Divergent Expansion, Borel Summability and 3-D Navier-Stokes Equation

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We describe how Borel summability of divergent asymptotic expansion can be expanded and applied to nonlinear partial differential equations (PDEs). While Borel summation does not apply for nonanalytic initial data, the present approach generates an integral equation applicable to much more general data.

We apply these concepts to the 3-D Navier-Stokes system and show how the integral equation approach can give rise to local existence proofs. In this approach, the global existence problem in 3-D Navier-Stokes, for specific initial condition and viscosity, becomes a problem of asymptotics in the variable p (dual to $1/t$ or some positive power of $1/t$). Furthermore, the errors in numerical computations in the associated integral equation can be controlled rigorously, which is very important for nonlinear PDEs such as Navier-Stokes when solutions are not known to exist globally.

Moreover, computation of the solution of the integral equation over an interval $[0, p_0]$ provides sharper control of its $p \to \infty$ behavior. Preliminary numerical computations give encouraging results.

Keywords: 3-D Navier-Stokes, Smooth Solution, Borel Summation

1. Introduction

It is well known that asymptotic expansions arising in applications are usually divergent. Their calculation is usually algorithmic, once proper scales are identified. Nonetheless, an algorithmically constructed consistent expansion does not guarantee existence of a solution to the problem in the first place.

Borel summation associates to a divergent asymptotic series an actual function, whose asymptotics is given by the series. Under some conditions, this association is an isomorphism (Écalle 1981*a*, *b*, 1985; Costin 1998) under all the usual algebraic operations, including differentiation and integration, between factorially divergent series and actual functions. This is similar to the isomorphism between locally convergent power series and analytic functions. In particular, if a series is a formal solution of a problem—an ordinary differential equation (ODE), partial differential equation (PDE), difference equation, etc., so will the actual function obtained by Borel summation be. Therefore, Borel summability of a formal series to the problem at hand ensures that an actual solution exists.

Furthermore, while the asymptotic series, say in a variable x, is only valid as $x \to \infty$, the Borel sum $f(x)$ has wider validity. In some concrete problems arising in differential equations the validity may even extend to $x = 0$. Thus, unlike the asymptotic series, its Borel sum is useful even when x is not so large.

By Borel summability of a *solution* to a differential equation (ODE or PDE), we mean Borel summability of its asymptotic expansion, usually in one large independent variable or parameter, which plays the role of x in the above discussion. For evolution PDEs, when the domain is $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ and the initial condition is analytic in a strip containing real x, a suitable choice of summation variable is an inverse power of t. We will apply this new method to the 3-D Navier-Stokes (NS) problem: find smooth function $v : \Omega \times \mathbb{R}^+ \to \mathbb{R}^3$, where $\Omega \subset \mathbb{R}^3$ such that it satisfies

$$v_t - \nu \Delta v = -P\left[(v \cdot \nabla) v\right] + f, \quad \text{and} \quad v(x, 0) = v^{[0]}(x), \tag{1.1}$$

with some smoothness condition on f and $v^{[0]}$. In the equation above, P is the Hodge projection to the space of divergence-free vector fields and ν the kinematic viscosity. Additionally, when the domain Ω is bounded, a no-slip boundary condition $v = 0$ on $\partial\Omega$ is physically appropriate for rigid boundaries. The mathematical complications of no-slip boundary conditions are avoided in the periodic case. The latter is less physical, yet it is widely studied since it is useful in understanding homogeneous isotropic fluid flows.

The global existence of smooth solutions of (1.1) for smooth initial conditions $v^{[0]}$ and forcing f remains a formidable open mathematical problem, even for $f = 0$, despite extensive research in this area (see for example monographs Temam 1986; Constantin & Foias 1988; Doering & Gibbon 1995; Foias *et al.* 2001). The problem is important not only in mathematics but it has wider impact, particularly if singular solutions exist. It is known (Beale *et al.* 1984) that the singularities can only occur if ∇v blows up. This means that near a potential blow-up time, the relevance of NS to model actual fluid flow becomes questionable, since the linear approximation in the constitutive stress-strain relationship, the assumption of incompressibility and even the continuum hypothesis implicit in derivation of NS become doubtful. As Trevor Stuart pointed out in the talk by S. Tanveer, the incompressibility hypothesis itself becomes suspect. In some physical problems (such as inviscid Burger's equation) the blow-up of an idealized approximation is mollified by inclusion of regularizing effects. It may be expected that if 3-D NS solutions exhibited blow-up, then the smallest time and space scales observed in fluid flow would involve parameters other than those present in NS. This can profoundly affect our understanding of small scale in turbulence. In fact, some 75 years back, Leray (1933, 1934*a*, *b*) was motivated to study weak solutions of 3-D NS, conjecturing that turbulence was related to blow-up of smooth solutions.

The typical method used in the mathematical analysis of NS, and of more general PDEs, is the so-called energy method. For NS, the energy method involves *a priori* estimates on the Sobolev $\mathbb{H}^m$ norms of v. It is known that if $\|v(\cdot, t)\|_{\mathbb{H}^1}$ is bounded, then so are all the higher order energy norms $\|v(\cdot, t)\|_{\mathbb{H}^m}$ if they are bounded initially. The condition on v has been further weakened (Beale *et al.* 1984) to $\int_0^t \|\nabla \times v(\cdot, t)\|_{L^\infty} \mathrm{d}t < \infty$. Prodi (1959) and Serrin (1963) have found a family of other controlling norms for classical solutions (Ladyzhenskaya 1967). For instance it is known that if

$$\int_0^T \|v(\cdot, t)\|_{L^\infty}^2 \,\mathrm{d}t < \infty,$$

then classical solution to 3-D NS exists in the interval $(0, T)$.

In this connection, it may be mentioned that the 3-D Euler equation, which is the idealized limit of Navier-Stokes with no viscosity, also has been subject of many investigations. Indeed, J. T. Stuart has found some ingenious explicit solutions that exhibit finite-time blow-up (Stuart 1987, 1998). The issue of blow-up for flows with finite energy, however, still remains open though there have been many investigations in this area and there is some numerical evidence for blow-up.

The Borel based method that we use for the NS problem is fundamentally different from the usual classical approaches to PDE. By Borel summing a formal small time expansion in powers of t:

$$v^{[0]}(x) + \sum_{m=1}^{\infty} t^m v^{[m]}(x), \tag{1.2}$$

we obtain an actual solution to 3-D NS problem in the form

$$v(x,t) = v^{[0]}(x) + \int_0^{\infty} \mathrm{e}^{-p/t} U(x,p)\,\mathrm{d}p \tag{1.3}$$

where $U(x,p)$ solves some integral equation (IE), whose solution is known to exist within the class of integrable functions in p that are exponentially bounded in p, uniformly in x. If the IE solution U does not grow with p or grows at most subexponentially, then global existence of NS follows. This new approach to global existence of 3-D Navier-Stokes and indeed to many other evolution PDEs is presented in this paper.

2. Borel Transforms and Borel Summability

We first mention some of the relevant concepts of Borel summation of formal series, leaving aside for now the context where such series arise.

Consider a formal series† $\tilde{f}(x) = \sum_{j=1}^{\infty} a_j x^{-j}$. Its Borel transform is *the formal, term by term, inverse Laplace transform*

$$\mathcal{B}[\tilde{f}](p) \equiv F(p) = \sum_{j=1}^{\infty} \frac{a_j p^{j-1}}{\Gamma(j)}. \tag{2.1}$$

If (2.1) has all of the following three properties:

i. a nonzero radius of convergence at $p = 0$,
ii. its analytic continuation $F(p)$ exists on $(0, \infty)$, and
iii. $\mathrm{e}^{-cp} F(p) \in L^1(0,\infty)$ for some $c \geq 0$, i.e. $\int_0^{\infty} \mathrm{e}^{-cp} |F(p)|\,\mathrm{d}p < \infty$,

then the Borel sum of $\tilde{f}$ is defined as the Laplace transform of F, i.e.

$$f(x) = \mathcal{L}[\mathcal{B}\tilde{f}](x) = \int_0^{\infty} \mathrm{e}^{-px} F(p)\,\mathrm{d}p. \tag{2.2}$$

The function $f(x)$ is clearly well defined and analytic in the complex half-plane $\mathrm{Re}\, x > c$. If the integral exists along a complex ray $(0, \infty \mathrm{e}^{\mathrm{i}\theta})$ for $\theta = -\arg x$, then the corresponding Laplace transform $\mathcal{L}_\theta$ provides the analytic continuation of $f(x)$ to other complex sectors.

† Borel transform also exists for series involving fractional powers of $1/x$.

Borel summability of a formal series means that properties (i)-(iii) are satisfied. It is clear from Watson's lemma (Wasow 1968; Bender & Orszag 1978) that if the Borel sum $f(x) \equiv \mathcal{L}_\theta \mathcal{B} \tilde{f}$ exists, then $f(x) \sim \tilde{f}$ for large x and that $\tilde{f}$ is a Gevrey-1 asymptotic series (Balser 1994); i.e. coefficients a_j diverge like $j!$, up to an algebraic factor.

3. Illustration of Borel Sum for Initial Value Problem

Consider first the heat equation

$$v_t = v_{xx}, \quad v(x,0) = v^{[0]}(x); \quad (v^{[0]} \text{ analytic}) \tag{3.1}$$

where we look for formal series solutions

$$v(x,t) = v^{[0]}(x) + \sum_{m=1}^{\infty} t^m v_m(x) \tag{3.2}$$

as in the Cauchy-Kowalewski approach, except the expansion is in t alone. We get

$$(m+1) v_{m+1}(x) = v_m''(x). \tag{3.3}$$

By induction,

$$v_m(x) = \frac{{v^{[0]}}^{(2m)}(x)}{m!}. \tag{3.4}$$

Assuming $v^{[0]}$ is analytic in a strip of width a containing $\mathbb{R}$ but is not entire, (3.2) diverges factorially since

$$v_m(x) = \frac{(2m)!}{m!(2\pi \mathrm{i})} \oint_{|\zeta - x| = a'} \frac{v^{[0]}(\zeta)}{(\zeta - x)^{2m+1}} \mathrm{d}\zeta; \quad a' < a \tag{3.5}$$

and $(2m)!/m!$ is a factorial up to a geometric factor. It is then easily checked that the Borel transform (formal inverse Laplace transform) of (3.2) is convergent in p in a ball around the origin. Indeed Gevrey estimates on ${v^{[0]}}^{(2m)}$ show that the radius of convergence of the Borel transform of (3.2) for any x is at least the analyticity width of $v^{[0]}$. However, it is not obvious whether conditions (ii) and (iii) in §2 for Borel summability are satisfied or not.

Instead, we substitute

$$v(x,t) = \frac{1}{\sqrt{t}} u(x,t) \tag{3.6}$$

in (3.1), Borel transform the resulting PDE in $1/t$ and write $\mathcal{L}^{-1} u = p^{-1/2} W(x, 2\sqrt{p})$. We get

$$W_{ss} - W_{xx} = 0 \;\; \text{implying} \;\; W = f_1(x+s) + f_2(x-s). \tag{3.7}$$

Retracing the transformations and using the initial condition, one obtains, from the Laplace transform of $p^{-1/2} W(x, 2\sqrt{p})$ in p, the well-known solution to the heat equation in terms of the heat kernel (see §3.0.1 in Costin & Tanveer 2004*a*). The above calculation shows that while (3.2) is only valid as an asymptotic expansion for $t \ll 1$, its Borel sum is valid for all t.

Though the heat equation is special in that explicit solutions are readily available, the analysis above shows that instead of Borel transforming formal series in t, it is better to apply Borel transform directly on the PDE itself and carry out a mathematical analysis on the resulting equation. This is indeed what can be accomplished for many PDE initial value problems (indeed other types of problems are also amenable to similar analysis). In the following section, we derive an integral equation that arises from Borel-transforming (1.1).

4. Navier-Stokes Equation and Integral Equation

We denote by '$\hat{\ }$' the Fourier transform, by '$\hat{*}$' the Fourier convolution ('$*$' is the Laplace convolution), and assume that the forcing is time independent. We also denote Fourier transform by $\mathcal{F}$ and its inverse by $\mathcal{F}^{-1}$†. For 2π periodic box problem, in the Fourier space, NS reads (see for e.g. Temam 1986; Constantin & Foias 1988; Doering & Gibbon 1995; Foias *et al.* 2001):

$$\hat{v}_t + |k|^2 \hat{v} = -\mathrm{i} k_j P_k \left[\hat{v}_j \hat{*} \hat{v}\right] + \hat{f}, \quad \hat{v}(k,0) = \hat{v}^{[0]}(k), \;\; \hat{v} = (\hat{v}_j)_{j=1,2,3} \tag{4.1}$$

where

$$P_k \equiv \left(1 - \frac{k(k\cdot)}{|k|^2}\right) \tag{4.2}$$

is the Fourier transform of Hodge projection P. We also follow the usual convention of summation over repeated indices. When $x \in \mathbb{T}^3[0,2\pi]^3$, we take $k = (k_1, k_2, k_3)$ an ordered triple of integers, i.e. $k \in \mathbb{Z}^3$, while if $x \in \mathbb{R}^3$ we would take $k \in \mathbb{R}^3$. Without loss of generality we can assume that average velocity and average force over a period is zero, implying $\hat{v}(0,t) = 0$ and $\hat{f}(0) = 0$.

We write $\hat{v} = \hat{v}^{[0]} + \hat{u}$ and apply the Borel transform in $1/t$ to the resulting equation; we get

$$\begin{aligned} p\hat{U}_{pp} + 2\hat{U}_p + |k|^2 \hat{U} = -\mathrm{i} k_j P_k \left[\hat{v}_j^{[0]} \hat{*} \hat{U} + \hat{U}_j \hat{*} \hat{v}^{[0]} + \hat{U}_j \overset{*}{*} \hat{U}\right] + \hat{v}^{[1]}(k)\delta(p) \\ =: -\mathrm{i} k_j \hat{H}_j(k,p) + \hat{v}^{[1]}(k)\delta(p). \end{aligned} \tag{4.3}$$

The solution to the homogeneous equation on the left side of (4.3) can be expressed in terms of the Bessel functions J_1 and Y_1. Using boundedness of $\hat{U}(k,p)$ at $p=0$ (which follows from $\hat{v}(k,0) = \hat{v}^{[0]}(k)$), one obtains the integral equation (see Costin & Tanveer 2006*a* for more details):

$$\hat{U}(k,p) = -\mathrm{i} k_j \int_0^p \mathcal{G}(z,z') \hat{H}_j(k,p')\, \mathrm{d}p' + \frac{2J_1(z)}{z} \hat{v}^{[1]}(k) \equiv \mathcal{N}\left[\hat{U}\right](k,p), \quad \text{where}$$

$$\mathcal{G}(z,z') = \frac{z'}{z}\left(J_1(z')Y_1(z) - Y_1(z')J_1(z)\right), \quad z = 2|k|\sqrt{p}, \quad z' = 2|k|\sqrt{p'}, \tag{4.4}$$

where $\overset{*}{*}$ denotes Fourier transform followed by Laplace transform and

$$\hat{v}^{[1]}(k) = -|k|^2 \hat{v}^{[0]} - \mathrm{i} k_j P_k \left[\hat{v}_j^{[0]} \hat{*} \hat{v}^{[0]}\right]. \tag{4.5}$$

† For periodic problem, $\mathcal{F}^{-1}$ is simply evaluation of a function based on its Fourier coefficients.

5. Results

(a) Overview of Results

We have proved that the integral equation (4.4) has a unique solution $\hat{U}(k,p)$ (precise statements and spaces of functions being considered are spelled out in §5 b) that is Laplace transformable in p and absolutely summable over $k \in \mathbb{Z}^3$. Therefore, it generates, through (1.3), a classical solution to (1.1) over some time interval. Furthermore, $U(x,p)$ is analytic in p for $p \geq 0$ (i.e. $p \in \mathbb{R}^+ \cup \{0\}$) when $v^{[0]}$ and f are analytic in x. Applying Watson's lemma to (1.3), the asymptotic nature of the formal expansion

$$v(x,t) \sim v^{[0]}(x) + \sum_{m=1}^{\infty} t^m v^{[m]}(x) \tag{5.1}$$

is confirmed for small t. Further, because U is analytic at $p=0$, the above series is divergent like $m!$ (up to geometric corrections in m), implying that a least term truncation of the above series will result in exponentially small errors for small t.

We now make an important point about the integral equation representation of Navier-Stokes solution. Though (1.3) is the Borel sum of the formal small time expansion (1.2) for analytic initial data $v^{[0]}(x)$ and forcing $f(x)$, the representation (1.3) transcends these restrictions. As stated in theorem 5.1, there is a solution $\hat{U}(k,p)$ satisfying (4.4), even when $\hat{v}^{[0]} \equiv \mathcal{F}\left[v^{[0]}\right]$, $\hat{f} \equiv \mathcal{F}[f]$ are only in l^1, i.e. have absolutely summable Fourier series. Through the Laplace transform representation (1.3), $U(x,p) \equiv \mathcal{F}^{-1}\left[\hat{U}(\cdot,p)\right](x)$ generates a classical solution to (1.1) for t in some time interval. Thus, while Borel summability does not make sense for nonanalytic initial data or forcing, the representation (4.4) and (1.3) continue to provide classical solutions to Navier-Stokes! Furthermore, if the solution $\hat{U}(k,p)$ to (4.4) does not grow with p, or grows at most subexponentially, then global existence of 3-D NS follows.

The existence interval $\left[0,\alpha^{-1}\right)$ for 3-D NS proved in theorem 5.1 is suboptimal. It does not take into account the fact that initial data $v^{[0]}$ and forcing f are real valued. (Blow-up of Navier-Stokes solution for particular complex initial data is known (Li & Sinai 2006).) Also, the estimates ignore possible cancellations in the integrals.

In the following we address the issue of sharpening the estimates, in principle arbitrarily well, based on more detailed knowledge of the solution of the IE on a p-interval $[0,p_0]$. This knowledge may come from, among others, a computer assisted set of estimates, or *a priori* information based on optimal truncation of asymptotic series. If this information shows that the solution is small for p towards the right end of the interval, then α can be shown to be small. This in turn results in longer times of guaranteed existence, possibly global existence for $f=0$ if this time exceeds T_c, the time after which it is known that a weak solution becomes classical again because of long term effect of viscosity.

To get a mathematical sense of how such estimates are possible from the integral equation (4.4), define

$$\hat{U}^{(a)}(k,p) = \hat{U}(k,p) \text{ for } \; p \leq p_0 \; \text{ and 0 otherwise.}$$

Define $\hat{W} = \hat{U} - \hat{U}^{(a)}$, where we see that $\hat{W}$ is nonzero only for $p > p_0$. Then, from (4.4) we have for $p > p_0$,

$$\hat{W}(k,p) = \hat{W}^{(0)}(k,p) - \mathrm{i}k_j \int_{p_0}^{p} \mathcal{G}(z,z') \hat{H}_j^{(w)}(k,p')\, \mathrm{d}p' \equiv \mathcal{N}^{(w)} \left[\hat{W}\right](k,p), \qquad (5.2)$$

where

$$\hat{H}_j^{(w)}(k,p) = P_k \left[\hat{v}_j^{[0]} \hat{*} \hat{W} + \hat{W}_j \hat{*} \hat{v}^{[0]} + \hat{W}_j \hat{*}_{*} \hat{U}^{(a)} + \hat{U}_j^{(a)} \hat{*}_{*} \hat{W} + \hat{W}_j \hat{*}_{*} \hat{W} \right], \qquad (5.3)$$

$$\hat{W}^{(0)}(k,p) = 2\frac{J_1(z)}{z} \hat{v}^{[1]}(k) - \mathrm{i}k_j \int_0^{\min(p,2p_0)} \mathcal{G}(z,z') \hat{H}_j^{(a)}(k,p')\, \mathrm{d}p', \qquad (5.4)$$

and

$$\hat{H}_j^{(a)} = \hat{v}_j^{[0]} \hat{*} \hat{U}^{(a)} + \hat{U}_j^{(a)} \hat{*} \hat{v}^{[0]} + \hat{U}_j^{(a)} \hat{*}_{*} \hat{U}^{(a)}. \qquad (5.5)$$

We note that if the calculated $\hat{U}^{(a)}$ is seen to rapidly decrease in some subinterval $[p_d, p_0]$, then the inhomogeneous term $\hat{W}^{(0)}$ in the integral equation (5.2) becomes small. For sufficiently large p_0, the factor $p^{-1/2}$ multiplying integral term in (5.2) is also small for $p \geq p_0$. This ensures contractivity of operator $\mathcal{N}^{(w)}$ at a smaller α. Precise statements on estimates on α-based $\hat{U}^{(a)}$ are given in theorem 5.3.

The results in theorem 5.3 rely on knowledge of $\hat{U}$ for $p \in [0, p_0]$. When the initial data and forcing are analytic, the formal series (1.2) can be useful in this respect since its Borel transform has a nonzero radius of convergence in the p-domain. However, this ball about the origin may not contain p_0 when p_0 is large.

A second approach towards knowing $\hat{U}^{(a)}$ is to rely on a discretization in p and a Galerkin projection to a finite number (say $8N^3$) of Fourier modes in k. This approach is attractive, even from the viewpoint of rigorous results, since the errors are completely controlled as we now argue.

Let $\mathcal{N}_\delta^{(N)}$ denote the discretized version of operator $\mathcal{N}$ and $\hat{U}_\delta^{(N)}$ denote the solution of the discretized equation, which can be calculated numerically. Then

$$\hat{U}_\delta^{(N)} = \mathcal{N}_\delta^{(N)} \left[\hat{U}_\delta^{(N)} \right]. \qquad (5.6)$$

The continuous solution $\hat{U}$ to (4.4), when plugged into the discretized system, satisfies

$$\hat{U} = \mathcal{N}_\delta^{(N)} \left[\hat{U} \right] + \hat{T}_E \qquad (5.7)$$

where $\hat{T}_E$ is the sum of truncation errors due to discretization in p and Galerkin projection on $[-N, N]^3$. This error can be expressed in terms of derivatives of $\hat{U}$ with respect to p and estimates on $k\hat{U}$. Each is available *a priori* from solutions of the integral equation (4.4). By subtracting (5.6) from (5.7), we obtain an equation for the error $\hat{U} - \hat{U}_\delta^{(N)}$. From the contractivity properties of $\mathcal{N}_\delta^{(N)}$, it follows, essentially by using the same arguments as for $\mathcal{N}$, that $\hat{U} - \hat{U}_\delta^{(N)}$ may be estimated in terms of the truncation error, which itself is *a priori* small for sufficiently small δ and sufficiently large N. So, in principle, $\hat{U}$ can be computed to any desired precision with rigorous error control. More details of this argument appear in Costin *et al.* 2008.

(i) *Acceleration*

We have already established that at most subexponential growth of $\|\hat{U}(\cdot,p)\|_{l^1}$ implies global existence of a classical solution to (1.1).

We now look for a converse: suppose (1.1) has a global solution, is it true that $\hat{U}(\cdot,p)$ always is subexponential in p? The answer is no in general. Any complex singularity τ_s in the right-half complex τ-plane of $v(x,\tau^{-1})$ produces exponential growth of $\hat{U}$ with rate $\mathrm{Re}\,\tau_s$ (and oscillation of $\hat{U}$ with frequency $\mathrm{Im}\,\tau_s$), as it is seen by looking at the asymptotics of the inverse Laplace transform.

However, when there is no forcing $f=0$, it can be proved (see theorem 5.4 for precise statements) that given a global classical solution of (1.1), there is a $c>0$ so that for any τ_s we have $|\arg\tau_s|>c$. This means that for sufficiently large n, the function $v(x,\tau^{-1/n})$ has no singularity in the right-half τ-plane. Then the inverse Laplace transform

$$U_{\mathrm{acc}}(x,q)=\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left\{v(x,\tau^{-1/n})-v^{[0]}(x)\right\}\mathrm{e}^{q\tau}\,\mathrm{d}\tau \tag{5.8}$$

can be shown to decay as $q\to\infty$, reflecting the exponential decay of $v(x,t)$ for large t.

This means that it is advantageous to find $U_{\mathrm{acc}}(x,q)$ so that the generalized Laplace transform representation

$$v(x,t)=v^{[0]}(x)+\int_0^\infty U_{\mathrm{acc}}(x,q)\mathrm{e}^{-q/t^n}\,\mathrm{d}q \tag{5.9}$$

gives a solution to (1.1). The transformation from $U(x,p)$ to $U_{\mathrm{acc}}(x,q)$ is referred to as acceleration and was first used in one variable by Écalle. Indeed, there is an integral transformation that directly relate U to U_{acc}, though this is not used in the analysis.

The resulting integral equation for $\hat{U}_{\mathrm{acc}}(k,q)$ has been analyzed (Costin *et al.* 2008) and results similar to theorems 5.1 and 5.3 hold. Indeed, preliminary numerical calculations, described in §5 *c* give encouraging results.

(*b*) *Some Theorems*

For analysis of the IE, it is convenient to define a number of different spaces of functions and corresponding norms.

Definition 5.1. We denote by $l^1(\mathbb{Z}^3)$ the set of functions $\hat{f}$ of an ordered integer triple $k=(k_1,k_2,k_3)$ (i.e. of $k\in\mathbb{Z}^3$) such that†

$$\|\hat{f}(k)\|_{l^1(\mathbb{Z}^3)}=\sum_{k\in\mathbb{Z}^3}|\hat{f}(k)|.$$

Also, for analytic functions $f(x)$ whose series coefficients are exponentially decaying functions in $\mathbb{Z}^3$, it is convenient to define in the Fourier space the $\|.\|_{\mu,\beta}$ norm:

$$\|\hat{f}\|_{\mu,\beta}=\sup_{k\in\mathbb{Z}^3}\left\{\mathrm{e}^{\beta|k|}(1+|k|)^{\mu}|\hat{f}(k)|\right\}.$$

† Since for NS the velocity and forcing have the property that $\hat{f}(0)=0=\hat{v}(0,t)$, the $k=0$ term is left out in the l^1 sum.

Definition 5.2. For $\alpha \geq 0$, we define the norm $\|.\|_1^{(\alpha)}$ for functions of (k,p), $k \in \mathbb{Z}^3$, p real, with $p \geq 0$ (i.e. $p \in \mathbb{R}^+ \cup \{0\}$):

$$\|\hat{U}\|_1^{(\alpha)} = \int_0^\infty \mathrm{e}^{-\alpha p} \left\{ \sum_{k \in \mathbb{Z}^3} |\hat{U}(k,p)| \right\} \mathrm{d}p = \int_0^\infty \mathrm{e}^{-\alpha p} \|\hat{U}(\cdot,p)\|_{l^1(\mathbb{Z}^3)} \, \mathrm{d}p. \tag{5.10}$$

Definition 5.3. We define $\mathcal{A}_1^{(\alpha)}$ to be the Banach space of functions $\hat{U}(k,p)$ that are $l^1(\mathbb{Z}^3)$ in $k \in \mathbb{Z}^3$ and absolutely integrable in p such that $\|\hat{U}\|_1^{(\alpha)} < \infty$.

Definition 5.4. For analytic initial condition $v^{[0]}$ and forcing f, it is also convenient to define for $\beta > 0$, $\mu > 3$, the following space $\mathcal{A}$ of functions of (k,p) that are bounded in k and continuous in $p \in \mathbb{R}^+$ so that

$$\|\hat{U}\| = \sup_{p \in \mathbb{R}^+} \mathrm{e}^{-\alpha p}(1+p^2) \left[\sup_{k \in \mathbb{Z}^3} \mathrm{e}^{\beta |k|} (1+|k|)^\mu |\hat{U}(k,p)| \right]$$
$$= \sup_{p \in \mathbb{R}^+} \mathrm{e}^{-\alpha p}(1+p^2) \|\hat{U}(\cdot,p)\|_{\mu,\beta} < \infty.$$

We have the following theorems:

Theorem 5.1. *If $|k|^2 \hat{v}^{[0]}, \hat{f} \in l^1$, then the integral equation (4.4) has a unique solution in the space $\mathcal{A}_1^{(\alpha)}$ for α large enough. Taking the Laplace transform relation:*

$$\hat{v}(k,t) = \hat{v}^{[0]}(k) + \int_0^\infty \hat{U}(k,p) \mathrm{e}^{-p/t} \, \mathrm{d}p, \tag{5.11}$$

$\hat{v}(k,t)$ satisfies the Navier-Stokes equation (4.1) in Fourier space. The generated Fourier series $v(x,t) = \mathcal{F}^{-1}[\hat{v}(\cdot,t)](x)$ is a classical solution to Navier-Stokes for $t \in (0, \alpha^{-1})$.

Outline of the proof: The detailed proof of the theorem is given in theorem 1 in Costin *et al.* 2008, though a more general IE is considered; the theorem 5.1 here corresponds to the special case $n = 1$ in Costin *et al.* 2008. The key feature of the proof is the boundedness of $|k|\mathcal{G}$ for $z, z' \in \mathbb{R}^+$ for $z' \leq z$ (i.e. for $p' \leq p$), which follows from the properties of J_1 and Y_1. Therefore,

$$\|\mathcal{N}[\hat{U}](\cdot,p)\|_{l^1(\mathbb{Z}^3)} \leq \frac{C}{\sqrt{p}} \int_0^p \left[\|\hat{v}^{[0]}\|_{l^1} \|\hat{U}(\cdot,s)\|_{l^1} + \|\hat{U}(\cdot,s)\|_{l^1} * \|\hat{U}(\cdot,s)\|_{l^1} \right] \mathrm{d}s + \|\hat{v}^{[1]}\|_{l^1}$$

and from the properties of Laplace convolutions we obtain

$$\|\mathcal{N}[\hat{U}]\|_1^{(\alpha)} \leq C\alpha^{-1/2} \|\hat{U}\|_1^{(\alpha)} \left(\|\hat{v}^{[0]}\|_{l^1} + \|\hat{U}\|_1^{(\alpha)} \right) + \frac{1}{\alpha} \|\hat{v}^{[1]}\|_{l^1},$$

and in a similar manner

$$\|\mathcal{N}[\hat{U}^{[1]}] - \mathcal{N}[\hat{U}^{[2]}]\|_1^{(\alpha)} \leq C\alpha^{-1/2} \|\hat{U}^{[1]} - \hat{U}^{[2]}\|_1^{(\alpha)} \left(\|\hat{v}^{[0]}\|_{l^1} + \|\hat{U}^{[1]}\|_1^{(\alpha)} + \|\hat{U}^{[2]}\|_1^{(\alpha)} \right).$$

It follows that for large enough α, $\mathcal{N}$ is contractive with respect to $\|.\|_1^{(\alpha)}$ in the ball of radius $2\alpha^{-1}\|\hat{v}^{[1]}\|_{l^1}$. The transformations are easily undone to obtain a classical solution to the 3-D NS equation for $t \in \left(0, \alpha^{-1}\right)$† that satisfies given initial condition. Conversely, since a smooth solution to (1.1) is known to be unique, its Fourier transform must be expressible as (5.11), implying the solution is analytic in time for $\mathrm{Re}\, t^{-1} > \alpha$ for some α. The inverse Laplace transform $\mathcal{L}^{-1}\left[\hat{v}(k,\tau^{-1}) - \hat{v}^{[0]}\right](p) = \hat{U}(k,p)$ must exist and satisfy IE (4.4). Therefore, the solution to (4.4) is unique, without any restriction on the ball size in the Banach space.

Remark 5.5. The main significance of theorem 5.1 is not that there exists smooth solution to 3-D Navier-Stokes locally in time. This has been a standard result for many years (see for instance Temam 1986; Constantin & Foias 1988; Doering & Gibbon 1995; Foias *et al.* 2001). The connection with the integral equation (4.4) is more significant. Its solution $\hat{U}(k,p)$ exists for $p \in \mathbb{R}^+$. If this solution does not grow with p or grows at most subexponentially, then 3-D NS will have global solution for the particular initial condition in question. So, in a sense the problem of global existence has become one of asymptotics. We will see later that this connection can be made stronger.

Theorem 5.2. *For $\beta > 0$ (analytic initial data) and $\mu > 3$, the solution $v(x,t)$ is Borel summable in $1/t$, i.e. there exists $U(x,p)$, analytic in a neighborhood of $\mathbb{R}^+$, exponentially bounded, and analytic in x for $|\mathrm{Im}\, x| < \beta$ so that*

$$v(x,t) = v^{[0]}(x) + \int_0^\infty U(x,p)\mathrm{e}^{-p/t}\,\mathrm{d}p.$$

Therefore, in particular, as $t \to 0$,

$$v(x,t) \sim v^{[0]}(x) + \sum_{m=1}^{\infty} t^m v^{[m]}(x)$$

with

$$|v^{[m]}(x)| \le m! A_0 B_0^m,$$

where A_0 and B_0 depend on $v^{[0]}$ and f only.

Remark 5.6. Borel summability and classical Gevrey-asymptotic results (Balser 1994) imply for small t that

$$\left| v(x,t) - v^{[0]}(x) - \sum_{m=1}^{m(t)} t^m v^{[m]}(x) \right| \le A_0\; m(t)^{1/2}\mathrm{e}^{-m(t)}$$

where $m(t) = \lfloor B_0^{-1} t^{-1} \rfloor$. Our bounds on B_0 are likely suboptimal. Formal arguments in the recurrence relation of $v^{[m+1]}$ in terms of $v^{[m]}$, $v^{[m-1]}$,…,$v^{[1]}$, indicate that B_0 only depends on β, but not on $\|\hat{v}^{[0]}\|_{\mu,\beta}$.

(i) *Sharper Estimates*

Let $\hat{U}(k,p)$ be the solution of (4.4) provided by theorem 5.1. Define

$$\hat{U}^{(a)}(k,p) = \begin{cases} \hat{U}(k,p) & (0,p_0] \subset \mathbb{R}^+ \\ 0 & \text{otherwise} \end{cases}, \tag{5.12}$$

† The solution immediately smooths out in x for $t > 0$.

$$\hat{U}^{(s)}(k,p) = -\mathrm{i}k_j \int_0^p \mathcal{G}(p,p';k)\hat{H}_j^{(a)}(k,p')\,\mathrm{d}p' + \hat{U}^0(k,p),$$
$$\hat{H}_j^{(a)}(k,p) = P_k\Big[\hat{v}_j^{[0]}\hat{*}\hat{U}^{(a)} + \hat{U}_j^{(a)}\hat{*}\hat{v}^{[0]} + \hat{U}_j^{(a)}\hat{*}\hat{U}^{(a)}\Big](k,p).$$

Using (5.12) we introduce the following functions of $\hat{U}^{(a)}(k,p)$, $\hat{v}^{[0]}$ and $\hat{f}$:

$$b := \alpha \int_{p_0}^{\infty} \mathrm{e}^{-\alpha p}\|\hat{U}^{(s)}(\cdot,p)\|_{l^1}\,\mathrm{d}p, \tag{5.13}$$

$$\epsilon_1 = C_1 + \int_0^{p_0} \mathrm{e}^{-\alpha p'} C_2(p')\,\mathrm{d}p'. \tag{5.14}$$

Finally, we let

$$C_0(k) = \sup_{p_0\le p'\le p}\Big\{|\mathcal{G}(p,p';k)|\Big\}, \qquad C_1 = 4\sup_{k\in\mathbb{Z}^3}\big\{|k|C_0(k)\big\}\|\hat{v}^{[0]}\|_{l^1},$$
$$C_2(q) = 4\sup_{k\in\mathbb{Z}^3}\big\{|k|C_0(k)\big\}\|\hat{U}^{(a)}(\cdot,p)\|_{l^1}, \qquad \epsilon = 2\sup_{k\in\mathbb{Z}^3}\big\{|k|C_0(k)\big\}.$$

Theorem 5.3. *(i) The exponential growth rate α of $\hat{U}$ is estimated in terms of the restriction of $\hat{U}$ to $[0,p_0]$ as follows.*

$$\text{If} \quad \alpha > \epsilon_1 + 2\sqrt{\epsilon b} \quad \text{then} \quad \int_0^{\infty} \|\hat{U}(\cdot,p)\|_{l^1}\mathrm{e}^{-\alpha p}\,\mathrm{d}p < \infty. \tag{5.15}$$

Remark 5.7. It was argued in Costin *et al.* 2008, in a slightly more general context, that the estimated $\epsilon_1 + \sqrt{2b\epsilon}$ is small if the solution $\hat{U}^{(a)}$ is small in some subinterval $[p_d, p_0]$. This implies a long interval $(0, \alpha^{-1})$ of guaranteed existence of a solution to (1.1).

As mentioned in §5 *a* , while subexponential growth of $\hat{U}$ guarantees global solution to (1.1), the converse is not true if there exist purely complex right half t-plane singularities. The following theorem shows however that a converse is true, for $f = 0$, *after suitable acceleration*. Thus the problem of global existence of (1.1) becomes a problem of asymptotics for solution $\hat{U}_{\mathrm{acc}}$.

Theorem 5.4. *Assume that $\hat{v}^{[0]} \in l^1(\mathbb{Z}^3)$, $\nu > 0$ and $\hat{f} = 0$ (zero forcing). If NS has a global classical solution, then there exists n large enough so that $U_{\mathrm{acc}}(x,q) = O(\mathrm{e}^{-Cq^{1/(n+1)}})$ as $q \to +\infty$, for some $C > 0$.*

The proof of this theorem is given in Costin *et al.* 2008.

Remark 5.8. Together with theorem 5.1, theorem 5.4 shows that global existence is equivalent to an asymptotic problem. The solution to NS exists for all time if and only if $\hat{U}_{\mathrm{acc}}$ decays for some $n \in \mathbb{Z}^+$.

(*c*) *Preliminary Numerical Results*

The computations described here are for $n = 1$ (unaccelerated case) and $n = 2$. We present the details elsewhere (Costin *et al.* 2008). The code is far from being optimized and the results are only presented over a modest interval in p or q.

(i) *Kida Flow*

We consider the Kida initial condition (Kida 1985)

$$v_1(x_1, x_2, x_3, 0) = \sin x_1(\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3).$$

The high degree of symmetry (preserved in time) lowers the number of computational operations. We computed the solution for $\nu = 0.1$ and no forcing, for a Reynolds number $Re = 20\pi$. This is not very large, but we were mainly interested in testing the concepts developed. We used $q_0 = 10$ and used 128 points in each space dimension, i.e. $N = 64$, and step size $\Delta q = 0.05$. To investigate the growth of the solution $\hat{U}$ with q, we computed the l^1-norm

$$\|\hat{U}(\cdot, q)\|_{l^1,2N} := \sum_{k\in[-N,N]^3} |\hat{U}(k, q)|$$

and plotted $\|\hat{U}_1(\cdot, q)\|_{l^1,128}$ vs. q in figure 1(b). For comparison we also included in figure 1(a) a plot of the solution to the original (unaccelerated) equation, i.e. $q = p$ case.

Singularities in the right half plane, if present, come in conjugate pairs because of reality of the solution. This exponential growth (mixed with oscillation) of $\hat{U}(k, p)$ is seen in figure 1(a). The oscillation parameters depend on k while the growth rate is virtually insensitive to k. By monitoring the oscillation against the growth rate of each of the modes, we predicted that acceleration with $n = 2$ would eliminate singularities in the right-half $\tau = 1/t^n$ plane. This expectation is confirmed in figure 1(b), in which it is seen that $\|\hat{U}(\cdot, q)\|_{l_1}$ decreases beyond some q. Actually, it has been shown in Costin *et al.* 2008 that if (1.1) has a global solution and an appropriate acceleration is used, then $\|\hat{U}(\cdot, q)\|_{l^1}$ is $O(\mathrm{e}^{-cq^{1/(n+1)}})$. Remarkably, though the interval of calculation is only modestly long, $[0, 10]$, this asymptotic trend is clear in figure 2. For large enough q, the low k modes dominate, while for smaller q more modes contribute to the norm, and this explains the damped oscillation present in figure 2. It is remarkable that a computation over a moderate q-interval can capture the large q-trend.

6. Discussion and Conclusion

We have shown here how Borel transform methods provide a different approach to the study of evolution PDEs, such as for 3-D incompressible Navier-Stokes. In this formulation, the PDE problem becomes equivalent to a nonlinear integral equation with unique solutions in $\mathbb{R}^+$. These solutions are smooth and analytic in the dual variable providing control of the errors. Furthermore, global existence becomes an asymptotic problem. We also illustrate how to obtain better asymptotic estimates if the solution for a finite interval $[0, p_0]$ is known. Preliminary numerical results are encouraging.

Regarding future work in this direction, we point out that (1) so far, our focus has been existence for fixed initial data and Reynolds number. While the upper bound estimates can be obtained for arbitrary initial data, we are yet to explore more optimal estimates in these settings. (2) A related question, which is interesting in its own right, is that whether Borel summation applies to incompressible Euler equations. As indicated by a preliminary calculation (not shown here), the small time expansion for Euler is likely convergent, which means Borel summation would result in infinite radius of convergence. Exploiting this aspect will be subject of future research.

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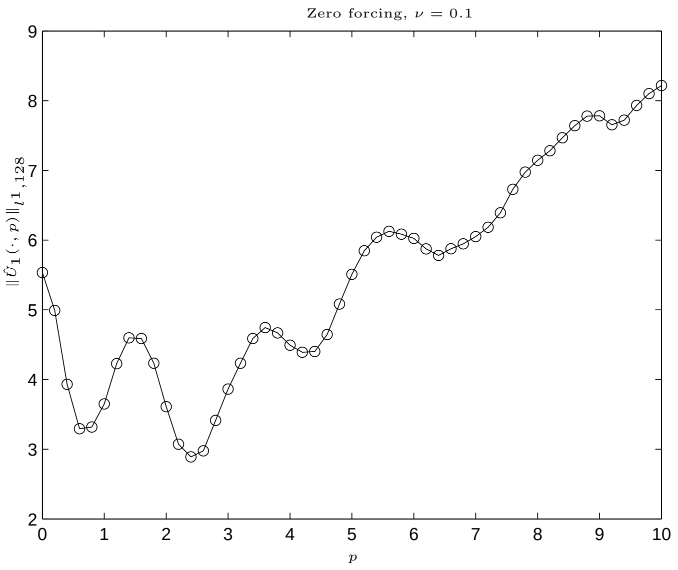
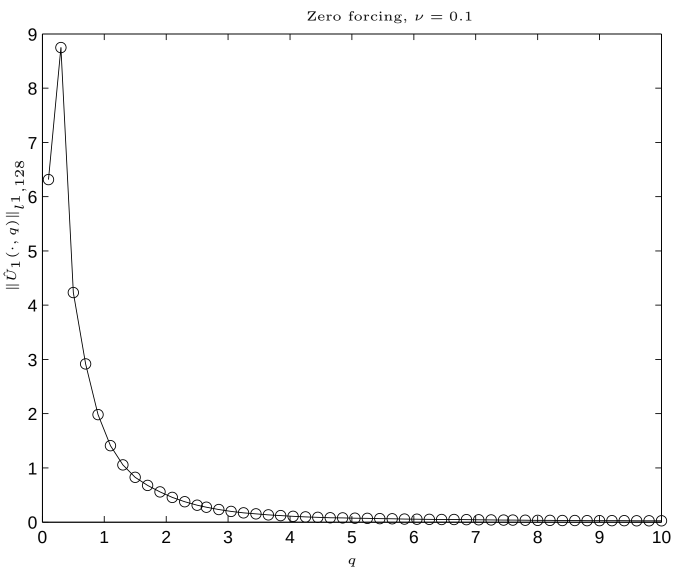
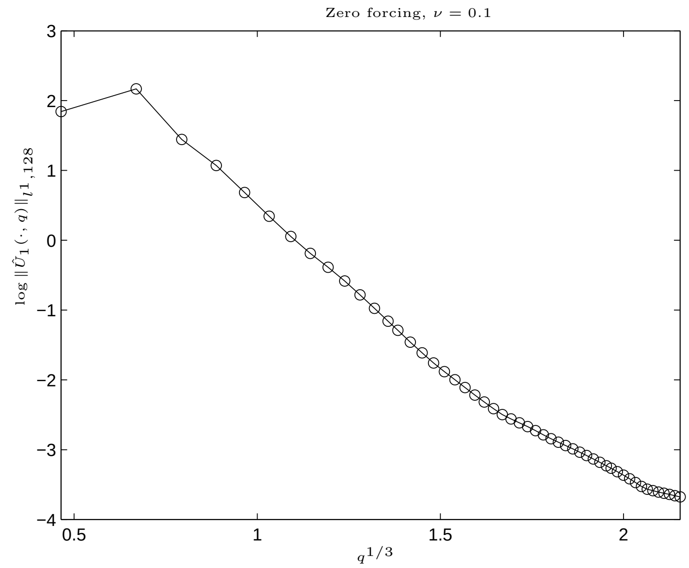


(a)

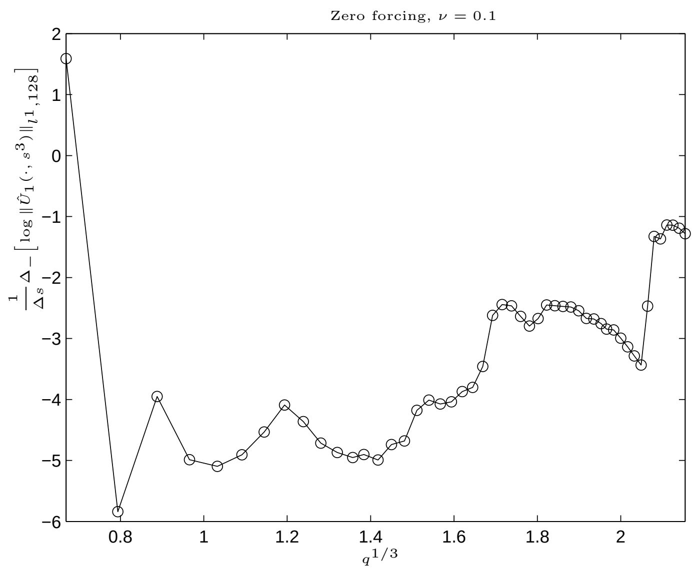


(b)

Figure 1. For zero forcing and $\nu = 0.1$: (a). The original (unaccelerated) equation, $\|\hat{U}_1(\cdot,p)\|_{l^1,128}$ vs. p. (b). Accelerated equation with $n = 2$, $\|\hat{U}_1(\cdot,q)\|_{l^1,128}$ vs. q.



(a)



(b)

Figure 2. Asymptotic behavior of $\|\hat{U}_1(\cdot, q)\|_{l^1,128}$. (a). $\log \|\hat{U}_1(\cdot, q)\|_{l^1,128}$ vs. $q^{1/3}$. (b) $\dfrac{1}{\Delta s}\Delta_-\Big[\log \|\hat{U}_1(\cdot, s^3)\|_{l^1,128}\Big]$ vs. s, where $s = q^{1/3}$ and Δ_- is the backward difference operator in s.

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