ON THE COMPLETE IONIZATION OF A PERIODICALLY PERTURBED QUANTUM SYSTEM

Dedicated to the memory of Jürgen Moser

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Abstract. We analyze the time evolution of a one-dimensional quantum system with zero range potential under time periodic parametric perturbation of arbitrary strength and frequency. We show that the projection of the wave function on the bound state vanishes, i.e. the system gets fully ionized, as time grows indefinitely.

1. Introduction and results

The ionization of atoms subjected to external time dependent perturbations is an issue of central importance in quantum mechanics which has attracted substantial theoretical and experimental interest [1], [2]. There exists by now a variety of theoretical methods, and a vast amount of literature, devoted to the subject. Beyond the celebrated Fermi's golden rule, approaches include higher order perturbation theory, semi-classical phase-space analysis, Floquet theory, complex dilation, some exact results for small fields and bounds for large fields and numerical integration of the time dependent Schrödinger equation [2]-[14]. Nevertheless there is apparently no complete analysis of the ionization of any periodically perturbed model with no restrictions on the amplitudes and frequencies of the perturbing field. This is not so surprising considering the very complex behavior we find in even the most elementary of such systems.

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To appear in Proceedings of the CRM meeting "Nonlinear Analysis and Renormalization Group"

In the present paper we show rigorously the full ionization, in all ranges of amplitudes and frequencies, of one of the simplest models with spatial structure which, with a different perturbing potential, is however frequently used as a model system [5], [6], [13]. As shown in the references, many features of systems with short range or rapidly decaying potentials can be expected to be well described, at least qualitatively, by delta function potentials.

The unperturbed Hamiltonian we consider is

$$\mathcal{H}_0 = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - g\,\delta(x), \quad g > 0, \quad -\infty < x < \infty. \tag{1.1}$$

$\mathcal{H}_0$ has a single bound state $u_b(x) = \sqrt{p_0}e^{-p_0|x|},\ p_0 = \frac{m}{\hbar^2}g$ with energy $-\hbar\omega_0 = -\hbar^2p_0^2/2m$ and a continuous uniform spectrum on the positive real line, with generalized eigenfunctions

$$u(k,x) = \frac{1}{\sqrt{2\pi}}\left(e^{ikx} - \frac{p_0}{p_0 + i|k|}e^{i|kx|}\right), \quad -\infty < k < \infty$$

and energies $\hbar^2k^2/2m$.

Beginning at $t = 0$, we apply a parametric perturbing potential, i.e. for $t \geq 0$ we have

$$\mathcal{H}(t) = \mathcal{H}_0 - g\,\eta(t)\delta(x) \tag{1.2}$$

and solve the time dependent Schrödinger equation for $\psi(x,t)$,

$$\begin{aligned}\psi(x,t) &= \theta(t)u_b(x)e^{i\omega_0 t}\\ &\quad + \int_{-\infty}^{\infty}\Theta(k,t)u(k,x)e^{-i\frac{\hbar k^2}{2m}t}dk \quad (t \geq 0)\end{aligned} \tag{1.3}$$

with initial values $\theta(0) = 1,\ \Theta(k,0) = 0$. This gives the survival probability $|\theta(t)|^2$, as well as the fraction of ejected electrons $|\Theta(k,t)|^2dk$ with (quasi-) momentum in the interval dk.

This problem can be reduced to the solution of an integral equation [15]. Setting

$$\theta(t) = 1 + 2i\int_0^t Y(s)ds \tag{1.4}$$

$$\Theta(k,t) = 2|k|/\left[\sqrt{2\pi}(1 - i|k|)\right]\int_0^t Y(s)e^{i(1+k^2)s}ds \tag{1.5}$$

$Y(t)$ satisfies the integral equation

(1.6)

$$Y(t) = \eta(t)\left\{1 + \int_0^t [2i + M(t-t')]Y(t')dt'\right\} = \eta(t)\Big(1 + (2i+M) * Y\Big)$$

where $\hbar, 2m$ and $\frac{g}{2}$ have been set equal to 1 (implying $p_0 = 1$, $\omega_0 = 1$),

$$M(s) = \frac{2i}{\pi}\int_0^\infty \frac{u^2 e^{-is(1+u^2)}}{1+u^2}du = \frac{1+i}{2\sqrt{2}\pi}\int_s^\infty \frac{e^{-iu}}{u^{3/2}}du$$

and

$$f * g = \int_0^t f(s)g(t-s)ds$$

Theorem 1. *When $\eta(t) = r\sin\omega t$ the survival probability $|\theta(t)|^2$ tends to zero as $t \to \infty$, for all $\omega > 0$ and $r \neq 0$.*

Note: For definiteness we assume in the following that $r > 0$.
The method of proof relies on the properties of the Laplace transform of Y, $y(p) = \mathcal{L}Y(p) = \int_0^\infty e^{-pt'}Y(t')dt'$ (note that $y(p) = \frac{i}{2}(1 - p\mathcal{L}\theta)$). In particular we need to show that $y(p)$ is bounded in the closed right half of the complex $p-$ plane. Before the proof we describe briefly some additional results on this model system, cf. [16].

1.1. **Further results not proven in the present paper.** (**1**) Theorem 1 generalizes to the case when $\eta(t)$ is a trigonometric polynomial:

$$\eta(t) = \sum_{j=1}^{K}[A_j \sin(j\omega t) + B_j \cos(j\omega t)], \tag{1.7}$$

where we assume $|A_K| + |B_K| \neq 0$.

(**2**) The detailed behavior of the system as a function of t, ω, and r is obtained from the singularities of $y(p)$ in the complex p-plane. We summarize them *for small* r; below $\frac{1}{2} < \delta < 1$.

At $p = \{in\omega - i : n \in \mathbb{Z}\}$, y has square root branch points and y is analytic in the right half plane and also in an open neighborhood $\mathcal{N}$ of the imaginary axis with cuts through the branch points. As $|\Im(p)| \to \infty$ in $\mathcal{N}$ we have $|y(p)| = O(r\omega|p|^{-2})$. If $|\omega - \frac{1}{n}| > \text{const}_n O(r^{2-\delta})$, $n \in \mathbb{Z}^+$, then for small r the function y has a unique pole $p_m = p_0 + im\omega$ in each of the strips $-m\omega > \Im(p) + 1 \pm O(r^{2-\delta}) > -m\omega - \omega$, $m \in \mathbb{Z}$. $\Re(p_m)$ is strictly independent of m and gives the exponential decay of θ.

The analytic structure of y is indicated in Figure 1 where the dotted lines represent (the square root) branch cuts and the dark circles are simple poles. The function Y is the inverse Laplace transform of y

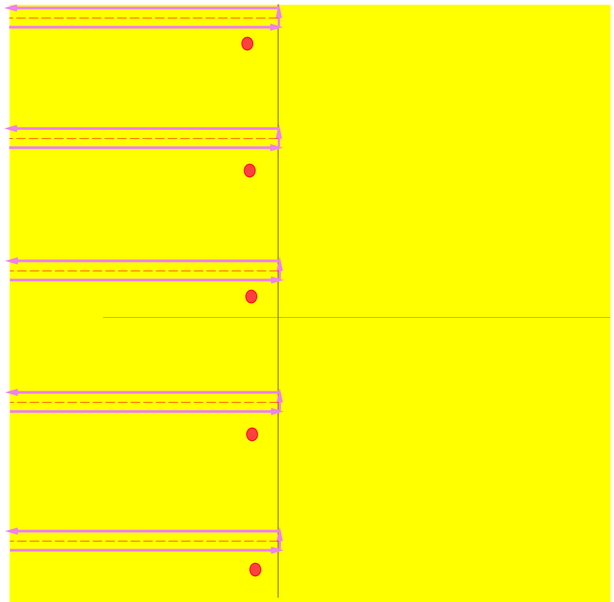

FIGURE 1. Singularities of y and relevant inverse Laplace contours.

$$Y(t) = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{pt} y(p)\, dp \tag{1.8}$$

where the contour of integration $\mathcal{C}$ can be initially taken to be the imaginary axis $i\mathbb{R}$, since y is continuous there and decays like p^{-2} for large p.

We then show that $\mathcal{C}$ can be pushed through the poles, collecting the appropriate residues, and along the branch cuts as shown in Figure 1. The residue at the pole p_m is $Const_m e^{(p_0+im\omega)t}$ while the (rapidly convergent) integral along the branch cut m is (as seen by standard Laplace integral techniques), a function whose large t behavior is $Const_m e^{im\omega t} t^{-3/2}$ (the t power is due to the $(p-im\omega)^{1/2}$ behavior of y at the branch points). Making this analysis carefully yields:

$$\theta(t) = e^{-\gamma(r;\omega)t} F_\omega(t) + \sum_{m=-\infty}^{\infty} e^{(mi\omega - i)t} h_m(t) \tag{1.9}$$

where F_ω is periodic of period $2\pi\omega^{-1}$ and

$$h_m(t) \sim \sum_{j=0}^{\infty} c_{m,j} t^{-3/2-j} \quad \text{as } t \to \infty,\ \arg(t) \in \Big(-\frac{\pi}{2} - \epsilon, \frac{\pi}{2} + \epsilon\Big)$$

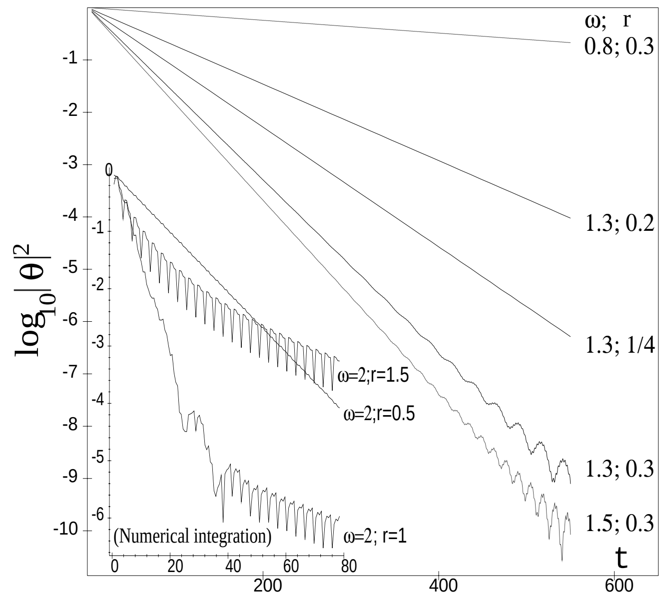

FIGURE 2. Plot of $\log_{10}|\theta(t)|^2$ vs. time in units of ω_0^{-1} for several values of ω and r. The main graph was calculated from (1.9) and the inset used numerical integration of (1.6).

Not too close to resonances, i.e. when $|\omega - n^{-1}| > O(r^{2-\delta})$, for all integer n, $|F_\omega(t)| = 1 \pm O(r^2)$ and its Fourier coefficients decay faster than $r^{|2m|}|m|^{-|m|/2}$. Also, the sum in (1.9) does not exceed $O(r^2 t^{-3/2})$ for large t, and the h_m decrease with m faster than $r^{|m|}$.

(**3**) By (1.9), for times of order $1/\Gamma$ where $\Gamma = 2\Re(\gamma)$, the survival probability for ω not close to a resonance decays as $\exp(-\Gamma t)$. This is illustrated in Figure 2 where it is seen that for $r \lesssim 1/2$ the exponential decay holds up to times at which the survival probability is extremely small. It follows from our analysis that for small r the final asymptotic behavior for $t \to \infty$, is $|\theta(t)|^2 = O(t^{-3})$ with many oscillations as described by (1.9). Note the slow decay for $\omega = .8$, when ionization requires the absorption of two photons.

(**4**) When r is larger the polynomial-oscillatory behavior starts sooner. Since the amplitude of the late asymptotic terms is $O(r^2)$ for small r, increased r yields higher late time survival probability. This phenomenon, sometimes referred to as atomic stabilization [12], [13], can be associated with the perturbation-induced probability of back-transitions to the well.

(**5**) Using the continued fraction representation (2.12) Γ can be calculated convergently for any ω and r.

The limiting behavior for small r of the exponent Γ is described as follows. Let n be the integer part of $\omega^{-1}+1$ and assume $\omega^{-1}\notin\mathbb{N}$. Then we have, for $T>0$ ($t=r^{-2n}T$),

$$\hat{\Gamma}=-T^{-1}\lim_{r\to 0}\ln\left|\theta(r^{-2n}T)\right|^2=\frac{2^{-2n+2}\sqrt{n\omega-1}}{n\omega\prod_{m<n}(1-\sqrt{1-m\omega})^2} \tag{1.10}$$

(**6**) The behavior of Γ is different at the resonances $\omega^{-1}\in\mathbb{N}$. For instance, whereas if ω is not close to 1, the scaling of Γ implied by (1.10) is r^2 when $\omega>1$ and r^4 when $\frac{1}{2}<\omega<1$, by taking $\omega-1=r^2/\sqrt{2}$ we find

$$-T^{-1}\lim_{\substack{r\to 0\\ \omega=1+r^2/\sqrt{2}}}\ln\left|\theta(r^{-3}T)\right|^2=\frac{2^{1/4}}{8}-\frac{2^{3/4}}{16}$$

2. Proof of Theorem 1

Lemma 2. *(i) $\mathcal{L}Y$ exists and is analytic in the right half plane $\mathbb{H}=\{p:\Re(p)>0\}$. Furthermore, $y(p)\to 0$ as $\Im(p)\to\pm\infty$ in $\mathbb{H}$.*
(ii) The function $y(p)$ satisfies (and is determined by) the functional equation

$$y=r\Big(T^{-}-T^{+}\Big)\Big(h_1+h_2y\Big) \tag{2.1}$$

with

$$\Big(T^{\pm}f\Big)(p)=f(p\pm i\omega),\ \ h_1(p)=-\frac{i}{2p}\ \ \text{and}\ \ h_2(p)=\frac{1}{2p}\Big(1+\sqrt{1-ip}\Big)$$

and by the boundary condition $y(p)\to 0$ as $\Im(p)\to\pm\infty$ in $\mathbb{H}$.

The branch of the square root is such that for $p\in\mathbb{H}$, the real part of $\sqrt{1-ip}$ is nonnegative and the imaginary part nonpositive.

Proof. (i) The time evolution of ψ is unitary and thus $|\langle\psi\,|\,u_b\rangle|=|\theta(t)|\le 1$. The stated analyticity is an immediate consequence of the elementary properties of the Laplace transform [2]. The asymptotic behavior follows then from the Riemann-Lebesgue lemma.

[2]See also the appendix for a proof of analyticity for $\Re(p)>p_0$ (all that is required in the subsequent analysis), relying only on the properties of the convolution equation.

(ii) We have in $\mathbb{H}$,

$$\mathcal{L}M = \lim_{a\downarrow 0} \frac{2i}{\pi} \int_0^\infty \mathrm{d}x e^{-px} \int_0^\infty \frac{u^2 e^{-i(x-ia)(1+u^2)}}{1+u^2} \mathrm{d}u \tag{2.2}$$

$$= \frac{i}{\pi} \int_{-\infty}^\infty \frac{u^2}{(1+u^2)(p+i(1+u^2))} \mathrm{d}u \tag{2.3}$$

For $\Re(p) > 0$ we push the integration contour through the upper half plane. At the two poles in the upper half plane u^2+1 equals 0 and ip respectively, so that

$$\frac{i}{\pi} \int_{-\infty}^\infty \frac{u^2}{(1+u^2)(p+i(1+u^2))} \mathrm{d}u \tag{2.4}$$
$$= \frac{i}{\pi} \left(\frac{(-1)}{(2i)(p)} \oint \frac{ds}{s} + \frac{u_0^2}{(ip)(2iu_0)} \oint \frac{ds}{s} \right) = -\frac{i}{p} + \frac{u_0}{p}$$

where u_0 is the root of $p+i(1+u^2) = 0$ in the *upper* half plane. Thus

$$\mathcal{L}M = -\frac{i}{p} + \frac{i\sqrt{1-ip}}{p} \tag{2.5}$$

with the branch satisfying $\sqrt{1-ip} \to 1$ as $p \to 0$ in $\mathbb{H}$. As p varies in $\mathbb{H}$, $1-ip$ belongs to the lower half plane $-i\mathbb{H}$ and then $\sqrt{1-ip}$ varies in the fourth quadrant.

For $\Re(p) > 0, \omega > 0$ we have

$$\mathcal{L}\left(e^{\pm i\omega} M\right) = -\frac{i}{p \mp i\omega} + \frac{i\sqrt{1-ip \mp \omega}}{p \mp i\omega} \tag{2.6}$$
$$(\text{with} \quad \sqrt{1-ip-\omega} = -i\sqrt{\omega - 1 + ip} \;\; \text{if} \;\; \omega > 1)$$

and relation (2.1) follows. ☐

After the substitution $y(p) = 2(\sqrt{1-ip}-1)e^{-\frac{\pi p}{2\omega}} v(p)$ we get

$$v(p-i\omega) + v(p+i\omega) = \frac{2}{r}(\sqrt{1-ip}-1)v(p) + \frac{i\omega}{\omega^2+p^2} \tag{2.7}$$

Remark 3. *It is clear that the functional equation (2.7) only links the points on one dimensional lattice* $\{p + i\mathbb{Z}\omega\}$*. It is convenient to take* p_0 *such that* $p = p_0 + in\omega$ *with* $\Re(p_0) = \Re(p)$ *and*

$$\Im(p_0) \in [0, \omega) \tag{2.8}$$

and write $v(p) = v(p_0 + in\omega) = v_n$ which transforms (2.7) to a recurrence relation:

$$v_{n+1} + v_{n-1} = \frac{2}{r}(\sqrt{1 - ip_0 + n\omega} - 1)v_n + \frac{i\omega}{\omega^2 + (p_0 + in\omega)^2} \tag{2.9}$$

where v_n depends parametrically on p_0. It will be seen that the asymptotic conditions as well as analyticity in p_0 determine the solution of (2.9) uniquely.

Remark 4. *The approach is based on a discrete analog of the Wronskian technique. The regularity of the bounded solution of (2.9) will be a consequence of the absence of a bounded solution of the homogeneous equation*

$$v_{n+1} + v_{n-1} = \frac{2}{r}(\sqrt{1 - ip_0 + n\omega} - 1)v_n = D_n v_n \tag{2.10}$$

a problem which we analyze first.

Proposition 5. *For p_0 satisfying (2.8) and $\Re(p_0) \geq 0$ (actually for any $p_0 \in \overline{\mathbb{H}} = \mathbb{H} \cup i\mathbb{R}$) there is no nonzero solution of (2.10) such that $v \in l_2(\mathbb{Z})$.*

Proof. To get a contradiction, assume $v \not\equiv 0$ is an $l_2(\mathbb{Z})$ solution of (2.10). Multiplying (2.10) by $\overline{v_n}$, and summing with respect to n from $-\infty$ to $+\infty$ we get

$$\begin{aligned} \sum_{n=-\infty}^{\infty} v_{n+1}\overline{v}_n + \sum_{n=-\infty}^{\infty} v_{n-1}\overline{v}_n \\ = 2\sum_{n=-\infty}^{\infty} \Re(v_n\overline{v}_{n+1}) = \sum_{n=-\infty}^{\infty} \frac{2}{r}\Big(\sqrt{1 - ip_0 + n\omega} - 1\Big)|v|_n^2 \end{aligned} \tag{2.11}$$

For $p_0 \in \overline{\mathbb{H}}$ the imaginary part of $\sqrt{1 - ip_0 + n\omega}$ is nonpositive, by Lemma 2, and is strictly negative for $n < 0$ large enough. Thus if for some such n, v_n is nonzero then the last sum in (2.11) has a strictly negative imaginary part, which is impossible since the left side is real. If on the other hand v_n is zero when n is large negative, then solving (2.10) for v_{n+1} in terms of the v_n, v_{n-1} it would follow inductively that $v \equiv 0$, contradicting the assumption.

□

Lemma 6. *(i) There is, up to multiplicative constants, a unique pair of solutions v^+ and v^- of (2.10) such that $v_n^{\pm} \to 0$ as $n \to \pm\infty$ (respectively). These solutions are related to convergent continued fractions representations:*

$$v^{\pm}_{n\mp1}/v^{\pm}_n =: \frac{1}{\rho^{\pm}_n} = D_n - \cfrac{1}{D_{n\pm1} - \cfrac{1}{D_{n\pm2}}\cdots} \tag{2.12}$$

(ii) We have the following estimates

$$\frac{1}{\rho^{\pm}_n} = \frac{1}{\tilde{\rho}^{\pm}_n} + O(n^{-3/2}) \quad (n \to \pm\infty) \tag{2.13}$$

where

$$\begin{aligned} \frac{1}{\tilde{\rho}^{+}_n} &= \frac{2}{r}\sqrt{n\omega} - \frac{2}{r} - \frac{r^2 - 2 + 2ip_0}{2r\sqrt{n\omega}} - \frac{r}{2\omega n} \quad (n > 0) \\ \frac{1}{\tilde{\rho}^{-}_n} &= -\frac{2i}{r}\sqrt{|n|\omega} - \frac{2}{r} + \frac{(2 - r^2)i + 2p_0}{2r\sqrt{|n|\omega}} + \frac{r}{2\omega|n|} \quad (n < 0) \end{aligned} \tag{2.14}$$

Let $\tilde{v}^{\pm}_n$ be solutions of the one step recurrences $\tilde{v}^{\pm}_n = \tilde{v}^{\pm}_{n\mp1}\tilde{\rho}^{\pm}_n$. Then

$$\begin{aligned} \ln \tilde{v}^{+}_n = &-\frac{1}{2} n \ln n + n \ln\left(\frac{r}{2}\sqrt{\frac{e}{\omega}}\right) \\ &+ 2\sqrt{\frac{n}{\omega}} + \left(\frac{2ip_0 + r^2 + \omega}{4\omega}\right)\ln n + o(1) \quad (n \to \infty) \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} \ln(\tilde{v}^{-}_n) = &-\frac{1}{2}|n| \ln |n| + |n| \ln\left(\frac{r}{2}\sqrt{\frac{e}{\omega}}\right) + i\pi|n| - 2i\sqrt{|n|/\omega} \\ &+ \left(\frac{2ip_0 + r^2 + \omega}{4\omega}\right)\ln |n| + o(1) \quad (n \to -\infty) \end{aligned} \tag{2.16}$$

and, for some constants $K^{\pm}$,

$$\ln(v^{\pm}_n) = \ln(\tilde{v}^{\pm}_n) + K^{\pm} + o(1) \tag{2.17}$$

($v^{\pm}_n$ decay roughly as $1/\sqrt{|n|!}$ for $n \to \pm\infty$, respectively).
(iii) Two special solutions of (2.10), v^+ and v^-, are well defined by:

(2.18)
$$v_n^+ = \tilde{v}_n^+ \prod_{j\ge n+1} \frac{\tilde{\rho}_j^+}{\rho_j^+} \text{ for } n > N, \text{ and } v_n^- = \tilde{v}_n^- \prod_{j\le n-1} \frac{\tilde{\rho}_j^-}{\rho_j^-} \text{ for } n < -N$$

if N is sufficiently large (this amounts to making a convenient choice of the free multiplicative constant in (i)). These functions do not depend on N. v^+ and v^- are linearly independent for $p_0 \in \overline{\mathbb{H}}$: their discrete Wronskian, defined by $W(v^+, v^-)_n = v_n^+ v_{n+1}^- - v_n^- v_{n+1}^+$, satisfies

$$W(v^+, v^-) = const \neq 0 \tag{2.19}$$

As functions of parameters, $v^\pm$ and $W(v^+, v^-)$ are analytic in $p_0 \in \mathbb{H}$. If $\omega \notin \{0, n^{-1} : n \in \mathbb{N}\}$ then $v^\pm$ and $W(v^+, v^-)$ are analytic in some neighborhood of $p_0 = 0$ as well. For any $\omega > 0$, $v_n^\pm$ are Lipschitz continuous of exponent at least $1/2$ in p_0, for $p_0 \in \mathbb{R}$.

Proof. (i) We look at v^+, the case of v^- being similar. Dropping the $^+$ superscript we have from (2.10)

$$\rho_n = \frac{1}{D_n - \rho_{n+1}} \tag{2.20}$$

To find the analytic properties of the solution ρ_n it is convenient to regard (2.20) as a contractive equation in the space $\ell^\infty(S_N)$ of sequences $\{\rho_j\}_{j>N}$ in the norm $\|\rho\|_\infty = \sup_{j>N} |\rho_j|$. Let N be large. The map $J : S_N \mapsto S_N$ defined by

$$J(\rho)_n = \frac{1}{D_n - \rho_{n+1}} \tag{2.21}$$

depends analytically on $p_0 \in \mathbb{H}$ and is Lipschitz continuous of exponent at least $1/2$ if $\Re(p_0) \ge 0$. In addition, if $\|\rho_j\|_\infty \le 1$ we have for sufficiently large $N = N(p_0, \omega, r)$

$$\|J(\rho)\|_\infty \le \frac{1}{\frac{2}{r}\left(\sqrt{|N\omega| - 1 - |p_0|} - 1\right) - 1} < \frac{|r|}{|\omega|^{1/2}} \frac{1}{\sqrt{N}} \tag{2.22}$$

Similarly,

(2.23)
$$\|J(\rho)-J(\rho')\|_\infty \le \frac{\|\rho-\rho'\|_\infty}{\left[\frac{2}{r}\left(\sqrt{|N\omega|-1-|p_0|}-1\right)-1\right]^2} < \frac{|r|^2}{N|\omega|}\|\rho-\rho'\|_\infty$$

for sufficiently large N which shows that J is contractive in the unit ball in $\ell^\infty(S_N)$. Thus, equation (2.21) has a unique solution in S_N, which depends analytically on $p_0 \in \mathbb{H}$ and is Lipschitz continuous of exponent at least 1/2 if $\Re(p_0) \ge 0$. This also implies the convergence of (2.12).

Note that given $\mathcal{K}_1 \subset \overline{\mathbb{H}}$ and $\mathcal{K}_2 \subset \mathbb{R}^+$ both compact, N can be chosen the same for all $p_0 \in \mathcal{K}_1$ and $r \in \mathcal{K}_2$.

(ii) From (2.22) it is seen that $|\rho_j| = O(j^{-1/2})$ for large j. Thus, we may write, for large j,

$$\frac{1}{\rho_j^+} = D_j - \cfrac{1}{D_{j+1} - \cfrac{1}{D_{j+2}+O(j^{-1/2})}} \tag{2.24}$$

The estimates (2.13) now follow by a straightforward calculation. Since $\ln v_n^+ = \ln v_N^+ + \sum_{j=N+1}^n \ln \rho_j^+$, the estimates follow from (2.24) and the Euler-Maclaurin summation formula.

(iii) As before, we only need to look at v^+. We take two compact sets $\mathcal{K}_1$ and $\mathcal{K}_2$, and choose N as in the note at the end of the proof of (i). Taking the log in the definition (2.18), the infinite sums are absolutely convergent. By standard measure theory, v_n^+ has the same analyticity properties in the interior of $\mathcal{K}_1\times\mathcal{K}_2$ and Lipschitz continuity in $\mathcal{K}_1\times\mathcal{K}_2$ as those of ρ^+, when $n > N$. Now, (2.10) easily implies that the same is true for $n \le N$ as well.

If f_n and g_n are solutions of (2.10) then $(g_{n+1}+g_{n-1})f_n - (f_{n+1}+f_{n-1})g_n = 0$ and thus $W_n(f,g) = f_n g_{n+1} - g_n f_{n+1} = \mathit{const}$. Thus, if $W_n(f,g)=0$ for some n then $W_n \equiv 0$ and $f \equiv \mathit{const}\, g$. The smoothness properties follow from the proof of (iii).

□

Proposition 7. *There exists a unique solution of (2.7) which is bounded as $\Im(p) \to \pm\infty$ in $\mathbb{H}$. This solution is analytic in $p \in \mathbb{H}$, and $v(p) = O(p^{-2})$ as $\Im(p) \to \pm\infty, p \in \mathbb{H}$.*

Proof. By analyticity and continuity $W(v^+, v^-)$ does not vanish for any $p \in \mathbb{H}$ and $r > 0, \omega > 0$. By Lemma 11 the function v defined through $v(p_0 + in\omega) = f_n$, where

$$f_n := W(v^+, v^-)^{-1} \left(v_n^+ \sum_{l=-\infty}^{n-1} v_l^- H_l + v_n^- \sum_{l=n}^{\infty} v_l^+ H_l \right) \tag{2.25}$$

and

$$H_n = \frac{i\omega \exp\left(\frac{\pi p_n}{2\omega}\right)}{p_n^2 + \omega^2} \tag{2.26}$$

has the required properties. Since no solution of the homogeneous equation is bounded on $\mathbb{Z}$, v is the unique solution with the desired properties. □

Note 8. *The link between y and f_n is*

$$y(p) = 2(\sqrt{1 - ip} - 1)e^{-\frac{\pi p}{2\omega}} f_n; \quad \text{for } p = p_0 + in\omega \tag{2.27}$$

Proposition 9. *The function $y(p)$ is analytic in the right half plane, Lipschitz continuous of exponent at least $1/2$ on the imaginary axis and $\lim_{p\to 0} y(p) = i/2$.*

Proof. Since W is analytic in $\mathbb{H}$, continuous and nonzero in $\overline{\mathbb{H}}$, W is bounded below in compact sets in $\overline{\mathbb{H}}$. Then, the smoothness properties of y derive easily from those of $q_n := W(v^+, v^-) f_n$ on which we concentrate now.

(a) For $n \geq 2$ we write, using (2.26),

$$\begin{aligned} q_n = v_n^+ \sum_{\substack{l=-\infty \\ l \neq \pm 1}}^{n-1} v_l^- H_l + v_n^- \sum_{l=n}^{\infty} v_l^+ H_l \\ + v_n^+ i\omega e^{\frac{\pi p_0}{2\omega}} \left(\frac{-iv_{-1}^-}{p_0(p_0 - 2i\omega)} + \frac{iv_1^-}{p_0(p_0 + 2i\omega)} \right) \end{aligned} \tag{2.28}$$

The last term in parenthesis can be rewritten, using also (2.10), as

$$\begin{aligned} \frac{i(v_1^- - v_{-1}^-)}{p_0^2 + 4\omega^2} + \frac{2\omega}{p_0^2 + 4\omega^2} \left(\frac{v_1^- + v_{-1}^-}{p_0} \right) \\ = \frac{i(v_1^- - v_{-1}^-)}{p_0^2 + 4\omega^2} + \frac{4\omega}{r(p_0^2 + 4\omega^2)} \frac{\sqrt{1 - ip_0} - 1}{p_0} v_0^- \end{aligned} \tag{2.29}$$

Thus we see that q_n is continuous as $\Re(p_0) \to 0$ and $\Im(p_0) \in [0, \omega)$ [cf. (2.8)], if $n \geq 2$. A very similar calculation shows the continuity of q_n if $n \leq -1$.

(b) By part (a), $y(p)$ is continuous as $\Re(p) \downarrow 0$ with $\Im(p) \geq 2$ or $\Im(p) < 0$. Now, (2.1) written in the form

$$rph_2(p)y(p) = rp\Big(h_1(p+2i\omega) - h_1(p)\Big) + rp\Big(y(p+i\omega) + h_2(p+2i\omega)y(p+2i\omega)\Big) \tag{2.30}$$

shows that $y(p)$ is Lipschitz continuous as $\Re(p) \downarrow 0$ if $\Im(p) > -2$ thus for all $\Im(p)$. The value of $y(0)$ is easily calculated using (2.30). □

Proposition 10. $1 + 2i\lim_{x\to\infty}\int_0^x Y(s)ds = 0.$

Proof. Indeed,

$$2\pi i\int_0^\infty Y(s)ds = \lim_{x\to\infty}\lim_{\delta\to 0^+}\left(\int_{-i\infty}^{-i\delta} + \int_{i\delta}^{i\infty}\right)\frac{e^{xp}}{p}(i/2 + (y(p) - i/2))dp = -\pi \tag{2.31}$$

□

3. Appendix

Lemma 11. *Equation (1.6) has a unique solution $Y \in L^1_{loc}(\mathbb{R}^+)$ and $|Y(x)| < Ke^{Cx}$ for some $K \in \mathbb{R}^+$ and $C \in \mathbb{R}$.*

Proof. Consider $L^1_{loc}[0,A]$ endowed with the norm $\|F\|_\nu := \int_0^A |F(s)|e^{-\nu s}ds$, where $\nu > 0$. If f is continuous and $F, G \in L^1_{loc}[0,A]$, a straightforward calculation shows that

$$\|fF\|_\nu < \|F\|_\nu \sup_{[0,A]}|f| \tag{3.1}$$

$$\|F * G\|_\nu < \|F\|_\nu\|G\|_\nu \tag{3.2}$$

$$\|F\|_\nu \to 0 \quad \text{as} \quad \nu \to \infty \tag{3.3}$$

where the last relation follows from the Riemann-Lebesgue lemma. The integral equation (1.6) can be written as

$$Y = r\eta + \mathcal{J}Y \text{ where } \quad \mathcal{J}F := r\eta(2i + M) * F \tag{3.4}$$

Since M is locally in L^1 and bounded for large x it is clear that for large enough C_2, and for any A, (1.6) is contractive if $\nu > C_2$. □

Acknowledgments. The authors would like to thank A. Soffer and M. Weinstein for interesting discussions and suggestions. Work of O. C. was supported by NSF Grant 9704968, that of J. L. L. and A. R. by AFOSR Grant F49620-98-1-0207 and NSF Grant DMR-9813268.

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