IONIZATION OF COULOMB SYSTEMS IN $\mathbb{R}^3$ BY TIME PERIODIC FORCINGS OF ARBITRARY SIZE

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Abstract. We analyze the long time behavior of solutions of the Schrödinger equation $i\psi_t = (-\Delta - b/r + V(t,x))\psi$, $x \in \mathbb{R}^3$, $r = |x|$, describing a Coulomb system in a spatially compactly supported time periodic potential $V(t,x)$. We show that either the system has a Floquet bound state or, starting with an initial state $\psi \in H^1$, the system delocalizes as $t \to \infty$, *i.e.* $\lim_{t\to\infty} \int_K |\psi(t,x)|^2 dx = 0$ for any compact set $K \subset \mathbb{R}^3$.

We show further that if $V(t,x) = \Omega(r)\cos\omega t$, with $\Omega(r) > 0$ for $r \leqslant R_0$ and $\Omega(r) = 0$ otherwise, then delocalization occurs for all Ω and ω.

Our method uses the analytic structure of the Laplace transform, in t, of ψ combined with rigorous WKB analysis generalized to infinite systems of ODEs.

Keywords: Ionization, delocalization, resonances, Floquet theory.

Contents

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1. Introduction

The long time behavior of solutions of the Schrödinger equation of a system with both discrete and continuous spectrum subjected to a time periodic potential, as occurs for an atom in an electromagnetic field, has been and continues to be the subject of intense investigation. Many general and powerful results have been obtained under various assumptions on the potentials, see [3, 4, 5, 6, 24, 34, 36, 37] and references therein. None of these results however prove or disprove delocalization (ionization) of a Coulomb–bound particle subject to time-periodic forcing of arbitrary amplitude and frequency. In fact, such results have only recently been obtained even for simple model systems, see [14, 15, 16, 42, 18, 31] and references cited there.

What experiments and simplified models show is that the behavior of a system with both discrete and continuous spectrum subjected to periodic external potential of arbitrary strength can be very complicated. For amplitudes where perturbation theory is not applicable, qualitative departures from the behavior at small fields are observed. There are even situations, see *e.g.* [15], where for small enough fields ionization occurs for all initial states while for fields of order one there exist localized time–*quasiperiodic* solutions of the Schrödinger equation, *i.e.* Floquet bound states.

Though such situations are rather exceptional, constructive methods are required to deal with arbitrary strength fields.

1.1. **Formulation of the problem.** In this paper we study the question of ionization for the nonrelativistic Coulomb atom interacting with time periodic, compactly supported potentials in three dimensions. That is, we consider the Schrödinger equation for a forced Hydrogen atom (more generally Rydberg atom) with a nucleus of charge qe and an electron with charge $-e$. Setting $b = qe^2 > 0$ we get

$$i\psi_t(t,x) = \Big(-\Delta - br^{-1} + V(t,x)\Big)\psi(t,x) =: (H_C + V(t,x))\psi(t,x) \tag{1.1}$$

where $r = |x|$, $x \in \mathbb{R}^3$, $V(t,x) = V(t + 2\pi/\omega, x)$ is compactly supported in x, L^2 in t and $\psi_0 = \psi(0,x) \in H^1$. We can assume, without further loss of generality, that $V(t,x) = 0$ for $r > 1$. We are interested in whether this system will delocalize (ionize) as $t \to \infty$.

Our results show that delocalization in the large time limit occurs iff the Floquet discrete spectrum is empty. We prove furthermore that this is indeed the case regardless of the amplitude or frequency of the forcing field, whenever the latter is harmonic in time, spherical, compactly supported and nonvanishing on its support.

As in our previous work [13] – [18] on systems with short range reference potentials, we convert questions about the time asymptotic behavior of the solution of the Schrödinger equation to analyticity ones of its time Laplace transform. We then use a modified Fredholm theory to prove a dichotomy: either there are bound Floquet states, or the system gets ionized.

Mathematically the Coulomb potential introduces a number of substantial difficulties due to its singular behavior at the origin and, more importantly, its very slow decay at infinity compared to the potentials considered before. This induces a complicated structure of the discrete spectrum of the unforced Hamiltonian with an accumulation of eigenvalues at zero. Consequently, the Laplace transform of ψ can have an essential singularity at zero.

These difficulties necessitate important modifications with respect to our previous analysis of ionization in simpler models, due to the essential singularity of ψ. To deal with these problems we develop new rigorous WKB methods for infinite systems of ODEs. The approach is constructive and suitable for further generalization.

1.2. **Connection with Floquet theory.** One of the simplest ways in which ionization may fail for a given $V(t,x)$ is the existence of a solution of the Schrödinger equation of the form

$$\psi(t,x) = e^{i\phi t}v(t,x) \text{ with } \phi \in \mathbb{R} \text{ and } v \in L^2(\mathbb{R}^3 \times [0, 2\pi/\omega]) \text{ time-periodic.} \tag{1.2}$$

Substitution in (1.1) leads to the following Floquet equation:

$$Kv = -\phi v;\ K := i\frac{\partial}{\partial t} - \Big(-\Delta - br^{-1} + V(t,x)\Big) \tag{1.3}$$

where the Floquet operator K is defined on $L^2(\mathbb{R}^3 \times [0, 2\pi/\omega])$.

Although it may seem surprising, the existence of a Floquet bound state is in fact –in a wide class of settings [24, 13, 14, 15, 18]– the only possibility for ionization to fail. As we will show this is also true for (1.1). This is due to the possibility of reformulating the ionization problem in terms of a compact operator whose spectrum is closely connected to the Floquet spectrum. Generic delocalization is then expected since L^2 solutions of the Schrödinger equation of the form (1.2) are unlikely. We prove that for $V(t,x)=\Omega(r)\cos\omega t$, $\Omega>0$ on $[0,1]$ and sufficiently smooth, they do not exist. Whether they exist for any Ω periodic in t and continuous is an open problem.

2. Main results

If the probability to find the particle in any compact set vanishes for large t, *i.e.*

$$\int_{r<a}|\psi(t,x)|^2dx\to 0 \quad as \quad t\to\infty \ \ \forall a\in\mathbb{R}^+ \tag{2.1}$$

we say that the system *ionizes*.

2.1. A general criterion of ionization.

We write (1.1) in the form

$$i\psi_t(t,x)=\Big(-\Delta-b/r+\sum_{j\in\mathbb{Z}}\Omega_j(x)e^{ij\omega t}\Big)\psi(t,x),\quad \psi(0,x)=\psi_0(x) \tag{2.2}$$

with $\Omega_{-j}=\overline{\Omega_j}$ compactly supported in x, say in the unit ball B_1, differentiable there, and sufficiently smooth in t. More precisely, we require that a weighted l_1 norm in j is finite,

$$\sum_{j\in\mathbb{Z}}\|\Omega_j\|_2(1+|j|)^{3/2}<\infty \tag{2.3}$$

where $\|\cdot\|_2$ is the usual L_2 norm in x. Consider the homogeneous partial differential system corresponding to the Laplace transform of (2.2), cf §5,

$$(-\Delta-b/r+\sigma+n\omega)v_n=\sum_{j\in\mathbb{Z}}\Omega_j(x)v_{n-j};\quad v\in\mathcal{H},\ \operatorname{Re}\sigma\in[0,\omega),n\in\mathbb{Z} \tag{2.4}$$

where the Hilbert space $\mathcal{H}$ is defined in §3 and

$$p=i(\sigma+n\omega),\ \text{with}^{(1)}\ \operatorname{Re}\sigma\in[0,\omega),\quad n\in\mathbb{Z} \tag{2.5}$$

Theorem 2.1. *(i) In the setting (1.1) ionization occurs* iff *(2.4) has only trivial solutions. This is true* iff *the discrete Floquet spectrum (see (1.3)) is empty.*

(ii) Furthermore, if there exists a nontrivial solution of (2.4), then

$$v_n(x)=0 \ \ \textit{for all} \ \ n<0 \ \textit{and}\ x\notin B_1 \tag{2.6}$$

The proof is given in §7.

Note 2.1. Equation (2.6) makes the second order system (2.4) formally overdetermined since the regularity of v_n imposes both Dirichlet and Neumann conditions on ∂B_1 for $n<0$. Nontrivial solutions are not, in general, expected to exist.

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(1) In some parts of the paper it is convenient to relax this restriction on σ.

2.2. **Ionization in the spherically symmetric case.** It is usually difficult to analyze the spectrum of the Floquet operator. In this paper we introduce a relatively general method, based on Theorem 2.1 (ii) and rigorous WKB analysis of infinite systems of ODEs to show that exceptional solutions of the Floquet problem do not exist in the case of harmonic, radially symmetric, compactly supported forcings and thus ionization occurs. More precisely, we have the following result.

Theorem 2.2. *For $V(t,x) = \Omega(r)\cos\omega t$, with $\Omega(r) = 0$ for $r > 1$, $\Omega(r) > 0$ for $r \le 1$ and $\Omega(r) \in C^\infty[0,1]$, ionization always occurs.*

Note 2.2. The condition $\Omega(1^-) \neq 0$ simplifies the arguments but our method can allow Ω to vanish no faster than algebraically at 1. Some conditions on $V(t,x)$ are almost certainly needed since there are simple models for which ionization fails, see [15], [31] and [35].

Note 2.3. The results can be extended to non-Coulomb systems where H_C is replaced by

$$H_W = -\Delta - b/r + W(r)$$

where b may be zero and $W(r)$ decays at least as fast as $r^{-1-\epsilon}$ for large r, $W \in L^\infty(\mathbb{R}^3)$. The crucial part where the specific shape of the Green's function of $H_C - ip$ is used is in Proposition 5.7 and in the continuity with respect to an extended variable X. With spherical symmetry, separation of variables and standard ODE arguments would relatively easily give the necessary decay and continuity properties of ψ.

3. Domain and Hilbert space

Let B be a ball of radius $r_0 \geqslant 1$ centered at zero (thus containing the support of V) and let χ be its characteristic function. In our setting, $-br^{-1} \in L^\infty + L^2$ and thus Theorem 5.4, [30] pp. 303, shows that, $-\Delta - br^{-1}$ restricted to C_0^∞ is essentially self-adjoint. Given a decomposition $-br^{-1} = q_0 + q_1$ where $q_0 \in L^\infty(\mathbb{R}^3)$ is say $-br^{-1}(1-\chi)$ and $q_1 = -br^{-1} - q_0$, we denote by Q_0 and Q_1 the maximal operators associated to q_0 and q_1 and set $Q = Q_0 + Q_1$. Then, [30], $H_C := -\Delta + Q$ is self-adjoint on $D(H_0) = D(-\Delta) = \mathcal{F}^{-1}D(k^2)\mathcal{F}$ where $\mathcal{F}$ is the Fourier transform and k is multiplication by the Fourier space variable.

Note 3.1. For every t the potential $H_C + V(t,x)$ satisfies the assumptions of Theorem X.71, [33] v.2 pp 290. Thus at all t the solution of the Schrödinger equation (1.1) is in the domain of $-\Delta$. This implies continuity in x of $\psi(t,x)$ and of its $t-$Laplace transform. It also follows that the unitary propagator is strongly differentiable in t.

3.1. **The Hilbert space $\mathcal{H}$.** To analyze the properties of a suitable integral version of (1.1) we use the Hilbert space(s)

$$\mathcal{H} = l^2_{3/2}(L^2(B)) \tag{3.1}$$

defined as the space of sequences $\{y_n\}_{n\in\mathbb{Z}}$, $y_n \in L^2(B)$ with

$$\|y\|^2_{\mathcal{H}} = \sum_{n\in\mathbb{Z}} (1+|n|)^3 \|y_n\|^2_{L^2(B)} < \infty$$

and B is any ball in $\mathbb{R}^3$.

Note 3.2. The space $\mathcal{H}$ is different from the Hilbert space $l^2(L^2(\mathbb{R}^d))$ used in Floquet theory. See also [18].

4. Approach and organization of the paper

The proof of ionization is obtained by representing the solution of the Schrödinger equation as an inverse Laplace transform of a function shown to be analytic in the open right half plane in $\mathbb{C}$, and in $L^1(i\mathbb{R})$; ionization follows essentially from the Riemann-Lebesgue lemma.

Since the construction consists of many steps with relatively intricate proofs, we begin by listing the main constituent results, relegating their proof to later sections when necessary. We include at the end of the paper an index of definitions.

The key ingredients for Theorem 2.2 are Propositions 6.3, 6.5, 6.6, 9.2 and Theorem 9.1. These allow us to conclude that for sufficiently large negative n, a nonzero solution of the homogeneous system (2.4) in the setting of Theorem (2.2) would have unacceptably singular behavior at $r = 0$.

The accumulation of eigenvalues at the top of the discrete spectrum of H_C, stemming from the slow decay of the potential, creates an essential singularity in the Laplace transform of ψ. Proposition 6.3 is essential to show that this singularity is integrable (see also Notes 5.9 and 6.2).

As shown in §14.2, part of the content of Theorem 9.1 could be formally obtained by WKB methods. The rigorous proof is delicate since one deals with an infinite coupled system of ODEs and furthermore the (usually transcendental) multiplicative constant in the asymptotics matters.

5. Some auxiliary results

Let $\psi_0 \in L^2$. The Laplace transform, in the sense of L^2, of the unitarily evolved ψ_0,

$$\hat{\psi}(\cdot, p) := \int_0^\infty \psi(\cdot, t) e^{-pt} dt \tag{5.1}$$

exists for $\operatorname{Re} p > 0$ and the map $p \to \psi(\cdot, p)$ is L^2 valued analytic in the right half plane

$$p \in \mathbb{H} = \{z : \operatorname{Re} z > 0\} \tag{5.2}$$

Simple calculations show that if ψ satisfies (2.2) then $\hat{\psi}$ satisfies the equation

$$(H_C - ip)\hat{\psi}(p, x) = -i\psi_0 - \sum_{j \in \mathbb{Z}} \Omega_j(x) \hat{\psi}(p - ij\omega, x) \tag{5.3}$$

Clearly, (5.3) couples $\hat{\psi}(p_1, x)$ with $\hat{\psi}(p_2, x)$ iff $(p_1 - p_2) \in i\omega\mathbb{Z}$. With the notation (2.5) we have we define[(2)] $y_n^{[0]}(\sigma, x) = \hat{\psi}(i(\sigma + n\omega), x)$. Eq. (5.3) now becomes a differential-difference system

$$(H_C + \sigma + n\omega) y_n^{[0]} = -i\psi_0 - \sum_{j \in \mathbb{Z}} \Omega_j(x) \left(S^{-j} y^{[0]} \right)_n \tag{5.4}$$

where S is the shift operator defined by

$$(Sy)_n = y_{n+1} \tag{5.5}$$

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(2)The superscript [0] is used since further transformations will be made later.

We are interested in regularity properties of $\hat{\psi}(p,x)$ with respect to p in the closed right half plane. These are most easily obtained from a Duhamel reformulation of the differential system (5.3). But this reformulation is not so straightforward, since H_C has eigenvalues, of increasing multiplicity, accumulating at zero.

Lemma 5.1 ([30], pp. 304, Theorem 5.7)**.** *The spectrum of $H_C = -\Delta - b/r$ is real (cf. §3) and its restriction to $\mathbb{R}^-$ consists of isolated eigenvalues with finite multiplicity.*

We proceed as follows. Recalling that χ is the characteristic function of a ball B with radius $r_0 > 1$, we define

$$\beta = \begin{cases} c > 0 \text{ if } 0 \leqslant n \leqslant n_c \\ 0 \text{ otherwise} \end{cases} \tag{5.6}$$

where n_c is chose sufficiently large so that $-\operatorname{Re}\sigma - n_c\omega$ is below the discrete spectrum of H_C by a fixed amount, say the same c as in (5.6).

Let

$$\mathcal{A}_\beta := H_C - i\beta\chi(r) - ip =: H_C - i\beta\chi(r) + \sigma + n\omega \tag{5.7}$$

Clearly $\mathcal{A}_\beta$ is defined on $D(H_0)$ and $\mathcal{A}_\beta^* = \mathcal{A}_{-\beta} + \sigma^* - \sigma$.(3) The dependence of β on p (through n_c and c, which, for simplicity of notation we do not indicate) is convenient for pushing the poles of the resolvent $\mathcal{A}_\beta^{-1}$ into the left half plane while allowing for an explicit formula throughout the continuous spectrum. We denote, with slight abuse of notation,

$$\mathcal{A}_0 = H_C - ip \tag{5.8}$$

Then $\mathcal{A}_0^{-1}$ is well defined for $-ip = \sigma + n\omega$ for all $n > n_c$, i.e., $\operatorname{Im} p > n_c\omega$.

Proposition 5.2. *For large $c > 0$ in (5.6), see also §11.9, $\mathcal{A}_\beta(p)$ is invertible for $\operatorname{Re} p > 0$ (that is, $\operatorname{Im}\sigma < 0$) and for $\operatorname{Re} p = 0, \operatorname{Im} p > 0$ ($\operatorname{Im}\sigma = 0$, $n\omega + \operatorname{Re}\sigma > 0$). The inverse $\mathfrak{R}_\beta$ is analytic in a neighborhood $\mathcal{D}$ of the set $\{p : \operatorname{Re} p > 0\} \cup \{p : \operatorname{Re} p = 0, \operatorname{Im} p > 0\}$.*

Proof. Note that $\mathcal{A}_\beta$ and $\mathcal{A}_\beta^*$ are densely defined, hence closed [33]. Once we show that $\mathcal{A}_\beta(p)$ is invertible in $\mathbb{H}$ and on the ray $\{p : \operatorname{Re} p = 0, \operatorname{Im} p > 0\}$, invertibility in $\mathcal{D}$ follows (the spectrum of the closed operator $H_C - i\beta\chi(r)$ is a closed set). Analyticity in $\mathcal{D}$ also follows readily (cf [33], vol. 1, Theorem VIII.2 , p. 254).

Invertibility. We first show below that no $ip \in i\mathcal{D}$ is an eigenvalue of $H_C - i\beta\chi(r)$. Assume we had $(H_C - i\beta\chi(r))\psi = ip\psi$, *i.e.,* $\mathcal{A}_\beta\psi = 0$; then, with $\sigma = \sigma_1 + i\sigma_2$ we have

$$\langle\psi, (-\Delta + (\sigma_1 + i\sigma_2) + n\omega - br^{-1})\psi\rangle + \langle\psi, -i\beta\chi\psi\rangle = 0 \tag{5.9}$$

The real part of (5.9) can only vanish if $n_c \geq n \geq 0$. In that case, we would have $\beta = c > 0$. Taking the imaginary part of (5.9) we get

$$0 = \sigma_2\langle\psi,\psi\rangle - c\langle\psi,\chi\psi\rangle \tag{5.10}$$

If $\sigma_2 < 0$ this immediately implies $\psi = 0$. If $\sigma_2 = 0$ we get

$$0 = c\langle\chi\psi,\chi\psi\rangle \tag{5.11}$$

(3)Indeed, since $\mathcal{A}_{-\beta} + \sigma^* - \sigma$ is well defined on $D(-\Delta)$, a larger domain for $\mathcal{A}_\beta^*$ would entail a larger domain for the already self-adjoint operator $-\Delta + Q$ on $D(-\Delta)$.

implying $\langle \chi\psi, \chi\psi\rangle = 0$, i.e., $\chi\psi = 0$. But $\chi\psi = 0$ implies $0 = \mathcal{A}_\beta\psi = \mathcal{A}_0\psi$. In spherical coordinates the equation $\mathcal{A}_0\psi = 0$ becomes a system of ODEs

$$\left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - br^{-1} + \sigma_1 + n\omega\right)\psi_{n,l,m} = 0 \tag{5.12}$$

Since $\chi\psi = 0$, the solution of (5.12) vanishes identically on $[0, r_0]$; but then, by standard ODE arguments the solution is identically zero. The rest of the proof of invertibility is given in §11.1. ∎

5.1. **Restriction to the ball B.**

Note 5.3. To study ionization, we only need to know $\psi(t, x)$ for x in a fixed (but arbitrary) ball B with radius greater than one. As before, let χ be the characteristic function of B. We shall therefore need to study the properties of $\chi\mathfrak{R}_\beta\chi$. This sandwiched operator (which preserves information about $L^2(\mathbb{R}^3)$ through boundary conditions on ∂B) is the one that we shall most often use below.

5.2. **Integral representation of the system.**

We now use the invertibility of $\mathcal{A}_\beta$ for $\mathrm{Re}\, p > 0$ ($\mathrm{Im}\,\sigma < 0$) to write (5.4) in the form

$$y_n^{[0]} = -i\mathfrak{R}_\beta\psi_0 + \mathfrak{R}_\beta\Big[-i\beta\chi y_n^{[0]} - \sum_{j\in\mathbb{Z}}\Omega_j(x)\left(S^{-j}y^{[0]}\right)_n\Big] \tag{5.13}$$

Restricting (5.13) to B, since Ω is supported there, we have

$$y_n^{[0]} = -i\chi\mathfrak{R}_\beta\psi_0 + \chi\mathfrak{R}_\beta\chi\Big[-i\beta y_n^{[0]} - \sum_{j\in\mathbb{Z}}\Omega_j(x)\left(S^{-j}y^{[0]}\right)_n\Big] \tag{5.14}$$

5.3. **Definition of $\mathfrak{C}$.**

Setting $Y^{[0]} = \{y_n^{[0]}\}_{n\in\mathbb{Z}}$, $Y_0^{[0]} = \{-i\chi\mathfrak{R}_\beta\psi_0\}_{n\in\mathbb{Z}}$ and

$$(\mathfrak{C}y)_n = \chi\mathfrak{R}_\beta\chi\Big[-i\beta y_n - \sum_{j\in\mathbb{Z}}\Omega_j(x)\left(S^{-j}y\right)_n\Big]$$

we write, compactly,

$$Y^{[0]} = Y_0^{[0]} + \mathfrak{C}Y^{[0]} \tag{5.15}$$

To ensure the decay in n of the inhomogeneous term and thus make it an element of $\mathcal{H}$, we take out several leading asymptotic terms in n: we write $Y^{[0]} = Y_0^{[0]} + \mathfrak{C}Y_0^{[0]} + \mathfrak{C}^2Y_0^{[0]} + \mathfrak{C}^3Y_0^{[0]} + Y$, and let $Y_0 = \mathfrak{C}^4Y_0^{[0]}$. It follows from (5.15) that

$$Y = Y_0 + \mathfrak{C}Y \tag{5.16}$$

Proposition 5.4. *For $\mathrm{Re}\, p$ large enough, (5.16) has a unique solution in $\mathcal{H}$. By construction, this gives rise to a solution of (5.4), and therefore corresponds to the Laplace transform of the unique L^2 solution of the initial value problem (2.2).*

Proof. This follows from the fact that $\mathfrak{C}$ has small norm for large $\mathrm{Re}\, p$. Indeed, the translation operator S is quite straightforwardly shown to be bounded in $\mathcal{H}$: using (2.3), the proof in Lemma 27 in [18] goes through without changes for the norm used in the present paper. By the second resolvent identity we have

$$\mathfrak{R}_\beta = \left(1 - i\beta\mathfrak{R}_0\chi\right)^{-1}\mathfrak{R}_0$$

Since $-\Delta - br^{-1}$ is self-adjoint, we have by the spectral theorem, for some $C > 0$ independent of p,

$$\|(-\Delta - br^{-1} - ip)^{-1}\|_{L^2(\mathbb{R}^3)} \leqslant C(\mathrm{Re}\, p)^{-1}$$

and thus $\|\mathfrak{R}_\beta\|_{L^2(B)} \leqslant C_1(1 + |\mathrm{Im}\,\sigma|)^{-1}$ where C_1 does not depend on n or σ. By the definition of $\mathcal{H}$ it follows immediately that $\|\mathfrak{C}\|_{\mathcal{H}} \leqslant C_2(1 + |\mathrm{Im}\,\sigma|)^{-1}$ for some $C_2 > 0$ independent of σ. ∎

Remark 5.5. As shown in §11.2 it suffices to prove ionization on a dense set of ψ_0; let this be $C_0^\infty(\mathbb{R}^3)$. With this choice, $Y_0 \in \mathcal{H}$, as it is easily seen from Proposition 5.7 below. For spherically symmetric V, a more symmetric choice of the dense set will be made.

Note 5.6. We know that the system (5.4) has a unique $L^2(\mathbb{R}^3)$ solution corresponding to $\hat{\psi}(p, x)$ if $\mathrm{Re}\, p > 0$. In the *spherically symmetric case*, the logic of the analysis is as follows. We prove that the integral equation possesses a unique solution for smaller values of $\mathrm{Re}\, p \geq 0$ also, and that its solution is analytic in p up to, but not including, the imaginary line, on which however it has an L^1. By uniqueness, this solution is nothing else but $\hat{\psi}$ and ionization follows from the Riemann-Lebesgue lemma.

Proposition 5.7. *As $|n| \to \infty$, $\|\chi\mathfrak{R}_\beta\chi\| = \|\chi\mathfrak{R}_0\chi\| = O(|n|^{-1/2}\log^{1/2}|n|)$ (recall that $\beta = 0$ if $|n|$ is large).*

The proof is given in §11.6. It relies, for $n < 0$, on the explicit form of the resolvent for H_C, see §11.3. Using spherical symmetry, the explicit Green's function could be avoided, but in view of future generalizations we prefer this more delicate approach.

5.4. **Compactness.**

Proposition 5.8. *$\mathfrak{C} : \mathcal{H} \to \mathcal{H}$ is compact for p in $\overline{\mathbb{H}}$.*

Proof. We first take $\mathrm{Re}\, p \geqslant 0$, $\mathrm{Im}\, p > 0$ not too large so that $\beta > 0$. We have, by adding and subtracting $i\epsilon$ with $\epsilon > 0$ from $\mathcal{A}_\beta$ and using the second resolvent formula (*i.e.* $A^{-1} - B^{-1} = B^{-1}(B - A)A^{-1}$ whenever everything is well defined), (5.17)

$$\chi\mathcal{A}_\beta^{-1}\chi =: \chi\mathfrak{R}_\beta\chi = \chi(-\Delta + i\epsilon)^{-1}\chi - \chi\mathfrak{R}_\beta(-br^{-1} - i\beta\chi - i\epsilon - ip)(-\Delta + i\epsilon)^{-1}\chi$$

The Green's function for $-\Delta + i\epsilon$ is given by ([33])

$$G(x, y) = \frac{1}{4\pi|x - y|}e^{-\frac{1}{\sqrt{2}}(1+i)\sqrt{\epsilon}|x-y|} \tag{5.18}$$

Now if $\|\phi_j\|_{L^2(B)} \leqslant 1$ then the functions $f_j = (-\Delta + i\epsilon)^{-1}\chi\phi_j$ are seen by straightforward calculation to be equicontinuous on the one point compactification of $\mathbb{R}^3$. A subsequence, without loss of generality assumed to be the f_j's themselves, converges in $L^2(\mathbb{R}^3)$ as well (to a function with exponential decay, since there is a $\delta_1 > 0$ small enough $e^{\delta_1|x|}(-\Delta + i\epsilon)^{-1}\chi\phi_j)$ is also equicontinuous on the compactification of $\mathbb{R}^3$). The first term in (5.17) is then compact by definition.

The functions f_j are equicontinuous with exponential decay, so that the sequence $\{g_j\}_{j\in\mathbb{N}}, g_j = (-br^{-1} - i\beta\chi - i\epsilon - ip)f_j$ is also convergent in $L^2(\mathbb{R}^3)$, to a function with exponential decay rate at least δ_1. Since $\mathfrak{R}_\beta$ is a bounded operator on $L^2(\mathbb{R}^3)$ $\mathfrak{R}_\beta g_j$ converges in $L^2(\mathbb{R}^3)$, and this case is completed.

For $\operatorname{Im} p \leqslant 0$ or larger $\operatorname{Im} p > 0$ we have $\beta = 0$. In this case, compactness follows from the explicit expression of the Green's function see §11.4, cf. also [29].

By Proposition 5.7, and the previous argument, $\mathfrak{C}$ is a norm limit of compact operators (the truncations of $\mathfrak{C}$ to the subspaces of $\mathcal{H}$ with vanishing components for $|n| > N$). ∎

Note 5.9. *Compactness versus regularity.* The term $-i\beta\mathcal{X}$ was introduced in §5.2 to ensure that the poles of the modified resolvent, $\mathfrak{R}_\beta$, are not in the closed right half p plane. Since $-i\beta\mathcal{X}$ is localized in x, the shifts in the poles created by the discrete spectrum of H_C are smaller as $p \to 0$ (the size of the orbitals of the Hydrogen atom grow when the energy approaches zero.) The resulting integral operators have an essential singularity at $p = 0$. If instead of $-i\beta\mathcal{X}$ we had added a nonlocal operator on both sides of (5.4) before inversion of the operator, it would have destroyed compactness and created technical complications (albeit, clearly, with no impact on ionization).

6. Ionization condition.

We consider the homogeneous equation (2.4), $v = \mathfrak{C}v$, associated to (5.16), and look for a nontrivial solution in $\mathcal{H}$. We multiply (2.4) by $\overline{v}_n$, integrate over the ball B (with radius $r_0 > 1$), sum over n (this is legitimate since $v \in \mathcal{H}$) and take the imaginary part of the resulting expression. Noting that

$$\overline{\sum_{j,n\in\mathbb{Z}} \Omega_j(x) v_{n-j}\overline{v_n}} = \sum_{j,n\in\mathbb{Z}} \Omega_{-j}\overline{v}_{n-j}v_n = \sum_{j,n\in\mathbb{Z}} \Omega_j \overline{v}_{n+j} v_n = \sum_{j,m\in\mathbb{Z}} \Omega_j(x)\overline{v_m}v_{m-j} \tag{6.1}$$

so the sum (6.1) is real, we get from (2.4)

$$\begin{aligned} 0 &= \operatorname{Im}\left(-\sigma\sum_{n\in\mathbb{Z}}\int_B |v_n(x)|^2 dx + \int_B \sum_{n\in\mathbb{Z}} dx\,\overline{v}_n\Delta v_n\right) \\ &= -\operatorname{Im}\sigma\sum_{n\in\mathbb{Z}}\int_B |v_n(x)|^2 dx + \frac{1}{2i}\int_{\partial B}\left(\sum_{n\in\mathbb{Z}} \overline{v}_n\nabla v_n - v_n\nabla\overline{v}_n\right)\cdot \mathbf{n}\, dS \end{aligned} \tag{6.2}$$

It is convenient to decompose v_n using spherical harmonics. We write

$$v_n = \sum_{l\geqslant 0, |m|\leqslant l} R_{n,l,m}(r) Y_l^m(\theta,\phi). \tag{6.3}$$

The last integral in (6.2), including the prefactor, then equals

$$\begin{aligned} &-2\pi\, i\, r_0^2 \sum_{n\in\mathbb{Z}}\sum_{m,l}\left[\overline{R}_{n,m,l}R'_{n,m,l} - \overline{R'}_{n,m,l}R_{n,m,l}\right] \\ &= -2\pi\, i\, r_0^2\sum_{n\in\mathbb{Z}}\sum_{m,l}\mathcal{W}[\overline{R}_{n,m,l}, R_{n,m,l}] \end{aligned} \tag{6.4}$$

where $\mathcal{W}[f,g]$ is the Wronskian of f and g. On the other hand, we have outside of B

$$\Delta v_n + br^{-1}v_n - (\sigma + n\omega)v_n = 0 \tag{6.5}$$

and then by (6.3), the $R_{n,l,m}$ satisfy for $r > r_0$ the equation

$$R'' + \frac{2}{r}R' + br^{-1}R - \frac{l(l+1)}{r^2}R = (\sigma + n\omega)R \tag{6.6}$$

where we have suppressed the subscripts. Let $g_{n,l,m} = rR_{n,l,m}$. Then for the $g_{n,l,m}$ we get

$$g'' - \left[\frac{l(l+1)}{r^2} + (\sigma + n\omega) - br^{-1}\right] g = 0 \tag{6.7}$$

Thus

$$\overline{R}R' = \frac{\overline{g}g'}{r^2} - \frac{|g|^2}{r^3} \tag{6.8}$$

and

$$r^2\mathcal{W}[\overline{R}, R] = \mathcal{W}[\overline{g}, g] =: \mathcal{W}_n. \tag{6.9}$$

Multiplying (6.7) by $\overline{g}$, and the conjugate of (6.7) by g and subtracting, we get for $r > r_0$,

$$\mathcal{W}_n' = (\sigma - \overline{\sigma})|g|^2 = 2i|g|^2 \mathrm{Im}\, \sigma \tag{6.10}$$

Remark 6.1. *Direct estimates using the Green's function representation (11.18) below imply that*

$$v_n(x) = \frac{e^{-\kappa_n r}}{r^{1+\frac{b}{2\kappa_n}}}\Big(c_n(\theta, \phi) + O(r^{-1})\Big) \text{ as } r \to \infty \tag{6.11}$$

with $c_n(\theta, \phi)$ independent of r and with

$$\kappa_n = \sqrt{-ip} = \sqrt{\sigma + n\omega} \tag{6.12}$$

(when $\mathrm{Re}\, p > 0$, $\mathrm{Im}\, \sigma < 0$, κ_n is in the fourth quadrant)

(i) We first take $\mathrm{Im}\sigma < 0$, to illustrate the argument. Using (6.11) we get

$$g \sim Ce^{-\kappa_n r} r^{-\frac{b}{2\kappa_n}}(1 + o(1)) \quad \text{as} \quad r \to \infty \tag{6.13}$$

There is a one-parameter family of solutions of (6.7) satisfying (6.13) and the asymptotic expansion can be differentiated [44]. We assume, to get a contradiction, that there exist n for which $g = g_n \neq 0$. For these n we have, using (6.13), differentiability of this asymptotic expansion and the definition of κ_n that

$$\frac{1}{2i} \lim_{r\to\infty} |g_n|^{-2}\mathcal{W}_n = -\mathrm{Im}\, \kappa_n > 0. \tag{6.14}$$

It follows from (6.10) and (6.14) that $\frac{1}{2i}\mathcal{W}_n$ is strictly positive for all $r > r_0$ and all n for which $g_n \neq 0$. This implies that the last term in (6.2) is a sum of nonnegative terms which shows that (6.2) cannot be satisfied nontrivially.

(ii): $\mathrm{Im}\, \sigma = 0$. For $n < 0$, we use Remark 6.1 (and differentiability of the asymptotic expansion as in Case (i)) to calculate $\mathcal{W}_n$ in the limit $r \to \infty$: $\mathcal{W}_n = 2i|c_n|^2|\kappa_n|(1 + o(1))$. Since for $\mathrm{Im}\, \sigma = 0$, $\mathcal{W}_n$ is constant, cf. (6.10), it follows that $\mathcal{W}_n = 2i|c_n|^2|\kappa_n|$ exactly. Thus, using (6.2) and (6.4) we have

$$v_n(x) = 0 \quad \text{for all} \quad n < 0 \text{ and } r > r_0 \tag{6.15}$$

For $r > r_0 > 1$ (where $V(t, x) = 0$) we have $\mathfrak{O}v_n = 0$, where $\mathfrak{O}$ is the elliptic operator $-\Delta - b/r + \sigma + n\omega$. The proof that $v_n(x) = 0$ for $r > 1$ then follows immediately from (6.15), by standard unique continuation results [26, 32, 43] (in fact, $\mathfrak{O}$ is analytic hypo-elliptic). See also Note 2.1.

Note 6.2. As a result of the choice explained in Note 5.9, the poles of the resolvent $\mathfrak{R}_\beta$ accumulate at $p=0$ from $-\mathbb{H}$, along a curve tangent to the positive imaginary p-axis. As a result, while being uniformly bounded, $\mathfrak{R}_\beta$ is not continuous along the imaginary p line at zero but oscillates without limit. Boundedness of $\chi\mathfrak{R}_\beta\chi$ (which is not difficult to prove) does not ensure boundedness of the solution. We do have however continuity in an extended parameter. Let $X := (\sigma, Z)$ with

$$Z = e^{\frac{i\pi b}{2\lambda}}; \qquad (\text{where } \lambda^2 = \sigma) \tag{6.16}$$

(The dependence of Z on λ reflects the nature of the behavior of the solution.) The resolvent is continuous in X and a useful Fredholm alternative can be applied.

Definition Let D_ϵ^+ a half disk of radius ϵ in the right half complex plane and let D be the unit disk.

Proposition 6.3. *For any $r_0 > 1$ (i.e. outside the support of Ω) we can choose a c in (5.6) (see §11.9 below) such that the following statement holds.*

The resolvent $\chi\mathfrak{R}_\beta\chi$ is analytic in $p = i(\sigma + n\omega)$ for $\operatorname{Re} p \geqslant 0$, $p \neq 0$. For $n = 0$ and ϵ small enough, it is continuous in X (cf. (6.16)) on the compact set $K = \overline{(-iD_\epsilon^+)} \times \overline{D}$; in particular, $\chi\mathfrak{R}_\beta\chi$ is bounded in K.

Proof. The existence of an analytic limit of $\chi\mathfrak{R}_\beta\chi$ as $i\mathbb{R}\setminus\{0\}$ is approached follows *e.g.* from [9] pp. 385. The behavior in X is more delicate and we obtain it in §11.9 from an explicit calculation.

6.1. **Regularity of the solutions.** The behavior of Y for $p \in i\mathbb{R}$ is crucial for ionization. Indeed, by Proposition 6.5 below, the inverse Laplace transform contour can be pushed to this line. We next show that analyticity of Y on $i\mathbb{R}\setminus\{0\}$ and boundedness at zero and decay at infinity imply ionization. This follows from Proposition 6.6 below. We first need two results.

Proposition 6.4. *Let C_X be a collection of bounded operators depending continuously on a parameter $X \in K$ where K is a compact set. Assume furthermore that $I - C_X$ is invertible for all $X \in K$. Then*

$$(I - C_X)^{-1} \tag{6.17}$$

is continuous in the operator norm in $X \in K$. In particular, since K is a compact set, there is a constant $\tilde{c}$ such that for all $X \in K$ we have

$$\|(I - C_X)^{-1}\| \leqslant \tilde{c} \tag{6.18}$$

and the solution of the equation

$$Y_X = Y_0 + C_X Y_X \tag{6.19}$$

is continuous in X.

This follows from the second resolvent formula, in a standard way. Indeed,

$$(I - C_X)^{-1} - (I - C_{X'})^{-1} = (I - C_{X'})^{-1}(C_X - C_{X'})(I - C_X)^{-1}$$

$$(I - C_{X'})^{-1}\Big[I + (C_X - C_{X'})(I - C_X)^{-1}\Big] = (I - C_X)^{-1} \tag{6.20}$$

We fix X and let $X' \to X$. Since $(I - C_X)^{-1}$ is bounded, then $\|(C_X - C_{X'})(I - C_X)^{-1}\| \to 0$ as $X' \to X$ and

$$I + (C_X - C_{X'})(I - C_X)^{-1} \tag{6.21}$$

is invertible when X and X' are close enough and $[I-(C_X-C_{X'})(I-C_X)^{-1}]^{-1} \to I$ in operator norm as $X' \to X$. Thus

$$(I - C_{X'})^{-1} \to (I - C_X)^{-1} \tag{6.22}$$

in operator norm, as $X \to X'$.

Proposition 6.5. *Y is analytic in p for* $\mathrm{Re}\, p > 0$*, and it is continuous in $X \in \overline{(iD_\epsilon^+)} \times \overline{D}$. In particular Y is bounded for $p \in \overline{iD_\epsilon^+}$. Furthermore, there exists a p–independent continuous $C(x) > 0$ such that $|\hat{\psi}(p,x)| \leqslant C(x)(1+|p|)^{-3/2}$ for $x \in B$.*

Proof. Regularity is a direct consequence of Propositions 6.3 and 6.4. The large p inequality follows from the fact that the norm $\|\cdot\|_{\mathcal{H}}$ is also continuous in $X \in K$. ∎

6.1.1. *The Fredholm alternative.* We can now formulate the ionization condition in terms of the Fredholm alternative.

Proposition 6.6. *If the homogeneous equation associated to (5.16) has no nontrivial solution, then the system ionizes as in (2.1).*

Proof. The function $\hat{\psi}(is;x)$, $s \in \mathbb{R}$, is analytic for $s \neq 0$ and, by Proposition 6.3, bounded at zero. It has sufficient decay at infinity by construction. Thus its inverse Fourier transform, $\psi(t;x)$, decays as $t \to \infty$. The required integral (2.1) also goes to zero as shown in §11.10. ∎

7. Proof of Theorem 2.1

It is easy to check that the discrete time-Fourier transform of the eigenvalue equation for the Floquet operator, Eq. (1.3), $Kv = -\phi v$, coincides with (2.4), the differential version of the homogeneous equation associated to (5.16).

Part (ii) is simply (6.15).

For part (i) we first note that the existence of a Floquet eigenfunction entails failure of ionization since it implies the existence of a solution of (1.1) for which the absolute value is time-periodic. In the opposite direction, the result follows from Proposition 6.6 since (6.15) and (6.11) show that a solution of the homogeneous system $Y = \mathfrak{C}Y$ is necessarily in $L^2(\mathbb{R}^3)$.

8. Spherically symmetric Ω

We consider the case $V(t,x) = \Omega(r)\cos\omega t$.

Remark 8.1. *(i) Define as usual the angular momentum operators*

$$-\mathbf{L}^2 = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\tan\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

and

$$L_z = -i\frac{\partial}{\partial\phi}$$

Let $P_{l,m}$ be the orthogonal projector on the space of functions for which $\mathbf{L}^2 = l(l+1)$, $l \in \mathbb{N}\cup\{0\}$ and $L_z = m \in \mathbb{Z}$, $|m| \leqslant l$; define $P_{l_0} = \sum_{l>l_0;m\leqslant l} P_{l,m}$.

(ii) It is enough to show ionization for initial conditions ψ_0 in a dense set (cf. §11.2). In fact, by linearity, and since for any l_0 there is a finite number of projectors $P_{l,m}$ with $l \leqslant l_0$ which in the spherically symmetric case commute with the propagator, it suffices to show ionization for $\psi_0 \in P_{l,m}\Big(C_0^\infty(\mathbb{R}^3)\Big)$ for all l and m.

(iii) With this choice of ψ_0, the function Y_0 defined below, see (8.4) is in $\mathcal{H}$ for $\operatorname{Re} p \geqslant 0$ as follows easily from Proposition 5.7.

We thus restrict the analysis to $P_{l,m}\Big(C_0^\infty(\mathbb{R}^3)\Big)$. By separation of variables in spherical coordinates, the system (5.4) reduces to an ODE system[(4)] which now takes the form

$$\mathcal{A}_{\beta,r} y_n^{[0]} = -i\psi_0 - i\beta\chi y_n^{[0]} - \Omega(r) \sum_{j=\pm 1} \Big(S^{-j} y^{[0]}\Big)_n \tag{8.1}$$

where

$$\mathcal{A}_{\beta,r} = -\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - \frac{b}{r} + \sigma + n\omega - i\beta\chi \tag{8.2}$$

Applying $\mathfrak{R}_\beta$ to both sides of (8.1) we get

$$y_n^{[0]} = \mathfrak{R}_\beta \Big[-i\psi_0 - i\beta\chi y_n^{[0]} - \Omega(r) \sum_{j=\pm 1} \Big(S^{-j} y^{[0]}\Big)_n \Big] \tag{8.3}$$

To ensure the needed decay with respect to n we proceed as for (5.16): we write $Y^{[0]} = Y_0^{[0]} + \mathfrak{C}_{lm} Y_0^{[0]} + \mathfrak{C}_{lm}^2 Y_0^{[0]} + \mathfrak{C}_{lm}^3 Y_0^{[0]} + Y$ and let $Y_0 = \mathfrak{C}_{lm}^4 Y_0^{[0]}$ to get[(5)]

$$Y = Y_0 + \mathfrak{C}_{lm} Y \tag{8.4}$$

Remark 8.2. *The operator $\mathfrak{C}_{lm} = P_{l,m}\mathfrak{C}$ is compact. This follows from Proposition 5.8.*

8.0.2. *The Fredholm alternative in the spherically symmetric case.* Let $v = \{v_n\}_{n\in\mathbb{Z}}$ and write the homogeneous equation associated to (8.4):

$$v = \mathfrak{C}_{lm} v, \quad v = v(r) \in \mathcal{H} \tag{8.5}$$

Then, by the Fredholm alternative for compact operators, (8.5) implies $v = 0$ iff (8.4) has a unique solution in $\mathcal{H}$.

9. Further analysis of (8.5)

In view of Proposition 6.6 we see that the homogeneous equation holds the necessary ionization information.

Proposition 9.1. *Assume there exists a nontrivial solution $v \in \mathcal{H}$ of (8.5). Then there exists some $n_0 \geq 0$ such that either* (**i**) *$v_{n_0}(1) \neq 0$; or* (**ii**) *$v_{n_0}(1) = 0$, but $v'_{n_0}(1) \neq 0$. By homogeneity, we can assume that $v_{n_0}(1) = 1$ in case* (**i**) *and $v'_{n_0}(1) = -\sqrt{\Omega(1)}$ in case* (**ii**).

If this was not the case, contractive mapping arguments show that the solution vanishes identically. Lemma 12.1 (where $v_n = r^{-1} g_n$) deals with the details.

Proposition 9.2. *There is no nonzero solution of (8.5) in $\mathcal{H}$.*

---

(4) To simplify the notation we drop some l and m indices.

(5) As in the previous footnote, the indices l, m are dropped in Y, etc.

The proof relies on the detailed behavior of a hypothetical nonzero solution of the homogeneous system, rewritten as a differential system using the conditions at $r = 1$ implied by Theorem 2.1; see §12.

Definition 9.3. *For $r \in (0, 1]$ we define the function $\mathfrak{s}$ by*

$$\mathfrak{s}(r) = \int_r^1 \sqrt{\Omega(\rho)}d\rho \tag{9.1}$$

By our assumptions $\Omega > 0$ and $r \to \mathfrak{s}(r)$ is a smooth change of variable.

Definition 9.4. *Let*

$$\Omega_0 = \Omega(0),\ \Omega_0' = \Omega'(0),\ \mathfrak{s}_0 = \mathfrak{s}(0),\ \alpha = \frac{2\sqrt{\Omega_0}}{\mathfrak{s}_0}, \tag{9.2}$$

We let $\tau = 0$ in case (**i**) *and $\tau = 1$ in case* (**ii**) *as specified in Proposition 9.1 and define*

$$m_k(r) = \frac{\mathfrak{s}^{2k+\tau}}{(2k+\tau)!}F_k(r) \tag{9.3}$$

where

$$F_k(r) = \Omega^{-1/4}(r)\Omega^{1/4}(1) \exp\left[\frac{1}{4}\int_1^r ds \frac{\omega\mathfrak{s}(s)}{\sqrt{\Omega(s)}}\right] \frac{H(\alpha k r)}{H(\alpha k)} \tag{9.4}$$

In (9.4) $H(\zeta) = H_0(\zeta) + k^{-1}\check{H}(\zeta; k, l)$, $H_0(\zeta) = \sqrt{\frac{2}{\pi}}e^\zeta\zeta^{1/2}K_{l+1/2}(\zeta)$ and $\check{H}(\zeta; k, l)$ is the unique solution of the integral equation

$$\begin{aligned}\check{H}(\zeta; k, l) = H_0(\zeta)\int_0^\zeta e^{-2s}G_0(s)\mathcal{R}(H_0 + k^{-1}\check{H})(s)ds \\ - G_0(\zeta)\int_{k\alpha}^\zeta e^{-2s}H_0(s)\mathcal{R}(H_0 + k^{-1}\check{H})(s)ds\end{aligned} \tag{9.5}$$

where $G_0(\zeta) = \pi^{-1/2}e^\zeta I_{l+1/2}(\zeta)$,

$$\Big(\mathcal{R}H\Big)(\zeta) = 2H'\left(-\frac{\omega}{2\alpha^2} + \frac{\Omega_0'(1+2\zeta)}{2\alpha\Omega_0} + \frac{\tau}{2}\right) - \frac{bH}{\alpha\zeta}$$

and $K_{l+1/2}$ and $I_{l+1/2}$ are the modified Bessel functions of order $l + 1/2$. It can be checked from (9.5) that $H = H_0 + k^{-1}\check{H}$ satisfies

$$H'' = 2H'\left(1 - \frac{\omega}{2k\alpha^2} + \frac{\Omega_0'(1+2\zeta)}{2k\alpha\Omega_0} + \frac{\tau}{2k}\right) + H\left(\frac{l(l+1)}{\zeta^2} - \frac{b}{\alpha\zeta k}\right) \tag{9.6}$$

with the following asymptotic condition:

$$H(\zeta) \sim 1 + \frac{l(l+1)}{2\zeta} + \frac{b}{2k\alpha}\log\zeta + O\left(\frac{\log\zeta}{k\zeta}, \frac{1}{\zeta^2}\right) (\zeta, k \to \infty, \zeta \leqslant k\alpha) \tag{9.7}$$

Remark 9.5. *From the expression (9.5) for $\check{H}$ it is seen that as $\zeta \to 0$ we have $\check{H}(\zeta; k, l) \sim \mathrm{const.}\zeta^{-l+1}$ for $l \neq 1$ and $\check{H} \sim \mathrm{const.} + \mathrm{const}\zeta\log\zeta$ for $l = 1$. For any τ, $\check{H}$ is less singular than H_0 at $\zeta = 0$.*

Corollary 9.6. *To leading order in k as $k \to \infty$ we have*

$$H(\zeta) \sim \sqrt{\frac{2}{\pi}} e^{\zeta} \zeta^{1/2} K_{l+1/2}(\zeta)$$

It follows that for small ζ,

$$H_0(\zeta) \ \sim 2^{-l} \zeta^{-l} (2l)!/l!$$

Theorem 9.1. *If there is a nontrivial solution to (8.5), then there exists a subsequence $k_j \to \infty$ such that for any $r \in [0,1]$,*

$$r^{l+1} v_{n_0 - k_j}(r) = i^{k_j} r^l m_{k_j}(r)[1 + R_{k_j}(r)], \text{where } \lim_{j\to\infty} R_{k_j}(r) = 0 \tag{9.8}$$

and n_0 is given in Lemma 12.1, with $u_j := r^l m_{k_j}(r)$ uniformly continuous in $r \in (0,1]$ and $u_j(0) \to const \neq 0$ as $j \to \infty$.

The proof is given in §13 . For a heuristic discussion see §14.2.

10. Proof of Theorem 2.2

Theorem 9.1 shows that $(r^{l+1} v_n)(0) = m_n \neq 0$ for a subsequence of $n < 0$. For $l > 0$ this means the solution of the homogeneous system is not in $\mathcal{H}$. The case $l = 0$ is special. The kernel of the operator $\mathfrak{R}_0$ can be written in the form

$$K(x,y) = \frac{\log|x-y|}{|x-y|} K_1(x,y) \tag{10.1}$$

where K_1 is uniformly continuous (as follows from [9] p. 385 and the explicit resolvent). It is then immediate to show that if $v_{n\pm1}$ behave like $r^{-\alpha_1}$, $\alpha_1 \leqslant 1$ near $r = 0$. But v_n satisfy an integral system which this entails that they are bounded at $r = 0$, a contradiction. Thus there is no admissible solution of the homogeneous system and the result follows from Theorem 2.1 (i). See also the remarks in §14.4.

11. Proof of auxiliary statements

11.1. **End of the proof of Proposition 5.2.** The absence of eigenvalues of $\mathcal{A}_\beta$ was shown in the main text. If $\operatorname{Im} p \geq n_c$ then $\mathcal{A}_\beta = \mathcal{A}_0$, and we are, by construction, outside the spectrum of $\mathcal{A}_0$. Thus we are left with analyzing the case $\operatorname{Im} p \in [0, n_c)$ for which $\beta = c > 0$.

(1) *The range of $\mathcal{A}_\beta$ is dense.* Indeed, the opposite would imply [(6)] $\operatorname{Ker}(\mathcal{A}_\beta^*) \neq 0$, which leads to the same contradiction as in Step 1 (note that $\mathcal{A}_\beta^*$ is simply $\mathcal{A}_\beta$ with the signs of β and σ_2 changed at the same time).

(2) *For any $p \in \mathcal{D}$ there is an $\epsilon > 0$ such that $\|\mathcal{A}_\beta \psi\| > \epsilon \|\psi\|$.*

(a) If $\operatorname{Im} \sigma < 0$ and $\|\psi\| = 1$ then $\|\mathcal{A}_\beta \psi\| \geqslant -\operatorname{Im} \sigma$ since

$$\|\mathcal{A}_\beta \psi\| \geqslant |\langle \mathcal{A}_\beta \psi, \psi \rangle| = |\langle \mathcal{A}_0 \psi, \psi \rangle - i\beta \langle \chi\psi, \chi\psi \rangle| \geqslant |\operatorname{Im} \sigma \langle \psi, \psi \rangle| \tag{11.1}$$

(b) Let now $\operatorname{Im} \sigma = 0$, and assume $\sigma + n\omega > 0$ is between two eigenvalues of $-H_C$, the distance to the nearest being $\delta > 0$. To get a contradiction, assume that $\|\psi_j\| = 1$ and $\|\mathcal{A}_\beta \psi_j\| = \epsilon_j \to 0$. Then

$$\begin{aligned} \epsilon_j = \|\mathcal{A}_\beta \psi_j\| &\geqslant |\langle \mathcal{A}_\beta \psi_j, \psi_j \rangle| \\ &= |\langle \mathcal{A}_0 \psi_j, \psi_j \rangle - i\beta \langle \chi\psi_j, \chi\psi_j \rangle| \geqslant c\, |\langle \chi\psi_j, \chi\psi_j \rangle| \to 0 \end{aligned} \tag{11.2}$$

(6) [30], p. 267

thus $\chi\psi_j \to 0$, and by the definition of $\mathcal{A}_\beta$ and $\mathcal{A}_0$ we get

$$\|\mathcal{A}_0\psi_j\| \to 0 \tag{11.3}$$

which is impossible, since our assumption and (11.3) imply noninvertibility of $H_C + \sigma + n\omega$ while $-\sigma - n\omega$ is outside the spectrum of H_C.

(c) In the last case, $\operatorname{Im}\sigma = 0, \sigma + n\omega \in \sigma_d(-H_C)$; then if we assume there is a sequence ψ_j, $\|\psi_j\| = 1$ such that $\|\mathcal{A}_\beta\psi_j\| \to 0$ as $j \to \infty$ we get

$$\begin{aligned} \|\mathcal{A}_\beta\psi_j\| \geqslant |\langle\mathcal{A}_\beta\psi_j, \psi_j\rangle| = |\langle\mathcal{A}_0\psi_j, \psi_j\rangle - i\beta\langle\chi\psi_j, \chi\psi_j\rangle| \\ \geqslant |\beta\langle\chi\psi_j, \chi\psi_j\rangle| \to 0 \end{aligned} \tag{11.4}$$

Since $\|\mathcal{A}_0\psi_j\| \leq \|\mathcal{A}_\beta\psi_j\| + \beta\|\chi\psi_j\|$, (11.4) implies $\|\mathcal{A}_0\psi_j\| \to 0$. On the other hand, with P the orthogonal projection on the finite dimensional eigenspace of H_C corresponding to the eigenvalue $-\sigma - n\omega$, we have $\mathcal{A}_0 P = 0 \Rightarrow \mathcal{A}_0 = \mathcal{A}_0(I - P)$ and then since $\mathcal{A}_0\psi_j \to 0$,

$$\|\mathcal{A}_0(I - P)\psi_j\| \to 0 \tag{11.5}$$

But by definition $\mathcal{A}_0$ is invertible on $(I - P)L^2(\mathbb{R}^3)$ and (11.5) then implies $\|(I - P)\psi_j\| \to 0$, i.e. $P\psi_j - \psi_j \to 0$. Since $\|\psi_j\| = 1$, $\|P\psi_j\| \to 1$. Then $P\psi_j$ is a bounded sequence in the finite dimensional space $PL^2(\mathbb{R}^3)$, hence we can extract a convergent subsequence, which we may without loss of generality assume to be $P\psi_j$ itself, $P\psi_j \to \psi$, $\|\psi\| = 1$, and also $\psi_j \to P\psi_j \to \psi$, thus $P\psi = \psi$. Therefore, $\mathcal{A}_0\psi = \mathcal{A}_0 P\psi = 0$. Also, since multiplication by $\beta\chi$ is a bounded operator we have $\beta\chi\psi_j \to \beta\chi\psi = 0$, since $\beta\chi\psi_j \to 0$. Therefore, $\|A_\beta\psi\| \leq \|\mathcal{A}_0\psi\| + \|\beta\chi\psi\| = 0$ in contradiction to the absence of eigenvalues.

(3) *Definition of the inverse.* This is standard: we let $\psi \in D(\mathcal{A}_\beta)$, $\mathcal{A}_\beta\psi = \phi$ and define $\mathfrak{R}_\beta\phi = \psi$. This is well defined since $\mathcal{A}_\beta\psi_1 = \mathcal{A}_\beta\psi_2$ entails, by Step 1, $\psi_1 = \psi_2$. By Step 2, $\mathfrak{R}_\beta$ is defined on a dense set. By Step 3, for any p there is an $\epsilon > 0$ such that $\|\mathfrak{R}_\beta\| < \epsilon^{-1}$. Thus $\mathfrak{R}_\beta$ extends by density to $L^2(\mathbb{R}^3)$ and by construction $\mathcal{A}_\beta\mathfrak{R}_\beta\phi = \phi$ whenever $\mathfrak{R}_\beta\phi \in D(\mathcal{A}_\beta)$. Conversely, if $\phi \in D(\mathcal{A}_\beta)$, and $\mathcal{A}_\beta\phi = u$ then $\mathfrak{R}_\beta u = \phi$ entailing $\mathfrak{R}_\beta\mathcal{A}_\beta\phi = \phi$ on the dense set $\mathrm{D}(\mathcal{A}_\beta)$.

11.2. **Proof of Remark 8.1.** Let $\psi \in L^2(\mathbb{R}^3)$ and $\epsilon > 0$ be arbitrary. Let U be the unitary operator associated with the Schrödinger evolution (1.1). Take a ϕ, in the dense set for which we have proved ionization, and such that $\|\phi - \psi\| < \epsilon/2$ and choose T_0 such that for all $t > t_0$ we have $\|U\phi\|_{L^2(B)} \leqslant \frac{1}{2}\epsilon$. We have for all $t > T_0$

$$\|U\psi\|_{L^2(B)} = \|U(\psi - \phi) + U\phi\|_{L^2(B)} \leqslant \|U(\psi - \phi)\|_{L^2(\mathbb{R}^3)} + \|U\phi\|_{L^2(B)} \leqslant \epsilon$$

11.3. **Specific behavior of Coulomb systems.** The retarded Green's functions $G = G_+$ is defined as the solution of the equation

$$\mathcal{A}_0 G(x, x'; k) = \delta(x - x') \tag{11.6}$$

in distributions, satisfying the radiation condition

$$G(x, x'; k) \sim F(\theta, \phi)e^{ikr}r^{-1-i\nu}; \text{ as } \quad r \to \infty \tag{11.7}$$

where

$$k = \sqrt{ip} \quad (\operatorname{Im} k > 0 \text{ if } \operatorname{Re} p > 0), \quad \nu = \frac{b}{2k} \tag{11.8}$$

Equivalently, G is the $\mathbb{R}^3\setminus\{0\}$ solution of (11.6) with zero right hand side, satisfying (11.7) and $|x - x'|G(x, x'; k) \to (4\pi)^{-1}$ as $x - x' \to 0$.

Proposition 11.1.

$$\mathfrak{R}_0\chi g = \int_B G(x,x';k)g(x')dx' \tag{11.9}$$

Proof. The function

$$f := \int_B G(x,x';k)g(x')dx' \tag{11.10}$$

solves, as can be checked, the equation

$$\mathcal{A}_0 f = \chi g \tag{11.11}$$

with the radiation condition (11.7). Such a solution is unique since the difference of two solutions satisfies the equation $\mathcal{A}_0 f = 0$ (with the radiation condition (11.7)). Multiplying by $G(x,x';k)$, integrating over a volume and passing to the limit where the volume approaches $\mathbb{R}^3$ we see that $f \equiv 0$. ∎

Special symmetries of the Coulomb potential $-b/r$ allow for a closed form of G (we note that the Green's function in [29] is defined as minus our Green's function) in terms of Whittaker functions W and $\mathfrak{M}$.

$$G(x;x';k) = \frac{\Gamma(1-i\nu)}{4\pi ik|x-x'|}\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right) W_{i\nu,\frac{1}{2}}(-ik\xi)\mathfrak{M}_{i\nu,\frac{1}{2}}(-ik\eta) \tag{11.12}$$

where $\operatorname{Im} k > 0$, $2k\nu = b$,

$$\xi = |x| + |x'| + |x-x'|; \quad \eta = |x| + |x'| - |x-x'| \tag{11.13}$$

The Whittaker functions are defined in terms the Kummer functions M and U by the relation, see [1] Chapter 13,

$$\mathfrak{M}_{\kappa,\mu}(z) = e^{-\frac{z}{2}} z^{\frac{1}{2}+\mu} M\left(\frac{1}{2}+\mu-\kappa, 1+2\mu, z\right), \ -\pi < \arg z \leqslant \pi \tag{11.14}$$

$$W_{\kappa,\mu}(z) = e^{-\frac{z}{2}} z^{\frac{1}{2}+\mu} U\left(\frac{1}{2}+\mu-\kappa, 1+2\mu, z\right), \ -\pi < \arg z \leqslant \pi$$

Let

$$I(z) = \int_0^{i\infty} e^{-zt}t^{-i\nu}(1+t)^{i\nu}dt, \quad \dot{I}(z) = -\int_0^{i\infty} e^{-zt}t^{1-i\nu}(1+t)^{i\nu}dt \tag{11.15}$$

$$J(z) = \int_0^1 e^{zt}t^{-i\nu}(1-t)^{i\nu}dt; \ \dot{J}(z) = \int_0^1 e^{zt}t^{1-i\nu}(1-t)^{i\nu}dt$$

The following integral representation follows from [1] Chapter 13, for the values we are interested in, $z_1 = -ik\xi$, $z_2 = -ik\eta$, $a = 1 - i\nu$, $b = 2$ (a different "b" than the one in our Coulomb potential)

$$\mathfrak{M}_{i\nu;\frac{1}{2}}(z) = \frac{e^{-\frac{1}{2}z}zJ(z)}{\Gamma(1-i\nu)\Gamma(1+i\nu)}; W_{i\nu;\frac{1}{2}}(z) = \frac{e^{-\frac{1}{2}z}zI(z)}{\Gamma(1-i\nu)} \tag{11.16}$$

in the regions where the integrals converge (in particular, $|\operatorname{Im}\nu| < 1$). For other values of ν of interest, the integrals can be replaced by appropriate contour integrals. For instance J would be replaced by

$$\left(1 - e^{-2\pi\nu}\right)^{-1} \oint_C e^{zt}t^{-i\nu}(1-t)^{i\nu}dt$$

where C is smooth simple curve encircling $[0,1]$, as it can be checked by calculating the jump across the cut of the integrand. It follows from these integral representations that the Green's function is analytic at any (small) $\sigma \neq 0$. It is also clear that for $k>0, \xi>0, \eta>0$ we have

$$|I(-ik\xi)| \leqslant \int_0^\infty e^{-k\xi s}|i^{-i\nu}|ds \leqslant \frac{e^{\frac{1}{2}\pi\nu}}{k\xi}; \tag{11.17}$$
$$|\dot{I}(-ik\xi)| \leqslant \frac{e^{\frac{1}{2}\pi\nu}}{(k\xi)^2}; \quad |J(-ik\eta)| \leqslant 1; \quad |\dot{J}(-ik\eta)| \leqslant \frac{1}{2}$$

11.4. **Expression of the Coulomb resolvent.** It can be checked that

$$G(x,x';k) = \frac{ik(\eta-\xi)IJ - k^2\xi\eta[I\dot{J} - J\dot{I}]}{\Gamma(1-i\nu)\Gamma(1+i\nu)} \frac{e^{\frac{ik}{2}(\xi+\eta)}}{4\pi|x-x'|} \tag{11.18}$$

Continuity away from $p=0$ is clear from the integral representation of the kernel.

11.5. **Proof of Proposition 5.7 for $n>0$.** For large positive n this is an easy consequence of the second resolvent formula. Indeed, with $\lambda^2 = \sigma + n\omega$ large and R_0 the resolvent of $-\Delta + \lambda^2$, then $\mathfrak{R}_0$, the resolvent of $-\Delta + \lambda^2 - br^{-1}$ we are looking for, should satisfy

$$R_0 - \mathfrak{R}_0 = \mathfrak{R}_0(-br^{-1})R_0 \Rightarrow \mathfrak{R}_0 = (I - br^{-1}R_0)^{-1}R_0 \tag{11.19}$$

For $\operatorname{Re}\lambda > 0$, the resolvent R_0 is well defined on $L^2(\mathbb{R}^3)$ by

$$(R_0 f)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-\lambda|x-x'|}}{|x-x'|} f(x')dx' = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{-\lambda|x'|}}{|x'|} f(x+x')dx' \tag{11.20}$$

and it follows that

$$\|R_0 f\| \leqslant \|f\| O(1/\operatorname{Re}\lambda) \tag{11.21}$$

for large $\operatorname{Re}\lambda$. Thus the right side of (11.19) is well defined and it is easy to check that it is the limit of our resolvent, for $\operatorname{Re}\lambda$ large.

11.6. **Proof of Proposition 5.7 for $n<0$.** The proof when $\sigma + n\omega$ is negative is more delicate; we are now in the spectrum of both the free problem and the Coulomb system, and no useful resolvent formula restricted to B is readily available, since the boundary condition on the solution is a condition at infinity (11.7) and not one natural to ∂B; we need a more detailed information on the Green's function.

Proposition 11.2. *We have*

$$\|\chi\mathfrak{R}_0\chi\| \leqslant 4\pi r_0^3 \Big[\sup_{(x',x'')\in B\times B}\int_B |G(x,x';k)G(x,x'';k)|dx\Big]^{1/2} \leqslant Ck^{-1/2}\log k \;\; as \;\; k\to\infty \tag{11.22}$$

Proof.

Note 11.3. *In J and $\dot{J}$ the functions*

$$(1+T)\int_0^1 e^{-iTt}t^{-i\nu}(1-t)^{i\nu}dt; \;\; \dot{J} = (1+T)\int_0^1 e^{-iTt}t^{1-i\nu}(1-t)^{i\nu}dt \tag{11.23}$$

are continuous in T and uniformly bounded on $\mathbb{R}^+$. Thus, with some constants C_1 and C_2 we have the estimates

$$|J(-ik\eta)| \leqslant C_1(1+k\eta)^{-1}; |\dot{J}(-ik\eta)| \leqslant C_2(1+k\eta)^{-1} \tag{11.24}$$

which are especially useful for large $k\eta$.

We note that the variable ξ is bounded away from zero in $\mathbb{R}^6 \setminus B_\epsilon$. If we use the bounds (11.17) and (11.24) in (11.12) we get the following estimate for the Green's function for some constant C and all x, x'.

$$|G(x;x';k)| \leqslant C|x-x'|^{-1} \tag{11.25}$$

Asymptotic expansions for large k.

11.7. **Case 1:** $|x-x'| \leqslant k^{-1}\log k$ **or** $|x-x''| \leqslant k^{-1}\log k$**.** We start with the estimates (11.25). Now

$$\frac{2C^2}{|x-x'||x-x''|} \leqslant \frac{C^2}{|x-x'|^2} + \frac{C^2}{|x-x''|^2} \tag{11.26}$$

and thus

$$|G(x,x';k)||G(x,x'';k)| \leqslant \frac{C^2}{2|x-x'|^2} + \frac{C^2}{2|x-x''|^2} \tag{11.27}$$

Finally, we obtain, over the set $P := \{|x-x'| \leqslant k^{-1}\log k$ and $|x-x''| \leqslant k^{-1}\log k\}$,

$$I_1 := \int_P |G(x,x';k)G(x,x'';k)|dx \leqslant 4\pi C^2 k^{-1}\log k \tag{11.28}$$

11.8. **Case 2:** $x \notin P, x \in B$**.** We start with J, which is given by

(11.29)

$$J(-ik\eta) = \int_0^1 e^{-ik\eta t} t^{-i\nu}(1-t)^{i\nu} dt = \int_0^1 e^{-ik\eta t}dt + \int_0^1 e^{-ik\eta t}\left[t^{-i\nu}(1-t)^{i\nu}-1\right]dt$$
$$:= J_0(-ik\eta) + J_1(-ik\eta)$$

Lemma 11.4. *If $x' \in B$ then η is bounded. For bounded η, we have, as $k \to \infty$,*

$$|(1+k\eta)J_1(-ik\eta)| \leqslant Ck^{-1}\log k \tag{11.30}$$

and

$$|(1+k\eta)\dot{J}_1(-ik\eta)| \leqslant Ck^{-1}\log k \tag{11.31}$$

where C does not depend on k or η.

Proof. The first statement follows immediately from (11.13). We prove (11.30), the proof of (11.31) being similar. If $|k\eta| \leq 1$ then we break the integral into integrals over $[0, k^{-1}]$, $[k^{-1}, 1-k^{-1}]$ and $[1-k^{-1}, 1]$. Since the integrand is bounded, the contribution of the first and third integrals is $O(k^{-1})$. As for the middle integral, we note that the integrand is $O(\nu\log t, \nu\log(1-t))$ hence the integral is $O(k^{-1}\log k)$. Let now $|k\eta| > 1$. It is convenient to split the interval into $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. By obvious symmetry, the two integrals can be treated similarly, so we focus on the first one. In this interval $f = (1-t)^{-i\nu}$ is analytic, and we retain that $f = 1 + i\nu^{-1}t + O(k^{-2})$. In $J_2 = \int_0^{1/2} \nu^{-1}t^{1-i\nu}e^{-ik\eta t}dt$ we integrate by parts (differentiating $t^{1-i\nu}$) and we get $J_2 = O(k^{-2}\eta^{-1})$. We are left with estimating $\int_0^{1/2}(t^{-i\nu}-1)e^{-ik\eta t}dt$. We deform the contour into $[0, -i\infty]$ and $[1/2, -i\infty]$. The integrand is analytic at $1/2$ and Watson's lemma applies yielding a contribution of order $O(k^{-2}\eta^{-1})$ of the second contour. The first contour integral evaluates explicitly as $const.\left\{(k\eta)^{1-i\nu}[\Gamma(1-i\nu)-1]+(k\eta)^{-1}[(k\eta)^{-i\nu}-1]\right\} = O((k^{-1}\eta^{-1})k^{-1}\log k)$. ∎

We now consider I which we write as

$$I = i^{1-i\nu}\int_0^\infty e^{-k\xi s}s^{-i\nu}(1+is)^{i\nu}ds \\ = i^{1-i\nu}(k\xi)^{i\nu-1}\int_0^\infty e^{-\tau}\tau^{-i\nu}\exp\left[i\nu\log\left(1+\frac{i\tau}{k\xi}\right)\right]d\tau \tag{11.32}$$

It is easy to check by dominated convergence that for $k\xi > \log k$ which holds in this section, the integral

$$\int_0^\infty e^{-\tau}\tau^{-i\nu}\exp\left[i\nu\log\left(1+\frac{i\tau}{k\xi}\right)\right]d\tau \tag{11.33}$$

can be expanded convergently in powers of k^{-1}. This leads to the uniform expansion of I

$$I = \frac{i}{\xi k} - \frac{b}{2\xi k^2}\left(\log(k\xi) - i\pi/2 + \gamma\right) + O(k^{-3}\log^2 k) := I_0 + I_1, \tag{11.34}$$

where γ is the Euler constant, I_0 is the contribution from zero Coulomb potential (the $b=0$ term), and I_1 is the remaining contribution. Therefore, we obtain the bound

$$k\xi|I_1| \leqslant Ck^{-1}\log k \tag{11.35}$$

A similar calculation yields for $\dot I$

$$\dot I = -\frac{1}{\xi^2k^2} - \frac{ib}{2\xi^2k^3}\left(\log(k\xi) + (\gamma-1) - i\frac{\pi}{2}\right) + O(k^{-4}\log^2 k) := \dot I_0 + \dot I_1 \tag{11.36}$$

This gives rise to the bounds

$$k^2\xi^2|\dot I_1| \leqslant Ck^{-1}\log k \tag{11.37}$$

It is convenient to write

$$ik(\eta-\xi)IJ - k^2\xi\eta(I\dot J - J\dot I) = ik(\eta-\xi)I_0J_0 - k^2\xi\eta(I_0\dot J_0 - J_0\dot I_0) \\ + ik(\eta-\xi)(I_1J + I_0J_1) - k^2\xi\eta\left(I_1\dot J + I_0\dot J_1 - J_1\dot I - J_0\dot I_1\right) \tag{11.38}$$

Then, using $x \in B, x \notin P$ implies $k\xi > \log k$, and the bounds (11.17), (11.24), (11.30), (11.31), (11.35), (11.37), together with Relations (11.18) and (11.38), $F_1 := G - G_0$ (G_0 is the Green's function in the absence of Coulomb potential *i.e.* for $b=0$) satisfies

$$|x-x'||F_1(x,x')| \leqslant Ck^{-1}\log k \tag{11.39}$$

Therefore,

$$\int_B |G_0(x,x')||F_1(x,x'')|dx \leqslant Ck^{-1}\log k \tag{11.40}$$

Using again (11.27) we get

$$\|\chi\mathfrak{R}_0\chi\|^2 \leqslant const.\int_B |G_0(x,x')||G_0(x,x'')|dx + O(k^{-1}\log k) = O(k^{-1}\log k) \tag{11.41}$$

where for the integral we used [2] formula (A2′) p. 203, retaining only the $\alpha = 0$ term there and keeping in mind the compactness of the domain. ▮

11.9. **Completion of the proof of Proposition 6.3.**

11.9.1. *Part (i), $\beta > 0$, corresponding to $n \in [0, n_c)$.* Since we have in our convention $\operatorname{Re}\sigma \in [0,\omega)$, and we are interested in the neighborhood of $-ip = \sigma + n\omega = 0$, we will take $n = 0$. The function $f = \mathfrak{R}_\beta g$ is the solution of the equation

$$\begin{aligned}\Big(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - \frac{b}{r} + \lambda^2 - ic\Big) f = g; \quad r \leqslant r_0\\ \Big(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{l(l+1)}{r^2} - \frac{b}{r} + \lambda^2\Big) f = 0; \quad r > r_0\end{aligned} \tag{11.42}$$

such that f decays at infinity, is regular at the origin and C^1 at $r = r_0$. We note $\lambda = \sqrt{\sigma} = \sqrt{-ip}$ is in the closure of the fourth quadrant for $\operatorname{Re} p \geqslant 0$. We let $\alpha = \sqrt{\lambda^2 - ic}$, $\kappa_1 = b/(2\alpha)$, $\kappa = b/(2\lambda)$, $\mu = 2l+1$ and define

$$m_1(s) := s^{-1}\mathfrak{M}_{\kappa_1,\mu/2}(2\alpha s); \quad w_1(s) := s^{-1}W_{\kappa_1,\mu/2}(2\alpha s); \quad w_2(s) := s^{-1}W_{\kappa,\mu/2}(2\lambda s) \tag{11.43}$$

For $r > r_0$ we have $f = \mathrm{B}w_2(r)$ while, for $r \leqslant r_0$ we must have

$$f = Am_1 + f_0 \tag{11.44}$$

where [7], pp. 25, [1], pp 505, 508

$$\frac{2\alpha\Gamma(1+\mu)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\mu - \kappa_1\right)} f_0 = w_1(r)\int_0^r s^2 m_1(s) g(s) ds + m_1(r)\int_r^{r_0} s^2 w_1(s) g(s) ds \tag{11.45}$$

The integral representations of the functions $\mathfrak{M}$ and W [(7)] entail immediately that the functions f_0, f and $\mathfrak{M}_{\kappa_1,\mu/2}(2\alpha r)$ depend analytically on λ for small λ.

11.9.2. *Matching.* We now impose the condition of regularity at $r = r_0$; Continuity of f and f' at $r_0 \geqslant 1$ imply that A defined in (11.44), is given by

$$A = \frac{f_0(r_0)w_2'(r_0) - f_0'(r_0)w_2(r_0)}{m_1'(r_0)w_2(r_0) - m_1(r_0)w_2'(r_0)} \tag{11.46}$$

The functions w_1 and m_1 depend analytically on λ for small λ (because of the presence of the factor $-ic$). As to w_2 we use [7] pp 105, (see also [41] §4.4.1 and 4.4.3), with $2\kappa = b/\lambda$. We have $\operatorname{Re}\kappa > 0, \operatorname{Im}\kappa > 0$, $0 < |\lambda^2 r/b| < 1$ in the region of interest) and with

$$c_1 = 2^{-1}b^{1/4}\kappa^{\kappa-1/2}e^{-\kappa},$$

$$r^{3/4}c_1^{-1}w_2(r) = \Big\{Z(1-i)e^{-2i\sqrt{br}} + Z^{-1}(1+i)e^{2i\sqrt{br}}\Big\}(1 + O(\lambda^{1/2}) \tag{11.47}$$

---

(7) [7], pp. 60 (1) and pp. 63 (5)

This asymptotic expansion is differentiable since we have by [1] p. 507

$$W'_{k,\mu}(z) = \left(\frac{1}{2} - \frac{k}{z}\right) W_{k,\mu}(z) - \frac{1}{z} W_{k+1,\mu}(z) \tag{11.48}$$

and thus

(11.49)
$$4r^{7/4} c_1^{-1} w_2' = \left\{ -(3+4i\sqrt{br})(1-i)e^{-2i\sqrt{br}} Z - (3-4i\sqrt{br})(1+i)Z^{-1} \right\} (1+O(\lambda^{1/2}))$$

11.9.3. *Estimating A.* Vanishing of the determinant, the denominator of A, would clearly mean the existence of an eigenvector of $\mathcal{A}_\beta$ which we ruled out already for all values of p except $\mathrm{Re}\, p = 0, \mathrm{Im}\, p < 0$. Multiplying the numerator and denominator of A by Z and defining $r_2 = i\sqrt{br_0}$, $n_1 = ie^{4i\sqrt{br_0}}$, $m = m_1(r_0), m' = m_1'(r_0), f = f(r_0), f' = f'(r_0)$ we get

$$A = -\frac{\left[(3+4r_2)Z^2 + (3-4r_2)n_1\right] f + 4r_0\left[Z^2 + n_1\right] f'}{\left[(3+4r_2)Z^2 + (3-4r_2)n_1\right] m + 4r_0\left[Z^2 + n_1\right] m'} \left\{1 + O(\lambda^{1/2})\right\} \tag{11.50}$$

11.9.4. *Part (ii), $\beta = 0$, corresponding to $n \notin [0, n_c)$.* Since $|p|$ is small, in our original convention, we have $n = -1$ and $\sigma - \omega$ small with Re $\sigma < \omega$. However, since $-ip = n\omega + \sigma$, this is equivalent to choosing $n = 0$, with $|\sigma|$ small, but with Re $\sigma < 0$. As before, we still have Im $\sigma \le 0$, corresponding to Re $p > 0$, implying tat $\sigma = \lambda^2$ is in the third quadrant. Also, in this case, since $c = 0$, $w_2(r) = w_1(r)$. By using the behavior of Whittaker functions near the origin [1], pp 505, 508, it follows that

$$\mathcal{W}(w_1, m_1)(r_0) \equiv m_1'(r_0) w_1(r_0) - m_1(r_0) w_1'(r_0) = C(\lambda, b) r_0^{-2}$$

where

$$C(\lambda, b) = \frac{2\lambda \Gamma(1+\mu)}{\Gamma\left(\frac{1}{2} + \frac{\mu}{2} - \kappa\right)},$$

It is to be noted that since λ^2 is in the third quadrant, and therefore $\kappa = \frac{b}{2\lambda}$ stays away from $\mathbb{R}^+$ and thus $C(\lambda, b) \ne 0$ for any small $\lambda \ne 0$ in this region. Furthermore $c = 0$ implies $\kappa_1 = \kappa$, $\alpha = \lambda$ in the representation (11.45); since $w_2 = w_1$, a simple calculation in 11.50 shows that $A = 0$. Therefore, we only need to bound $f_0(r)$ independently of λ for $r \in [0, r_0]$. From (11.45) we now obtain

$$f_0(r) = \frac{1}{C(\lambda, b)} \left(w_1(r) \int_0^r s^2 m_1(s) g(s) ds + m_1(r) \int_r^{r_0} s^2 w_1(s) g(s) ds \right)$$

In this case, using [7] p. 105, in the regime $\lambda^{-1} \gg r \gg 1$, for $\sigma = \lambda^2 \to 0$ in the third quadrant we have

(11.51)
$$w_1(r) = C_1(\lambda, b) \frac{1}{\sqrt{r}} \sqrt{\frac{1}{\pi\sqrt{br}}} \exp\left(2i\sqrt{br} - i\left(l + \frac{1}{2}\right)\pi - i\frac{\pi}{4}\right)(1 + O(\lambda^{1/2}))$$

$$m_1(r) = C_2(\lambda, b) \frac{1}{\sqrt{r}} \sqrt{\frac{1}{\pi\sqrt{br}}} \cos\left(2\sqrt{br} - \left(l + \frac{1}{2}\right)\pi - \frac{\pi}{4}\right)(1 + O(\lambda^{1/2})), \tag{11.52}$$

for some nonzero $C_1(\lambda, b), C_2(\lambda, b)$. We further note that Equation (11.42) with $c = 0$ is regularly perturbed with respect to λ for any fixed r. Therefore, the solutions w_1, m_1 must approach, as $\lambda \to 0$, the $\lambda = 0$ solutions of (11.42), which are linear combinations of $r^{-1/2}J_{2l+1}(2\sqrt{br})$ and $r^{-1/2}Y_{2l+1}(2\sqrt{br})$ (where J_{2l+1} and Y_{2l+1} are the usual Bessel functions). The large r matching conditions (11.51) and (11.52) imply that

$$w_1(r) \sim C_1(\lambda,b)r^{-1/2}H^{(1)}_{2l+1}(2\sqrt{br})$$

$$m_1(r) \sim C_2(\lambda,b)r^{-1/2}J_{2l+1}(2\sqrt{br})$$

(where $H^{(1)}$ is the type one Hankel function, [1]). Therefore, the expression for $f_0(r)$ for $0 \le r \le r_0$ simplifies to

$$f_0(r) \sim \frac{C_1(\lambda,b)C_2(\lambda,b)}{C(\lambda,b)}\Big(r^{-1/2}H^{(1)}_{2l+1}(2\sqrt{br})\int_0^r s^{3/2}J_{2l+1}(2\sqrt{bs})ds + r^{-1/2}J_{2l+1}(2\sqrt{br})\int_r^{r_0} s^{3/2}H^{(1)}_{2l+1}(2\sqrt{bs})ds\Big) \tag{11.53}$$

Since

$$C(\lambda,b)/r^2 = \mathcal{W}(w_1,m_1) \sim C_1(\lambda,b)C_2(\lambda,b)\mathcal{W}\left(r^{-1/2}H^{(1)}_{2l+1}, r^{-1/2}J_{2l+1}\right)$$

It follows that $C_1(\lambda,b)C_2(\lambda,b)/C(\lambda,b)$ has an upper bound completely independent of λ since $J_{2l+1}(2\sqrt{br})$ and $H_{2l+1}(2\sqrt{br})$ are independent of λ. Thus, $f_0(r)$ is bounded independent of λ.

Proposition 11.5. *The denominator in (11.50) cannot vanish if $|p|$ is small and $p \in \overline{\mathbb{H}}$, i.e for small $\sigma \in \overline{-i\mathbb{H}}$. A is continuous in $(Z,p) \in \overline{D} \times \overline{iD_\epsilon^+}$.*

Proof. We note that since m, m', f and f' all have finite limit as $\lambda \to 0$, it is clear that to the leading order in λ, the dependence of A only comes through Z.

Lemma 11.6. *For large enough c the denominator of A in (11.50) is nonzero for all Z inside the closed unit disk.*

Proof. It can be checked using the asymptotics of $\mathfrak{M}$, [1] Chapter 13, or directly from the defining differential equation, that for large $c > 0$ we have $m'/m \sim \sqrt{c}e^{-i\pi/4}$. For the denominator to vanish we must have

$$-\frac{r_2}{r_0}\frac{Z^2-n_1}{Z^2+n_1} = \frac{m'}{m} + \frac{3}{4r_0} \tag{11.54}$$

The left side of (11.54) is a linear fractional transformation in $z = Z^2$. It is easy to check that if z is on the unit circle, the left side is purely real, while for $z = 0$ it equals i. It thus maps the unit disk into the upper half plane, whereas the right side is, as we have seen, in the open lower half plane for large c. ∎

The only thing remaining to prove is continuity. But this is manifest now from (11.50), the fact that m_1 and f_0 are analytic in λ for small λ. ∎

11.10. **Interchange of limits.** We have

$$4\pi^2|\psi(t,x)|^2 = \int_{-\infty}^{\infty} e^{its}\hat{\psi}(is;x)ds \int_{-\infty}^{\infty} e^{-its'}\overline{\hat{\psi}(is';x)}ds' \leqslant \int_{-\infty}^{\infty}|\hat{\psi}(is;x)|ds\int_{-\infty}^{\infty}|\hat{\psi}(is';x)|ds' \leqslant C(x)^2\left|\int_{-\infty}^{\infty}(1+|p|)^{-3/2}dp\right|^2 \leqslant c \tag{11.55}$$

for some $c \in \mathbb{R}^+$. This is because by Note 3.1 $\hat{\psi}(p, x)$ is continuous in x implying that $C(x)$ (see Proposition 6.5) is continuous too. Now we can apply dominated convergence to conclude $\lim_{t\to\infty} \int_B |\psi(t,x)|^2 dx = \int_B \lim_{t\to\infty} |\psi(t,x)|^2 dx = 0$.

12. Proof of Proposition 9.2

We seek to show that the only solution to the homogeneous system

$$Y = \mathfrak{C}_{l,m} Y \tag{12.1}$$

in the space $\mathcal{H}$ is $Y = 0$. Equation (12.1) implies that the components of $Y = \left\{r^{-1} g_n\right\}_{n\in\mathbb{Z}}$ satisfy the differential-difference system

$$\frac{d^2}{dr^2} g_n - \left(-br^{-1} + n\omega + \sigma + \frac{l(l+1)}{r^2}\right) g_n = i\Omega\left(g_{n+1} - g_{n-1}\right) \tag{12.2}$$

First, we notice that for $n < 0$, Theorem 2.1 implies that $g_n(r) = 0$ for $r \geqslant 1$. Thus $g_n(1) = 0$, $g_n'(1) = 0$ for all $n < 0$.

Lemma 12.1. *If $Y \neq 0$, then there exists some $n_0 \geqslant 0$ so that either $g_{n_0}(1) \neq 0$ or $g_{n_0}'(1) \neq 0$. In the sequel, we shall define n_0 to be the smallest such integer.*

Proof. To get a contradiction, assume the statement is false. The functions g_n are in the domain of Δ, thus, in particular for any n, g_n is continuous. Thus, the set $Z_n := \{r : g_n(r) = 0\}$ is closed and so is the (possibly empty) left connected component of 1 in Z_n, call it K_n. Let

$$K = \bigcap_{n\in\mathbb{Z}} K_n$$

Assume to get a contradiction that K is nonempty: let then $K = [a, 1]$. If $a = 0$, then $Y \equiv 0$ since $g_n(1) = 0$, $g_n'(1) = 0$ imply $g_n(r) = 0$ for $r > 1$. Then $Y \neq 0$ implies $a > 0$. We first take $0 < a < 1$. We write the differential equation for $g_n(r)$ in integral form and use the conditions $g_n(a) = 0 = g_n'(a)$, since g_n vanishes on $[a, 1]$:

$$e^{\sqrt{n\omega}r} g_n(r) = \int_r^a \left(\frac{\left[1 - e^{-2\sqrt{n\omega}(s-r)}\right]}{2\sqrt{n\omega}}\right) e^{\sqrt{n\omega}s} \left\{\left[\frac{l(l+1)}{s^2} + \sigma + \tilde{V}(s)\right] g_n(s) - i\Omega(s)\left(g_{n-1}(s) - g_{n+1}(s)\right)\right\} ds \tag{12.3}$$

Consider the Banach space of sequences

$$\{g_n(r)\}_{n=-\infty}^{\infty}$$

in the norm

$$\sup_{n\in\mathbb{Z}, r\in[a-\epsilon,a]} \left|e^{\sqrt{n\omega}r} g_n(r)\right|$$

It is easy to see that the rhs of (12.3) is a contractive mapping if ϵ is small enough and then $g_n(r) = 0$ for $r \in [a-\epsilon, a]$ contradicting the definition of a. The same is true if $a = 1$, since $g_n(1) = 0$ and $g_n'(1) = 0$ would imply, with the same proof as before, that $g_n = 0$ for $r \in [1-\epsilon, 1]$, for some $\epsilon > 0$, contradicting the definition of a. ∎

13. Proof of Theorem 9.1

13.1. **Outline.** In the proof we transform the differential system to an integral system. The fact that there are two competing potentially large variables, k and $1/r$ makes it necessary to resort to rigorous matched asymptotics. We state the main steps, proved in §13.2.

Lemma 13.1. *For any $\epsilon_1 > 0$ and $\gamma \in (0,1)$, there exists $C_3 > 0$ independent of k and ϵ_1 so that for $k \geqslant k_0 = C_3\epsilon_1^{-1}$, and for $r \in [\epsilon_1, 1]$,*

$$\|h_k'\|_\infty \leqslant C_4 k_0 \left(\frac{k_0}{k}\right)^\gamma \tag{13.1}$$

where C_4 is independent of ϵ_1 and k and

$$g_{n_0-k}(r) =: i^k m_k(r) h_k(r) \tag{13.2}$$

Definition 13.2. *We denote $L_\epsilon = \alpha C_3 \left(\dfrac{C_4 C_3}{\gamma\epsilon}\right)^{\frac{1}{\gamma}}$ and $\zeta = \alpha k r$.*

Finally, in what follows, c_* is a positive "generic" constant, the value of which is immaterial.

Lemma 13.3. *For $\zeta \in [L_\epsilon, k\alpha]$ we have (note that $\zeta = k\alpha$ implies $r = 1$)*

$$|h_k(r) - 1| \leqslant \epsilon \tag{13.3}$$

Definition 13.4. *Let $\tilde{h}_k(\zeta) = h_k(\zeta/(\alpha k))$.*

Lemma 13.5. *There exists a subsequence $S = \{\tilde{h}_{k_j}\}_{j\in\mathbb{N}}$ that converges to a continuous function $\tilde{h}$ for $\zeta \in [0, L_\epsilon]$. The limiting function $\tilde{h}$ satisfies*

$$|\tilde{h}(\zeta) - 1| \leqslant 2\epsilon \text{ for any } \zeta \in [0, L_\epsilon]$$

In particular $\lim_{j\to\infty} |h_{k_j}(r) - 1| \leqslant 2\epsilon$ and $\tilde{h}(\zeta) = 1$ for all $r \in [0,1]$.

The proof of Theorem 9.1 now follows from the definition of h_k, Definition 9.4, Corollary 9.6 and the last equality in Lemma 13.5.

13.2. **Proofs of Lemmas 13.1, 13.3 and 13.5.**

Lemma 13.6. *For any $j, k \in \mathbb{N} \cup \{0\}$ we have at $r = 1$:*

$$\frac{\partial^{j+\tau} g_{n_0-k}}{\partial \mathfrak{s}^{j+\tau}}\Big|_{\mathfrak{s}=0} = \delta_{j,2k}\ i^k \qquad \text{for } 0 \leqslant j \leqslant 2k$$

Proof. In case $\tau = 0$ (**i**) note that (12.2) may be rewritten as

$$(g_{n_0-k})_{\mathfrak{s}\mathfrak{s}} - \frac{\Omega'}{2\Omega^{3/2}}(g_{n_0-k})_{\mathfrak{s}} + \frac{Q_k}{\Omega} g_{n_0-k} = i\,(g_{n_0-k+1} - g_{n_0-k-1}) \tag{13.4}$$

Since $g_{n_0-k}(1) = 0 = g'_{n_0-k}(1)$ for all $k \geqslant 1$, while $g_{n_0}(1) = 1$, the statement follows from (13.4) for any $0 \leqslant j \leqslant 2$, if $2k \geqslant j$. Assuming the statement holds for some $j \geqslant 2$ for $2k \geqslant j$, we prove it for $(j+1)$ for $2k \geqslant (j+1)$.

Taking $(j-1)$ $\mathfrak{s}-$derivatives of (13.4) at $\mathfrak{s} = 0$, we obtain

$$\frac{\partial^{j+1} g_{n_0-k}}{\partial \mathfrak{s}^{j+1}} = i\frac{\partial^{j-1}}{\partial \mathfrak{s}^{j-1}} g_{n_0-(k-1)} - i\frac{\partial^{j-1}}{\partial \mathfrak{s}^{j-1}} g_{n_0-(k+1)} + L$$

where L is a linear combination of derivatives of g_{n_0-k} up to order j, which are all zero since $2k \geqslant (j+1) > j$. The first two terms on the rhs give a contribution of

$ii^k\delta_{(j-1),2(k-1)} + 0$ since $2k \geqslant (j+1)$ implies $2(k-1) \geqslant (j-1)$ and $2(k+1) > (j-1)$ completing the inductive step.

In case $\tau = 1$ (**ii**) Since $g_{n_0}(1) = 0$ and $g_{n_0-k}(1) = 0 = g'_{n_0-k}(1)$ for all $k \geqslant 1$, it follows from (13.4) that $g''_{n_0-k} = 0$ for all $k \geqslant 1$ implying the conclusion for $j = 0$ and $j = 1$. By taking an additional derivative of (13.4) with respect to $\mathfrak{s}$ and evaluating at $\mathfrak{s} = 0$, we obtain

$$\frac{\partial^3 g_{n_0-k}}{\partial \mathfrak{s}^3} = i\delta_{2,2k} \frac{\partial}{\partial \mathfrak{s}} g_{n_0}|_{\mathfrak{s}=0} = i\delta_{2,2k} \frac{g'_{n_0}(1)}{-\sqrt{\Omega(1)}} = i\delta_{2,2k}$$

so the statement holds for $j = 2$ and any k with $2k \geqslant j$. The rest of the proof is very similar to that for $\tau = 0$. ∎

Lemma 13.7. *The system (12.2) for $n = n_0 - k$ with $k \geqslant 1$ is equivalent to the following set of integral equations:*

$$g_{n_0-k}(r) = i \int_r^1 \Omega(s) \left(g_{n_0-k+1}(s) - g_{n_0-k-1}(s) \right) G_k(r,s) ds \quad k \geqslant 1 \tag{13.5}$$

where

$$G_k(r,s) = \frac{\psi_{1,k}(r)\psi_{2,k}(s) - \psi_{2,k}(r)\psi_{1,k}(s)}{W_k} \tag{13.6}$$

where $\psi_{1,k}$, $\psi_{2,k}$ are two independent solutions of

$$\mathcal{L}_k \psi = 0 \text{ ; and } W_k = \psi_{1,k}(r)\psi'_{2,k}(r) - \psi_{2,k}(r)\psi'_{1,k}(r) \tag{13.7}$$

is the Wronskian, which is independent of r because of the form of the differential operator $\mathcal{L}_k$ defined by:

$$\mathcal{L}_k \psi = \psi'' + \left\{ \frac{b}{r} - (n_0 - k)\omega - \sigma - \frac{l(l+1)}{r^2} \right\} \psi \tag{13.8}$$

Proof. The proof simply follows from the well-known variation of parameter formula, the two boundary conditions at $r = 1$ and $g_{n_0-k}(1) = g'_{n_0-k}(1) = 0$. ∎

Lemma 13.8. *For $k \geqslant 1$, $m_k(r)$ (note Definition 9.4) satisfies*

$$\mathcal{L}_k m_k - \Omega m_{k-1} = \frac{j_k}{\mathfrak{s}} m_k \text{ ; with } m_k(1) = 0 \text{ , } m'_k(1) = 0 \tag{13.9}$$

where $j_k(r)$ is given by (14.1) and (14.2) in the Appendix. Furthermore, for any $r \in [0,1]$, $|j_k| \leqslant c_*$ for a constant c_* independent of k, while for $r \geqslant k^{-1}$, $|j'_k(r)| \leqslant k^{-1}C_1 r^{-2} + C_2$.

Proof. The proof simply follows from the form of $j_k = \mathfrak{s}[\mathcal{L}_k m_k - \Omega m_{k-1}]/m_k$ ((14.1) and (14.2) in the Appendix), using the differential equation (9.6) for $H(\zeta)$ and the definition (9.2) of α. Careful examination shows that the $O(k^2)$ and $O(k)$ terms drop out as $k \to \infty$; we are left with a finite limit for any $\zeta \in (0, k\alpha]$. In fact, even as $\zeta \downarrow 0$, we still obtain a finite limit independent of k as $k \to \infty$. Also taking the $r-$ derivative of j_k (see Appendix) and using the asymptotics of $H(\zeta)$ for large ζ we get the bounds we stated. ∎

Lemma 13.9. *For $k \geqslant 1$, $h_k(r)$ defined in (13.2) satisfies the system of differential equations:*

$$h_k'' + 2h_k' \frac{m_k'}{m_k} + \left(\frac{\Omega m_{k-1}}{m_k} + \frac{j_k}{\mathfrak{s}}\right) h_k = \Omega \left(\frac{m_{k-1}}{m_k} h_{k-1}(r) + \frac{m_{k+1}}{m_k} h_{k+1}(r)\right), \tag{13.10}$$

and the system of integral equations (13.5) is equivalent, for $k \geqslant 1$, to

$$\begin{aligned} h_k(r) = &\int_r^1 \frac{\Omega(s) m_{k-1}(s)}{m_k(r)} G_k(r,s) h_{k-1}(s) ds \\ &+ \int_r^1 \frac{\Omega(s) m_{k+1}(s)}{m_k(r)} G_k(r,s) h_{k+1}(s) ds := \mathcal{A}_k h_{k-1} + \mathcal{B}_k h_{k+1} \end{aligned} \tag{13.11}$$

Proof. This simply follows by substituting $g_{n_0-k}(r) = i^k m_k(r) h_k(r)$ into (12.2) and (13.5), and using

$$\frac{m_k''}{m_k} + Q_k = \frac{\Omega m_{k-1}}{m_k} + \frac{j_k}{\mathfrak{s}},$$

which follows from Lemma 13.8. ∎

Remark 13.10. *When $r \in [\epsilon, 1]$, where $\epsilon \geqslant C_2 k^{-1}$ for C_2 large but independent of k, it is convenient to rewrite $\mathcal{A}_k$ and $\mathcal{B}_k$ in (13.11) in terms of $\mathfrak{s}$ (see (9.1)). Furthermore, changing the variable of integration from s to $t = \mathfrak{s}(s)/\mathfrak{s}(r)$, we obtain*

$$[\mathcal{A}_k h_{k-1}](\mathfrak{s}) = 2k(2k-1+2\tau) \int_0^1 t^{2k-2+\tau} T_k(\mathfrak{s}, t) h_{k-1}(\mathfrak{s}t) dt \tag{13.12}$$

where

$$T_k(\mathfrak{s}, t) = \frac{\sqrt{\Omega(r(\mathfrak{s}t))} F_{k-1}(r(\mathfrak{s}t)}{\mathfrak{s} F_k(r(\mathfrak{s}))} G_k(r(\mathfrak{s}), r(\mathfrak{s}t)) \tag{13.13}$$

and

$$\begin{aligned} [\mathcal{B}_k h_{k+1}](\mathfrak{s}) = &\frac{\mathfrak{s}^3}{(2k+2)(2k+1+2\tau)} \\ &\times \int_0^1 \sqrt{\Omega(r(\mathfrak{s}t)} t^{2k+2+\tau} \frac{F_{k+1}(r(\mathfrak{s}t)}{F_k(r(\mathfrak{s}))} G_k(r(\mathfrak{s}), r(\mathfrak{s}t)) h_{k+1}(\mathfrak{s}t) dt \end{aligned} \tag{13.14}$$

In evaluating $\mathcal{A}_k$ for large k, it is useful to calculate the Taylor expansion of $T_k(\mathfrak{s}, t)$ and its $\mathfrak{s}$ derivative at $t = 1$:

$$\begin{aligned} T_k = (1-t) + &\left(-\frac{k}{4} f_1 + \frac{f_2}{r^2}\right)\left(\frac{2}{3}(1-t)^3 - \frac{(1-t)^2}{k}\right) \\ &+ O\left(\frac{(1-t)^4}{r^3}, \frac{(1-t)^3}{kr^3}, \frac{(1-t)^3}{r}, \frac{(1-t)^2}{kr}\right) \end{aligned} \tag{13.15}$$

$$\begin{aligned} \frac{\partial T_k}{\partial \mathfrak{s}} = &\left(-\frac{k}{4} f_1' + \frac{f_3}{r^3}\right)\left(\frac{2}{3}(1-t)^3 - \frac{(1-t)^2}{k}\right) \\ &+ O\left(\frac{(1-t)^4}{r^4}, \frac{(1-t)^3}{kr^4}, \frac{(1-t)^3}{r^2}, \frac{(1-t)^2}{kr^2}\right) \end{aligned} \tag{13.16}$$

where

$$f_1(\mathfrak{s}) = \frac{\omega \mathfrak{s}^2}{\Omega(r(\mathfrak{s}))} \tag{13.17}$$

$$f_2(\mathfrak{s}) = \frac{l(l+1)\mathfrak{s}^2}{4\Omega} \tag{13.18}$$

$$f_3(\mathfrak{s}) = \frac{l(l+1)\mathfrak{s}^2}{2\Omega^{3/2}} \tag{13.19}$$

When $r \in [0,\epsilon]$, for small ϵ, it is sometimes more convenient to express $\mathcal{A}_k$ in terms of $\zeta = k\alpha r$. For that purpose, we define

$$Q(\zeta) = -2k\log\left[1 - \frac{\mathfrak{s}(0)-\mathfrak{s}}{\mathfrak{s}(0)}\right] - \log\left[\left(\frac{\Omega(0)}{\Omega(r)}\right)^{1/4}\exp\left(\frac{1}{4}\int_0^r ds\frac{\omega\mathfrak{s}(s)}{\sqrt{\Omega(s)}}\right)\right] \tag{13.20}$$

where we recall the relation (9.1) between $\mathfrak{s}$ and $r = \frac{\zeta}{k\alpha}$, $\zeta \in [0, k\alpha\epsilon]$. A series expansion in k^{-1} leads

$$Q(\zeta) = \zeta - \frac{\zeta}{k}\left(\frac{\omega}{2\alpha^2} - \frac{\Omega'(0)}{4\Omega(0)\alpha}\right) + \frac{\zeta^2}{4k}\left(1 + \frac{\Omega'(0)}{\alpha\Omega(0)}\right) + O\left(\frac{\zeta^3}{k^2}\right) \tag{13.21}$$

We choose $\epsilon_1 - \epsilon = C_2 k^{-1}\log k$, with C_2 large but independent of k, and define δ_1 by

$$(1-\delta_1)\mathfrak{s}\left(\frac{\zeta}{k\alpha}\right) = \mathfrak{s}(\epsilon_1) \tag{13.22}$$

Since $\frac{\zeta}{k\alpha} \in [0,\epsilon]$, it follows that for some constant $\hat{C}_2$,

$$\delta_1 \geqslant 1 - \frac{\mathfrak{s}(\epsilon_1)}{\mathfrak{s}(\epsilon)} \geqslant \hat{C}_2\frac{\log k}{k}, \tag{13.23}$$

It is convenient to rewrite

$$\begin{aligned} [\mathcal{A}_k h_{k-1}](\zeta) = &\left(1 - \frac{1}{2k}\right) \\ &\times \int_\zeta^{k\alpha\epsilon_1} e^{-Q(\eta)+Q(\zeta)}\left(1 + \frac{a_1}{k}\right)\frac{H(\eta(1-k^{-1}))}{H(\zeta)}\mathcal{G}(\zeta,\eta)h_{k-1}(\eta(1-k^{-1}))d\eta \\ &+ (2k+\tau)(2k+\tau-1)\int_{\epsilon_1}^1\left[\frac{\mathfrak{s}(s)}{\mathfrak{s}(r)}\right]^{2k-2}\Omega(s)G_k\left(r(\mathfrak{s}), r(\mathfrak{s}t)\right)\frac{F_{k-1}(s)}{\mathfrak{s}^2F_k(r)}h_{k-1}(\mathfrak{s}(s))ds \\ &=: [\mathcal{A}_k^0 h_{k-1}](\zeta) + [\mathcal{A}_k^1 h_{k-1}](r) \end{aligned} \tag{13.24}$$

where $\mathcal{G}(\zeta,\eta)$ is defined by

$$\mathcal{G}(\zeta,\eta) = k^2\alpha^2 G_k(r(\zeta), r(\eta))r_\zeta(\eta), \tag{13.25}$$

while

$$\begin{aligned} a_1(\eta,\zeta) &= k\left[\frac{\mathfrak{s}^2(0)\Omega(\eta/(k\alpha))}{\mathfrak{s}^2(\eta/(k\alpha))\Omega(0)} - 1\right] \text{ in case } ((\mathbf{i})) \\ a_1(\eta,\zeta) &= k\left[\frac{\mathfrak{s}^2(0)\Omega(\eta/(k\alpha))}{\mathfrak{s}(\eta/(k\alpha))\mathfrak{s}(\zeta/(k\alpha))\Omega(0)} - 1\right] \text{ in case } ((\mathbf{ii})) \end{aligned} \tag{13.26}$$

while for large k and $\zeta \in (0, k\epsilon_1\alpha]$ we have

$$\begin{aligned} a_1(\eta,\zeta) &= \left(1 + \frac{\Omega'(0)}{\alpha\Omega(0)}\right)\eta + O(\eta^2/k), \text{ in case } ((\mathbf{i})) \\ a_1(\eta,\zeta) &= \frac{1}{2}(\zeta+\eta) + \frac{\Omega'(0)}{\alpha\Omega(0)}\eta + O(\eta^2/k, \zeta^2/k), \text{ in case } ((\mathbf{ii})) \end{aligned} \tag{13.27}$$

Similarly, for $k\alpha r = \zeta \in (0, k\epsilon_1\alpha)$, defining

$$b_1(\eta,\zeta) = k\left[\frac{\mathfrak{s}^2(\eta/(k\alpha))\Omega(\eta/(k\alpha))}{\mathfrak{s}^2(0)\Omega(0)} - 1\right] \text{ in case}((\mathbf{i})), \tag{13.28}$$

$$b_1(\eta,\zeta) = k\left[\frac{\mathfrak{s}^3(\eta/(k\alpha))\Omega(\eta/(k\alpha))}{\mathfrak{s}(\zeta/(k\alpha))\mathfrak{s}^2(0)\Omega(0)} - 1\right] \text{ in case}((\mathbf{ii}))$$

we have

$$\begin{aligned}[\mathcal{B}_k h_{k+1}](\zeta) &= \frac{\Omega(0)\mathfrak{s}^2(0)}{\alpha^2 k^2(2k+1+2\tau)(2k+2)} \\ &\times \int_\zeta^{k\alpha\epsilon_1} e^{-Q(\eta)+Q(\zeta)}\left(1+\frac{b_1}{k}\right)\frac{H(\eta(1+k^{-1})}{H(\zeta)}\mathcal{G}(\zeta,\eta)h_{k+1}(\eta(1+k^{-1}))d\eta \\ &+ \frac{\mathfrak{s}^2}{(2k+2)(2k+1+2\tau)} \\ &\times \int_0^{1-\delta_1}\sqrt{\Omega(r(\mathfrak{s}t))}G_k\left(r(\mathfrak{s}),r(\mathfrak{s}t)\right)t^{2k+2+\tau}\frac{F_{k+1}(r(\mathfrak{s}t))}{F_k(r(\mathfrak{s}))}h_{k+1}(\mathfrak{s}t)dt \\ &=: [\mathcal{B}_k^0 h_{k+1}] + [\mathcal{B}_k^1 h_{k+1}]\end{aligned} \tag{13.29}$$

Lemma 13.11. *For $k \geqslant 2$ and $k_1 = k-1, k$ or $k+1$, $G_k(r,s)F_{k_1}(s)/F_k(r)$ (see definitions (9.4) and (13.6)) satisfies the following bounds*

(1) *If $r \in (0,1)$ and $s \in (r, r+\delta)$, where $\delta = \min\left\{C_2 k^{-1}\log k, 1-r\right\}$,*

$$\left|G_k(r,s)\frac{F_{k_1}(s)}{F_k(r)}\right| \leqslant \frac{c_*}{k^{1/2}}, \quad \left|\frac{\partial}{\partial r}\left(G_k(r,s)\frac{F_{k_1}(s)}{F_k(r)}\right)\right| < c_* k^{1/2}$$

(2) *If $r \in (0,1)$, $\delta = C_2 k^{-1}\log k$ with $r+\delta < 1$, then for $s \in (r+\delta, 1)$,*

$$\left|G_k(r,s)\frac{F_{k_1}(s)}{F_k(r)}\right| < c_* k^{l/2-1/2}, \left|\frac{\partial}{\partial r}\left(G_k(r,s)\frac{F_{k_1}(s)}{F_k(r)}\right)\right| < c_* k^{l/2+1/2}$$

Proof. It suffices to find bounds for $G_k(r,s)H(\alpha k_1 s)/H(\alpha k r)$ since the other functions involved are regular everywhere for $r, s \in [0,1]$, see (9.4). We first consider $k \to +\infty$.

It is easily verified that $\mathcal{G}(\zeta,\eta)$, defined in (13.26), is the Green's function for

$$\mathcal{L} := \psi \mapsto \Psi'' - \frac{l(l+1)}{\zeta^2}\Psi + \frac{\Psi}{k}\left[\frac{\omega}{\alpha^2} + \frac{b}{\alpha\zeta}\right] + \frac{\Psi}{k^2\alpha^2}\left[\sigma + n_0\omega\right] \tag{13.30}$$

and is given by

$$\mathcal{G}(\zeta,\eta) := k\alpha G_k(r(\zeta), r(\eta)) = \frac{\Psi_1(\zeta)\Psi_2(\eta) - \Psi_2(\zeta)\Psi_1(\eta)}{W}, \tag{13.31}$$

where Ψ_1, Ψ_2 are two independent solution of $\mathcal{L}\Psi = 0$ and $W = \Psi_1(\zeta)\Psi_2'(\zeta) - \Psi_2(\zeta)\Psi_1'(\zeta)$ is their constant Wronskian.

Standard asymptotic results show there exist two independent solutions Ψ_1, Ψ_2 such that for large k, we have uniformly in $z \in [0, \sqrt{\omega k}]$

$$\Psi_1 \sim -\frac{2^l l!}{(2l)!}\sqrt{\frac{\pi z}{2}}Y_{l+1/2}(z) \text{ ; where } z = \sqrt{\frac{\omega}{\alpha^2 k}}\zeta = \sqrt{\omega k}r \tag{13.32}$$

$$\Psi_2 \sim \frac{2^{-l-1}(2l+2)!}{(l+1)!}\sqrt{\frac{\pi z}{2}}J_{l+1/2}(z) \tag{13.33}$$

The Wronskian W is asymptotic, for large k, to $(2l+1)\sqrt{\omega}/\sqrt{\alpha^2 k}$. The expressions (13.32) and (13.33) may also be used to determine the asymptotics of Ψ_1' and Ψ_2'. Using (13.31), (13.32), (13.33) and (9.4) and the bounds on W, it follows that

$$\left|\frac{F_{k_1}(s)}{F_k(r)}G_k(r,s)\right| \leqslant \frac{c_*|zz'|^{1/2}}{k^{1/2}}\left|\frac{H\left(\alpha\sqrt{\frac{k_1}{\omega}}z'\right)}{H\left(\alpha\sqrt{\frac{k}{\omega}}z\right)} \times \left\{Y_{l+1/2}(z)J_{l+1/2}(z') - J_{l+1/2}(z)Y_{l+1/2}(z')\right\}\right| \tag{13.34}$$

where $z' = \eta\sqrt{\omega}/\sqrt{\alpha^2 k} = \sqrt{\omega k}s$. A similar bound holds for

$$\left|\frac{\partial}{\partial r}\left\{\frac{F_{k_1}(s)}{F_k(r)}G_k(r,s)\right\}\right|$$

We now prove part 1. We break this case up into two cases: (a) $r \in [k^{-2/3}, 1]$ and (b) $r \in [0, k^{-2/3}]$. In case (a), we note that $s \in [r, r+\delta]$ implies s/r and therefore z'/z are $O(1)$. The function H in (13.34) is close to 1 because its argument is large. Furthermore, note that $\sqrt{z}Y_{l+1/2}(z)$ and $\sqrt{z}J_{l+1/2}(z)$ are bounded for large z, while they are asymptotic to constant multiples of z^{-l} and z^{l+1} for small z. Using (13.34), part 1 of the Lemma follows on inspection for case (a). For case (b), (13.34) further simplifies since z,z' is small and

$$\frac{H(k_1\eta/k)}{H(\zeta)}G_k(r(\zeta), r(\eta)) = \frac{H(k_1\eta/k)}{k\alpha H(\zeta)}\mathcal{G}(\zeta,\eta) \sim \frac{H(k_1\eta/k)}{k\alpha H(\zeta)(2l+1)}\left[\eta^{l+1}\zeta^{-l} - \zeta^{l+1}\eta^{-l}\right] \tag{13.35}$$

If $\zeta \in [\log k, \alpha k^{1/3}]$, then $\eta \in [\zeta, C_2 \log k + \alpha k^{1/3}]$; therefore $[\eta/\zeta]^l$ is bounded in this regime and

$$\left|\frac{H(k_1\eta/k)}{H(\zeta)}G_k(r(\zeta), r(\eta))\right| = \left|\frac{H(k_1\eta/k)}{k\alpha H(\zeta)}\mathcal{G}(\zeta,\eta)\right| \leqslant \frac{c_*}{k^{1/2}}$$

The same inequality holds if $\zeta \in [0, \log k]$, since $\eta \in [\zeta, (C_2+1)\log k]$ since in this regime $\zeta^{-l}/H(\zeta)$ is bounded and the logarithmic growth in k of terms involving η can be bounded by, say, $k^{1/2}$. The bounds on derivatives follow in a similar manner using $\frac{d}{dr} = k\alpha\frac{d}{d\zeta}$.

Part 2 (which is only relevant for $r+\delta \leqslant 1$) follows similarly on careful inspection of (13.34), from the asymptotic behavior in different regimes of z and z'.

If k is bounded, since the coefficients are regular, all solutions to $\mathcal{L}_k\psi$ are regular except possibly at $r=0$. We note that the asymptotics for $\psi_1(r)$, $\psi_2(r)$ as $r \to 0$ is similar to the asymptotics of $\Psi_1(\zeta)$ and $\Psi_2(\zeta)$ as $k \to +\infty$ for $\zeta = O(1)$. Hence $\psi_1(r) \sim \text{const}.r^{-l}$, while $\psi_2(r) \sim \text{const}.r^{l+1}$ near $r=0$. Now, for small r, $H(\alpha kr) \sim \text{const}.r^{-l}$ as well for small r. Using these, the Lemma follows when k is bounded, noting that the dependence on k is immaterial. ∎

Lemma 13.12. *The operator $\mathcal{A}_k^1$ defined in (13.24) satisfies for $r \in (0,\epsilon]$ the bounds $|[\mathcal{A}_k^1 f](r)| \leqslant c_* k^{l/2+2}(1-\delta_1)^{2k-2}\|f\|_\infty \leqslant c_* k^{-4}\|f\|_\infty$ and $|\frac{d}{dr}[\mathcal{A}_k^1 f](r)| \leqslant c_* k^{l/2+3}(1-\delta_1)^{2k-2}\|f\|_\infty \leqslant c_* k^{-3}\|f\|_\infty$ for some constant $c_* > 0$.*

Proof. Consider $\mathcal{A}_k^1$ given by (13.24). We note that $\mathfrak{s}^{-2}\Omega(s)$ and its $r-$derivative are bounded, while $G_k(s,r)F_k(s)/F_k(r)$ and its r-derivative are bounded by $c_* k^{l/2-1/2}$ and $c_* k^{l/2+1/2}$ respectively for any τ (cf. Lemma 13.11). Further $|\mathfrak{s}(s)/\mathfrak{s}(r)| \leqslant (1-\delta_1)$ and from (13.23), we have

$$(1-\delta_1)^{2k-2} \leqslant \exp\left[-C_2 \log k\right]$$

and hence for large enough C_2, the lemma follows. ∎

Remark 13.13. *Since for $r \in (0,\epsilon]$, the bound in Lemma 13.12 on $\mathcal{A}_k^1$ is $O(k^{-4})$, we will see later that $\mathcal{A}_k$ is dominated by $\mathcal{A}_k^0$ as $k \to \infty$.*

Lemma 13.14. *Define $\mathcal{G}_0(\zeta,\eta) = \lim_{k\to\infty} \mathcal{G}(\zeta,\eta)$ and $H_0(\zeta) = \lim_{k\to\infty} H(\zeta)$, where $\zeta, \eta \ll k^{1/2}$ as $k \to \infty$. Then,*

$$\int_\zeta^\infty e^{-\eta+\zeta} \mathcal{G}_0(\zeta,\eta) \frac{H_0(\eta)}{H_0(\zeta)} d\eta = 1 \tag{13.36}$$

$$\int_\zeta^\infty e^{-\eta+\zeta} \mathcal{G}_{0\zeta}(\zeta,\eta) \frac{H_0(\eta)}{H_0(\zeta)} d\eta = -1 + \frac{H_0'(\zeta)}{H_0(\zeta)} \tag{13.37}$$

Proof. Recall that in the proof of Lemma 13.11, we had for $\zeta, \eta \ll k^{1/2}$,

$$\mathcal{G}_0(\zeta,\eta) = \lim_{k\to\infty} \mathcal{G}(\zeta,\eta) = \frac{\eta^{l+1}\zeta^{-l} - \zeta^{l+1}\eta^{-l}}{2l+1} \tag{13.38}$$

and $H_0(\zeta) = \lim_{k\to\infty} H(\zeta) = \zeta^{1/2} e^{\zeta} K_{l+1/2}(\zeta)$. Now, using the modified Bessel function equation, it is easily verified that $p(\zeta) = e^{-\zeta} H_0(\zeta)$ satisfies

$$p'' - \frac{l(l+1)}{\zeta^2} p = p$$

with $p(\zeta) \sim e^{-\zeta}$ as $\zeta \to \infty$. Using variation of parameters to invert the left hand side of the above equation, and using the boundary conditions at ∞ we obtain

$$p(\zeta) = \int_\zeta^\infty \mathcal{G}_0(\zeta,\eta) p(\eta) d\eta$$

Dividing through by $p(\zeta)$, the first identity in the Lemma follows. By differentiating the first identity with respect to ζ, and using the first identity in the resulting expression, we obtain the second identity. ∎

Lemma 13.15.

$$\Big| |\mathcal{A}_k[1](r) - 1 \Big| = \Big| \int_r^1 \Omega(s) \frac{m_{k-1}(s)}{m_k(r)} G_k(r,s) ds - 1 \Big| \leqslant \frac{c_*}{k^2}, \tag{13.39}$$

$$\Big| \frac{d}{dr} \mathcal{A}_k[1](r) \Big| = \Big| \frac{d}{dr} \int_r^1 \Omega(s) \frac{m_{k-1}(s)}{m_k(r)} G_k(r,s) ds \Big| \leqslant \frac{c_*}{k^2} + \frac{c_*}{k^3 r^2} \text{ for } kr \geqslant 1, \tag{13.40}$$

while for any $r \in [0,1]$,

$$(2k)(2k-1+2\tau) \int_r^1 \Omega(s) \frac{[\mathfrak{s}(s)]^{2k-2+\tau}}{[\mathfrak{s}(r)]^{2k+\tau}} \Big| \frac{\partial}{\partial r} \left(G_k(r,s) \frac{F_{k-1}(s)}{F_k(r)} \right) \Big| ds \leqslant c_* k \tag{13.41}$$

Proof. We note from Lemma 13.8 that

$$\mathcal{L}_k m_k - \Omega m_{k-1} = \frac{j_k(r)}{\mathfrak{s}} m_k \tag{13.42}$$

where $j_k(r) = O(1)$ as $k \to +\infty$ for any $r \in [0,1]$. We can check that $m_k(1) = 0$, $m_k'(1) = 0$ for $k \geqslant 1$ and thus

$$m_k(r) = \int_r^1 G_k(r,s) \left\{ \Omega(s) m_{k-1}(s) + \frac{j_k(s)}{\mathfrak{s}(s)} m_k(s) \right\} ds \tag{13.43}$$

Therefore,

$$\int_r^1 G_k(r,s) \frac{\Omega(s) m_{k-1}(s)}{m_k(r)} ds \quad = \quad 1 \quad - \quad \int_r^1 G_k(r,s) \frac{j_k(s) m_k(s)}{\mathfrak{s}(s) m_k(r)} ds \tag{13.44}$$

First, we choose $\hat{\delta} = \left(\frac{5}{2} + \frac{l}{2}\right) \frac{\mathfrak{s}(r) \log k}{(2k+\tau)\sqrt{\Omega(r)}}$. We define $\hat{\delta}_1$ so that $\hat{\delta}_1 \mathfrak{s}(r) = \mathfrak{s}(r+\hat{\delta})$. It is clear that $1 - \hat{\delta}_1 \sim \frac{(5+l)\log k}{4k+2\tau}$ for large k. The bounds on $\frac{F_k(s)}{F_k(r)} G_k(r,s)$ in Lemma 13.11, and the fact that

$$k^{l/2+1/2} \frac{(1-\hat{\delta}_1)^{2k+1+\tau}}{2k+1+\tau} \leqslant \frac{1}{k^4}$$

give

$$\begin{aligned} \Big| \int_{r+\hat{\delta}}^1 G_k(r,s) \frac{j_k(s) m_k(s)}{\mathfrak{s}(s) m_k(r)} ds \Big| &\leqslant \Big| \int_0^{1-\hat{\delta}_1} t^{2k+\tau} \\ &\times \frac{F_k(r(\mathfrak{s}t)}{\sqrt{\Omega(r(\mathfrak{s}t))} F_k(r(\mathfrak{s})} G_k(r(\mathfrak{s}), r(\mathfrak{s}t)) j_k(r(\mathfrak{s}t)) dt \Big| \leqslant \frac{c_*}{k^4} \|j_k\|_\infty \leqslant \frac{c_*}{k^4} \end{aligned} \tag{13.45}$$

Now, consider the contribution from $\int_r^{r+\hat{\delta}}$. There are again two cases: (i) $1 \geqslant r \geqslant k^{-2/3}$ and (ii) $0 < r \leqslant k^{-2/3}$.

In the first case, it is convenient note from the Taylor expansion of $G_k(r,s)$ for small $s-r$ that $G_k = (s-r) + O((s-r)^3 Q_k) = \sqrt{\Omega(r)}\mathfrak{s}(1-t) + O(k^{4/3}(1-t)^3, (1-t)^2)$. Hence,

$$\Big| \int_r^{r+\hat{\delta}} G_k(r,s) \frac{j_k(s) m_k(s)}{\mathfrak{s}(s) m_k(r)} ds \Big| \leqslant c_* \|j_k\|_\infty \int_{1-\hat{\delta}_1}^1 t^{2k+\tau-1}(1-t) dt \leqslant \frac{c_*}{k^2} \tag{13.46}$$

For case (ii), we rewrite the integral in terms of $\zeta = k\alpha r$, to obtain

$$\begin{aligned} &\Big| \int_r^{r+\hat{\delta}} G_k(r,s) \frac{j_k(s) m_k(s)}{\mathfrak{s}(s) m_k(r)} ds \Big| \\ &\leqslant \frac{c_*}{k^2} \|j_k\|_\infty \int_\zeta^{\zeta + k\alpha\hat{\delta}} e^{-Q(\eta)+Q(\zeta)} \mathcal{G}(\zeta,\eta) \frac{H(\eta)}{H(\zeta)} d\eta \\ &\leqslant \frac{c_*}{k^2} \int_\zeta^{\zeta+k\alpha\delta} d\eta e^{-\eta+\zeta} \mathcal{G}_0(\zeta,\eta) \frac{H_0(\eta)}{H_0(\zeta)} \\ &\leqslant \frac{c_*}{k^2} \int_\zeta^\infty d\eta e^{-\eta+\zeta} \mathcal{G}_0(\zeta,\eta) \frac{H_0(\eta)}{H_0(\zeta)} \leqslant \frac{c_*}{k^2} \end{aligned} \tag{13.47}$$

by Lemma 13.6. So, using (13.44) and (13.47), the first part follows.

To prove (13.40), we first note that for $rk \geqslant c_*$ with large $c_* > 0$, Taylor expansion (similar to that of $T(\mathfrak{s}, t)$) gives rise to

$$
\begin{aligned}
(13.48)\quad U_1(\mathfrak{s}, t) &:= \frac{F_k(r(\mathfrak{s}t)}{\sqrt{\Omega(r(\mathfrak{s}t))}F_k(r(\mathfrak{s})} G_k(r(\mathfrak{s}), r(\mathfrak{s}t)) \\
&= f_4(\mathfrak{s})(1-t) + O\left((1-t)^2, k(1-t)^3, \frac{(1-t)^3}{r^2}, \frac{(1-t)^2}{kr^2}\right)
\end{aligned}
$$

for some f_4 differentiable at $r = 0$ (the precise form of f_4 is unimportant), while

$$
\frac{\partial}{\partial \mathfrak{s}} U_1(\mathfrak{s}, t) = f_4'(\mathfrak{s})(1-t) + O\left((1-t)^2, k(1-t)^3, \frac{(1-t)^3}{r^3}, \frac{(1-t)^2}{kr^3}\right)
$$

Therefore, it follows that

$$
\begin{aligned}
(13.49)\quad & \frac{d}{dr} \int_r^{r+\hat{\delta}} G_k(r, s) \frac{j_k(s) m_k(s)}{\mathfrak{s}(s) m_k(r)} ds = -\sqrt{\Omega(r(\mathfrak{s}))} \Big\{ \\
& \times \int_{1-\hat{\delta}_1}^1 t^{2k+\tau-1} U_1(\mathfrak{s}, t) \frac{\partial}{\partial \mathfrak{s}} j_k(r(\mathfrak{s}t)) dt - \int_{1-\hat{\delta}_1}^1 t^{2k+\tau-1} \frac{\partial}{\partial \mathfrak{s}} U_1(\mathfrak{s}, t) j_k(r(\mathfrak{s}t)) dt \Big\}
\end{aligned}
$$

Since Lemma 13.8 implies $|j_k(r)| < c_*$ and $|j_k'(r)| < c_* + c_*/(kr^2)$ for $kr \geqslant 1$, it follows from the local expansion of $U_1(\mathfrak{s}, t)$ and its $\mathfrak{s}$-derivative in (13.49) that

$$
\Big| \frac{d}{dr} \int_r^{r+\hat{\delta}} G_k(r, s) \frac{j_k(s) m_k(s)}{\mathfrak{s}(s) m_k(r)} ds \Big| \leqslant \frac{c_*}{k^3 r^2} + \frac{c_*}{k^2}
$$

Differentiating (13.44) and using the equation above, (13.40) follows, since the contribution from $\int_{r+\hat{\delta}}^1$ is easily checked to be subdominant by transforming to t, using the bounds on G_k from Lemma 13.11 and the smallness of $(1-\hat{\delta}_1)^{2k+\tau}$.

We now prove the last part of the Lemma. By differentiating (13.44) with respect to r we see that

$$
\begin{aligned}
(13.50)\quad (2k+\tau)(2k+\tau-1) & \int_r^1 \Omega(s) \frac{[\mathfrak{s}(s)]^{2k-2+\tau}}{[\mathfrak{s}(r)]^{2k+\tau}} \frac{\partial}{\partial r} \left(G_k(r, s) \frac{F_{k-1}(s)}{F_k(r)} \right) ds \\
&= (2k+\tau) \frac{\mathfrak{s}'(r)}{\mathfrak{s}(r)} \int_r^1 \Omega(s) \frac{m_{k-1}(s)}{m_k(r)} G_k(r, s) ds \\
&\quad - (2k+\tau) \frac{\mathfrak{s}'(r)}{\mathfrak{s}(r)} \int_r^1 \frac{m_k(s)}{m_k(r)} G_k(r, s) j_k(s) ds \\
&\quad\quad + \int_r^1 \frac{j_k(s)}{\mathfrak{s}(s)} \left[\frac{\mathfrak{s}(s)}{\mathfrak{s}(r)} \right]^{2k+\tau} \frac{\partial}{\partial r} \left(G_k(r, s) \frac{F_k(s)}{F_k(r)} \right) ds
\end{aligned}
$$

Using the first part of Lemma, besides equations (13.45) and (13.46), it is clear that the contribution from the first two terms on the right side $\leqslant c_* k$. We are left with the last term on the right side. As before we break up into separate contribution from $\int_{r+\hat{\delta}}^1$ and $\int_r^{r+\hat{\delta}}$. The contribution from $\int_{r+\hat{\delta}}^1$ is easily shown to be small, using Lemma 13.11. For $\int_r^{r+\hat{\delta}}$, we note that since $s \in (r, r+\hat{\delta})$, again by Lemma 13.11, it follows that

$$\Big| \int_r^{r+\hat\delta} \frac{j_k(s)}{\mathfrak{s}(s)} \left[\frac{\mathfrak{s}(s)}{\mathfrak{s}(r)}\right]^{2k+\tau} \frac{\partial}{\partial r}\left(G_k(r,s)\frac{F_k(s)}{F_k(r)}\right) ds\Big| \leqslant c_* k^{1/2} \int_r^1 \left[\frac{\mathfrak{s}(s)}{\mathfrak{s}(r)}\right]^{2k+\tau-1} \frac{ds}{\mathfrak{s}(r)} \leqslant \frac{c_*}{k^{1/2}}$$

Therefore, (13.50) implies

$$(2k+\tau)(2k+\tau-1)\Big|\int_r^{r+\hat\delta} \Omega(s)\frac{[\mathfrak{s}(s)]^{2k-2}}{[\mathfrak{s}(r)]^{2k}}\frac{\partial}{\partial r}\left(G_k(r,s)\frac{F_{k-1}(s)}{F_k(r)}\right) ds\Big| \leqslant c_* k$$

Now, we note that for $r+\hat\delta \geqslant s \geqslant r \geqslant 0$ we have

$$\frac{\partial}{\partial r}\left(G_k(r,s)\frac{F_{k-1}(s)}{F_k(r)}\right) = \partial_r G_k(r,s)\frac{F_{k-1}(s)}{F_k(r)} - G_k(r,s)\frac{F_{k-1}(s)F_k'(r)}{F_k^2(r)}$$

We see that for large k, $\partial_r G_k(r,s) \sim -1 < 0$ for $r \geqslant k^{-2/3}$, $r \leqslant s \leqslant r+\hat\delta$. The same is true for $0 \leqslant r \leqslant k^{-2/3}$ since $\partial_r G_k(r(\zeta), s(\eta)) = \frac{1}{k\alpha}\mathcal{G}_\zeta(\zeta,\eta)$ and $\mathcal{G}_\zeta(\zeta,\eta) \sim -\frac{l}{2l+1}\left(\frac{\eta}{\zeta}\right)^{l+1} - \frac{l+1}{2l+1}\left(\frac{\eta}{\zeta}\right)^{-l} < 0$. Therefore,

$$(13.51)\quad (2k+\tau)(2k+\tau-1)\int_r^{r+\hat\delta} \Omega(s)\frac{[\mathfrak{s}(s)]^{2k-2}}{[\mathfrak{s}(r)]^{2k}}\Big|\frac{\partial}{\partial r}\left(G_k(r,s)\frac{F_{k-1}(s)}{F_k(r)}\right)\Big| ds$$
$$\leqslant (2k+\tau)(2k+\tau-1)\Big|\int_r^{r+\hat\delta} \Omega(s)\frac{[\mathfrak{s}(s)]^{2k-2}}{[\mathfrak{s}(r)]^{2k}}\frac{\partial}{\partial r}\left(G_k(r,s)\frac{F_{k-1}(s)}{F_k(r)}\right) ds\Big|$$
$$+\Big|\frac{2F_k'(r)}{F_k(r)}\Big|\int_r^{r+\hat\delta}\Omega(s)\frac{m_{k-1}(s)}{m_k(r)}G_k(r,s)ds$$
$$\leqslant c_* k + \Big|\frac{2F_k'(r)}{F_k(r)}\Big|[\mathcal{A}_k[1](r) + \frac{c_*}{k^2}] \leqslant c_* k$$

∎

Lemma 13.16. *For any $f \in L^\infty[0,1]$,*

$$(13.52)\qquad \|\mathcal{A}_k f\|_\infty \leqslant \left(1+\frac{c_*}{k^2}\right)\|f\|_\infty$$

$$(13.53)\qquad \left\|\frac{d}{dr}[\mathcal{A}_k f](r)\right\|_\infty \leqslant c_* k\|f\|_\infty$$

Proof. Consider the expression for $\mathcal{A}_k f$ from (13.11). We break up the integral contribution $\int_r^{r+\delta} + \int_{r+\delta}^1$, where $\delta = C_2 k^{-1}\log k$, with C_2 large enough so that

$$(1-\delta_1)^{2k-2+\tau} \leqslant \frac{c_*}{k^{l/2+7/2}}$$

where δ_1 is determined by $\mathfrak{s}(r+\delta) = \mathfrak{s}(r)(1-\delta_1)$. If $r+\delta \geqslant 1$, then it is not necessary to consider $\int_{r+\delta}^1$ and it suffices to consider the entire integral $\int_r^1$ at the same time. ¿From (9.3) and Lemma 13.11, part (2), transforming the integration variable to t, it follows that

$$\Big|\int_{r+\delta}^1 \Omega(s)\frac{m_k(s)}{m_k(r)}G_k(r,s)f(s)ds\Big| \leqslant \frac{c_*}{k^2}\|f\|_\infty$$

Now, consider the contribution from $\int_r^{r+\delta}$ to $\mathcal{A}_k f$ (we replace upper limit $r+\delta$ by 1 if $r+\delta>1$). We consider two distinct cases. (i) $r \in [k^{-2/3}, 1]$, (ii) $r \in (0, k^{-2/3}]$. ¿From the the positivity of $T_k(\mathfrak{s}, t)$ in (13.15) for $t \in (1-\delta_1, 1)$ when $\delta_1 = O\left(\frac{1}{k}\log k\right)$ for case (i) and and the simplification (13.35) for $r \in [k^{-2/3}, 1]$, it follows that $G_k(r,s) \geqslant 0$ for $s \in [r, r+\delta)$. Therefore,

$$\|\mathcal{A}_k f\|_\infty \leqslant \|f\|_\infty \left\{ \left\{ \int_r^{r+\delta} \frac{\Omega(s) m_{k-1}(s)}{m_k(r)} G_k(r,s) \right\} + \frac{c_*}{k^2} \right\} \tag{13.54}$$

¿From the argument in the last paragraph (with $f=1$), we have

$$\Big| \int_{r+\delta}^1 \Omega(s) \frac{m_{k-1}(s)}{m_k(r)} G_k(r,s) ds \Big| \leqslant \frac{c_*}{k^2}$$

Hence

$$\int_r^{r+\delta} \Omega(s) \frac{m_{k-1}(s)}{m_k(r)} G_k(r,s) ds = \int_r^1 \Omega(s) \frac{m_{k-1}(s)}{m_k(r)} G_k(r,s) ds + O\Big(\frac{1}{k^2}\Big) \tag{13.55}$$

Using Lemma 13.14 in (13.54), the first part of the Lemma follows.

For the second part, we have

$$\begin{aligned} &\frac{d}{dr} \int_r^1 \Omega(s) \frac{m_{k-1}(s)}{m_k(r)} G_k(r,s) f(s) ds \\ &\quad = (2k+\tau)(2k+\tau-1) \int_r^1 \frac{[\mathfrak{s}(s)]^{2k-2+\tau}}{[\mathfrak{s}(r)]^{2k+\tau}} \frac{\partial}{\partial r} \left(G_k(r,s) \frac{F_{k-1}(s)}{F_k(r)} \right) f(s) ds \\ &\qquad\qquad - 2k \frac{\mathfrak{s}'(r)}{\mathfrak{s}(r)} [\mathcal{A}_k f](r) \end{aligned} \tag{13.56}$$

By the first part of Lemma, the last term on the right bounded by $c_* k \|f\|_\infty$. On the other hand,

$$\begin{aligned} &(2k+\tau)(2k+\tau-1) \Big| \int_r^1 \Omega(s) \frac{[\mathfrak{s}(s)]^{2k-2+\tau}}{[\mathfrak{s}(r)]^{2k+\tau}} \frac{\partial}{\partial r} \left(G_k(r,s) \frac{F_{k-1}(s)}{F_k(r)} \right) f(s) ds \Big| \\ &\quad \leqslant \|f\|_\infty \int_r^1 \Omega(s) \frac{[\mathfrak{s}(s)]^{2k+\tau-2}}{[\mathfrak{s}(r)]^{2k+\tau}} \Big| \frac{\partial}{\partial r} \left(G_k(r,s) \frac{F_{k-1}(s)}{F_k(r)} \right) \Big| ds \leqslant c_* k \|f\|_\infty \end{aligned} \tag{13.57}$$

by Lemma 13.15. ∎

Lemma 13.17. *For any $f \in \mathcal{L}_\infty[0,1]$,*

$$\|\mathcal{B}_k f\|_\infty \leqslant \frac{c_*}{k^2} \|f\|_\infty$$

$$\Big\| \frac{d}{dr} [\mathcal{B}_k f](r) \|_\infty \leqslant \frac{c_*}{k^2} \Big\| f \|_\infty$$

Proof. As before, we choose $\delta = C_2 k^{-1} \log k$ large C_2 independent of k. Using Lemma 13.11, it follows that

$$\Big| \int_{r+\delta}^1 \frac{\Omega(s) m_{k+1}(s)}{m_{k(r)}} G_k(r,s) f(s) ds \Big| \leqslant c_* (1-\delta_1)^{2k+2} k^{l/2-5/2} \|f\|_\infty \leqslant \frac{c_*}{k^4} \|f\|_\infty$$

$$(13.58)\quad \Big|\int_{r+\delta}^{1}\frac{\partial}{\partial r}\left\{\frac{\Omega(s)m_{k+1}(s)}{m_{k(r)}}G_k(r,s)\right\}f(s)ds\Big| \leqslant c_*(1-\delta_1)^{2k+2}k^{l/2-3/2}\|f\|_\infty \leqslant \frac{c_*}{k^3}\|f\|_\infty$$

Now consider the contribution from $\int_r^{r+\delta}$. Lemma 13.11 implies

$$\Big|\int_r^{r+\delta}\frac{\Omega(s)m_{k+1}(s)}{m_{k(r)}}G_k(r,s)f(s)ds\Big| \leqslant c_*\frac{\|f\|_\infty}{k^2}\int_0^1 t^{2k+2+\tau}dt \leqslant \frac{c_*\|f\|_\infty}{k^2}$$

$$(13.59)\quad \Big|\int_r^{r+\delta}\frac{\partial}{\partial r}\left\{\frac{\Omega(s)m_{k+1}(s)}{m_{k(r)}}G_k(r,s)\right\}f(s)ds\Big| \leqslant \frac{c_*\|f\|_\infty}{k}\int_0^1 t^{2k+2+\tau}dt \leqslant \frac{c_*\|f\|_\infty}{k^2}$$

∎

Lemma 13.18. *Given $h_{k_0}\in L^\infty(0,1)$ we have, with c_* is independent of k,*

$$(13.60)\quad \|h_k\|_\infty < c_*$$

Proof. Define $r_k = \mathcal{B}_k h_{k+1}$. Note that

$$(13.61)\quad h_k = \mathcal{A}_k\left(\mathcal{A}_{k-1}h_{k-2}+r_{k-1}\right)+r_k$$

By induction, we get after $k-k_0$ steps

$$(13.62)\quad h_k = \mathcal{A}_k\mathcal{A}_{k-1}..\mathcal{A}_{k_0+1}h_{k_0}+\mathcal{B}_kh_{k+1}+\sum_{m=1}^{k-k_0-1}\left(\prod_{j=1}^{m}\mathcal{A}_{k-j+1}\right)\mathcal{B}_{k-m}h_{k-m+1}$$

We write this abstractly as

$$(13.63)\quad \mathfrak{h}=\mathfrak{h}^0+\mathfrak{N}\mathfrak{h}$$

where

$$\mathfrak{h}_k^0=\mathcal{A}_k\mathcal{A}_{k-1}..\mathcal{A}_{k_0+1}h_{k_0}\ ;\ [\mathfrak{N}\mathfrak{h}]_k=\mathcal{B}_kh_{k+1}+\sum_{m=1}^{k-k_0-1}\left(\prod_{j=1}^{m}\mathcal{A}_{k-j+1}\right)\mathcal{B}_{k-m}h_{k-m+1}$$

Lemmas 13.16 and 13.17 imply that the linear operator $\mathfrak{N}$ on the space $l^\infty\left(L^\infty(0,1)\right)$, of sequences $\mathfrak{h}=\{h_k\}_{k=k_0+1}^\infty$ in the norm

$$(13.64)\quad \|\mathfrak{h}\|=\sup_{k\geqslant k_0+1}\|h_k\|_\infty,$$

satisfies

$$(13.65)\quad |[\mathcal{N}\mathfrak{h}]_k| \leqslant \left(\frac{c_*}{k_0^2}+c_*\sum_{m=1}^{k-k_0-1}\left\{\prod_{j=1}^{m}\left[1+\frac{c_*}{(k-j+1)^2}\right]\right\}\frac{1}{(k-m)^2}\right)\|\mathfrak{h}\|_\infty<\nu\|\mathfrak{h}\|_\infty$$

where $\nu<1$ can be chosen independent of k for k_0 large enough. Thus, $\mathfrak{N}$ is contractive and it follows that there is a unique solution $\mathfrak{h}$ in this space. ∎

Lemma 13.19. $\|\frac{d}{dr}h_k\|_\infty\leqslant c_*k$.

Proof. Lemmas 13.17 and 13.16 imply

$$|h_k'(r)| \leqslant |\frac{d}{dr}[\mathcal{A}_k h_{k-1}](r)| + |\frac{d}{dr}[\mathcal{B}_k h_{k+1}](r)| \leqslant c_* k$$

since h_k is bounded from Lemma 13.18. ∎

Lemma 13.20. *For all $k \geqslant 1$, $h_k(1) = 1$.*

Proof. In case (**i**), simple computation using $g_{n_0-k} = i^k m_k h_k$ shows that

$$\frac{\partial^{2k}}{\partial \mathfrak{s}^{2k}}|_{\mathfrak{s}=0} g_{n_0-k} = i^k h_k(1)$$

since from the differential equation satisfied by h_k, all the derivatives exist at $\mathfrak{s} = 0$ ($r = 1^-$) by the assumptions on Ω. Therefore, using Lemma 13.6 for $j = 2k$ we get

$$i^k = \frac{\partial^{2k}}{\partial \mathfrak{s}^{2k}}|_{\mathfrak{s}=0} g_{n_0-k} = i^k h_k(1)$$

Hence the result follows in case (**i**). In case (**ii**), using Lemma 13.7, a similar computation involving $g_{n_0-k} = i^k m_k h_k$ shows that

$$i^k = \frac{\partial^{2k+1}}{\partial \mathfrak{s}^{2k+1}}|_{\mathfrak{s}=0} g_{n_0-k} = i^k h_k(1)$$

Hence $h_k(1) = 1$ in this case as well for all $k \geqslant 1$. ∎

Definition 13.21. *It is convenient to define*

$$\hat{T}_k(\mathfrak{s}, s) = s^{-2k+1-\tau} \int_0^s t^{2k-2+\tau} \mathfrak{s} \frac{\partial}{\partial \mathfrak{s}} T_k(\mathfrak{s}, t) dt, \tag{13.66}$$

where $T_k(\mathfrak{s}, t)$ is defined in (13.13).

Lemma 13.22. *For $s \in (0, \delta)$, where $\delta = k^{-1} \log k$, and $r(\mathfrak{s}) \geqslant k^{-1} C_2$ for large enough C_2, the asymptotic behavior of $\hat{T}_k$ for large k is given by*

$$\begin{aligned} \hat{T}_k(\mathfrak{s}, s) = \mathfrak{s} S_k(\mathfrak{s}) - \frac{\mathfrak{s} f_1'(\mathfrak{s})}{12}(1-s)^3 + \frac{\mathfrak{s} f_3(\mathfrak{s})}{3kr^3}(1-s)^3 \\ + O\left(\frac{(1-s)^4}{kr^4}, \frac{(1-s)^3}{k^2 r^4}, \frac{(1-s)^2}{k^3 r^4}, \frac{(1-s)}{k^4 r^3}, \frac{(1-s)^3}{kr^2}, \frac{(1-s)^2}{k^2 r^2}\right) \end{aligned} \tag{13.67}$$

$$S_k(\mathfrak{s}) = \frac{\partial}{\partial \mathfrak{s}} \int_0^1 t^{2k-2} T_k(\mathfrak{s}, t) dt \tag{13.68}$$

Proof. This simply follows by integrating the asymptotic expansion of $T_k(\mathfrak{s}, t)$ in (13.15) from $t = 1$ to s and using $\hat{T}_k(\mathfrak{s}, 1) = \mathfrak{s} S_k(\mathfrak{s})$. ∎

Proof of Lemma 13.1

¿From Lemma 13.17, it follows that

$$\left\| \frac{d}{d\mathfrak{s}}[\mathcal{B}_k h_{k+1}] \right\|_\infty \leqslant \frac{c_*}{k^2} \|h_{k+1}\|_\infty \leqslant \frac{c_*}{k^2}$$

where we applied Lemma 13.18. Further, we note that

$$\begin{aligned}(13.69)\quad & \frac{1}{(2k+\tau)(2k+\tau-1)}\frac{d}{d\mathfrak{s}}\mathcal{A}_k h_{k-1}(\mathfrak{s}) \\ &= \int_0^1 t^{2k+\tau-2}\frac{\partial T_k}{\partial \mathfrak{s}}(\mathfrak{s},t)h_{k-1}(\mathfrak{s}t)dt + \int_0^1 t^{2k+\tau-1}T_k(\mathfrak{s},t)h'_{k-1}(\mathfrak{s}t)dt\end{aligned}$$

We note that we may write

$$\begin{aligned}(13.70)\quad & \int_0^1 t^{2k+\tau-2}\frac{\partial T_k}{\partial \mathfrak{s}}(\mathfrak{s},t)h_{k-1}(\mathfrak{s}t)dt = h_{k-1}(\mathfrak{s})S_k(\mathfrak{s}) - \int_0^1 t^{2k+\tau-2}\mathfrak{s}\frac{\partial T_k}{\partial \mathfrak{s}}(\mathfrak{s},t) \\ &\times \int_t^1 h'_{k-1}(\mathfrak{s}s)ds = h_{k-1}(\mathfrak{s})S_k(\mathfrak{s}) - \int_0^1 h'_{k-1}(\mathfrak{s}s)\left[\int_0^s t^{2k+\tau-2}\mathfrak{s}\frac{\partial T_k}{\partial \mathfrak{s}}(\mathfrak{s},t)dt\right]ds \\ &= h_{k-1}(\mathfrak{s})S_k(\mathfrak{s}) - \int_0^1 h'_{k-1}(\mathfrak{s}s)s^{2k-1+\tau}\hat{T}_k(\mathfrak{s},s)ds = (2k+\tau-1)S_k(\mathfrak{s}) \\ &\times \int_0^1 s^{2k-2+\tau}h_{k-1}(\mathfrak{s}s)ds - \int_0^1 s^{2k-1+\tau}[\hat{T}_k(\mathfrak{s},s)-\hat{T}_k(\mathfrak{s},1)]h'_{k-1}(\mathfrak{s}s)ds\end{aligned}$$

Therefore,

$$\begin{aligned}(13.71)\quad & \frac{\frac{d}{d\mathfrak{s}}\mathcal{A}_k[h_{k-1}](\mathfrak{s})}{(2k+\tau)(2k+\tau-1)} = \int_0^1 [T_k(\mathfrak{s},s)-\hat{T}_k(\mathfrak{s},s)+\mathfrak{s}S_k(\mathfrak{s})]s^{2k+\tau-1} \\ &\times h'_{k-1}(\mathfrak{s}s)ds + (2k+\tau-1)S_k(\mathfrak{s})\int_0^1 s^{2k+\tau-2}h_{k-1}(\mathfrak{s}s)ds\end{aligned}$$

We note that

$$(2k+\tau)(2k+\tau-1)S_k(\mathfrak{s}) = \frac{\partial}{\partial \mathfrak{s}}\left[\mathcal{A}_k[1](\mathfrak{s})\right] = O\left(\frac{1}{k^3\epsilon_1^2},\frac{1}{k^2}\right)$$

and that $(2k+\tau-1)\int_0^1 s^{2k+\tau-2}h_{k-1}(\mathfrak{s}s)ds$ has a bound independent of k. Further, combining the asymptotic expansion (13.15) with Lemma 13.22, it follows that for all k for which $k\epsilon_1$ is sufficiently large,

$$\begin{aligned}(13.72)\quad & T_k(\mathfrak{s},s) - [\hat{T}_k(\mathfrak{s},s)-\mathfrak{s}S_k(\mathfrak{s})] = (1-s) + \left(-\frac{kf_1}{4}+\frac{f_2}{r^2}\right) \\ &\times \Big[-\frac{(1-s)^2}{k}+\frac{2}{3}(1-s)^3\Big] - \mathfrak{s}\left(-\frac{f_1'}{12}+\frac{f_3}{3kr^3}\right)(1-s)^3 \\ &+ O\Big(\frac{(1-s)^4}{kr^4},\frac{(1-s)^3}{k^2r^4},\frac{(1-s)^2}{k^3r^4},\frac{(1-s)}{k^4r^3},\frac{(1-s)^3}{kr^2},\frac{(1-s)^2}{k^2r^2}, \\ &\qquad \frac{(1-s)^4}{r^3},\frac{(1-s)^3}{kr^3},\frac{(1-s)^3}{r},\frac{(1-s)^2}{kr}\Big)\end{aligned}$$

¿From (13.72), it is clear that for $s\in(1-\delta,1)$, for $r\epsilon_1$ sufficiently large, $T_k(\mathfrak{s},s) - \hat{T}_k(\mathfrak{s},s)+\mathfrak{s}S_k(\mathfrak{s})>0$. Further, we note that since $\frac{\mathfrak{s}f_3}{3kr^3}(1-s)^3$ is positive and has a lower bound greater in size than any term that follows it on the right of (13.72). Thus, if we define

$$(13.73)\qquad M_k = \sup_{r(\mathfrak{s})\in[\epsilon_1,1]}|h_k'(\mathfrak{s})|$$

(13.74)
$$|h_k'(\mathfrak{s})| \leqslant (2k+\tau)(2k+\tau-1)M_{k-1}\left\{\int_{1-\delta_1}^1 s^{2k+\tau-1}\left[(1-s)+\left(-\frac{kf_1}{4}+\frac{f_2}{r^2}\right)\right.\right.$$
$$\left.\left.\times\ [-\frac{1}{k}(1-s)^2+\frac{2}{3}(1-s)^3]+\mathfrak{s}\frac{f_1'}{12}(1-s)^3\right]ds\right\}+\frac{c_*}{k^2}+\frac{c_*}{k^3\epsilon_1^2}$$

Therefore, since $\int_{1-\delta}^1 s^{2k-1}[-\frac{1}{k}(1-s)^2+\frac{2}{3}(1-s)^3]ds = O(\frac{1}{k^5})$ it follows that

$$M_k \leqslant M_{k-1}\left(\frac{2k-1+\tau}{2k+1+\tau}+\frac{c_*}{k^2}+\frac{c_*}{k^3\epsilon_1^2}\right)+\frac{c_*}{k^2}+\frac{c_*}{k^3\epsilon_1^2} \tag{13.75}$$

Now, choose

$$k_0(\epsilon_1)=\frac{C_2}{\epsilon_1}, \tag{13.76}$$

with C_2 large enough, but independent of ϵ_1 so that for $k \geqslant k_0$,

$$\left(\frac{2k+\tau-1}{2k+\tau+1}+\frac{c_*}{\epsilon_1^2k^3}+\frac{c_*}{k^2}\right)\leqslant\left(\frac{k-1}{k}\right)^\gamma$$

Then for $k \geqslant k_0$,

$$M_k \leqslant \left(\frac{k-1}{k}\right)^\gamma M_{k-1}+\frac{c_*}{k^2}+\frac{c_*}{k^3\epsilon_1^2} \tag{13.77}$$

This implies,

$$M_k \leqslant \left(\frac{k_0}{k}\right)^\gamma M_{k_0}+\frac{c_*}{k^\gamma}\sum_{j=k_0}^k\frac{1}{j^{2-\gamma}}+\frac{c_*}{k^\gamma}\sum_{j=k_0}\frac{1}{j^{3-\gamma}\epsilon_1^2} \tag{13.78}$$
$$\leqslant c_*\frac{k_0^{1+\gamma}}{k^\gamma}+\frac{c_*}{k^\gamma k_0^{1-\gamma}}+\frac{c_*}{k^\gamma k_0^{2-\gamma}\epsilon_1^2}$$

The Lemma follows by substituting the relation between k_0 and ϵ_1 and noting that the latter terms are bounded by a multiple of the first.

Proof of Lemma 13.3 By Lemma 13.1, and using the relation between k_0 and ϵ_1, it follows that

$$|h_k'(\epsilon_1)| \leqslant \frac{C_4C_3^{1+\gamma}}{k^\gamma\epsilon_1^{1+\gamma}}$$

for $k \geqslant C_3{\epsilon_1}^{-1}=k_0$. Using $h_k(1)=1$, it follows that

$$|h_k(r)-1| \leqslant \int_r^1 |h_k'(r')|dr' \leqslant \frac{C_4C_3^{1+\gamma}}{\gamma(kr)^\gamma}$$

It follows that when

$$\alpha kr \geqslant \left(\frac{C_4C_3^{1+\gamma}\alpha^\gamma}{\gamma\epsilon}\right)^{1/\gamma}=L_\epsilon \quad \text{then} \quad |h_k(r)-1|\leqslant\epsilon$$

Proof of Lemma 13.5: For $\zeta \in [0,L_\epsilon]$, using the *a priori* boundedness of h_k in k and bounds on h_k' from Lemma 13.19, we note that both $\tilde{h}_k(\zeta):=h_k(r(\zeta))$ and its derivative $(\tilde{h}_k)_\zeta$ are bounded independently of k. Hence the sequence $\{\tilde{h}_k\}_{k\geqslant 2}$ is bounded and equicontinuous. Therefore, by Ascoli-Arzelà's theorem, there exists a subsequence $\tilde{h}_{k_j}(\zeta)$ that converges to a continuous function $\tilde{h}$. The first part of the Lemma is complete.

We know from Lemma 13.3 that

$$|\tilde{h}_k(\zeta) - 1| \leqslant \epsilon \text{ for } \zeta \in [L_\epsilon, \alpha k] \text{ for sufficiently large } k \tag{13.79}$$

Now, let $\tilde{h}_{k,j}$ be a subsequence that converges to $\tilde{h}$ for $\zeta \in [0, L_\epsilon]$. Let ζ_m and ζ_M denote values of $\zeta \in [0, L_\epsilon]$ where the limiting function $\tilde{h}$ attains a minimum m and a maximum M respectively within this interval. By continuity and applying (13.79) at $\zeta = L_\epsilon$, it is clear that $M \geqslant 1 - \epsilon$ and $m \leqslant 1 + \epsilon$. We will assume $M > m$, as otherwise there is nothing to prove. Since $\tilde{h}$ is continuous, there exists an interval $[a, b] \subset [0, L_\epsilon]$ containing ζ_m so that

$$m \leqslant \tilde{h}(\eta) < \frac{1}{2}(M + m) < M \text{ for } \eta \in [a, b] \tag{13.80}$$

It is convenient to break up the operator $\mathcal{A}_k^0$ (see (13.24)), for $\hat{L} > L_\epsilon$ independent of k, as follows:

$$\begin{aligned}\left[\mathcal{A}_k^0 f\right](\zeta) &= \left(1 - \frac{1}{2k}\right) \\ &\times \left[\int_\zeta^{\hat{L}} + \int_{\hat{L}}^{k\alpha\epsilon_1}\right] e^{-Q(\eta)+Q(\zeta)} \left(1 + \frac{a_1}{k}\right) \frac{H(\eta(1-k^{-1})}{H(\zeta)} \mathcal{G}(\zeta, \eta) f(\eta(1-k^{-1}))d\eta \\ &=: [\mathcal{A}_k^{00} f](\zeta) + [\mathcal{A}_k^{01} f](\zeta)\end{aligned} \tag{13.81}$$

We will denote by K the kernel of $\mathcal{A}_k^{00}$ (or $\mathcal{A}_k^{01}$),

$$[\mathcal{A}_k^{00} f](\zeta) = \int_\zeta^{\hat{L}} K(\zeta, \eta) f(\eta) d\eta \tag{13.82}$$

For any fixed ζ and η,

$$\lim_{k\to\infty} K(\zeta, \eta) = K_0(\zeta, \eta) = e^{-\eta+\zeta} \frac{H_0(\eta)}{H_0(\zeta)} \mathcal{G}_0(\zeta, \eta) \tag{13.83}$$

We note that $\eta \geqslant \zeta$ on our interval and then $\mathcal{G}_0 \geqslant 0$ (see (13.38)); $\mathcal{G}_0$ can vanish only if $\eta = \zeta$. Furthermore, by (13.80) we have $\zeta_M \notin [a, b]$. We can then define

$$J = \frac{3M}{(b-a)K_m}, \text{ where } K_m = \min_{\eta\in[a,b]} K_0(\zeta_M, \eta) > 0$$

Note that $Q(\eta) \sim \eta$ for large k and aside from the exponential term, the integrand in (13.81) is algebraically bounded. We can thus choose $\hat{L} > L_\epsilon$ large enough independently of k, so that

$$|[\mathcal{A}_k^{01} f](\zeta)| \leqslant \frac{\epsilon}{J} \|f\|_{\infty, [L_\epsilon, k\alpha\epsilon_1]} \tag{13.84}$$

We take, if necessary, a subsequence of $\tilde{h}_{k_j}$ that converges for $\zeta \in [0, \hat{L}]$ (applying Ascoli-Arzelà); for simplicity, we will use the same notation $h_{k,j}$ for the subsequence. It is clear that this subsequence will converge to $\tilde{h}(\zeta)$ for $\zeta \in [0, L_\epsilon]$. Therefore, we can use the same notation for $\tilde{h}$ for limit function in the extended $[0, \hat{L}]$ interval. We note that (13.79) implies

$$|\tilde{h}(\zeta) - 1| \leqslant \epsilon \text{ for } \zeta \in [L_\epsilon, \hat{L}] \tag{13.85}$$

We denote the maximum and minimum values of $\tilde{h}$ in this enlarged interval $[0, \hat{L}]$ by $\tilde{M}$ and $\tilde{m}$ and the corresponding maximum and minimum points by $\zeta_{\tilde{M}}$ and $\zeta_{\tilde{m}}$. If both $\zeta_{\tilde{M}}$, $\zeta_{\tilde{m}} \in [L_\epsilon, \hat{L}]$, there is nothing to prove since (13.85) implies $|\tilde{M} - \tilde{m}| \leqslant 2\epsilon$ and hence

$$|\tilde{h}(\zeta) - 1| \leqslant 2\epsilon \text{ for } \zeta \in [0, \hat{L}]$$

So, we will assume that either

(1) Case (i): $\zeta_{\tilde{M}} \in [0, L_\epsilon]$, in which case we may take $\zeta_{\tilde{M}} = \zeta_M$ and $\tilde{M} = M$, or
(2) Case (ii): $\zeta_{\tilde{m}} \in [0, L_\epsilon]$, in which case $\tilde{m} = m$ and we may take $\zeta_{\tilde{m}} = \zeta_m$.

We consider Case (i) first. For sufficiently large k_j

$$[\mathcal{A}^{00}_{k,j} \tilde{h}(\zeta_M)] = \int_{\eta \in [0,\hat{L}]-[a,b]} K(\zeta_M, \eta) \tilde{h}(\eta) d\eta + \int_a^b K(\zeta, \eta) \tilde{h}(\eta) d\eta$$
$$\leqslant M \int_{\eta \in [0,\hat{L}]-[a,b]} K(\zeta_M, \eta) d\eta + \frac{1}{2}(M+m) \int_a^b K(\zeta_M, \eta) d\eta = M \int_0^{\hat{L}}$$
$$\times K(\zeta_M, \eta) d\eta - \frac{1}{2}(M-m) \int_a^b K(\zeta_M, \eta) d\eta \leqslant M \mathcal{A}^{00}_{k_j}[1](\zeta_M) - \frac{(b-a)}{3}(M-m) K_m$$

Since (see (13.24) and (13.81))

$$\mathcal{A}_{k_j}[1] = \mathcal{A}^{00}_{k_j}[1] + \mathcal{A}^{01}_{k_j}[1] + \mathcal{A}^{1}_{k_j}[1]$$

Lemmas 13.15, 13.12 and (13.84) imply that for any $\epsilon_2 > 0$ and any correspondingly large k_j we have

$$[\mathcal{A}^{00}_{k_j}[1]](\zeta_M) \leqslant 1 + \frac{\epsilon}{J} + \epsilon_2$$

Hence, for sufficiently large k_j

$$[\mathcal{A}^{00}_{k_j} \tilde{h}](\zeta_M) \leqslant M \left(1 + \frac{\epsilon}{J} + \epsilon_2\right) - \frac{K_m}{3}(M-m)(b-a) \tag{13.86}$$

Now, there exists N so that if $j \geqslant N$,

$$\|\tilde{h}_{k_j} - \tilde{h}\|_{\infty, [0,\hat{L}]} < \epsilon_2$$

and

$$\Lambda_j = \mathcal{A}_{k_{j+1}} ... \mathcal{A}_{k_j+1}$$

satisfies

$$\|\Lambda_j - I\|_\infty \leqslant \epsilon_2$$

while

$$r_{j+1} := B_{k_{j+1}} + \sum_{m=1}^{k_{j+1}-k_j-1} \prod_{l=1}^{m} \mathcal{A}_{k_{j+1}-l+1} B_{k_{j+1}-m}$$

satisfies the estimate

$$|r_{j+1}| < \epsilon_2$$

Therefore, from the relation

$$\tilde{h}_{k_{j+1}} = \Lambda_j \mathcal{A}_{k_j} \tilde{h}_{k_j} + r_{j+1}$$

it follows that

$$\tilde{h}_{k_{j+1}}(\zeta_M) \geqslant \tilde{h}(\zeta_M) - \epsilon_2 = M - \epsilon_2$$

On the other hand, at $\zeta = \zeta_M$

$$\Lambda_j \mathcal{A}_{k_j} \tilde{h}_{k_j} + r_{j+1} \leqslant (1+\epsilon_2)\left[M(1 + \frac{\epsilon}{J} + \epsilon_2) + \epsilon_2 - \frac{K_m}{3}(M-m)(b-a)\right] + \epsilon_2$$

Thus,

$$M-\epsilon_2 \leqslant (1+\epsilon_2)\left[M(1+\frac{\epsilon}{J}+\epsilon_2)+\epsilon_2-\frac{K_m}{3}(M-m)(b-a)\right]+\epsilon_2$$

This is true for any ϵ_2, hence as $\epsilon_2 \downarrow 0$. Thus,

$$M \leqslant \left[M\left(1+\frac{\epsilon}{J}\right)-\frac{K_m}{3}(M-m)(b-a)\right]$$

However, from the definition of J, this implies $M-m \leqslant 2\epsilon$ contradicting the assumption. Case (ii) where $\tilde{\zeta}_m=\zeta_m \in [0,L_\epsilon]$ is similarly proved, applying the argument to $-\tilde{h}$, which has a maximum at ζ_m. Thus $|\tilde{h}(\zeta)-1| \leqslant 2\epsilon$ for $\zeta \in [0,L_\epsilon]$.

Combining the above result with Lemma (13.3), we obtain that there exists a subsequence $\left\{h_{k_j}\right\}_{j=1}^{\infty}$ so that for any $r \in [0,1]$,

$$\lim_{j\to\infty} |h_{k,j}(r)-1| \leqslant 2\epsilon$$

Since this holds for any $\epsilon > 0$, it follows that

$$\lim_{j\to\infty} h_{k,j}(r)=1$$

Therefore, the limiting function $\tilde{h}$ equals one for any $\zeta \in [0,\alpha k]$, *i.e.* for $r \in [0,1]$.

14. Appendix

14.1. **Calculation of j_k.** The remainder $j_k=\mathfrak{s}[\mathcal{L}_k m_k-\Omega m_{k-1}]/m_k$ in case ((**i**)) is

$$\begin{aligned}
(14.1)\quad j_k = k^2\alpha^2\mathfrak{s}\Bigg[&\frac{4\Omega}{\alpha^2\mathfrak{s}^2}\Big(1-\frac{H(\alpha k)H(\zeta-\zeta/k)}{H(\alpha(k-1))H(\zeta)}\Big)+\frac{H''(\zeta)}{H(\zeta)}-\frac{l(l+1)}{\zeta^2}\\
&-2\frac{H'(\zeta)}{H(\zeta)}\Bigg]+k\alpha^2\mathfrak{s}\Bigg\{-\frac{2\Omega}{\alpha^2\mathfrak{s}^2}\Big(1-\frac{H(\alpha k)H(\zeta-\zeta/k))}{H(\alpha(k-1))H(\zeta)}\Big)\\
&+\frac{b}{\alpha\zeta}+\left[\frac{\omega\mathfrak{s}}{2\sqrt{\Omega}\alpha}-\frac{\Omega'}{2\alpha\Omega}+k\Big(2-\frac{4\sqrt{\Omega}}{\alpha\mathfrak{s}}\Big)\right]\frac{H'(\zeta)}{H(\zeta)}\Bigg\}\\
&+\frac{5\mathfrak{s}\Omega'^2}{16\Omega^2}-\frac{\omega\mathfrak{s}^2\Omega'}{4\Omega^{3/2}}-\frac{\mathfrak{s}\Omega''}{4\Omega}-\frac{\omega\mathfrak{s}}{4}+\frac{\omega^2\mathfrak{s}^3}{16\Omega}\\
&-(\omega n_0+\sigma)\mathfrak{s}-\frac{\Omega\mathfrak{s}^3H(\zeta+\zeta/k))H(\alpha k)}{(2k+1)(2k+2)H(\zeta)H(\alpha(k+1))}
\end{aligned}$$

In case ($\boldsymbol{\beta}$) we obtain

$$(14.2)\quad j_k = k^2\alpha^2\mathfrak{s}\Big[\frac{4\Omega}{\alpha^2\mathfrak{s}^2}\Big(1-\frac{H(\alpha k)H(\zeta-\zeta/k)}{H(\alpha(k-1))H(\zeta)}\Big)+\frac{H''(\zeta)}{H(\zeta)}-\frac{l(l+1)}{\zeta^2} - 2\frac{H'(\zeta)}{H(\zeta)}\Big] + k\alpha^2\mathfrak{s}\Bigg\{-\frac{H'(\zeta)}{H(\zeta)}+\frac{2\Omega}{\alpha^2\mathfrak{s}^2}\Big(1-\frac{H(\alpha k)H(\zeta-\zeta/k)}{H(\alpha(k-1))H(\zeta)}\Big) + \frac{b}{\alpha\zeta}+\left[\frac{\omega\mathfrak{s}}{2\sqrt{\Omega}\alpha}-\frac{\Omega'}{2\alpha\Omega}+k\left(2-\frac{4\sqrt{\Omega}}{\alpha\mathfrak{s}}\right)\right]\frac{H'(\zeta)}{H(\zeta)}\Bigg\} + \frac{5\mathfrak{s}\Omega'^2}{16\Omega^2}-\frac{\omega\mathfrak{s}^2\Omega'}{4\Omega^{3/2}}-\frac{\mathfrak{s}\Omega''}{4\Omega}-\frac{3\omega\mathfrak{s}}{4}+\frac{\omega^2\mathfrak{s}^3}{16\Omega} - (\omega n_0+\sigma)\mathfrak{s}-\frac{\Omega\mathfrak{s}^3H(\zeta(1+k^{-1}))H(\alpha k)}{(2k+2)(2k+3)H(\zeta)H(\alpha(k+1))}$$

Examination of the differential equation for $H(\zeta)$ shows that for $\zeta = k\alpha r = O(1)$, the $O(k^2)$ and $O(k)$ terms cancel out and we are left an $O(1)$ contribution. When $\zeta = k\alpha r \gg 1$, from the asymptotics of $H(\zeta)$ for large ζ, it follows from close examination of the terms that $j_k(r) = O(1/\zeta)$ and therefore $j_k'(r) = O((kr^2)^{-1})$.

14.2. **Heuristics: how to obtain the asymptotics of g_k formally.** We take $V(t,x) = \Omega(r)\cos(\omega t)$. Recall from Lemma 13.6 that for case (**i**),

$$\partial_{\mathfrak{s}}^j|_{\mathfrak{s}=0}g_{n_0-k} = i^k\delta_{j,2k} \text{ for } j\leqslant 2k$$

This suggests that at least locally, near $\mathfrak{s}=0$, *i.e.* $r=1$, $g_{n_0-k}\sim i^k\mathfrak{s}^{2k}/(2k)!$. To obtain the behavior for more general $\mathfrak{s}$, we seek a generalization of the above form

$$g_{n_0-k}(r)\sim i^k\frac{\mathfrak{s}(r)^{2k}}{(2k)!}f(r)$$

calculate the remainder R_k defined as

$$R_k = [\mathcal{L}_k g_{n_0-k} - i\Omega(r)(g_{n_0-k+1}+g_{n_0-k-1})]/g_{n_0-k}$$

and consider the asymptotics for $k\gg 1$. We find that for $r=O(1)$, the $O(k^2)$ terms cancel out; however, the $O(k)$ term involves a first order linear differential differential equation for $f(r)$. Requiring that the $O(k)$ contribution vanishes, we obtain (aside from a multiplicative constant)

$$f(r) = \Omega^{-1/4}(r)\Omega^{1/4}(1)\exp\left[\frac14\int_1^r ds\frac{\omega\mathfrak{s}(s)}{\sqrt{\Omega(s)}}\right]$$

Therefore, at a formal level we conclude that as $k\to\infty$ for $r=O(1)$

$$g_{n_0-k}(r)\sim i^k\frac{\mathfrak{s}(r)^{2k}}{(2k)!}\Omega^{-1/4}(r)\Omega^{1/4}(1)\exp\left[\frac14\int_1^r ds\frac{\omega\mathfrak{s}(s)}{\sqrt{\Omega(s)}}\right]$$

The multiplicative constant is fixed to ensure the normalization condition $g_{n_0-k}(1) = 1$. However, the above formal expression cannot be valid uniformly as $r\to 0$ since R_k, defined above, contains an $O(r^{-2})$ term originating from the $r^{-2}l(l+1)$ term in $\mathcal{L}_k$. When $r = O(k^{-1})$, we must look for a correction factor tending to 1 as $kr \to \infty$. Therefore, this suggests we seek

$$g_{n_0-k}(r) \sim i^k \frac{\mathfrak{s}(r)^{2k}}{(2k)!} \Omega^{-1/4}(r) \Omega^{1/4}(1) \exp\left[\frac{1}{4}\int_1^r ds \frac{\omega \mathfrak{s}(s)}{\sqrt{\Omega(s)}}\right] \frac{H(\alpha k r)}{H(\alpha k)}$$

for some suitable function H (α is some constant to be determined later). The remainder R_k is recalculated in terms of ζ and we demand that the $O(k^2)$ and $O(k)$ terms vanish as $k \to \infty$. This can be done by requiring $H(\zeta)$ to satisfy (9.6) for $\tau = 0$, where $\alpha = 2\sqrt{\Omega(0)}/\mathfrak{s}(0)$. The remainder j_k (defined as $R_k/(f(r)H(\zeta))$, rewritten in terms of ζ is shown explicitly in (14.1) and it is seen that the $O(k^2)$ and $O(k)$ terms cancel when $\zeta = O(1)$ provided (9.6) is satisfied with $\tau = 0$. Similar considerations leads to the formal asymptotics for case ($\boldsymbol{\beta}$). Thus

$$g_{n_0-k} \sim i^k m_k$$

and it is indeed the asymptotic behavior that has been proved in earlier sections.

14.3. **Generalization.** In fact, the same asymptotic arguments hold more generally if

$$V(t,x) = \sum_{j=-M}^{M} e^{ij\omega t} \Omega_j(r)$$

with $\Omega_j(r)$ satisfying the conditions we used for Ω. We substitute for $r = O(1)$

$$g_{n_0-k}(r) = \frac{c_*}{\Gamma(2k/M+1)} \exp\left[k \log f_0(r) + \sum_{j=1}^{M} k^{1-j/M} f_j(r)\right]$$

and calculate the error term R_k as before. By demanding $O(k^{2-2j/M})$ vanish for $j = 0, .., M$, we obtain $(M+1)$ first order differential equations for f_j. To leading order, we find

$$f_0(r) = \left[\int_r^1 \sqrt{\Omega_{-M}(s)} ds\right]^{2/M}$$

The expressions for $f_j(r)$ for $j \geqslant 1$ are more complicated and involve some arbitrary constants that have to be determined from information for small k at $r = 1$. Again because of the presence of $r^{-2}l(l+1)$ in $\mathcal{L}_k$, the remainder is $O(r^{-2})$, which is $O(k^2)$ when $r = O(k^{-1})$. Once again we correct the above expression

$$g_{n_0-k}(r) \sim c_* \exp\left[k \log f_0(r) + \sum_{j=1}^{M} k^{1-j/M} f_j(r)\right] \frac{H(\alpha k r)}{\Gamma(2k/M+1)} \tag{14.3}$$

Then, to leading order if $\zeta = O(1)$, we find $H(\zeta) \sim H_0(\zeta)$ where

$$H_0'' - 2H_0' - \frac{l(l+1)}{\zeta^2} H_0 = 0$$

provided we choose $\alpha = 2\sqrt{\Omega_{-M}(0)}/\mathfrak{s}(0)$ where

$$\mathfrak{s}(r) = \int_r^1 \sqrt{\Omega_{-M}(s)}$$

As for $M=1$, we have to require $H_0(\zeta)\sim 1$ as $\zeta\to\infty$. This leads to

$$H_0(\zeta)=\sqrt{\frac{2}{\pi}}e^{\zeta}\zeta^{1/2}K_{l+\frac{1}{2}}(\zeta)$$

For nonzero g_{n_0-k}, the constant multiple in (14.3) is expected to be nonzero. On the other hand, the asymptotic behavior as $\zeta\downarrow 0$, $H_0(\zeta)\sim c_*\zeta^{-l}$ implies that g_{n_0-k}/r has unacceptable behavior as $r\downarrow 0$, implying the only solution to the homogeneous problem should be $g_n=0$.

14.4. **Further remarks on the asymptotics.**

Remark 14.1. Note that a weaker statement than Theorem 9.1 suffices to complete the proof of Theorem 2.2. For instance, it suffices to show that for sufficiently large j, $|R_{k,j}|<1$, where

$$r^{l+1}v_{n_0-k_j}(r)=i^{k_j}r^l m_{k_j}(r)[1+R_{k_j}(r)]$$

Remark 14.2. Stronger results than those in Lemma 13.5 hold. Noting that for any integer $q\geqslant 0$ we have

$$\|\mathcal{A}_{k_j+q}...\mathcal{A}_{k_j+2}\mathcal{A}_{k_j+1}\mathcal{A}_{k_j}[\tilde{h}-1]\|_\infty\leqslant\prod_{q'=0}^{\infty}\left(1+\frac{c_*}{(k_j+q')^2}\right)\|\tilde{h}-1\|_\infty$$

while

$$\mathcal{A}_{k_j+q}...\mathcal{A}_{k_j+2}\mathcal{A}_{k_j+1}\mathcal{A}_{k_j}[1]=1+O(k_j^{-1})$$

and the fact that $\|\mathcal{B}_{k_j+q}\tilde{h}\|_\infty\leqslant c_*k_j^{-2}$, it follows that the sequence $\tilde{h}_k$, satisfying

$$\tilde{h}_k=\mathcal{A}_k\tilde{h}_{k-1}+\mathcal{B}_k\tilde{h}_{k+1},$$

has the property $\lim_{k\to\infty}\tilde{h}_k=1$. Indeed, this is in accordance to the the heuristic arguments presented in §14.2. While these results completely justify the formal asymptotics, they are not needed in Theorem 9.1 and their detailed proofs are omitted.

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