MOVABLE SINGULARITIES OF SOLUTIONS OF DIFFERENCE EQUATIONS IN RELATION TO SOLVABILITY, AND STUDY OF A SUPERSTABLE FIXED POINT

OVIDIU COSTIN AND MARTIN KRUSKAL

Abstract. We overview applications exponential asymptotics and analyzable function theory to difference equations, in defining an analog of the Painlevé property for them and we sketch the conclusions with respect to the solvability properties of first order autonomous ones. It turns out that if the Painlevé property is present the equations are explicitly solvable and in the contrary case, under further assumptions, the integrals of motion develop singularity barriers. We apply the method to the logistic map $x_{n+1} = ax_n(1 - x_n)$ where it turns out that the only cases with the Painlevé property are $a = -2, 0, 2$ and 4 for which explicit solutions indeed exist; in the opposite case an associated conjugation map develops singularity barriers.

1. Introduction

We present an outline of new methods for determining the position and the type of singularities of a certain kind of solutions of difference equations [17] and use this information to perform Painlevé analysis on them. The approach relies on advances in exponential asymptotics and the theory of analyzable functions [10, 11, 23, 13, 14, 15, 16, 18, 20, 21, 25, 26, 34, 45]. The main concepts are discussed first.

Analyzable functions. Introduced by Jean Écalle, these are mostly analytic functions which at singular points are completely described by *transseries*, much in the same way as analytic functions are represented at regular points by convergent series. In contrast with analytic functions (which are not closed under division) and with meromorphic functions (which fail to be stable under integration and composition) analyzable ones are conjectured to be closed under all operations whence the grand picture of this theory, that all functions of natural origin are analyzable. In particular, solutions of many classes of differential and difference equations have been shown to be analyzable.

Transseries. Also introduced by J. Ecalle, transseries represent the "ultimate" generalization of Taylor series. Transseries are formal asymptotic combinations of power series, exponentials and logarithms and contain a wealth of information not only about local but also about global behavior of functions. One of the simplest nontrivial examples of a transseries is

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[1] In the sense stemming from Stokes original papers and the one favored in exponential asymptotics literature, *Stokes* lines are those where a small exponential is purely real; on an *antistokes* line the exponential becomes purely oscillatory .

$$\sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} + Ce^{-x} \qquad (x \to +\infty) \tag{1.1}$$

What distinguishes the expression (1.1) from a usual asymptotic expansion is the presence beyond all orders of the principal series of an exponentially small term which cannot be captured by classical Poincaré asymptoticity. More general transseries arising in generic ordinary differential and difference equations are doubly infinite series whose terms are powers of x multiplied by exponentials (see e.g. (2.3)). These transseries can be determined algorithmically and usually diverge factorially, but can be shown to be Borel summable in a suitable sense in some sector. In this sector the function to which the transseries sums has good analyticity properties and on the edges of the sector typically singularities appear.

The correspondence between the formal universe and actual functions. Envisaged by Écalle to be a "total" isomorphism implemented by generalized Borel summation, the correspondence has been established rigorously in a number of problems including ODEs, difference equations and PDEs.

2. Difference equations: the isolated movable singularity property (IMSP)

The problem of integrability has been and is a subject of substantial research. In the context of differential equations there exist numerous effective criteria of integrability, among which a crucial role is played by the Painlevé test (for extensive references see e.g. [33]).

An analog of the Painlevé property for difference equations turns out to be more delicate to define.

In the context of solvability, various methods have been proposed by Joshi [38], Ablowitz et al. [1], Ramani and Grammaticos (see references in [36]), Conte and Musette [12]. See also [1] for a comparative discussion of the various approaches in the literature. None of these is a proper extension of the Painlevé test. One difficulty resides in continuing the solutions $x(n)$ of a difference equation, which are defined on a *discrete set*, to the complex plane of the independent variable n in a natural and effective fashion. The embedding of $x(n)$ must be done in such a way that *properties are preserved*. It is important for the effectiveness of the analysis that this embedding $x(n) \sqsubset x(z)$ is natural, constructive and unique under proper conditions.

It is of course crucial that we are given infinitely many values of the function $x(n)$; since the accumulation point of n is infinity, the behavior of x at ∞ is key, and then the question boils down to when infinitely many values determine function, and in which class.

To illustrate a point we start with a rather trivial example. Assume that $x(n)$ is expressed as a convergent power series in powers of $1/n$

$$x_n = \sum_{k=1}^{\infty} c_k n^{-k}$$

We would then naturally define $x(n) \sqsubset x(z)$ by

$$x(z) = \sum_{k=1}^{\infty} c_k z^{-k}$$

for large enough z. Uniqueness is ensured by the analyticity at ∞ of $x(z)$. Still by analyticity, $\sqsubset$ preserves all properties.

We cannot rely merely on analytic continuation since there are very few classes of equations where solutions are given by convergent power series. However, as discovered by Écalle and proved in detail very recently in a general setting by Braaksma [11], a wide class of arbitrary order difference equations admit formal solutions as Borel summable *transseries*. The class considered by Braaksma is of the form

$$\mathbf{x}_{n+1} = \mathbf{G}(\mathbf{x}_n, n) = \left(\hat{\Lambda} + \frac{1}{n}\hat{A}\right)\mathbf{x}_n + \mathbf{g}(n, \mathbf{x}_n) \tag{2.1}$$

where $\mathbf{G}$ analytic at ∞ in n and at 0 in x_n, under genericity assumptions [11]. In particular a nonresonance condition is imposed

$$\mu_m = \mathbf{k} \cdot \boldsymbol{\mu} \mod 2\pi i \tag{2.2}$$

with $\mathbf{k} \in \mathbb{N}^n$ iff $\mathbf{k} = \mathbf{e}_m$. Transseries solutions for these equations have many similarities to transseries solutions of differential equations. With some m_1, $0 \le m_1 \le n$,

$$\tilde{\mathbf{x}}(n) = \sum_{\mathbf{k} \in \mathbb{N}^{m_1}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} n} n^{\mathbf{k}\cdot\mathbf{a}} \tilde{\mathbf{y}}_{\mathbf{k}}(n) \tag{2.3}$$

In (2.3) $\boldsymbol{\mu} = (\mu_1, ..., \mu_{m_1})$ and $\mathbf{a} = (a_1, ..., a_{m_1})$ depend only on the recurrence (they are the eigenvalues of $\hat{\Lambda}, \hat{A}$, respectively), $\mathbf{C}$ is a free parameter and $\tilde{\mathbf{y}}_{\mathbf{k}}$ are formal series in negative integer powers of n, independent on $\mathbf{C}$. The number m_1 is chosen so that all the exponentials in (2.3) tend to zero in the chosen sector.

Braaksma showed that $\tilde{\mathbf{y}}_{\mathbf{k}}(n)$ are Borel summable uniformly in $\mathbf{k}$. Let $\mathbf{Y}_{\mathbf{k}} = \mathcal{L}^{-1}\tilde{\mathbf{y}}_{\mathbf{k}}(n)$. Then $\mathbf{Y}_{\mathbf{k}}(p)$ are analytic in a neighborhood of $\mathbb{R}^+$ (in fact in a larger sector). Defining

$$\mathbf{y}_{\mathbf{k}} = \int_0^{\infty} e^{-np}\mathbf{Y}_{\mathbf{k}}(p)dp \tag{2.4}$$

we have uniform estimates $|\mathbf{y}_{\mathbf{k}}| < A^{\mathbf{k}}$ and thus the series

$$\mathbf{x}(n) = \sum_{\mathbf{k} \in \mathbb{N}^{m_1}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} n} n^{\mathbf{k}\cdot\mathbf{a}} \mathbf{y}_{\mathbf{k}}(n) \tag{2.5}$$

is classically convergent for large enough n. Braaksma showed that $\mathbf{x}(n)$ is an actual solution of (2.1). It is natural to replace n by z in (2.4) and define:

$$\mathbf{x}(z) = \sum_{\mathbf{k} \in \mathbb{N}^{m_1}} \mathbf{C}^{\mathbf{k}} e^{-\mathbf{k}\cdot\boldsymbol{\mu} z} z^{\mathbf{k}\cdot\mathbf{a}} \mathbf{y}_{\mathbf{k}}(z) \tag{2.6}$$

If z and all constants are real and $\mu_i < 0$, the functions (2.6) are special cases of Écalle's *analyzable* functions. As explained before we are allowing for z and constants to be complex, under restriction $\Re(\mathbf{k} \cdot \boldsymbol{\mu} z) > 0$.

It is crucial that the values of $\mathbf{y}(\mathbf{n})$ for all large enough n uniquely determine the expansion. In [17] it is shown that under suitable conditions, *two distinct analyzable functions cannot agree on a set of points accumulating at infinity.* Below is a simplified version of a theorem in [17].

Theorem 1 ([17])**.** *Assume*

$$(\mathbf{Z}\cdot\boldsymbol{\mu}=0 \mod 2\pi i \quad \textit{with} \quad \mathbf{Z}\in\mathbb{Z}^p)\Leftrightarrow\mathbf{Z}=0$$

and $\mathbf{x}(z)=\sum_{\mathbf{k}\in\mathbb{N}^{m_1}}\mathbf{C}^{\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\mu}z}z^{\mathbf{k}\cdot\mathbf{a}}\mathbf{y}_{\mathbf{k}}(z)$. *If* $\mathbf{x}(n)=0$ *for all large enough* $n\in\mathbb{N}$, *then* $\mathbf{x}(z)$ *is identically zero.*

Analyzable functions behave in most respects as analytic functions. Among the common properties, particularly important is the uniqueness of the extension from sets with accumulation points, implying the principle of permanence of relations. Under the assumptions [11] that ensure Borel summability, we have the following.

Definition 2. *If*

$$\mathbf{x}(n)=\sum_{\mathbf{k}\in\mathbb{N}^{m_1}}\mathbf{C}^{\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\mu}n}n^{\mathbf{k}\cdot\mathbf{a}}\mathbf{y}_{\mathbf{k}}(n)$$

then we call

$$\mathbf{x}(z)=\sum_{\mathbf{k}\in\mathbb{N}^{m_1}}\mathbf{C}^{\mathbf{k}}e^{-\mathbf{k}\cdot\boldsymbol{\mu}z}z^{\mathbf{k}\cdot\mathbf{a}}\mathbf{y}_{\mathbf{k}}(z)$$

the analyzable embedding of $\mathbf{x}(n)$ *from* $\mathbb{N}$ *to a sector in* $\mathbb{C}$.

Having now a suitable procedure of analytic continuation, we can define the isolated movable singularity property, an extension of the Painlevé property to difference equations, in a natural fashion. In analogy with the case of differential equations we require that all solutions are free from "bad" movable singularities.

Definition 3. *A difference equation has the IMSP if all* movable *singularities of all its solutions are* **isolated.**

Notes. (i) We use the common convention that isolated singularity exclude branch points, clusters of poles and barriers of singularities.

(ii) To determine the singularities, transasymptotic matching methods introduced for differential equations in [18], can be extended with little changes to difference equations.

3. Classification of some difference equations with IMSP. Solvability.

We look at autonomous difference equations of the form $x_{n+1}=G(x_n)$ where G is meromorphic and has attracting fixed points. A more general analysis is given in [17]. We then write

$$x_{n+1}=G(x_n):=ax_n+F(x_n) \tag{3.1}$$

and restrict for simplicity to the case $F(0)=F'(0)=0$ and $0<|a|<1$. There is a one-parameter family of solutions presented as simple transseries *convergent* for large enough n, of the form

$$x_n = x_n(C) = \sum_{k=1}^{\infty} e^{nk \ln a} C^k D_k \tag{3.2}$$

for given values of D_k, independent of C. The analyzable embedding of x, cf. Definition 2, reads

$$x(z) = x(z; C) = \sum_{k=1}^{\infty} e^{zk \ln a} C^k D_k \tag{3.3}$$

which is analytic for large enough z. To look for the IMSP, we find the properties of $x(z)$ beyond the domain of convergence of (3.3), and find the singular points of $x(z)$.

Note. Because equation (3.1) is nonlinear, although (3.3) has one continuous parameter, there may be more solutions. This issue is addressed in ([17]).

3.1. **Embedding versus properties of the conjugation map.** By the Poincaré conjugation theorem applied to $x_{n+1} = G(x_n)$ there exists a unique map ϕ with the properties

$$\phi(0) = 0, \quad \phi'(0) = 1 \text{ and } \phi \text{ analytic at } 0 \tag{3.4}$$

and such that $x_n = \phi(X_n)$ implies $X_{n+1} = aX_n$. The map ϕ is a conjugation map of x_{n+1} with its linearization X_{n+1}. We have $X_n = a^n X_0 = Ca^n$.

We obtain an extension of x from $\mathbb{N}$ to $\mathbb{C}$ through

$$x_n = \phi(Ca^n) \ \sqsubset \ x(z) := \phi(Ca^z) \tag{3.5}$$

Then the conjugation map satisfies the equation

$$\phi(az) = G(z) = a\phi(z) + F(\phi(z))$$

As expected by the uniqueness of the embedding in Definition 2 we have the following.

Lemma 4. *(i) For equations of the type (3.1) under the given assumptions, the embeddings through analyzability and via the conjugation map, cf. (3.5) agree.*

(ii) The solutions $x(z; C)$ have only isolated movable singularities iff ϕ has only isolated singularities.

Corollary *A necessary condition for (3.1) to have the IMSP is that the conjugation map ϕ, around every stable fixed point of G and of those of its iterates which are meromorphic (of the form $G^{[m]} = G \circ \cdots \circ G$ m times) extends analytically into the complex plane, except for isolated singularities.*

3.2. **Autonomous equations with the IMSP.** In ([17]) we classify the equations of the form (3.1) with respect to the IMSP. In a way analog to the case of ODEs of first order, only Riccati equations have this property. On the other hand, these equations are explicitly solvable, and thus autonomous equations of the first order which have the IMSP can be solved in closed form.

Theorem 5. *The equation (3.1) under the given assumptions* fails *to have the IMSP* unless*, for some $c \in \mathbb{C}$,*

$$G(z) = \frac{az}{1+cz} \tag{3.6}$$

In case (3.6 we have: $1/x_n = a^{-n}(C - c/(a-1)) + c/(a-1)$

It is also very interesting to see that in the case of failure of the IMSP the analytic properties of the solutions preclude the existence of nice constants of motion. This is discussed in the next section.

3.3. **Case of failure of IMSP.** We illustrate this situation when G polynomial. The surprising conclusion is that for polynomial G without the IMSP, constants of motion (defined as functions $C(x,n)$ constant along trajectories) develop barriers of singularities along some fractal closed curves $\partial\mathcal{K}_p$, see below. In a neighborhood of the origin, the function ϕ is invertible, and thus for small x we derive from $x_n = \phi(a^n C)$ that $C = a^{-n}Q(x_n)$ where $Q = \phi^{-1}$.

In Fig. 1. we depict the Julia set $\partial\mathcal{K}_p$ of a simple map. In that case, in the compact set bounded by $\mathcal{K}_p$ consists in the initial conditions for which the solution of the iteration converges to zero. The Julia set is a closed curve of nontrivial fractal dimension. For a comprehensive discussion of Julia sets and iterations of rational maps see [5].

Theorem 6. *Assume G is a polynomial map with an attracting fixed point at the origin. Denote by $\mathcal{K}_p$ (cf. Fig. 1) the maximal connected component of the origin in the Fatou set of G.*

Then the domain of analyticity of Q is $\mathcal{K}_p$, and $\partial\mathcal{K}_p$ is a barrier of singularities of Q.

The proofs of these results can be found in [17]. The logistic map discussed in relative detail in the next section represents a very simple illustration of some of the relevant phenomena.

4. Analysis of the logistic map at the superstable fixed point infinity

We show that the equation $x_{n+1} = ax_n(1-x_n)$ has the IMSP iff $a \in \{-2, 0, 2, 4\}$. The case $a = 0$ needs no analysis. Otherwise, taking $y = 1/x$ we get

$$y_{n+1} = \frac{y_n^2}{a(y_n - 1)} \tag{4.1}$$

For small y_0, the leading order form of equation (4.1) is $y_{n+1} = -a^{-1}y_n^2$ whose solution is $-y_0^{2^n}a^{-2^n+1}$. It is then convenient to seek solutions of (4.1) in the form $y_n = F(y_0^{2^n}a^{-2^n})$ whence the initial condition implies $F(0) = 0$, $F'(0) = -a$. Denoting $y_0^{2^n}a^{-2^n} = z$, the functional relation satisfied by F is

$$F(z^2) = \frac{F(z)^2}{a(F(z)-1)} = H(F(z)) \quad (F(0) = 0,\ F'(0) = -a) \tag{4.2}$$

Lemma 7. *(i) There exists a unique analytic function F in the neighborhood of the origin satisfying (4.2) and such that $F(0) = 0$ and $F'(0) = -a$. This F has only isolated singularities in $\mathbb{C}$ if and only if $a \in \{-2, 2, 4\}$. In the latter case, the equations (4.2) and (4.1) can be solved explicitly (see (4.5), (4.8) and (4.9)). If*

$F(z) \neq 4/a$ in D_1 then F is analytic in D_1, otherwise F has infinitely many branch points in D_1, of the form $z^{2^n} \in \{a_1, ..., a_M\}$; $M < \infty$, accumulating at S^1, and F is analytic in D_1 with cuts at the branch points.

If $a \notin \{-2, 2, 4\}$ then the unit circle S^1 is a barrier of singularities of F.

(ii) If $a \notin \{-2, 2, 4\}$ then $\partial\mathcal{K}_p$ is a barrier of singularities of the constant of motion $Q = F^{-1}$ (cf. Theorem 6).

4.1. **Proof of Lemma 7.**

4.1.1. *Analyticity at zero.* We write

$$F(z) = \frac{a}{2}F(z^2) - \frac{1}{2}\sqrt{a^2F(z^2)^2 - 4aF(z^2)} \tag{4.3}$$

(with the choice of branch consistent with $F(0) = 0, F'(0) = -a$), and take $F(z) = -az + h(z)$. This leads to the equation for h

$$h(z) = az + \frac{a}{2}(h(z^2) - az^2) - \frac{1}{2}\sqrt{4a^2z^2 - 4ah(z^2) + a^2(h(z^2) - az^2)^2} = \mathcal{N}(h) \tag{4.4}$$

It is straightforward to show that, for small ϵ, $\mathcal{N}$ is contractive in the ball of radius $|a|^2$ in the space of functions of the form $h(z) = z^2u(z)$, where u is analytic in the disk $D = \{z : \|z\| < \epsilon\}$ with the norm $\|h\| = \sup_D |u|$. The corresponding F is analytic for small z and is a conjugation map between (4.1) and its small-y approximation.

4.1.2. We first determine the singularities of F in D_1.

We let B_{r_0} be the open disk of radius r_0 centered at the origin and $B^c_{r_0}$ be its closure.

For small r_0 both sides of (4.3) are well defined and analytic in $B^c_{r_0}$. In the annulus $A = B^c_{r_0} \setminus B_{r_0^2}$, F has no zeros, otherwise by (4.3) F would have an infinite number of zeros accumulating at zero. We let $0 < \min_F = \min\{|F(z)| : z \in A\}$. We can choose r_0 small enough so that $4/a > \max_F = \max\{|F(z)| : z \in A\}$.

The right side of (4.2) remains analytic in any region in which $F(z^2) \notin \{0, 4/a\}$. As before, F cannot be zero.

Let r_1 be the largest r so that we have $F \neq 4/a$ in B_{r_1}. If $r_1 = 1$ then F is analytic in D_1.

Otherwise, we let $r(n) = r_1^{1/2^n}$. Since F is analytic in $B^c_{r_1}$, there are finitely many points $\{z_{11}, ..., z_{1m_1}\} = S_1 \subset B^c_{r_1}$, such that $F(z_i) = 4/a$. We draw radial cuts from the points $S_{1,\mathbb{N}} = \{z : z^{2^n} \in S_1, n = 0, 1, ...\}$ to the unit circle, and let C_1 be their union. Then, by (4.3), F has analytic continuation in $B^c_{r(1)} \setminus C_1$. Thus F has analytic continuation $B^c_{r(2)} \setminus C_1$, except possibly on the set of points in $z \in B^c_{r(2)} \setminus C_1$ such that $F(z^2) = H(F(z)) = 4/a$, that is, except for those $z \in B^c_{r(2)} \setminus C_1$ such that $F(z^4) = H(4/a)$. Since $z^4 \in B^c_{r(0)}$ there are finitely many such points, $\{z_{21}, ..., z_{2m_2}\} = S_2$. We make cuts at the points $S_{2,\mathbb{N}} = \{z : z^{2^n} \in S_2, n = 0, 1, ...\}$ and denote by C_2 their union with C_1. Now we see that F can be inductively analytically continued to $B_{r(m)} \setminus C_m$, where a new set S_m is defined by the condition $H^{m-1}(4/a) = F(z^{2^m})$. Since F is analytic in $B^c_{r(0)}$, for every n there are finitely many such points. But for large enough n we have $|H^m(z_m)| < \min_F$

and there are no more exceptional points. The total number of of points in $\cup S_m$ is finite.

If there is one branch point, then there are infinitely many. Indeed, there is analytic continuation to any neighborhood of a branch point z_0, and if $z_1^2 = z_0$ then z_1 is a branch point too, otherwise by F would be, by (4.2), single-valued at z_0. In this case S^1 is a singularity barrier since the points in $S_{j,\mathbb{N}}$ accumulate towards S^1.

If F is not analytic in D_1, then the proposition is proved.

4.1.3. Likewise, if F is analytic in D_1 but not meromorphic in $D_1 \cup S^1$, then S^1 is a bariier of singularities. Indeed, if $z_1 \in S^1$ is a non-meromorphic point of F, then so are all the points on S^1 such that $z^{2^n} = z_1$.

4.1.4. Now we assume F is analytic in D_1 and mermorphic in $D_1 \cup S^1$.

We claim that unless $a = -2$, the point $z = 1$ is a singular point of F. A Taylor series expansion $F = \sum_{k=0}^{\infty} c_k (z-1)^k$ gives $c_0 = 0$ or $c_0 = a/(a-1)$. It is straightforward to see that $c_0 = 0$ implies $c_k = 0$ for all k, that is $F \equiv 0$, which is not possible.

Therefore $c_0 = a/(a-1)$ in which case direct calculation shows that, unless $a = -2$, all c_k for all large k are zero, which is not possible since a nontrivial polynomial does not satisfy (4.2), because of blow-up of the right side when $F = 1$.

For $a = -2$, (4.2) has the explicit solution

$$F(z) = \frac{2z}{z^2 + z + 1} \tag{4.5}$$

It remains to look at the cases when $z = 1$ is a singular point of F.

4.1.5. Now $z = 1$ is a pole; we obtain from (4.2)

$$\lim_{z \to 1} \frac{aF(z^2)}{F(z)} = 1 \tag{4.6}$$

If $F(z) = \sum_{k=-p}^{\infty} c_k (z-1)^k$ is the Laurent series of F at $z = 1$, then for the coefficient of the highest order pole to be nonzero, we must have

$$a = 2^p, \quad p \in \mathbb{N} \tag{4.7}$$

For $a = 2$, (4.2) has the explicit solution

$$F(z) = \frac{2z}{z - 1} \tag{4.8}$$

For $a = 4$ the solution of (4.2) is

$$F(z) = -\frac{4z}{(z-1)^2} \tag{4.9}$$

4.1.6. We are left with the cases $a = 2^p, p > 2$.

We know that $F(z) \neq 0$ for $z \in D_1$; furthermore, $F(z) \neq 0$ in S^1 as well, otherwise $F(z_0) = 0$ implies that $F(z) = 0$ for all $z \in S^1$ such that $z^{2^n} = z_0$ and $F \equiv 0$. Thus $zg = z/F$ is analytic in $D_1 \cup S^1$. The function g satisfies the relation

$$g(z^2) = ag(z)(1 - g(z)) \tag{4.10}$$

From this and the analiticity of zg in $D_1 \cup S^1$ it follows that zg is entire. If g is bounded, then g is linear. Otherwise $M(r) = \max\{|g(z)| : |z| = r\} \to \infty$, while the recurrence gives $M(2^{2^n}) \leq (2aM(2))^{2^n} = B^{2^n}$. With $2^{2^n} = b$ we have $M(b) \leq b^{\ln B/\ln 2}$ and thus $Q(z) = zg(z)$ is a polynomial.

If z is not on the unit circle, then $g' \neq 0$ otherwise g' would have infinitely many zeros. Since $(g(z^2))' = ag'(1 - 2g)$ we also have $g \neq 1/2$ outside the unit circle. Unless g is constant, there is a $z_0 \in S^1$ with $g(z_0) = 1/2$, and thus $g(z_0^2) = a/4$.

For $a > 5$ it is easy to check that $H^{(m)}(4/a) \to 0$. Thus $g(z_0^{2^m}) \to \infty$, a contradiction.

(ii) Since F is invertible near the origin, Q is analytic near the origin, where it satisfies the relation $Q^2(z) = Q(z^2/(az - 1))$. The rest of the proof follows the lines of (i). $\square$

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Department of Mathematics, Busch Campus-Hill Center, Rutgers University, 110 Frelinghuysen Rd, Piscataway, NJ 08854, e-mail: costin@math.rutgers.edu