$$

It follows from lemma 1.15 that for large enough x_n $\mathbf{y}(x)$ is not analytic inside $\mathcal{C}$ either. Since the radius of $\mathcal{C}$ can be taken $o(1)$ Theorem 3.101 (iii) follows.

Note. In many cases the singularity of $\mathbf{y}$ is of the *same type* as the singularity of $\mathbf{F}_0$. See §3.2 for further comments.

In the following we will make rigorous these intuitive arguments and then proceed to explore further properties and consequences.

3.2 Examples

3.2a Equation (3.26)

1. We look again at ; we easily see that We see that

$$\lambda = 1, \quad \alpha = 1/5, \text{ and thus } \xi = C x^{1/5} \mathrm{e}^{-x} \quad (3.108)$$

Finding the two-scale expansion (3.103)

Having the second scale given by (3.108) and all the conditions of Theorem **??** satisfied, the simplest way to calculate the functions F_k in $\tilde{y} = \sum_{k=0}^{\infty} x^{-k} F_k(\xi)$ is by substituting $y = \tilde{y}$ in (3.29) and solving the differential equations, as in the proof of Theorem 3.101 (i); the equation for $F_0(\xi)$ is, cf. (3.91),

$$\xi F_0' = F_0(1 + 3F_0 + 3F_0^2); \qquad F_0'(0) = 1 \quad (3.109)$$

and, cf. (3.92),

$$\xi F_k' = (3F_0 + 1)^2 F_k + R_k(F_0, ..., F_{k-1}) \\ (\text{ for } k \geq 1 \text{ and where } R_1 = \frac{3}{5} F_0^3) \quad (3.110)$$

The first term F_0 of the expansion of u is then given by

$$\xi = \xi_0 F_0(\xi)(F_0(\xi) + \omega_0)^{-\theta} (F_0(\xi) + \overline{\omega_0})^{-\overline{\theta}} \quad (3.111)$$

with $\xi_0 = 3^{-1/2}\exp(-\frac{1}{6}\pi\sqrt{3})$, $\omega_0 = \frac{1}{2} + \frac{i\sqrt{3}}{6}$ and $\theta = \frac{1}{2} + i\frac{\sqrt{3}}{2}$. The functions F_k, $k \ge 1$ can also be obtained in closed form, order by order.

By Theorem **??**, the relation $y \sim \tilde{y}$ holds in the sector

$$S_{\delta_1} = \{x \in \mathbb{C} : \arg(x) \ge -\frac{\pi}{2} + \delta,\ |Cx^{1/5}e^{-x}| < \delta_1\}$$

for some $\delta_1 > 0$ and any small $\delta > 0$.

Theorem 3 insures that $y \sim \tilde{y}$ holds in fact on a larger region, surrounding singularities of F_0 (and thus of y). To apply this result we need the surface of analyticity of F_0 and an estimate for the location of its singularities.

Lemma 3.112 *(i) The function F_0 is analytic on the universal covering $\mathcal{R}_\Xi$ of $\mathbb{C} \setminus \Xi$ where*

$$\Xi = \{\xi_p = (-1)^{p_1}\xi_0 \exp(p_2\pi\sqrt{3}) : p_{1,2} \in \mathbb{Z}\} \tag{3.113}$$

and its singularities are algebraic of order $-1/2$, located at points lying above Ξ.

(ii) (The first Riemann sheet) The function F_0 is analytic in $\mathbb{C} \setminus \Big((-\infty, \xi_0] \cup [\xi_1, \infty)\Big)$.

(iii) The Riemann surface associated to F_0 is represented in Fig. 2.

Proof

Singularities of F_0. The RHS of (3.109) is analytic except at $F_0 = \infty$, thus F_0 is analytic except at points where $F_0 \to \infty$. From (3.111) it follows that $\lim_{F_0\to\infty} \xi \in \Xi$ and (i) follows straightforwardly; in particular, as $\xi \to \xi_p \in \Xi$ we have $(\xi - \xi_p)^{1/2}F_0(\xi) \to \sqrt{-\xi_p/6}$.

(ii) We now examine on which sheets in $\mathcal{R}_\Xi$ these singularities are located, and start with a study of the first Riemann sheet (where $F_0(\xi) = \xi + O(\xi^2)$ for small ξ). Finding which of the points ξ_p are singularities of F_0 on the first sheet can be rephrased in the following way. On which constant phase (equivalently, steepest ascent/descent) paths of $\xi(F_0)$, which extend to $|F_0| = \infty$ in the plane F_0, is $\xi(F_0)$ uniformly bounded?

Constant phase paths are governed by the equation $\Im(\mathrm{d}\ln\xi) = 0$. Thus, denoting $F_0 = X + iY$, since $\xi'/\xi = \left(F_0 + 3F_0^2 + 3F_0^3\right)^{-1}$ one is led to the *real* differential equation $\Im(\xi'/\xi)\mathrm{d}X + \Re(\xi'/\xi)\mathrm{d}Y = 0$, or

$$Y(1 + 6X + 9X^2 - 3Y^2)\mathrm{d}X - (X + 3X^2 - 3Y^2 + 3X^3 - 9XY^2)\mathrm{d}Y = 0 \tag{3.114}$$

We are interested in the field lines of (3.114) which extend to infinity. Noting that the singularities of the field are $(0, 0)$ (unstable node, in a natural parameterization) and $P_\pm = (-1/2, \pm\sqrt{3}/6)$ (stable foci, corresponding to $-\overline{\omega_0}$ and $-\omega_0$), the phase portrait is easy to draw (see Fig. 2) and there are only two curves starting at $(0,0)$ so that $|F_0| \to \infty$, ξ bounded, namely $\pm\mathbb{R}^+$, along which $\xi \to \xi_0$ and $\xi \to \xi_1$, respectively.

(iii) Thus Fig. 2 encodes the structure of singularities of F_0 on $\mathcal{R}_\Xi$ in the following way. A given class $\gamma \in \mathcal{R}_\Xi$ can be represented by a curve composed of rays and arcs of circle. In Fig. 2, in the F_0-plane, this corresponds to a curve γ' composed of constant phase (dark gray) lines or constant modulus (light gray) lines. Curves in $\mathcal{R}_\Xi$ terminating at singularities of F_0 correspond in Fig 2. to curves so that $|F_0| \to \infty$ (the four dark gray separatrices $S_1, ..., S_4$). Thus to calculate where, on a particular Riemann sheet of $\mathcal{R}_\Xi$, is F_0 singular, one needs to find the limit of ξ in (3.111), as $F_0 \to \infty$ along along γ' followed by S_i. This is straightforward, since the branch of the complex powers $\theta, \overline{\theta}$, is calculated easily from the index of γ' with respect to $P_\pm$. □

Theorem 3.101 can now be applied on relatively compact subdomains of $\mathcal{R}_\Xi$ and used to determine a uniform asymptotic representation $y \sim \tilde{y}$ in domains surrounding singularities of $y(x)$, and to obtain their asymptotic location. Going back to the original variables, similar information on $u(z)$ follows. For example, using Theorem 3.101 for the first Riemann sheet (cf. Lemma 3.112 (ii))

$$\mathcal{D} = \{|\xi| < K \mid \xi \notin (-\infty, \xi_1) \cup (\xi_0, +\infty)\ ,\ |\xi - \xi_0| > \epsilon, |\xi - \xi_1| > \epsilon, \}$$

(for any small $\epsilon > 0$ and large positive K) the corresponding domain in the z-plane is shown in Fig. 3.

In general, we fix $\epsilon > 0$ small, and some $K > 0$ and define $\mathcal{A}_K = \{z : \arg z \in \left(\frac{3}{10}\pi - 0, \frac{9}{10}\pi + 0\right),\ |\xi(z)| < K\}$ and let $\mathcal{R}_{K,\Xi}$ be the universal covering of $\Xi \cap \mathcal{A}_K$ and $\mathcal{R}_{z;K,\epsilon}$ the corresponding Riemann surface in the z plane, with $\epsilon-$ neighborhoods of the points projecting on $z(x(\Xi))$ deleted.

Proposition 3.115 *(i) The solutions $u = u(z; C)$ described in the beginning of §3.2 have the asymptotic expansion*

$$u(z) \sim z^{1/3}\left(1 + \frac{1}{9}z^{-5/3} + \sum_{k=0}^{\infty} \frac{F_k\left(C\xi(z)\right)}{z^{5k/3}}\right) \qquad (\text{as } z \to \infty;\quad z \in \mathcal{R}_{z;K,\epsilon}) \quad (3.116)$$

where

$$\xi(z) = x(z)^{1/5}\mathrm{e}^{-x(z)},\ \text{and}\ x(z) = -\frac{9}{5}z^{5/3} \qquad (3.117)$$

(ii) In the "steep ascent" strips $\arg(\xi) \in (a_1, a_2)$, $|a_2 - a_1| < \pi$ starting in $\mathcal{A}_K$ and crossing the boundary of $\mathcal{A}_K$, the function u has at most one singularity, when $\xi(z) = \xi_0$ or ξ_1, and $u(z) = z^{1/3}\mathrm{e}^{\pm 2\pi i/3}(1 + o(1))$ as $z \to \infty$ (the sign is determined by $\arg(\xi)$).

(iii) The singularities of $u(z; C)$, for $C \neq 0$, are located within $O(\epsilon)$ of the punctures of $\mathcal{R}_{z;K,0}$.

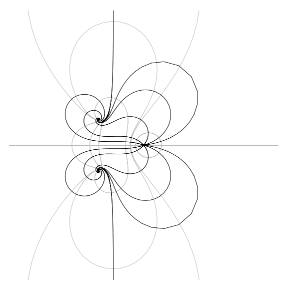


Figure 3.1: The dark lines represent the phase portrait of (3.114), as well as the lines of steepest variation of $|\xi(u)|$. The light gray lines correspond to the orthogonal field, and to the lines $|\xi(u)| = const.$

Applying Theorem 3.101 to (3.29) it follows that for $n \to \infty$, a given solution y is singular at points $\tilde{x}_{p,n}$ such that $\xi(\tilde{x}_{p,n})/\xi_p = 1 + o(1)$ ($|\tilde{x}_{p,n}|$ large).

Now, y can only be singular if $|y| \to \infty$ (otherwise the r.h.s. of (3.29) is analytic). If $\tilde{x}_{p,n}$ is a point where y is unbounded, with $\delta = x - \tilde{x}_{p,n}$ and $v = 1/y$ we have

$$\frac{\mathrm{d}\delta}{\mathrm{d}v} = vF_s(v, \delta) \tag{3.118}$$

where F_s is analytic near $(0, 0)$. It is easy to see that this differential equation has a unique solution with $\delta(0) = 0$ and that $\delta'(0) = 0$ as well.

The result is then that the singularities of u are also algebraic of order $-1/2$.

Proposition 3.119 *If z_0 is a singularity of $u(z; C)$ then in a neighborhood of z_0 we have*

$$u = \pm\sqrt{-1/2}(z - z_0)^{-1/2} A_0((z - z_0)^{1/2}) \tag{3.120}$$

where A_0 is analytic at zero and $A_0(0) = 1$.

Notes. 1. The local behavior near a singularity could have been guessed by local Painlevé analysis and the method of dominant balance, with the standard ansatz near a singularity, $u \sim Const.(z - z_0)^p$. Our results however are **global**:

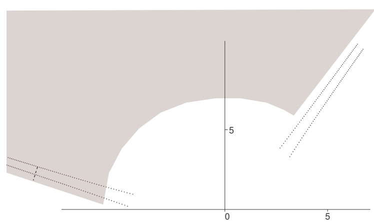


Figure 3.2: Singularities on the boundary of S_{trans} for (3.26). The gray region lies in the projection on $\mathbb{C}$ of the Riemann surface where (3.116) holds. The short dotted line is a generic cut delimiting a first Riemann sheet.

Proposition 3.115 gives the behavior of *a fixed* solution at infinitely many singularities, and gives the **position** of these singularities as soon as C_1 (or the position of only one of these singularities) is known (and in addition show that the power behavior ansatz is correct in this case).

2. By the substitution $y = v/(1+v)$ in (3.29) we get

$$v' = -v - 27\,\frac{v^3}{1+v} - 10\,v^2 + \frac{1}{5t}v + g^{[1]}(t^{-1}, v) \tag{3.121}$$

where $g^{[1]}$ is a now an $O(t^{-2}, v^{-2})$ polynomial of total degree 5. The singularities of v are at the points where $v(t) = -1$.

3.2b $\mathbf{P_I}$.

Proposition 3.122 below shows, in (i), how the constant C beyond all orders is associated to a truncated solution $y(z)$ of $\mathrm{P_I}$ for $\arg(z) = \pi$ (formula (3.123)) and gives the position of one array of poles z_n of the solution associated to C (formula (3.124)), and in (ii) provides uniform asymptotic expansion to all orders of this solution in a sector centered on $\arg(z) = \pi$ and one array of poles (except for small neighborhoods of these poles) in formula (3.126).

Proposition 3.122 *(i) Let y be a solution of (3.31) such that $y(z) \sim \sqrt{-z/6}$ for large z with $\arg(z) = \pi$. For any $\phi \in (\pi, \pi + \frac{2}{5}\pi)$ the following limit determines the constant C (which does not depend on ϕ in this range) in the transseries $\tilde{y}$ of y:*

$$\lim_{\substack{|z|\to\infty \\ \arg(z)=\phi}} \xi(z)^{-1} \left(\sqrt{\frac{6}{-z}}\, y(z) - \sum_{k \le |x(z)|} \frac{\tilde{y}_{0;k}}{z^{5k/2}} \right) = C \tag{3.123}$$

(Note that the constants $\tilde{y}_{0;k}$ do not *depend on C). With this definition, if $C \neq 0$, the function y has poles near the antistokes line* $\arg(z) = \pi + \frac{2}{5}\pi$ *at all points z_n, where, for large n*

$$z_n = -\frac{(60\pi i)^{4/5}}{24} \left(n^{\frac{4}{5}} + i L_n n^{-\frac{1}{5}} + \left(\frac{L_n^2}{8} - \frac{L_n}{4\pi} + \frac{109}{600\pi^2} \right) n^{-\frac{6}{5}} \right) + O\left(\frac{(\ln n)^3}{n^{\frac{11}{5}}} \right) \tag{3.124}$$

with $L_n = \frac{1}{5\pi} \ln\left(\frac{\pi i C^2}{72} n \right)$, or, more compactly,

$$\xi(z_n) = 12 + \frac{327}{(-24 z_n)^{5/4}} + O(z_n^{-5/2}) \quad (z_n \to \infty) \tag{3.125}$$

(ii) Let $\epsilon \in \mathbb{R}^+$ and define

$$\mathcal{Z} = \{ z : \arg(z) > \frac{3}{5}\pi; |\xi(z)| < 1/\epsilon;\ |\xi(z) - 12| > \epsilon \}$$

(the region starts at the antistokes line $\arg(z) = \frac{3}{5}\pi$ *and extends slightly beyond the next antistokes line,* $\arg(z) = \frac{7}{5}\pi$*). If $y \sim \sqrt{-z/6}$ as $|z| \to \infty$, $\arg(z) = \pi$, then for $z \in \mathcal{Z}$ we have*

$$y \sim \sqrt{\frac{-z}{6}} \left(1 - \frac{1}{8\sqrt{6}(-z)^{5/2}} + \sum_{k=0}^{\infty} \frac{30^k H_k(\xi)}{(-24z)^{5k/4}} \right) \qquad (|z| \to \infty,\ z \in \mathcal{Z}) \tag{3.126}$$

The functions H_k are rational, and $H_0(\xi) = \xi(\xi/12 - 1)^{-2}$. The expansion (3.126) holds uniformly in the sector $\pi^{-1}\arg(z) \in (3/5, 7/5)$ and also on one of its sides, where H_0 becomes dominant, down to an $o(1)$ distance of the actual poles of y if z is large.

Proof. We prove the corresponding statements for the normal form (3.35). To go back to the variables of (3.31) mere substitutions are needed, which we omit.

Most of Proposition 3.122 is a direct consequence of Theorems 1 and 2. For the one-parameter family of solutions which are small in the right half plane we then have

$$h \sim \sum_{k=0}^{\infty} x^{-k} H_k(\xi(x)) \tag{3.127}$$

As in the first example we find H_k by substituting (3.127) in (3.35).

The equation of H_0 is

$$\xi^2 H_0'' + \xi H_0' = H_0 + \frac{1}{2} H_0^2$$

The general solution of this equation are the Weierstrass elliptic functions of $\ln \xi$, as expected from the general knowledge of the asymptotic behavior of the Painlevé solutions (see [11]). For our special initial condition, H_0 analytic at zero and $H_0(\xi) = \xi(1 + o(1))$, the solution is a degenerate elliptic function, namely,

$$H_0(\xi) = \frac{\xi}{(\xi/12 - 1)^2}$$

Next, one of the two free constants in the general solution H_1 is determined by the condition of analyticity at zero of H_1 (this constant multiplies terms in $\ln \xi$). It is interesting to note that the remaining constant is only determined in the *next* step, when solving the equation for H_2! This pattern is typical (see §3.1k). Continuing this procedure we obtain successively:

$$H_1 = \left(216\,\xi + 210\,\xi^2 + 3\,\xi^3 - \frac{1}{60}\,\xi^4\right)(\xi - 12)^{-3} \tag{3.128}$$

$$H_2 = \left(1458\xi + 5238\xi^2 - \frac{99}{8}\xi^3 - \frac{211}{30}\xi^4 + \frac{13}{288}\xi^5 + \frac{\xi^6}{21600}\right)(\xi - 12)^{-4} \tag{3.129}$$

We omit the straightforward but quite lengthy inductive proof that all H_k are rational functions of ξ. The reason the calculation is tedious is that this property holds for (3.35) but *not* for its generic perturbations, and the last potential obstruction to rationality, successfully overcome by (3.35), is at $k = 6$. On the positive side, these calculations are algorithmic and are very easy to carry out with the aid of a symbolic language program.

In the same way as in Example 1 one can show that the corresponding singularities of h are double poles: all the terms of the corresponding asymptotic expansion of $1/h$ are *analytic* near the singularity of h! All this is again straightforward, and lengthy because of the potential obstruction at $k = 6$. We prefer to rely on an existing direct proof, see [5].

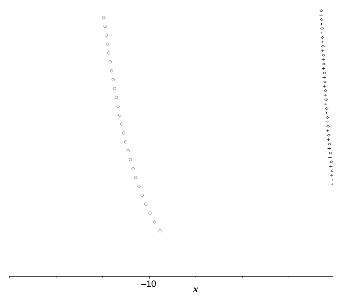


Figure 3.3: Poles of (3.35) for $C = -12$ ($\diamond$) and $C = 12$ ($+$), calculated via (3.130). The light circles are on the second line of poles for to $C = -12$.

Let ξ_s correspond to a zero of $1/h$. To leading order, $\xi_s = 12$, by Theorem 3.101 (iii). To find the next order in the expansion of ξ_s one substitutes $\xi_s = 12 + A/x + O(x^{-2})$, to obtain

$$1/h(\xi_s) = \frac{(A - 109/10)^2}{12^3 x^2} + O(1/x^3)$$

whence $A = 109/10$ (because $1/h$ is analytic at ξ_s) and we have

$$\xi_s = 12 + \frac{109}{10x} + O(x^{-2}) \tag{3.130}$$

Given a solution h, its constant C in ξ for which (3.127) holds can be calculated from asymptotic information in any direction above the real line by near least term truncation, namely

$$C = \lim_{\substack{x\to\infty \\ \arg(x)=\phi}} \exp(x) x^{1/2} \left(h(x) - \sum_{k \le |x|} \frac{\tilde{h}_{0,k}}{x^k} \right) \tag{3.131}$$

(this is a particular case of much more general formulas [9]) where $\sum_{k>0} \tilde{h}_{0,k} x^{-k}$ is the common asymptotic series of all solutions of (3.35) which are small in the right half plane.

□

General comments. 1. The expansion scales, x and $x^{-1/2}e^{-x}$ are crucial. Only for this choice one obtains an expansion which is valid both in S_{trans} and near poles of (3.35). For instance, the more general second scale $x^a e^{-x}$ introduces logarithmic singularities in H_j, except when $a \in -\frac{1}{2} + \mathbb{Z}$. With these logarithmic terms, the two scale expansion would only be valid in an $O(1)$ region in x, what is sometimes called a "patch at infinity", instead of more than a sector. Also, $a \in -\frac{1}{2} - \mathbb{N}$ introduces obligatory singularities at $\xi = 0$ precluding the validity of the expansion in S_{trans}. The case $a \in -\frac{1}{2} + \mathbb{N}$ produces instead an expansion valid in S_{trans} but not near poles. Indeed, the substitution $h(x) = g(x)/x^n$, $n \in \mathbb{N}$ has the effect of changing α to $\alpha + n$ in the normal form. This in turn amounts to restricting the analysis to a region far away from the poles, and then all H_j will be entire. In general we need thus to make (by substitutions in (1.91)) $a = \alpha$ minimal compatible with the assumptions (a1) and (a2), as this ensures the widest region of analysis.

2. The pole structure can be explored beyond the first array, in much of the same way: For large ξ induction shows that $H_n \sim Const_n.\xi^n$, suggesting a reexpansion for large ξ in the form

$$h \sim \sum_{k=0}^{\infty} \frac{H_k^{[1]}(\xi_2)}{x^k};\ \xi_2 = C^{[1]}\xi x^{-1} = C\, C^{[1]} x^{-3/2} e^{-x} \tag{3.132}$$

By the same techniques it can be shown that (3.132) holds and, by matching with (3.127) at $\xi_2 \sim x^{-2/3}$, we get $H_0^{[1]} = H_0$ with $C^{[1]} = -1/60$. Hence, if x_s belongs to the first line of poles, i.e. $\xi(x_s) = \xi_s$ cf. (3.130), the second line of poles is given by the condition

$$x_1^{-3/2} e^{-x_1} = -60 \cdot 12C$$

i.e., it is situated at a logarithmic distance of the first one:

$$x_1 - x_s = -\ln x_s + (2n+1)\pi i - \ln(60) + o(1)$$

(see Fig. 4). Similarly, on finds $x_{s,3}$ and in general $x_{s,n}$. The second scale for the $n-$th array is $x^{-n-1/2}e^{-x}$.

The expansion (3.127) can be however matched directly to an *adiabatic invariant*-like expansion valid throughout the sector where h has poles, similar to the one in [10]. In this language, the successive expansions of the form (3.132) pertain to the separatrix crossing region. We will not pursue this issue here.

3.2c The Painlevé equation P2

This equation reads:

$$y'' = 2y^3 + xy + \alpha \tag{3.133}$$

(Incidentally, this example also shows that for a given equation distinct solution manifolds associated to distinct asymptotic behaviors may lead to different normalizations.) After the change of variables

$$x = (3t/2)^{2/3}; \quad y(x) = x^{-1}(t\,h(t) - \alpha)$$

one obtains the normal form equation

$$h'' + \frac{h'}{t} - \left(1 + \frac{24\alpha^2 + 1}{9t^2}\right) h - \frac{8}{9}h^3 + \frac{8\alpha}{3t}h^2 + \frac{8(\alpha^3 - \alpha)}{9t^3} = 0 \tag{3.134}$$

and

$$\lambda_1 = 1, \quad \alpha_1 = -1/2; \; \xi = \frac{e^{-t}}{\sqrt{t}}; \quad \xi^2 F_0'' + \xi F_0' = F_0 + \frac{8}{9}F_0^3$$

The initial condition is (always): F_0 analytic at 0 and $F_0'(0) = 1$. This implies

$$F_0(\xi) = \frac{\xi}{1 - \xi^2/9}$$

Distinct normalizations (and sets of solutions) are provided by

$$x = (At)^{2/3}; \;\; y(x) = (At)^{1/3}\left(w(t) - B + \frac{\alpha}{2At}\right)$$

if $A^2 = -9/8, B^2 = -1/2$. In this case,

$$\begin{aligned} w'' + \frac{w'}{t} + w\left(1 + \frac{3B\alpha}{tA} - \frac{1 - 6\alpha^2}{9t^2}\right) w \\ - \left(3B - \frac{3\alpha}{2tA}\right) w^2 + w^3 + \frac{1}{9t^2}\left(B(1 + 6\alpha^2) - t^{-1}\alpha(\alpha^2 - 4)\right) \end{aligned} \tag{3.135}$$

so that

$$\lambda_1 = 1, \alpha_1 = -\frac{1}{2} - \frac{3}{2}\frac{B\alpha}{A}$$

implying

$$\xi^2 F_0'' + \xi F_0' - F_0 = 3BF_0^2 - F_0^3$$

and, with the same initial condition as above, we now have

$$F_0 = \frac{2\xi(1 + B\xi)}{\xi^2 + 2}$$

The first normalization applies for the manifold of solutions such that $y \sim -\frac{\alpha}{x}$ (for $\alpha = 0$ y is exponentially small and behaves like an Airy function) while the second one corresponds to $y \sim -B - \frac{\alpha}{2}x^{-3/2}$.

3.2d Singularity analysis

We now focus on singularities of $\mathbf{y}(x)$ and their connection with singularities of $\mathbf{F}_0$.

Definitions (cf. Figure 1)

Since

$$\tilde{\mathbf{y}}_{\mathbf{0}} = \sum_{r=2}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{0};r}}{x^r}, \quad (\ |x| \to \infty) \tag{3.136}$$

we have $\mathbf{F}_0(0) = 0$. Both $\mathbf{F}_0$ and $\mathbf{y}$ turn out to be analytic in S_{δ_1} (Theorems **??**(i) and 3.101(i)); the interesting region is then $\mathcal{E} \setminus S_{\delta_1}$ (containing the light grey region in Figure 1).

Denote by $\mathcal{P}$ a polydisk

$$\mathcal{P} = \{(x^{-1}, \mathbf{y}) : |x^{-1}| < \rho_1, |\mathbf{y}| < \rho_2\} \tag{3.137}$$

where $\mathbf{g}$ is analytic and continuous up to the boundary.

Let Ξ be a *finite* set (possibly empty) of points in the ξ-plane. This set will consist of singular points of $\mathbf{F}_0$ thus we assume $\mathrm{dist}(\Xi, 0) \geq \delta_1$.

Denote by $\mathcal{R}_\Xi$ the Riemann surface above $\mathbb{C} \setminus \Xi$. More precisely, we assume that $\mathcal{R}_\Xi$ is realized as equivalence classes of simple curves $\Gamma : [0,1] \mapsto \mathbb{C}$ with $\Gamma(0) = 0$ modulo homotopies in $\mathbb{C} \backslash \Xi$.

Let $\mathcal{D} \subset \mathcal{R}_\Xi$ be *open, relatively compact, and connected*, with the following properties:

(1) $\mathbf{F}_0(\xi)$ is analytic in an $\epsilon_{\mathcal{D}}$–neighborhood of $\mathcal{D}$ with $\epsilon_{\mathcal{D}} > 0$,

(2) $\sup_{\mathcal{D}} |\mathbf{F}_0(\xi)| := \rho_3$ with $\rho_3 < \rho_2$

(3) $\mathcal{D}$ contains $\{\xi : |\xi| < \delta_1\}$.[7]

It is assumed that there is an upper bound on the length of the curves joining points in $\mathcal{D}$: $d_{\mathcal{D}} = \sup_{a,b \in \mathcal{D}} \inf_{\Gamma \subset \mathcal{D}; a,b \in \Gamma} \mathrm{length}(\Gamma) < \infty$.

We also need the x-plane counterpart of this domain.

Let $R > 0$ (large) and let $X = \xi^{-1}(\Xi) \cap \{x \in \mathcal{E} : |x| > R\}$.

Let Γ be a curve in $\mathcal{D}$. There is a countable family of curves γ_N in the x-plane with $\xi(\gamma_N) = \Gamma$. The curves are smooth for $|x|$ large enough and satisfy

$$\gamma_N(t) \ = \ 2N\pi i + \alpha_1 \ln(2\pi i N) - \ln \Gamma(t) + \ln C_1 + o(1) \quad (N \ \to \ \infty) \tag{3.138}$$

(For a proof see [4]).

To preserve smoothness, we will restrict to $|x| > R$ with R large enough, so that along (a smooth representative of) each $\Gamma \in \mathcal{D}$, the branches of ξ^{-1} are analytic.

If the curve Γ is a smooth representative in $\mathcal{D}$ we then have $\xi^{-1}(\Gamma) = \cup_{N \in \mathbb{N}} \gamma_N$ where γ_N are smooth curves in $\{x : |x| > 2R\} \backslash X$.

[7] Conditions (2),(3) can be typically satisfied since $\mathbf{F}_0(\xi) = \xi + O(\xi^2)$ and $\delta_1 < \rho_2$ (see also the examples in §3.2); borderline cases may be treated after choosing a smaller δ_1.

We define $\mathcal{D}_x$ as the equivalence classes modulo homotopies in $\{x \in \mathcal{E} : |x| > R\} \setminus X$ (with ∞ fixed point) of those curves γ_N which are completely contained in $\mathcal{E} \cap \{x : |x| > 2R\}$.
Noting that $\left|\hat{M}(\xi, \xi_{1,0})\right| \mathrm{d}|s|$ is a finite measure along Γ, the proof is virtually identical to the proof of Theorem 3.101.

3.2e Appendix

Taking $\mathcal{L}^{-1}$ in (3.48) we get, with

$$\mathbf{D_j}(*\mathbf{Y}) = \sum_{\mathbf{l} \geq \mathbf{j}} \binom{\mathbf{l}}{\mathbf{j}} \left[\mathbf{G_l} * \mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})} + \mathbf{g}_{0,\mathbf{l}} * \mathbf{Y}_0^{*(\mathbf{l}-\mathbf{j})} \right]$$

$$\left(-p + \hat{\Lambda} - \mathbf{k} \cdot \boldsymbol{\lambda}\right) \mathbf{Y_k} + \left(\hat{B} + \mathbf{k} \cdot \boldsymbol{\alpha}\right) 1 * \mathbf{Y_k} + \mathbf{D_j}(*\mathbf{Y}) = \mathbf{T_k}(*\mathbf{Y}) \qquad (3.139)$$

$\mathbf{T_k}$ is now a *convolution* polynomial,

$$\mathbf{T_k}(*\mathbf{Y}) = \mathbf{T}\left(\mathbf{Y}_0, \{\mathbf{Y_{k'}}\}_{0 \prec \mathbf{k'} \prec \mathbf{k}}\right)$$

$$\mathbf{T_k}(*\mathbf{Y}) = \sum_{\mathbf{j} \leq k;\ |\mathbf{j}| > 1} \mathbf{D_j}(*\mathbf{Y}) * \sum_{(\mathbf{i}_{mp};\mathbf{k})} {\prod_{m=1}^{n_1}}^{*} {\prod_{p=1}^{j_m}}^{*} \left(\mathbf{Y}_{\mathbf{i}_{mp}}\right)_m \qquad (3.140)$$

where $\binom{\mathbf{l}}{j} = \prod_{j=1}^{n} \binom{l_i}{j_i}$, $(\mathbf{v})_m$ means the component m of $\mathbf{v}$, and $\sum_{(\mathbf{i}_{mp}:\mathbf{k})}$ stands for the sum over all vectors $\mathbf{i}_{mp} \in \mathbb{N}^n$, with $p \leq j_m, m \leq n$, such that $\mathbf{i}_{mp} > 0$ and $\sum_{m=1}^{n} \sum_{p=1}^{j_m} \mathbf{i}_{mp} = \mathbf{k}$;

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