Borel summability in differential equations

For this purpose we use (5.33) and (5.34) in order to derive the behavior of \( Y \). It is a way of exploiting what Écalle has discovered in more generality, bridge equations.

We start with exploring a relatively trivial, nongeneric possibility, namely that \( y_0^+ = y_0^- =: y_0 \). (This is not the case for our equation, though we will not prove it here; we still analyze this case since it may occur in other equations.)

Since in this case

\[
y_0^\pm = \int_0^{\infty e^{\pm i\epsilon}} Y(p)e^{-px} dp = y_0
\]

we have \( y \sim \tilde{y}_0 \) in a sector of arbitrarily large opening. By inverse Laplace transform arguments, \( Y \) is analytic in an arbitrarily large sector in \( \mathbb{C} \setminus \{0\} \).

On the other hand, we already know that \( Y \) is analytic at the origin, and it is thus entire, of exponential order at most one. Then, \( \tilde{y}_0 \) converges.

Exercise 5.45 Complete the details in the argument above.

We now consider the generic case \( y^+ \neq y^- \). Then there exists \( S \neq 0 \) so that

\[
y^+ = \int_0^{\infty e^{+i\epsilon}} e^{-px} Y^-(p) dp = y^- + \sum_{k=1}^{\infty} S^k e^{-kx} x^k \varphi_k^- (x) \tag{5.46}
\]

Thus

\[
\int_{\infty e^{-i\epsilon}}^{\infty e^{+i\epsilon}} e^{-px} Y(p) dp = \int_1^{\infty} e^{-px} (Y^+(p) - Y^-(p)) dp = \sum_{k=1}^{\infty} S^k e^{-kx} x^k \varphi_k^- (x) \tag{5.47}
\]

In particular, we have (since \( \Phi \) is continuous, this means (for \( p \neq 1 \))

\[
\frac{1}{x} \int_1^{\infty} e^{-px} (Y^+(p) - Y^-(p)) dp = Se^{-x} \int_0^{\infty e^{-i\epsilon}} e^{-px} \Phi_1 (p) dp + O(x^2 e^{-2x})
\]

\[
= S \int_1^{\infty e^{-i\epsilon}} e^{-px} \Phi_1 (p - 1) dp + O(x^2 e^{-2x}) \tag{5.48}
\]

Then, by Proposition 2.22, \( \int_0^{\infty} Y^+ = \int_0^{\infty} Y^- + S \Phi_1 (p - 1) \) on \((1, 2)\). (It can be checked that \( \int Y \) has lateral limits on \((1, 2)\), by looking at the convolution equation in a focusing space of functions continuous up to the boundary.)

Since \( \Phi_1 \) is continuous, this means (for \( p \neq 1 \)) \( \int_0^{\infty} Y^+ = S \Phi_1 (p - 1) + \int_0^{\infty} Y^- + SY_1(p - 1) \), or yet, \( Y^+ (1 + s) = Y^- (1 + s) + SY_1(s) \) everywhere in the right half plane where \( Y^- (1 + s) + SY_1(s) \) is analytic, in particular in the fourth quadrant. Thus the analytic continuation of \( Y \) from the upper plane along a curve passing between 1 and 2 exists in the lower half-plane; it equals the continuation of two functions along a straight line not crossing any singularities. The proof proceeds by induction, reducing the number of crossings at the expense of using more of the functions \( Y_2, Y_3, \ldots \).

This analysis can be adapted to general differential equations, and it allows for finding the resurgence structure (singularities in \( p \)) by constructing and solving Riemann-Hilbert problems, in the spirit above.