Transseries

G. A. Edgar

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Contents

Abstract

An attempt to write consistent definitions and terminology. And notation. But probably not the best way for learning.

Introduction

Starred (**Example*) are examples to illustrate the definitions, but make use of the later notation from Section 3.

The correspondence between multi-indices and transmonomials reverses the ordering. This means terminology that seems right on one side may seem to be backward on the other side. For example, I change my mind on whether $\mathbf{J_m}$ should be called a ***filter*** or an ***ideal***. Even with conventional asymptotic series, larger terms are written to the left, smaller terms to the right, reversing the convention for a number line.

1 Multi-indices

Begin with a positive integer n. The set $\mathbb{Z}^n$ of n-tuples of integers is a group under componentwise addition. For notation, avoiding subscripts, if $\mathbf{k} \in \mathbb{Z}^n$ and $1 \le i \le n$, let's write $\mathbf{k}[i]$ for the ith component of $\mathbf{k}$. The partial order $\le$ is defined by: $\mathbf{k} \le \mathbf{p}$ iff $\mathbf{k}[i] \le \mathbf{p}[i]$ for all i. And $\mathbf{k} < \mathbf{p}$ iff $\mathbf{k} \le \mathbf{p}$ and $\mathbf{k} \ne \mathbf{p}$. Element $\mathbf{0} = (0, 0, \cdots, 0)$ is the identity for addition.

Write $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ including 0. Subset $\mathbb{N}^n$ is closed under addition.

Definition 1.1. A set $\mathbf{J} \subseteq \mathbb{Z}^n$ is a ***filter*** iff it is upward-saturated: if $\mathbf{m} \in \mathbf{J}$ and $\mathbf{k} \geq \mathbf{m}$, then $\mathbf{k} \in \mathbf{J}$. The filter ***generated*** by a set $E \subseteq \mathbb{Z}^n$ is

$$\mathbf{J}(E) = \{\, \mathbf{k} \in \mathbb{Z}^n : \mathbf{k} \geq \mathbf{p} \text{ for some } \mathbf{p} \in E \,\}.$$

The ***strict filter generated*** by E is

$$\mathbf{J}^*(E) = \{\, \mathbf{k} \in \mathbb{Z}^n : \mathbf{k} > \mathbf{p} \text{ for some } \mathbf{p} \in E \,\}.$$

The ***principal filter*** of $\mathbf{m}$ is the set $\mathbf{J_m} = \mathbf{J}(\{\mathbf{m}\}) = \{\, \mathbf{k} \in \mathbb{Z}^n : \mathbf{k} \geq \mathbf{m} \,\}$. The ***strict principal filter*** of $\mathbf{m}$ is

$$\mathbf{J}^*_\mathbf{m} = \mathbf{J}^*(\{\mathbf{m}\}) = \mathbf{J_m} \setminus \{\mathbf{m}\} = \{\, \mathbf{k} \in \mathbb{Z}^n : \mathbf{k} > \mathbf{m} \,\}.$$

Note $\mathbf{J_m}$ is the translate of $\mathbb{N}^n$ by $-\mathbf{m}$. That is, $\mathbf{J_m} = \{\, \mathbf{k} - \mathbf{m} : \mathbf{k} \in \mathbb{N}^n \,\}$. And $\mathbb{N}^n = \mathbf{J_0}$. Translation preserves order.

Proposition 1.2. *The set $\mathbf{J_m}$ is well-partially-ordered in the sense: if $E \subseteq \mathbf{J_m}$ and $E \neq \varnothing$, then there is a minimal element: $\mathbf{k}_0 \in E$ and $\mathbf{k} < \mathbf{k}_0$ holds for no element $\mathbf{k} \in E$.*

Proof. Because translation preserves order, it suffices to do the case of $\mathbf{J_0} = \mathbb{N}^n$. First, $\{\, \mathbf{k}[1] : \mathbf{k} \in E \,\}$ is a nonempty subset of $\mathbb{N}$, so it has a least element, say m_1. Then $\{\, \mathbf{k}[2] : \mathbf{k} \in E, \mathbf{k}[1] = m_1 \,\}$ is a nonempty subset of $\mathbb{N}$, so it has a least element, say m_2. Continue. Then $\mathbf{k}_0 = (m_1, \cdots, m_n)$ is minimal in E. □

Proposition 1.3. *Let $E \subseteq \mathbf{J_m}$ be infinite. Then there is a sequence $\mathbf{k}_j \in E$, $j \in \mathbb{N}$, with $\mathbf{k}_0 < \mathbf{k}_1 < \mathbf{k}_2 < \cdots$.*

Proof. Enough to do the case $\mathbb{N}^n$. By induction on n. True if $n = 1$. Consider the set $\widetilde{E} \subseteq \mathbb{Z}^{n-1}$ defined by $\{\, (\mathbf{k}[1], \mathbf{k}[2], \cdots, \mathbf{k}[n-1]) : \mathbf{k} \in E \,\}$. Case 1: $\widetilde{E}$ is finite. Then for some $\mathbf{p} \in \widetilde{E}$, the set $E' = \{\, k \in \mathbb{N} : (\mathbf{p}[1], \cdots, \mathbf{p}[n-1], k) \in E \,\}$ is infinite. Choose an increasing sequence $k_j \in E'$ to get the increasing sequence in E.

Case 2: $\widetilde{E}$ is infinite. By induction hypothesis, there is a strictly increasing sequence $\mathbf{p}_j \in \widetilde{E}$. So there is a sequence $\mathbf{k}_j \in E$ that is increasing in every coordinate except possibly the last. If some last coordinate occurs infinitely often, use it to get an increasing sequence in E. If not, choose a subsequence of these last coordinates that increases. □

Proposition 1.4. *Let $E \subseteq \mathbf{J_m}$. Then the set $\operatorname{Mag} E$ of all minimal elements of E is finite. For every $\mathbf{k} \in E$, there is $\mathbf{k}_0 \in \operatorname{Mag} E$ with $\mathbf{k}_0 \leq \mathbf{k}$.*

Proof. No two minimal elements are comparable, so $\operatorname{Mag} E$ is finite by Prop. 1.3. If $E = \varnothing$, then $\operatorname{Mag} E = \varnothing$ vacuously satisfies this. Suppose $E \neq \varnothing$. Then $\operatorname{Mag} E \neq \varnothing$ satisfies the required conclusion by Prop. 1.2. □

Convergence of sets

Write $\triangle$ for the symmetric difference operation on sets. We will define convergence a sequence of sets $E_j \subseteq \mathbb{Z}^n$ (or indeed any infinite collection $(E_i)_{i\in I}$ of sets). But we define convergence to $\varnothing$, and then let $E_j \to E$ mean $E_j \triangle E \to \varnothing$.

Definition 1.5. Let I be an infinite index set, and for each $i \in I$, let $E_i \subseteq \mathbb{Z}^n$. We say the family $(E_i)_{i\in I}$ is ***point-finite*** iff each $\mathbf{p} \in \mathbb{Z}^n$ belongs to E_i for only finitely many i. Let $\mathbf{m} \in \mathbb{Z}^n$. We write $E_i \xrightarrow{\mathbf{m}} \varnothing$ iff $E_i \subseteq \mathbf{J_m}$ for all i and (E_i) is point-finite. We write $E_i \to \varnothing$ iff there exists $\mathbf{m}$ such that $E_i \xrightarrow{\mathbf{m}} \varnothing$. Furthermore, write $E_i \xrightarrow{\mathbf{m}} E$ iff $E_i \triangle E \xrightarrow{\mathbf{m}} \varnothing$ and write $E_i \to E$ iff $E_i \triangle E \to \varnothing$.

This type of convergence is metrizable when restricted to any $\mathbf{J_m}$. But there is no preferred choice of metric.

Notation 1.6. For $\mathbf{k} = (k_1, k_2, \cdots, k_n)$, define $|\mathbf{k}| = k_1 + k_2 + \cdots + k_n$.

Proposition 1.7. *Let $\mathbf{m} \in \mathbb{Z}^n$. For $E, F \subseteq \mathbf{J_m}$, define*

$$d(E, F) = \sum_{\mathbf{k}\in E\triangle F} 2^{-|\mathbf{k}|}.$$

Then for any sets $E_i \subseteq \mathbf{J_m}$, we have $E_i \to E$ if and only if $d(E_i, E) \to 0$. And d is a metric on subsets of $\mathbf{J_m}$.

Dominating

Definition 1.8. Let E, F be subsets of $\mathbb{Z}^n$. We say E ***dominates*** F iff for every $\mathbf{k} \in F$, there is $\mathbf{p} \in E$ with $\mathbf{p} < \mathbf{k}$. Equivalently, in terms of the filters:

$$F \subseteq \mathbf{J}^*(E).$$

This may seem backward. But correspondingly in the realm of transmonomials, we will say larger monomials dominate smaller ones.

It's transitive: If E_1 dominates E_2 and E_2 dominates E_3, then E_1 dominates E_3. Every E dominates $\varnothing$. Note $\{\mathbf{m}\}$ dominates E if and only if $E \subseteq \mathbf{J}^*_\mathbf{m}$.

Proposition 1.9. *Let E, F be subsets of $\mathbf{J_m}$. Then E dominates F if and only if $\operatorname{Mag} E$ dominates $\operatorname{Mag} F$.*

Proof. Assume E dominates F. Let $\mathbf{k} \in \operatorname{Mag} F$. Then $\mathbf{k} \in F$, so there is $\mathbf{k}_1 \in E$ with $\mathbf{k}_1 < \mathbf{k}$. Then there is $\mathbf{k}_0 \in \operatorname{Mag} E$ with $\mathbf{k}_0 \leq \mathbf{k}_1$. So $\mathbf{k}_0 < \mathbf{k}$.

Conversely, assume $\operatorname{Mag} E$ dominates $\operatorname{Mag} F$. Let $\mathbf{k} \in F$. Then there is $\mathbf{k}_1 \in \operatorname{Mag} F$ with $\mathbf{k}_1 \leq \mathbf{k}$. So there is $\mathbf{k}_0 \in \operatorname{Mag} E$ with $\mathbf{k}_0 < \mathbf{k}_1$. Thus $\mathbf{k}_0 \in E$ and $\mathbf{k}_0 < \mathbf{k}$. ∎

Proposition 1.10. *If E dominates F, then $\operatorname{Mag} E$ and $\operatorname{Mag} F$ are disjoint.*

Proof. Assume E dominates F. If $\mathbf{k} \in \operatorname{Mag} F$, then $\mathbf{k} \in F$, so there is $\mathbf{k}_1 \in E$ with $\mathbf{k}_1 < \mathbf{k}$. So even if $\mathbf{k} \in E$, it is not minimal. ∎

Proposition 1.11. *Let $E_j \subseteq \mathbf{J_m}$, $j \in \mathbb{N}$, be an infinite sequence such that E_j dominates E_{j+1} for all j. Then the sequence (E_j) is point-finite; $E_j \to \varnothing$.*

Proof. Let $\mathbf{p} \in \mathbf{J_m}$. Then $F = \{\, \mathbf{k} \in \mathbf{J_m} : \mathbf{k} < \mathbf{p} \,\}$ is finite. But the sets $F \cap \operatorname{Mag} E_j$ are disjoint, and for every j with $\mathbf{p} \in E_j$, the set $F \cap \operatorname{Mag} E_j$ is nonempty. Therefore, $\mathbf{p} \in E_j$ for only finitely many j. □

Proposition 1.12. *Let $E_i \subseteq \mathbf{J_m}$ be a point-finite family. Assume E_i dominates F_i for all i. Then (F_i) is also point-finite.*

Proof. Let $\mathbf{p} \in \mathbf{J_m}$. Then $F = \{\, \mathbf{k} \in \mathbf{J_m} : \mathbf{k} < \mathbf{p} \,\}$ is finite. But the sets $F \cap \operatorname{Mag} E_j$ are disjoint, and for every j with $\mathbf{p} \in F_j$, the set $F \cap \operatorname{Mag} E_j$ is nonempty. Therefore, $\mathbf{p} \in F_j$ for only finitely many j. □

2 Abstract transseries

We begin with an abelian totally ordered group $\mathcal{G}$. The operation is written multiplicatively, the identity is 1, the order relation is $\gg$ and read "far larger than". This is a "strict" order relation; that is, $g \gg g$ is false. We use the field $\mathbb{R}$ of real numbers as "values", but as far as this section is concerned, any field would work. Later we do need real numbers as values.

2.1 Without generators

Write $\mathbb{R}^{\mathcal{G}}$ for the set of functions $T \colon \mathcal{G} \to \mathbb{R}$. For $T \in \mathbb{R}^{\mathcal{G}}$ and $g \in \mathcal{G}$, we will use square brackets $T[g]$ for the value of T at g—because later we will want to use round brackets $T(x)$ in another sense.

Definition 2.1. The ***support*** of a function $T \in \mathbb{R}^{\mathcal{G}}$ is

$$\operatorname{supp} T = \{\, g \in \mathcal{G} : T[g] \neq 0 \,\}.$$

Let $\Gamma \subseteq \mathcal{G}$. We say T is ***supported by*** Γ if $\operatorname{supp} T \subseteq \Gamma$.

Notation 2.2. In fact, T will usually be written as a formal combination of group elements. That is:

$$T = \sum_{g \in \Gamma} a_g g, \qquad a_g \in \mathbb{R}$$

will be used for the function T with $T[g] = a_g$ for $g \in \Gamma$ and $T[g] = 0$ otherwise. The set Γ may or may not be the actual support of T.

Definition 2.3. If $c \in \mathbb{R}$, then $c\,1 \in \mathbb{R}^{\mathcal{G}}$ is called a "constant" and identified with c. (That is, $T[1] = c$ and $T[g] = 0$ for all $g \neq 1$.) If $g_0 \in \mathcal{G}$, then $1\,g_0 \in \mathbb{R}^{\mathcal{G}}$ is called a "transmonomial" (or simply "monomial") and identified with g_0. (That is, $T[g_0] = 1$ and $T[g] = 0$ for all $g \neq g_0$.)

In all cases of interest to us, the support will be ***well ordered*** (according to the converse of $\gg$). That is, for all $\Gamma \subseteq \operatorname{supp}(T)$, if $\Gamma \neq \varnothing$, then there is $g_0 \in \Gamma$ such that for all $g \in \Gamma$, if $g \neq g_0$, then $g_0 \gg g$.

Proposition 2.4. *Let $\Gamma \subseteq \mathfrak{G}$ be well ordered for the converse of $\gg$. Every infinite subset in Γ contains an infinite strictly decreasing sequence $g_1 \gg g_2 \gg \cdots$. There is no infinite strictly increasing sequence in Γ.*

Definition 2.5. Let $T \neq 0$ be

$$T = \sum_{g \in \Gamma} a_g g, \qquad a_g \in \mathbb{R},$$

with $g_0 \in \Gamma$, $g_0 \gg g$ for all other $g \in \Gamma$, and $a_{g_0} \neq 0$. Then the ***magnitude*** of T is $\operatorname{mag} T = g_0$ and the ***dominance*** of T is $\operatorname{dom} T = a_{g_0} g_0$. We say T is ***positive*** if $a_{g_0} > 0$ and write $T > 0$. We say T is ***negative*** if $a_{g_0} < 0$ and write $T < 0$. We say T is ***large*** if $\operatorname{mag} T \gg 1$ (or $T = 0$). We say T is ***small*** if $\operatorname{mag} T \ll 1$ (or $T = 0$). We say T is ***purely large*** if $g \gg 1$ for all $g \in \operatorname{supp} T$.

Definition 2.6. ***Addition*** is defined by components. $(S + T)[g] = S[g] + T[g]$. The union of two well ordered sets is well ordered. ***Scalar multiples*** aT are also defined by components.

Notation 2.7. We say $S > T$ if $S - T > 0$. For nonzero S and T we say $S \gg T$ iff $\operatorname{mag} S \gg \operatorname{mag} T$, and we say $S \asymp T$ iff $\operatorname{mag} S = \operatorname{mag} T$.

Proposition 2.8. *Every T may be written uniquely in the form $T = L + c + s$, where L is purely large, c is a constant, and s is small.*

Definition 2.9. ***Multiplication*** is defined by convolution (as suggested by the formal sum notation).

$$\sum_{g \in \mathfrak{G}} a_g g \cdot \sum_{g \in \mathfrak{G}} b_g g = \sum_{g \in \mathfrak{G}} \left(\sum_{g_1 g_2 = g} a_{g_1} b_{g_2} \right) g,$$
$$\text{or} \qquad (ST)[g] = \sum_{g_1 g_2 = g} S[g_1] T[g_2]$$

Products are defined at least for S, T with well ordered support.

Proposition 2.10. *If $\Gamma_1, \Gamma_2 \subseteq \mathfrak{G}$ are well ordered sets (for the reverse of $\gg$), then*

$$\Gamma = \{\, g_1 g_2 : g_1 \in \Gamma_1, g_2 \in \Gamma_2 \,\}$$

is also well ordered. For every $g \in \Gamma$, the set $\{\, (g_1, g_2) : g_1 \in \Gamma_1, g_2 \in \Gamma_2, g_1 g_2 = g \,\}$ is finite.

Proof. Let $\Gamma' \subseteq \Gamma$ be nonempty. Assume Γ' has no greatest element. Then there exist $g_j \in \Gamma_1$ and $g'_j \in \Gamma_2$ with $g_1 g'_1 \ll g_2 g'_2 \ll \cdots$. Because Γ_1 is well ordered, taking a subsequence we may assume $g_1 \gg g_2 \gg \cdots$. But then $g'_1 \ll g'_2 \ll \cdots$, so Γ_2 is not well ordered.

Suppose $(g_1, g_2), (g_3, g_4) \in \Gamma_1 \times \Gamma_2$ with $g_1 g_2 = g = g_3 g_4$. If $g_1 \neq g_3$, then $g_2 \neq g_4$. If $g_1 \gg g_3$, then $g_2 \ll g_4$. Any infinite subset of a well ordered set contains an infinite strictly decreasing sequence, but the other well ordered set contains no infinite strictly increasing sequence. ∎

Proposition 2.11. *The set of all* $T \in \mathbb{R}^{\mathcal{G}}$ *with well ordered support is an algebra over* $\mathbb{R}$ *with the operations defined.*

In algebra, this is called the Malcev–Neumann construction. In fact this is a field. That proof uses the Joe Kruskal theorem (?). But we don't need that result.

Proposition 2.12. *Every nonzero* T *with well ordered support may be written uniquely in the form* $T = (\operatorname{dom} T) \cdot (1 + s)$ *where* s *is small.*

Proposition 2.13. *The set of all purely large* T *(including* 0*) is a group under addition. The set of all small* T *is a group under addition. The set of all purely large* T *(with well ordered support) is closed under multiplication. The set of all small* T *(with well ordered support) is closed under multiplication.*

Definition 2.14. ***Series*** are (provisionally) defined by components. If I is an index set, and for each $i \in I$ we are given some $T_i \in \mathbb{R}^{\mathcal{G}}$, then the series

$$T = \sum_{i \in I} T_i$$

is defined iff the family $(\operatorname{supp} T_i)$ of supports is point-finite. Of course, even if $\operatorname{supp} T_i$ is well ordered for all i, it will not follow that $\operatorname{supp} T$ is well ordered.

Definition 2.15. ***Limits*** are (provisionally) defined by components. (And the topology used for the set $\mathbb{R}$ of values is discrete.) That is: Suppose for all $n \in \mathbb{N}$, T_n is given. If, for all $g \in G$ there is $n_g \in \mathbb{N}$ such that $T_n[g]$ is the same for all $n \geq n_0$, then $T = \lim T_n$ is defined by $T[g] = T_{n_g}[g]$. Again, this is not enough to insure $\operatorname{supp} T$ well ordered—We will re-define limits again later. For general infinite index set I, define $T_i \to 0$ iff the family $(\operatorname{supp} T_i)$ is point-finite. And $T_i \to T$ iff $T_i - T \to 0$.

2.2 With generators

Some definitions will depend on a finite set of "generators". We will keep track of the set of generators more than is customary. But it is useful for the proofs, and essential for Costin's fixed-point theorem (Prop. 2.47).

Notation 2.16. $\mathcal{G}^{\text{small}} = \{\, g \in \mathcal{G} : g \ll 1 \,\}$.

We begin with a finite set $\boldsymbol{\mu} \subset \mathcal{G}^{\text{small}}$. If convenient, we may number the elements of $\boldsymbol{\mu}$ in order, $\mu_1 \gg \mu_2 \gg \cdots \gg \mu_n$.

Notation 2.17. Let $\boldsymbol{\mu} = \{\mu_1, \cdots, \mu_n\} \subseteq \mathcal{G}^{\text{small}}$. For any multi-index $\mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{Z}^n$, define $\boldsymbol{\mu}^{\mathbf{k}} = \mu_1^{k_1} \cdots \mu_n^{k_n}$.

If $\mathbf{k} > \mathbf{p}$, then $\boldsymbol{\mu}^{\mathbf{k}} \ll \boldsymbol{\mu}^{\mathbf{p}}$. Also $\boldsymbol{\mu}^{\mathbf{0}} = 1$. If $\mathbf{k} > \mathbf{0}$ then $\boldsymbol{\mu}^{\mathbf{k}} \ll 1$ (but not in general conversely).

**Example* 2.18. Let $\boldsymbol{\mu} = \{x^{-1}, e^{-x}\}$. Then $1 \gg \mu_1^{-1}\mu_2 = xe^{-x}$, even though $(-1, 1) \not> (0, 0)$.

**Example* 2.19. The correspondence $\mathbf{k} \mapsto \boldsymbol{\mu}^{\mathbf{k}}$ may fail to be injective. Let $\boldsymbol{\mu} = \{x^{-1/3}, x^{-1/2}\}$. Then $\mu_1^3 = \mu_2^2$.

Proposition 2.20. *Let $\boldsymbol{\mu}$ and $\mathbf{m}$ be given. The principal filter of $\mathbf{m}$ in $\mathbb{Z}^n$ defines a set in $\mathcal{G}$ by*

$$\Gamma^{\boldsymbol{\mu},\mathbf{m}} = \left\{ \boldsymbol{\mu}^{\mathbf{k}} : \mathbf{k} \geq \mathbf{m} \right\}.$$

Then $\Gamma^{\boldsymbol{\mu},\mathbf{m}}$ is well ordered in $\mathcal{G}$.

Proof. Let $F \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$ be nonempty. Define $E = \left\{ \mathbf{k} \in \mathbf{J_m} : \boldsymbol{\mu}^{\mathbf{k}} \in F \right\}$. Then the set $\operatorname{Mag} E$ of minimal elements of E is finite. So $\max \left\{ \boldsymbol{\mu}^{\mathbf{k}} : \mathbf{k} \in \operatorname{Mag} E \right\}$ is the greatest element of F. □

Proposition 2.21. *Given $\boldsymbol{\mu}, \mathbf{m}, g$, there are only finitely many $\mathbf{k} \in \mathbf{J_m}$ with $\boldsymbol{\mu}^{\mathbf{k}} = g$.*

Proof. Suppose there are infinitely many $\mathbf{k} \in \mathbf{J_m}$ with $\boldsymbol{\mu}^{\mathbf{k}} = g$. By Prop. 1.3, this includes $\mathbf{k}_1 < \mathbf{k}_2$. But $\boldsymbol{\mu}^{\mathbf{k}_1} \gg \boldsymbol{\mu}^{\mathbf{k}_2}$. □

The map $\mathbf{k} \mapsto \boldsymbol{\mu}^{\mathbf{k}}$ may not be one-to-one, but it is finite-to-one. So: if $(g_i)_{i\in I}$ is a family of monomials in $\Gamma^{\boldsymbol{\mu},\mathbf{m}}$, define $E_i = \left\{ \mathbf{k} \in \mathbf{J_m} : \boldsymbol{\mu}^{\mathbf{k}} = g_i \right\}$, then $(\operatorname{supp} g_i)$ is point-finite if and only if (E_i) is point-finite.

Definition 2.22. ***Transseries*** generated by $\boldsymbol{\mu}$.

$$\begin{aligned} \mathcal{T}^{\boldsymbol{\mu},\mathbf{m}} &= \left\{ T \in \mathbb{R}^{\mathcal{G}} : \operatorname{supp} T \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}} \right\}, \\ \mathcal{T}^{\boldsymbol{\mu}} &= \bigcup_{\mathbf{m}\in\mathbb{Z}^n} \mathcal{T}^{\boldsymbol{\mu},\mathbf{m}}, \\ \mathcal{T}^{\mathcal{G}} &= \bigcup_{\boldsymbol{\mu}} \mathcal{T}^{\boldsymbol{\mu}}. \end{aligned}$$

In this union, all finite sets $\boldsymbol{\mu}$ are allowed, so all values of n are allowed. But each transseries is generated only by a finite set $\boldsymbol{\mu}$. Each is supported by one of the well ordered sets $\Gamma^{\boldsymbol{\mu},\mathbf{m}}$.

If $\boldsymbol{\mu} \subseteq \widetilde{\boldsymbol{\mu}}$, then $\mathcal{T}^{\boldsymbol{\mu}} \subseteq \mathcal{T}^{\widetilde{\boldsymbol{\mu}}}$ in a natural way. If $\mathcal{G}$ is a subgroup of $\widetilde{\mathcal{G}}$ and inherits the order, then $\mathcal{T}^{\mathcal{G}} \subseteq \mathcal{T}^{\widetilde{\mathcal{G}}}$ in a natural way.

**Example* 2.23. The series

$$\sum_{j=1}^{\infty} x^{1/j} = x^1 + x^{1/2} + x^{1/3} + x^{1/4} + \dots,$$

despite having well ordered support, does not belong to $\mathcal{T}^{\mathcal{G}}$. It is not finitely generated.

Manifestly small

Definition 2.24. If g may be written in the form $\boldsymbol{\mu}^{\mathbf{k}}$ with $\mathbf{k} > \mathbf{0}$, then g is ***μ*-*small***, written $g \ll^{\boldsymbol{\mu}} 1$. [For emphasis, ***manifestly μ-small***.] Let $\mathcal{G}^{\boldsymbol{\mu}\text{ small}}$ be the set of all $\boldsymbol{\mu}$-small transmonomials. Or: $\mathcal{G}^{\boldsymbol{\mu}\text{ small}} = \Gamma^{\boldsymbol{\mu},\mathbf{0}} \setminus \{1\}$. If $\operatorname{supp} T \subseteq \mathcal{G}^{\boldsymbol{\mu}\text{ small}}$ then T is ***μ-small***, written $T \ll^{\boldsymbol{\mu}} 1$.

Definition 2.25. ***Limits*** of transseries. Let $T_j, T \in \mathfrak{T}^{\mathcal{G}}$ for $j \in \mathbb{N}$. Then: $T_j \xrightarrow{\boldsymbol{\mu},\mathbf{m}} T$ means: $\operatorname{supp} T_j \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$ for all j, and $T_j \to T$ in the componentwise sense of Definition 2.15. $T_j \xrightarrow{\boldsymbol{\mu}} T$ means there exists $\mathbf{m}$ such that $T_j \xrightarrow{\boldsymbol{\mu},\mathbf{m}} T$. $T_j \to T$ means there exists $\boldsymbol{\mu}$ such that $T_j \xrightarrow{\boldsymbol{\mu}} T$.

It seems there is no reason for a countable index set, so use this also for other infinite index sets. The non-provisional definition: $T_i \to 0$ iff there is $\boldsymbol{\mu}, \mathbf{m}$ so that $\operatorname{supp} T_i \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$ for all i, and the family $(\operatorname{supp} T_i)$ is point-finite. Also, $T_i \to T$ iff $T_i - T \to 0$.

**Example* 2.26. The sequence $(x^j)_{j\in\mathbb{N}}$ converges (to 0) in the sense of Definition 2.15, but not in this new sense. It is not contained in any well ordered $\Gamma^{\boldsymbol{\mu},\mathbf{m}}$.

Proposition 2.27 (Continuity)**.** *Let I be an index set. If $S_i \to S$ and $T_i \to T$, then $S_i + T_i \to S + T$ and $S_i T_i \to ST$.*

Proof. We may increase $\boldsymbol{\mu}$ and decrease $\mathbf{m}$ to arrange $S_i \xrightarrow{\boldsymbol{\mu},\mathbf{m}} S$ and $T_i \xrightarrow{\boldsymbol{\mu},\mathbf{m}} T$ for the same $\boldsymbol{\mu}, \mathbf{m}$. Then $S_i + T_i \xrightarrow{\boldsymbol{\mu},\mathbf{m}} S + T$. and $S_i T_i \xrightarrow{\boldsymbol{\mu},\mathbf{p}} ST$ for $\mathbf{p} = 2\mathbf{m}$. To see this: let $g \in \Gamma^{\boldsymbol{\mu},\mathbf{p}}$. There are finitely many pairs $g_1, g_2 \in \Gamma^{\boldsymbol{\mu},\mathbf{k}}$ such that $g_1 g_2 = g$ (Prop. 2.10). So there is a single finite $I_0 \subseteq I$ outside of which $S_i[g_1] = S[g_1]$ and $T_i[g_2] = T[g_2]$ for all such g_1, g_2. For such i, we also have $(S_i T_i)[g] = (ST)[g]$. ◻

Definition 2.28. ***Series*** of transseries. Let $T_i, T \in \mathfrak{T}^{\mathcal{G}}$ for i in some index set I. Then

$$T = \sum_{i\in I} T_i$$

means: there exist $\boldsymbol{\mu}$ and $\mathbf{m}$ such that all $\operatorname{supp} T_i \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$ for all i; for all g, the set $I_g = \{\, i \in I : T_i[g] \neq 0 \,\}$ is finite; and $T[g] = \sum_{i\in I_g} T_i[g]$.

Proposition 2.29. *If $T \in \mathfrak{T}^{\mathcal{G}}$, then the "formal combination of group elements" that specifies T in fact converges to T in this sense as well.*

Note we have the "Freshman" (or ultrametric) Cauchy criterion: Series $\sum T_i$ converges if and only if $T_i \to 0$.

Proposition 2.30. *Let $s \in \mathfrak{T}^{\boldsymbol{\mu}}$ be $\boldsymbol{\mu}$-small. Then $(s^j)_{j\in\mathbb{N}} \xrightarrow{\boldsymbol{\mu}} 0$.*

Proof. Every transmonomial in $\operatorname{supp} s$ can be written in the form $\boldsymbol{\mu}^{\mathbf{k}}$ with $\mathbf{k} > \mathbf{0}$. The product of two of these is again one of these. Let $g_0 \in \mathcal{G}$. If g_0 is not $\boldsymbol{\mu}$-small, then $g_0 \in \operatorname{supp}(s^j)$ for no j. So assume g_0 is $\boldsymbol{\mu}$-small. Then there are just finitely many $\mathbf{p} > \mathbf{0}$ such that $g_0 = \boldsymbol{\mu}^{\mathbf{p}}$. Let

$$N = \max\{\, |\mathbf{p}| : \mathbf{p} > \mathbf{0}, \boldsymbol{\mu}^{\mathbf{p}} = g_0 \,\}.$$

Now let $j > N$. Since every $g \in \operatorname{supp} s$ is $\boldsymbol{\mu}^{\mathbf{k}}$ with $|\mathbf{k}| \geq 1$, we see that every element of $\operatorname{supp}(s^j)$ is $\boldsymbol{\mu}^{\mathbf{k}}$ with $|\mathbf{k}| \geq j$. So $g_0 \notin \operatorname{supp}(s^j)$. This shows the family $(\operatorname{supp}(s^j))$ is point-finite. ☐

Proposition 2.31. *Let $T \in \mathfrak{T}^{\boldsymbol{\mu}}$ be small. Then there is a (possibly larger) finite set $\widetilde{\boldsymbol{\mu}} \subseteq \mathfrak{G}^{\text{small}}$ such that T is manifestly $\widetilde{\boldsymbol{\mu}}$-small.*

Proof. Let $T \in \mathfrak{T}^{\boldsymbol{\mu},\mathbf{m}}$. If $T = 0$, there is nothing to do, so assume $T \neq 0$. Then $\operatorname{supp} T \neq \varnothing$. Define $E = \left\{\, \mathbf{k} \in \mathbf{J}_{\mathbf{m}} : \boldsymbol{\mu}^{\mathbf{k}} \in \operatorname{supp} T \,\right\}$. By Prop. 1.4, $\operatorname{Mag} E$ is finite. Let $\widetilde{\boldsymbol{\mu}} = \boldsymbol{\mu} \cup \left\{\, \boldsymbol{\mu}^{\mathbf{k}} : \mathbf{k} \in \operatorname{Mag} E \,\right\}$. Note $\widetilde{\boldsymbol{\mu}} \subset \mathfrak{G}^{\text{small}}$. Now for any $g \in \operatorname{supp} T$, there is $\mathbf{p} \in E$ with $\boldsymbol{\mu}^{\mathbf{p}} = g$, and then there is $\mathbf{k} \in \operatorname{Mag} E$ with $\mathbf{p} \geq \mathbf{k}$, so that $\widetilde{g} = \boldsymbol{\mu}^{\mathbf{k}} \in \widetilde{\boldsymbol{\mu}}$ and $g = \widetilde{g}\boldsymbol{\mu}^{\mathbf{p}-\mathbf{k}}$ which is manifestly $\widetilde{\boldsymbol{\mu}}$-small. ☐

Perhaps call $\widetilde{\boldsymbol{\mu}} \setminus \boldsymbol{\mu}$ the ***addendum***, or ***smallness addendum*** for T. [Costin suggests: $\widetilde{\boldsymbol{\mu}}$ is the ***resolution*** of T; $\widetilde{\boldsymbol{\mu}} \setminus \boldsymbol{\mu}$ the ***$\boldsymbol{\mu}$-spawn*** of T]

**Example* 2.32. The corresponding statement for purely large T is false. The transseries

$$T = \sum_{j=0}^{\infty} x^{-j} e^{x}$$

is purely large, but there is no finite set $\boldsymbol{\mu} \subseteq \mathfrak{G}^{\text{small}}$ and multi-index $\mathbf{m}$ such that all $x^{-j}e^{x}$ have the form $\boldsymbol{\mu}^{\mathbf{k}}$ with $\mathbf{m} \leq \mathbf{k} < \mathbf{0}$. This is because the set $\{\, \mathbf{k} : \mathbf{m} \leq \mathbf{k} < \mathbf{0} \,\}$ is finite.

Proposition 2.33. *Let $T \in \mathfrak{T}^{\mathfrak{G}}$ be small. Then $(T^j)_{j\in\mathbb{N}} \to 0$.*

Proof. First, $T \in \mathfrak{T}^{\boldsymbol{\mu}}$ for some $\boldsymbol{\mu}$. Then T is manifestly $\widetilde{\boldsymbol{\mu}}$-small for some $\widetilde{\boldsymbol{\mu}} \supseteq \boldsymbol{\mu}$. Therefore $T^j \xrightarrow{\widetilde{\boldsymbol{\mu}}} 0$, so $T^j \to 0$. ☐

Proposition 2.34. *Let $\sum_{j=0}^{\infty} c_j z^j$ be a power series (even one with radius of convergence zero). If s is a small transseries, then $\sum_{j=0}^{\infty} c_j s^j$ converges.*

Proof. Use Prop. 2.31. We need to add the smallness addendum of s to $\boldsymbol{\mu}$. ☐

Proposition 2.35. *Let $s_1, \cdots, s_m$ be $\boldsymbol{\mu}$-small transseries. Let $p_1, \cdots, p_m \in \mathbb{Z}$. Then the family*

$$\left\{ \operatorname{supp}\left(s_1^{j_1} s_2^{j_2} \cdots s_m^{j_m}\right) : j_1 \geq p_1, \ldots j_m \geq p_m \right\}$$

is point-finite. That is, all multiple series of the form

$$\sum_{j_1=p_1}^{\infty} \sum_{j_2=p_2}^{\infty} \cdots \sum_{j_m=p_m}^{\infty} c_{j_1 j_2 \ldots j_m} s_1^{j_1} \cdots s_m^{j_m}$$

are $\boldsymbol{\mu}$-convergent.

Proof. An induction on m shows that we may assume $p_1 = \cdots = p_m = 1$, since the series with general p_i and the series with all $p_i = 1$, differ from each other by a finite number of series with fewer summations. So assume $p_1 = \cdots = p_m = 1$.

Let $g_0 \in \mathcal{G}$. If g_0 is not $\boldsymbol{\mu}$-small, then $g_0 \in \operatorname{supp}\left(s_1^{j_1} \cdots s_m^{j_m}\right)$ for no $j_1, \cdots, j_m$. So assume g_0 is $\boldsymbol{\mu}$-small. There are finitely many $\mathbf{k} > \mathbf{0}$ so that $\boldsymbol{\mu}^{\mathbf{k}} = g_0$. Let

$$N = \max\left\{\, |\mathbf{k}| : \mathbf{k} > \mathbf{0}, \boldsymbol{\mu}^{\mathbf{k}} = g_0 \,\right\}.$$

Each s_i has the form $\boldsymbol{\mu}^{\mathbf{k}}$ with $|\mathbf{k}| \geq 1$. So if $j_1 + \cdots + j_m > N$, we have $g_0 \notin \operatorname{supp}\left(s_1^{j_1} \cdots s_m^{j_m}\right)$. □

Proposition 2.36. *Let $T \in \mathfrak{T}^{\boldsymbol{\mu}}$ be nonzero. Then there is a (possibly larger) finite set $\widetilde{\boldsymbol{\mu}} \subseteq \mathcal{G}^{\mathrm{small}}$ and $S \in \mathfrak{T}^{\widetilde{\boldsymbol{\mu}}}$ such that $ST = 1$. The set $\mathfrak{T}^{\mathcal{G}}$ of all $\mathcal{G}$-transseries is a field.*

Proof. Write $T = a\boldsymbol{\mu}^{\mathbf{k}}(1+s)$, where $a \in \mathbb{R}, a \neq 0$, $\mathbf{k} \in \mathbb{Z}^n$, and s is small. Then the inverse S is:

$$S = a^{-1}\boldsymbol{\mu}^{-\mathbf{k}} \sum_{j=0}^{\infty} (-1)^j s^j.$$

Now a^{-1} is computed in the reals. For the series, use Prop. 2.34. Let $\widetilde{\boldsymbol{\mu}}$ be $\boldsymbol{\mu}$ plus the smallness addendum for s. □

We will call $\widetilde{\boldsymbol{\mu}} \setminus \boldsymbol{\mu}$ the ***inversion addendum*** for T.

$\boldsymbol{\mu}$-order

Proposition 2.37. *The set $\Gamma^{\boldsymbol{\mu},\mathbf{m}}$ is well-partially-ordered for $\gg^{\boldsymbol{\mu}}$. That is: if $E \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$, then there is a $\boldsymbol{\mu}$-maximal element: $g_0 \in E$ and $g \gg^{\boldsymbol{\mu}} g_0$ for no $g \in E$.*

Proof. If $\mathbf{p}$ is minimal in $\left\{\, \mathbf{k} \in \mathbf{J_m} : \boldsymbol{\mu}^{\mathbf{k}} \in E \,\right\}$, then $\boldsymbol{\mu}^{\mathbf{p}}$ is $\boldsymbol{\mu}$-maximal in E. □

Proposition 2.38. *Let $E \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$ be infinite. Then there is a sqeuence $g_j \in E$, $j \in \mathbb{N}$, with $g_0 \gg^{\boldsymbol{\mu}} g_1 \gg^{\boldsymbol{\mu}} g_2 \gg^{\boldsymbol{\mu}} \cdots$.*

Proposition 2.39. *Let $E \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$. Then the set $\operatorname{Mag}^{\boldsymbol{\mu}} E$ of maximal elements of E is finite. For every $g \in E$ there is $g_0 \in \operatorname{Mag}^{\boldsymbol{\mu}} E$ with $g \ll^{\boldsymbol{\mu}} g_0$.*

Definition 2.40. Let $E, F \subseteq \mathcal{G}$. We say E ***$\boldsymbol{\mu}$-dominates*** F iff for all $g \in F$ there exists $\widetilde{g} \in E$ such that $\widetilde{g} \gg^{\boldsymbol{\mu}} g$. We say S ***$\boldsymbol{\mu}$-contracts to*** T iff $\operatorname{supp} S$ $\boldsymbol{\mu}$-dominates $\operatorname{supp} T$.

If s is $\boldsymbol{\mu}$-small, then T $\boldsymbol{\mu}$-contracts to Ts.

Proposition 2.41. *Let $E, F \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$. Then E $\boldsymbol{\mu}$-dominates F if and only if $\operatorname{Mag}^{\boldsymbol{\mu}} E$ $\boldsymbol{\mu}$-dominates $\operatorname{Mag}^{\boldsymbol{\mu}} F$.*

If E $\boldsymbol{\mu}$-dominates F, then $\operatorname{Mag}^{\boldsymbol{\mu}} E$ and $\operatorname{Mag}^{\boldsymbol{\mu}} F$ are disjoint.

Proposition 2.42. *Let $E_j \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$, $j \in \mathbb{N}$, be an infinite sequence such that E_j $\boldsymbol{\mu}$-dominates E_{j+1} for all j. Then the sequence (E_j) is point-finite.*

Proposition 2.43. *Let $E_i \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$ be a point-finite family. Assume E_i $\boldsymbol{\mu}$-dominates F_i for all i. Then (F_i) is also point-finite.*

Contraction

Definition 2.44. Let J be linear from some subspace of $\mathfrak{T}^{\boldsymbol{\mu}}$ to itself. Then we say J is ***$\boldsymbol{\mu}$-contractive*** iff T $\boldsymbol{\mu}$-contracts to $J(T)$ for all T in the subspace.

Definition 2.45. Let J be possibly non-linear from some subset of $\mathfrak{T}^{\boldsymbol{\mu}}$ to itself. Then we say J is ***$\boldsymbol{\mu}$-contractive*** iff $S - T$ $\boldsymbol{\mu}$-contracts to $J(S) - J(T)$ for all S, T in the subset.

There is an easy way to define a linear $\boldsymbol{\mu}$-contractive map J on $\mathfrak{T}^{\boldsymbol{\mu},\mathbf{m}}$. If J is defined on all monomials $g \in \Gamma \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$ and g contracts to $J(g)$ for them, then the family $(\operatorname{supp} J(g))$ is point-finite by Prop. 2.43, so

$$J\left(\sum c_g g\right) = \sum c_g J(g)$$

$\boldsymbol{\mu}$-converges and defines J on the span.

**Example* 2.46. The set $\boldsymbol{\mu}$ of generators is important. We cannot simply replace "$\boldsymbol{\mu}$-small" by "small" in the definitions. Suppose $J(x^{-j}) = x^j e^{-x}$ for all $j \in \mathbb{N}$, and $J(g) = gx^{-1}$ for all other monomials. Then $g \gg J(g)$ for all g. But $J(\sum x^{-j})$ evaluated pointwise is not a legal transseries. Or: Define $J(x^{-j}) = e^{-x}$ for all $j \in \mathbb{N}$, and $J(g) = gx^{-1}$ for all other monomials. Again $g \gg J(g)$ for all g, but the family $\operatorname{supp} J(x^{-j})$ is not point-finite.

Proposition 2.47. (i) *If J is linear and $\boldsymbol{\mu}$-contractive on $\mathfrak{T}^{\boldsymbol{\mu},\mathbf{m}}$, then for any $T_0 \in \mathfrak{T}^{\boldsymbol{\mu},\mathbf{m}}$, the fixed-point equation $T = J(T) + T_0$ has a unique solution $T \in \mathfrak{T}^{\boldsymbol{\mu},\mathbf{m}}$.* (ii) *If $A \subseteq \mathfrak{T}^{\boldsymbol{\mu},\mathbf{m}}$ is closed, and $J\colon A \to A$ is $\boldsymbol{\mu}$-contractive on A, then $T = J(T)$ has a unique solution in A.*

Proof. (i) follows from (ii), since if J is linear and $\boldsymbol{\mu}$-contractive, then $\widetilde{J}$ defined by $\widetilde{J}(T) = J(T) + T_0$ is $\boldsymbol{\mu}$-contractive.

(ii) First note J is $\boldsymbol{\mu}$-continuous: Assume $T_j \xrightarrow{\boldsymbol{\mu}} T$. Then $T_j - T \xrightarrow{\boldsymbol{\mu}} 0$, so $(\operatorname{supp}(T_j - T))$ is point-finite. But $\operatorname{supp}(T_j - T)$ $\boldsymbol{\mu}$-dominates $\operatorname{supp}(J(T_j) - J(T))$, so $(\operatorname{supp}(J(T_j) - J(T))$ is also point-finite by Prop. 2.43. And so $J(T_j) \xrightarrow{\boldsymbol{\mu}} J(T)$.

Existence: Define $T_{j+1} = J(T_j)$. We claim T_j is $\boldsymbol{\mu}$-convergent. The sequence $E_j = \operatorname{supp}(T_j - T_{j+1})$ satisfies: E_j $\boldsymbol{\mu}$-dominates E_{j+1} for all j, so (Prop. 2.42) (E_j) is point-finite, which means $T_j - T_{j+1} \xrightarrow{\boldsymbol{\mu}} 0$ and therefore (by Freshman Cauchy) T_j $\boldsymbol{\mu}$-converges. Difference preserves $\boldsymbol{\mu}$-limits, so the limit T satisfies $J(T) = T$.

Uniqueness: if T_1 and T_2 were two different solutions, then $J(T_1) - J(T_2) = T_1 - T_2$, which contradicts $\boldsymbol{\mu}$-contractivity. □

3 Transseries as $x \to \infty$

We recursively construct the group $\mathfrak{G}$ to be used.

3.1 Without logs

A dummy symbol "x" appears in the notation. When we think of a transseries as describing behavior as $x \to \infty$, then x is supposed to be a large parameter. When we write "compositions" involving transseries, x represents the identity function. But usually it is just a convenient symbol.

Definition 3.1. Group $\mathcal{G}_0$ is isomorphic to the reals with addition and the usual ordering. To fit our applications, we write the group element corresponding to $b \in \mathbb{R}$ as x^b. Then $x^a x^b = x^{a+b}$; $x^0 = 1$; x^{-b} is the inverse of x^b; $x^a \ll x^b$ iff $a < b$.

Log-free transseries of level zero are those defined from this group as in Definition 2.22. Write $\mathcal{T}_0 = \mathcal{T}^{\mathcal{G}_0}$. Then the set of purely large transseries in $\mathcal{T}_0$ (including 0) is closed under addition.

Definition 3.2. Group $\mathcal{G}_1$ consists of ordered pairs (b, L) but written $x^b e^L$, where $b \in \mathbb{R}$ and $L \in \mathcal{T}_0$ is purely large. Define the group operations: $(x^b e^L)\,(x^{\tilde b} e^{\widetilde L}) = x^{b+\tilde b}\, e^{L+\widetilde L}$. Define order lexicographically: $(x^b e^L) \gg (x^{\tilde b} e^{\widetilde L})$ iff either $L > \widetilde L$ or $\{L = \widetilde L$ and $b > \tilde b\}$. Identify $\mathcal{G}_0$ as a subgroup of $\mathcal{G}_1$, where x^b is identified with $x^b e^0$.

Log-free transseries of level 1 are those defined from this group as in Definition 2.22. Write $\mathcal{T}_1 = \mathcal{T}^{\mathcal{G}_1}$. We may identify $\mathcal{T}_0$ as a subset of $\mathcal{T}_1$. Then the set of purely large transseries in $\mathcal{T}_1$ (including 0) is closed under addition.

Definition 3.3. Suppose log-free transmonomials $\mathcal{G}_N$ and log-free transseries $\mathcal{T}_N$ of level N have been defined. Group $\mathcal{G}_{N+1}$ consists of ordered pairs (b, L) but written $x^b e^L$, where $b \in \mathbb{R}$ and $L \in \mathcal{T}_N$ is purely large. Define the group operations: $(x^b e^L)\,(x^{\tilde b} e^{\widetilde L}) = x^{b+\tilde b}\, e^{L+\widetilde L}$. Define order $(x^b e^L) \gg (x^{\tilde b} e^{\widetilde L})$ iff either $L > \widetilde L$ or $\{L = \widetilde L$ and $b > \tilde b\}$. Identify $\mathcal{G}_N$ as a subgroup of $\mathcal{G}_{N+1}$ recursively.

Log-free transseries of level $N + 1$ are those defined from this group as in Definition 2.22. Write $\mathcal{T}_{N+1} = \mathcal{T}^{\mathcal{G}_{N+1}}$. We may identify $\mathcal{T}_N$ as a subset of $\mathcal{T}_{N+1}$.

Definition 3.4. The group of log-free transmonomials is

$$\mathcal{G}_* = \bigcup_{N \in \mathbb{N}} \mathcal{G}_N.$$

The space of log-free transseries is

$$\mathcal{T}_* = \bigcup_{N \in \mathbb{N}} \mathcal{T}_N.$$

In fact, $\mathcal{T}_* = \mathcal{T}^{\mathcal{G}_*}$ because each individual transseries is finitely generated.

A set $\boldsymbol{\mu}$ is ***recursively complete*** if for every transmonomial $x^b e^L$ in $\boldsymbol{\mu}$, we also have $\operatorname{supp} L \subseteq \boldsymbol{\mu}$. Of course, given any finite set $\boldsymbol{\mu} \subseteq \mathcal{G}^{\text{small}}$, there is a recursively complete finite set $\widetilde{\boldsymbol{\mu}} \supseteq \boldsymbol{\mu}$. Call $\widetilde{\boldsymbol{\mu}} \setminus \boldsymbol{\mu}$ the ***completion addendum*** of $\boldsymbol{\mu}$.

Properties

Proposition 3.5. *Let T be a log-free transseries. If $T \gg 1$, then there exists a real number $c > 0$ such that $T \gg x^c$. If $T \ll 1$, then there exists a real number $c < 0$ such that $T \ll x^c$.*

Proof. Let $\operatorname{mag} T = x^b e^L$. If $L = 0$, then $b > 0$, so take $c = b/2$. If $L > 0$, $T \gg x^1$, since $\gg$ is defined lexicographically. The other case is similar. □

Proposition 3.6. *Let $L > 0$ be purely large of level N and not $N-1$, let $b \in \mathbb{R}$, and let T be of level N. Then $x^b e^L \gg T$.*

Proof. By induction on the level. Let $\operatorname{mag} T = x^{b_1} e^{L_1}$. So $L_1 \in \mathfrak{T}_{N-1}$, and therefore by the induction hypothesis $\operatorname{dom}(L - L_1) = \operatorname{dom}(L) > 0$. So $L > L_1$ and $x^b e^L \gg x^{b_1} e^{L_1}$. □

Derivative

Definition 3.7. ***Derivative*** (notations $'$, ∂, $\mathcal{D}$) is defined recursively. $(x^a)' = ax^{a-1}$, where we may need the addendum of generator x^{-1}. If ∂ has been defined for $\mathfrak{G}_N$, define termwise for $\mathfrak{T}_N$. (See the next proposition for the proof that this makes sense.) Then, if ∂ has been defined for $\mathfrak{T}_N$, define it on $\mathfrak{G}_{N+1}$ by

$$\left(x^b e^L\right)' = b x^{b-1} e^L + x^b L' e^L.$$

For the ***derivative addendum*** $\widetilde{\boldsymbol{\mu}}$: begin with $\boldsymbol{\mu}$, add the completion addendum of $\boldsymbol{\mu}$, and add x^{-1}.

Proposition 3.8. *Let $\boldsymbol{\mu}$ be given. Let $\widetilde{\boldsymbol{\mu}}$ be as described.* (i) *If $T_i \xrightarrow{\boldsymbol{\mu}} T$ then $T_i' \xrightarrow{\widetilde{\boldsymbol{\mu}}} T'$.* (ii) *If $\sum T_i$ is $\boldsymbol{\mu}$-convergent, then $\sum T_i'$ is $\widetilde{\boldsymbol{\mu}}$-convergent and $\left(\sum T_i\right)' = \sum T_i'$.* (iii) *If $\Gamma \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$, then $\sum_{g\in\Gamma} a_g g'$ is $\widetilde{\boldsymbol{\mu}}$-convergent.*

Proof. (iii) is stated equivalently: the family $(\operatorname{supp} g')$ is point-finite. Or: as g ranges over $\Gamma^{\boldsymbol{\mu},\mathbf{m}}$, we have $g' \xrightarrow{\widetilde{\boldsymbol{\mu}}} 0$.

Proof by induction on the level.

Say $\mu_1 = x^{-b_1} e^{-L_1}, \cdots, \mu_n = x^{-b_n} e^{-L_n}$, and $\mathbf{k} = (k_1, \cdots, k_n)$. Then

$$\begin{aligned}\left(\boldsymbol{\mu}^{\mathbf{k}}\right)' &= \left(x^{-k_1 b_1 - \cdots - k_n b_n} e^{-k_1 L_1 - \cdots - k_n L_n}\right)' \\ &= (-k_1 b_1 - \cdots - k_n b_n) x^{-1} \boldsymbol{\mu}^{\mathbf{k}} + (-k_1 L_1' - \cdots - k_n L_n') \boldsymbol{\mu}^{\mathbf{k}}.\end{aligned}$$

So if $T = \sum_{\mathbf{k} \geq \mathbf{m}} a_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}}$, then summing the above transmonomial result, we get

$$T' = x^{-1} T_0 + L_1' T_1 + \cdots + L_n' T_n,$$

where $T_0, \cdots, T_n$ are transseries with the same support as T, and therefore they exist in $\mathfrak{T}^{\boldsymbol{\mu},\mathbf{m}}$. Derivatives $L_1', \cdots, L_n'$ exist by induction hypothesis. So T' exists.

... □

Proposition 3.9. *There is no $T \in \mathcal{T}_*$ with $T' = x^{-1}$.*

Proof. In fact, we show: If $g \in \mathcal{G}_*$, then $x^{-1} \notin \operatorname{supp} g'$. This suffices since

$$\operatorname{supp} T' \subseteq \bigcup_{g \in \operatorname{supp} T} \operatorname{supp} g'.$$

Proof by induction on the level. If $g = x^b$, then $g' = bx^{b-1}$ and $x^{-1} \notin \operatorname{supp} g'$. If $g = x^b e^L$ with L of level $N-1$, then $g' = (bx^{-1} + L')e^L$. Now by the induction hypothesis, $bx^{-1} + L' \neq 0$, so (by Prop. 3.6) g' is far larger than x^{-1} if $L > 0$ and far smaller than x^{-1} if $L < 0$. □

Proposition 3.10. (a) *If $g_1 \gg g_2$, $g_1 \neq 1$, and $g_2 \neq 1$, then $g_1' \gg g_2'$.* (b) *If* $\operatorname{mag} T \neq 1$, *then* $T' \asymp (\operatorname{mag} T)'$ *and* $\operatorname{dom}(T') = \operatorname{dom}((\operatorname{dom} T)')$. (c) *If* $\operatorname{mag} T_1 \neq 1$ *and $T_1 \gg T_2$, then $T_1' \gg T_2'$.*

Proof. (a) If $g_1 = x^{b_1} e^{L_1} \gg g_2 = x^{b_2} e^{L_2}$, then $L_2 < L_1$ or $\{L_2 = L_1$ and $b_2 < b_1\}$. Then $g_1' = (b_1 x^{-1} + L_1') x^{b_1} e^{L_1}$ and $g_2' = (b_2 x^{-1} + L_2') x^{b_2} e^{L_2}$. By Prop. 3.9 the factors $(b_1 x^{-1} + L_1')$ and $(b_2 x^{-1} + L_2')$ are not zero. If $L_2 < L_1$, then by Prop. 3.6 $g_1' \gg g_2'$. If $L_2 = L_1$, then $L_2' = L_1'$ and $x^{b_1 - 1} \gg x^{b_2 - 1}$, so we get $g_1' \gg g_2'$.

(b), (c) follow from (a). □

Proposition 3.11. (i) *If $s \ll 1$, then $s' \ll 1$.* (ii) *If $T \gg 1$ and $T > 0$, then $T' > 0$.* (iii) *If $T \gg 1$ and $T < 0$, then $T' < 0$.* (iv) *If $T \gg 1$ then $xT' \gg 1$.* (v) *If $T \gg 1$, then $T^2 \gg T'$.*

Proof. (i) Assume $s \ll 1$. Then $s \ll x^c$ for some $c < 0$, and $s' \ll cx^{c-1} \ll 1$.

(ii) Assume $T \gg 1$ and $T > 0$. Let $\operatorname{dom} T = ax^b e^L$. So $T' \asymp (bx^{-1} + L')x^b e^L$. If $L > 0$, then this is far larger than 1 by Prop. 3.6. If $L = 0$, $b > 0$, then $\operatorname{dom} T' = abx^{b-1} > 0$. In both cases, $xT' \gg 1$. That's (iv). (iii) is similar.

(v) Again $\operatorname{dom} T = ax^b e^L$, so $\operatorname{dom} T^2 = a^2 x^{2b} e^{2L}$. We claim this is far larger than $(bx^{-1} + L')x^b e^L$. If $L \neq 0$, this is true by Prop. 3.6. If $L = 0, b > 0$ this is true because $2b > b - 1$. □

If $T \gg x^2$, then $T' \gg 1$. In particular, if $T \gg 1$ and T is of level ≥ 1, then $T' \gg 1$.

Proposition 3.12. *If $L \neq 0$ is purely large, then* $\operatorname{dom}(ax^b e^L)' = ax^b e^L \operatorname{dom} L'$.

Proof. Since $L \gg 1$, there is $c > 0$ with $L \gg x^c$, so $L' \gg x^{c-1} \gg x^{-1}$. So $(ax^b e^L)' = ax^b e^L (bx^{-1} + L') \asymp ax^b e^L L'$. □

Proposition 3.13. *If $T' = 0$, then T is a constant.*

Proof. Assume $T' = 0$. Write $T = L + c + s$. If $L \neq 0$ then $\operatorname{mag} T' = \operatorname{mag} L' \gg 1$, so $T' \neq 0$. If $L = 0$ and $s \neq 0$, then $\operatorname{mag} T' = \operatorname{mag} s' \ll 1$ so $L' \neq 0$. Therefore $L = c$. □

The set $\mathcal{T}_N$ is a differential field with constants $\mathbb{R}$.

Compositions

Definition 3.14. We define T^b, where $T \in \mathfrak{T}_*$ is positive, and $b \in \mathbb{R}$. First, write $T = cx^a e^L(1+s)$ as usual, with $c > 0$. Then define $T^b = c^b x^{ab} e^{bL}(1+s)^b$. Constant c^b, with $c > 0$, is computed in the reals. Next, x^{ab} is a transseries, but may require addendum of a generator. Also, $(1+s)^b$ is a convergent binomial series, again we may require the smallness addendum for s. Finally, since L is purely large, so is bL, and thus e^{bL} is a transseries, but may require addendum of a generator.

Definition 3.15. We define e^T, where $T \in \mathfrak{T}_*$. Write $T = L+c+s$, with L purely large, c a constant, and s small. Then $e^T = e^L e^c e^s$. Constant e^c is computed in the reals—note that $e^T > 0$. Next, e^s is a convergent power series; we may need the smallness addedum for s. And of course e^L is a transseries, but may not already be a generator, so e^L or e^{-L} may be required as addendum.

Of course, if T is purely large, then this definition of e^T agrees with the notation e^T used before.

Definition 3.16. Let $T_1, T_2 \in \mathfrak{T}_*$ with T_2 positive and large (but not necessarily purely large). We want to define the ***composition*** $T_1 \circ T_2$. This is done by induction on the level of T_1. When $T_1 = x^b e^L$ is a transmonomial, define $T_1 \circ T_2 = T_2^b\, e^{L \circ T_2}$. Both T_2^b and $e^{L\circ T_2}$ may require addenda. And $L \circ T_2$ exists by the induction hypothesis. In general, when $T_1 = \sum c_g g$, define $T_1 \circ T_2 = \sum c_g (g \circ T_2)$. The next proposition is required. If $T_1 \gg 1$, then $T_1 \circ T_2 \gg 1$. If $T_1 \ll 1$, then $T_1 \circ T_2 \ll 1$.

Proposition 3.17. *Let $\boldsymbol{\mu}, \mathbf{m}$ and $T_2 \in \mathfrak{T}_*$ be given with* $\operatorname{supp} T_2 \subseteq \Gamma^{\boldsymbol{\mu},\mathbf{m}}$, $T_2 \gg 1$. *Then there exist $\widetilde{\boldsymbol{\mu}}$ and $\widetilde{\mathbf{m}}$ so that $g \circ T_2 \in \mathfrak{T}^{\widetilde{\boldsymbol{\mu}},\widetilde{\mathbf{m}}}$ for all $g \in \Gamma^{\boldsymbol{\mu},\mathbf{m}}$, and the family* $\big(\operatorname{supp}(g \circ T_2)\big)$ *is point-finite.*

Proof. First, add the completion addendum of $\boldsymbol{\mu}$. Now for all these generators $\{\mu_1, \cdots, \mu_{n'}\}$, write $\mu_i = x^{-b_i} e^{-L_i}$, $1 \le i \le n'$. Arrange the list so that for all i, $\operatorname{supp} L_i \subseteq \{\mu_1, \cdots, \mu_{i-1}\}$. Then take the μ_i in order. Each $T_2^{-b_i}$ may require an addendum. Each $L_i \circ T_2$ may require an addendum, which has been added before. So all $\mu_i \circ T_2$ exist. They are small. Add smallness addenda for these (Is that needed?). So finally we get $\widetilde{\boldsymbol{\mu}}$.

Now for each $\mu_i \in \boldsymbol{\mu}$, we have $\mu_i \circ T_2$ is $\widetilde{\boldsymbol{\mu}}$-small. So by Prop. 2.35 we have $(g \circ T_2)_{g \in \Gamma^{\boldsymbol{\mu},\mathbf{m}}} \xrightarrow{\widetilde{\boldsymbol{\mu}}} 0$. ∎

Example 3.18. For composition $T_1 \circ T_2$, we need T_2 large. Example: $T_1 = \sum_{j=0}^{\infty} x^{-j}$, $T_2 = x^{-1}$ small. Then $T_1 \circ T_2 = \sum_{j=0}^{\infty} x^j$ is not a valid transseries.

3.2 With logs

Transseries with logs are obtained by composing with log on the right.

Notation 3.19. If $m \in \mathbb{N}$, we write formally $\log_m$ to represent the m-fold composition of the natural logarithm with itself. $\log_0$ will have no effect. Sometimes we may write $\log_m = \exp_{-m}$, especially when $m < 0$.

Definition 3.20. Let $M \in \mathbb{N}$. A transseries with depth M is a formal expression $Q = T \circ \log_M$, where $T \in \mathfrak{T}_*$.

We identify transseries of depth M as a subset of transseries of depth $M+1$ by identifying $T \circ \log_M$ with $(T \circ \exp) \circ \log_{M+1}$. Composition on the right with exp is defined in Def. 3.16. Using this idea, we define operations on transseries from the operations in $\mathfrak{T}_*$.

Definition 3.21. Let $Q_j = T_j \circ \log_M$, where $T_j \in \mathfrak{T}_*$. Define $Q_1 + Q_2 = (T_1 + T_2) \circ \log_M$; $Q_1 Q_2 = (T_1 T_2) \circ \log_M$; $Q_1 > Q_2$ iff $T_1 > T_2$; $Q_1 \gg Q_2$ iff $T_1 \gg T_2$; $Q_j \to Q_0$ iff $T_j \to T_0$; $\sum Q_j = (\sum T_j) \circ \log_M$; $Q_1^b = (T_1^b) \circ \log_M$; $\exp(Q_1) = (\exp(T_1)) \circ \log_M$; and so on.

Definition 3.22. ***Transseries.***

$$\begin{aligned}
\mathfrak{T}_{NM} &= \{\, T \circ \log_M : T \in \mathfrak{T}_N \,\}, \\
\mathfrak{T}_{*M} &= \bigcup_{N \in \mathbb{N}} \mathfrak{T}_{NM} = \{\, T \circ \log_M : T \in \mathfrak{T}_* \,\}, \\
\mathfrak{T}_{**} &= \bigcup_{M \in \mathbb{N}} \mathfrak{T}_{*M}.
\end{aligned}$$

When $M < 0$ we also write $\mathfrak{T}_{*M}$. So $\mathfrak{T}_{*,-1} = \{\, T \circ \exp : T \in \mathfrak{T}_* \,\}$.

If $T = \sum c_g g$ we may write $T \circ \log_M$ as a series

$$\Big(\sum c_g g\Big) \circ \log_M = \sum c_g (g \circ \log_M).$$

Simplifications along these lines may be carried out: $\exp(\log x) = x$; $e^{b \log x} = x^b$; etc. As usual we sometimes use x as a variable and sometimes as the identity function. On monomials we can write

$$(x^b e^L) \circ \log = (\log x)^b e^{L \circ \log}$$

but just consider this an abbreviation?

Definition 3.23. $Q \in \mathfrak{T}_{**}$ has ***exact depth*** M iff $Q = T \circ \log_M$, $T \in \mathfrak{T}_*$ and T cannot be written in the form $T = T_1 \circ \exp$ for $T_1 \in \mathfrak{T}_*$. This will also make sense for negative M.

Definition 3.24. ***Logarithm***. If $T \in \mathfrak{T}_*$, $T > 0$, write $T = a x^b e^L (1+s)$ as usual. Define $\log T = \log a + b \log x + L + \log(1+s)$. Now $\log a$, $a > 0$ is computed in the reals. $\log(1+s)$ is a series. The term $b \log x$ gives this depth 1; if $b = 0$ then we remain log-free.

For general $Q \in \mathfrak{T}_{**}$: if $Q = T \circ \log_M$, then $\log(Q) = \log(T) \circ \log_M$, which could have depth $M+1$.

Definition 3.25. Differentiation is done as expected from the usual rules.

$$\big(T \circ \log\big)' = (T' \circ \log) \cdot x^{-1} = \big(T' e^{-x}\big) \circ \log\,.$$

So ∂ maps $\mathfrak{T}_{*M}$ into itself.

Check usual properties.
We now have an antiderivative for x^{-1}.

$$\big(\log x\big)' = \big(x \circ \log\big)' = \big(1 \cdot e^{-x}\big) \circ \log = (x^{-1}) \circ \exp \circ \log = x^{-1}.$$

Aside. Would it be better to write these out as if they were functions? Should

$$T = x^{-1/2} e^{x^2 - 2x} \circ \log$$

be written as

$$T(x) = (\log x)^{-1/2} e^{(\log x)^2 - 2(\log x)} = (\log x)^{-1/2} e^{(\log x)^2} x^{-2}$$

and let this be understood as an abbreviation? Or should we use some symbol other than x for the dummy identity function?

$$T = \Box^{-1/2} e^{\Box^2 - 2\Box} \circ \log .$$

Contraction

Contraction (for the fixed-point theorem) is formulated for a particular $\boldsymbol{\mu}$. So to apply it in $\mathfrak{T}_{**}$, either we will have to convert problems to $\mathfrak{T}_*$, or else write out what to do with generating sets involving logs.

Integral

This is an example where we convert the problem to a log-free case to apply the contraction argument. The general integration problem (3.29) is reduced to one (3.26) where contraction can be easily applied.

Proposition 3.26. *Let $T \in \mathfrak{T}_*$ with $T \gg 1$. Then there is $S \in \mathfrak{T}_*$ with $S' = e^T$.*

Proof. Either T is positive or negative. We will do the positive case, the negative one is similar. If

$$S = \frac{e^T}{T'}\,(1 + \Delta),$$

where Δ satisfies

$$\Delta = \frac{T''}{(T')^2} + \frac{T''}{(T')^2}\Delta - \frac{\Delta'}{T'},$$

then it is a computation to see that $S' = e^T$. So it suffices to exhibit an appropriate $\boldsymbol{\mu}$ and show that the linear map $J\colon \mathfrak{T}^{\boldsymbol{\mu},\mathbf{0}} \to \mathfrak{T}^{\boldsymbol{\mu},\mathbf{0}}$ defined by

$$J(\Delta) = \frac{T''}{(T')^2}\Delta - \frac{\Delta'}{T'}$$

is $\boldsymbol{\mu}$-contractive, then apply Prop. 2.47(i).

Say T is of exact level N, so e^T is of exact level $N+1$. By Prop. 3.11, $T'' \ll (T')^2$ and $xT' \gg 1$. So $T''/(T')^2$ and $1/(xT')$ are small. Let $\boldsymbol{\mu}$ be the least set of generators including x^{-1}, the generators for T, the inversion addendum for T', the smallness addenda for $T''/(T')^2$ and $1/(xT')$, and is recursively complete. Check: all generators in $\boldsymbol{\mu}$ are (at most) of level N. (That is, none of the addenda mentioned will increase the level.) And all derivatives T', T'' belong to $\mathfrak{T}^{\boldsymbol{\mu}}$. If $g \in \Gamma^{\boldsymbol{\mu},\mathbf{0}}$ then $g' \in \mathfrak{T}^{\boldsymbol{\mu},\mathbf{0}}$. So for this $\boldsymbol{\mu}$, the function J maps $\mathfrak{T}^{\boldsymbol{\mu},\mathbf{0}}$ into itself.

Since J is linear, we just have to check that it $\boldsymbol{\mu}$-contracts monomials $g \in \operatorname{supp}\Delta$. Now $T''/(T')^2$ is $\boldsymbol{\mu}$-small so g $\boldsymbol{\mu}$-contracts to $(T''/(T')^2)g$. For the second term: If $g = x^b e^L$ with L of level $N-1$, then

$$\frac{g'}{T'} = \frac{bx^{b-1}e^L + L'x^b e^L}{T'} = \frac{bx^{-1} + L'}{T'}\, g = \frac{b + xL'}{xT'}\, g.$$

But $xT' \gg 1$ has exact level N while $b + xL'$ has level $N-1$, and thus the factor $(b + xL')/(xT')$ is $\boldsymbol{\mu}$-small. [Wait: do I need an explicit addendum for $(b + xL')/(xT')$?] So g $\boldsymbol{\mu}$-contracts to g'/T'. □

Definition 3.27. We say $x^b e^L \in \mathfrak{G}_*$ is ***power-free*** iff $b = 0$. We say $T \in \mathfrak{T}_*$ is power-free iff all transmonomials in $\operatorname{supp} T$ are power-free.

Since $(x^b e^L) \circ \exp = e^{bx} e^{L \circ \exp}$, it follows that all $T \in \mathfrak{T}_{*,-1}$ are power-free.

Proposition 3.28. *Let $T \in \mathfrak{T}_*$ be a power-free transseries. Then there is $S \in \mathfrak{T}_*$ with $S' = T$.*

Proof. For monomials $g = e^L$ with large L, write $\mathcal{P}(g)$ for the transseries constructed in Prop. 3.26 with $\mathcal{P}(g)' = g$. Then we must show that the family $(\operatorname{supp} \mathcal{P}(g))$ is point-finite, so we can define $\mathcal{P}\big(\sum c_g g\big) = \sum c_g \mathcal{P}(g)$. For large L we have $xL' \gg 1$ (Prop. 3.11), so the formula

$$\frac{\mathcal{P}(e^L)}{x} = \frac{e^L}{xL'}(1 + \Delta)$$

shows that e^L contracts to $\mathcal{P}(e^L)/x$. So the family of all of these $\operatorname{supp} \mathcal{P}(e^L)/x$ is point-finite and thus the family of $\operatorname{supp} \mathcal{P}(e^L)$ is point-finite. [Do we need x^{-1} to be a generator?] □

Proposition 3.29. *Let $Q \in \mathfrak{T}_{**}$. Then there exists $\mathcal{P}(Q) \in \mathfrak{T}_{**}$ with $\mathcal{P}(Q)' = Q$.*

Proof. Say $Q \in \mathfrak{T}_{*M}$. Then $Q = T_1 \circ \log_{M+1}$, where $T_1 \in \mathfrak{T}_{*,-1}$. Let $T = T_1 \cdot \exp_{M+1} \cdot \exp_M \cdots \exp_2 \cdot \exp_1$. Now T is power-free, so by Prop. 3.28, there exists $S \in \mathfrak{T}_*$ with $S' = T$. Then let $\mathcal{P}(Q) = S \circ \log_{M+1}$ and check that $\mathcal{P}(Q)' = Q$. Note that $\mathcal{P}(Q) \in \mathfrak{T}_{*,M+1}$. □