A RATIONAL WORLD?

P1–P12 guarantee the existence of rational numbers but not that of real numbers. How does a world in which only rational numbers exist "look like"? What becomes of (analytic) plane geometry be if the plane $\mathbb{R}^2$ is replaced by $\mathbb{Q}^2$, the set of all pairs of rationals?

The geometry of lines is not too different from that in "our world" where "real numbers exist" [1]. We can define lines in the same way, by the equation $ax + by + c = 0$ (now everything is in $\mathbb{Q}$: a, b, c, x, y are all rational numbers). As usual two lines coincide if one equation is equivalent to the other. For instance the line $2ax + 2ay + 2c = 0$ is the same as the line $ax + by + c = 0$. Check that two lines which are not parallel still intersect, that for any two distinct points there is one and only one line and only one joining the two points.

How about circles? Take first the center at $(0, 0)$ and the radius $R = 1$. Then a circle in the rational world is still naturally defined as the set of points $(x, y) \in \mathbb{Q}^2$ s.t.

$$x^2 + y^2 = 1 \tag{1}$$

Does this contain any points? Yes, clearly $(1, 0), (0, 1), (-1, 0), (0, -1)$ are on the circle.

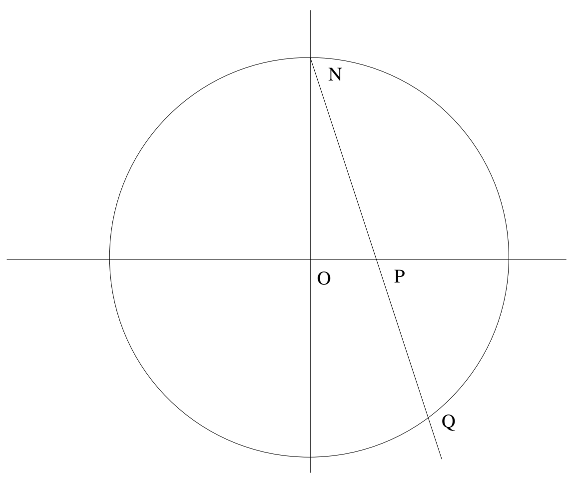

FIGURE 1. A circle and a special line

But there are many more, in fact infinitely many. Look at Fig. 1, where the points are $N(0, 1)$, $P(r, 0)$ where of course $r \in \mathbb{Q}$. You need to solve the exercises below correctly and in the order presented to get points.

Exercise 1 (10p)**.** Take any $r \in (0, 1)$. Write the equation of the segment NQ in point-slope form. Intersect it with the circle (1). Show that the (x, y) coordinates of Q are rational numbers. In $\mathbb{Q}^2$, the line through NP intersects the circle indeed.

Note 1. *There is an interesting connection between Exercise 1 and the Weierstrass substitution in trigonometric integrals, that we will study later.*

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[1] There are lots of things that would need a serious discussion in the statement "real numbers exist in our world", but for the sake of brevity, we assume we believe in that.

Thus there are infinitely many points on the circle, and there are points on it arbitrarily close to each other: the points form a "continuum" inasmuch as a "rational" person can define a continuum. This would look like a circle, for sure.

Exercise 2 (20p)**.** Show that a circle with arbitrary center (in $\mathbb{Q}^2$) and arbitrary radius (in $\mathbb{Q}$) also has a continuum of points and that a line that passes through a point in the circle nontangentially, passes though a second point as well, just as in our world.

Exercise 3 (10p)**.** Take now a second circle of radius one with center at $(0, d)$ where $d < 2$. In our world, $\mathbb{R}^2$, the circles would intersect. But do they *necessarily* intersect in $\mathbb{Q}^2$?

Exercise 4 (10p)**.** If $d = 1$, the point of intersection *if it exists* together with the two centers, $(0, 0)$ and $(1, 0)$ would form an equilateral triangle. However, does this equilateral triangle exist in $\mathbb{Q}^2$?

Exercise 5 (20p)**.** Is there *any* equilateral triangle in $\mathbb{Q}^2$?

You can reflect on whether the results of the exercises above prove that $\mathbb{R}$ exists in our world, and in fact whether such a proof is even possible. But these questions are not mathematical ones...