Borel summability in differential equations

\[ AC_\gamma((\psi \ast \psi)) = AC_\gamma(\psi) \ast AC_\gamma(\psi) \text{ if } \arg(\gamma(t)) = \text{const} \neq 0 \]  

(5.266)

(Since \( \psi \) is analytic along such a line). The notation \( \ast_\gamma \) means (2.21) with \( p = \gamma(t) \).

Note though that, suggestive as it might be, (5.266) is incorrect if the condition stated there is not satisfied and \( \gamma \) is a path that crosses the real line (see §5.11a)! From (5.266) and (5.264), we get

\[
(\psi \ast \psi)^{-} = \psi^{-} \ast \psi^{-} = \psi^{+} \ast \psi^{+} + \sum_{k=1}^{\infty} \left( \mathcal{H} \sum_{m=0}^{k} \psi_{m}^{+} \ast \psi_{k-m}^{+} \right) \circ \tau^{-k} = (\psi \ast \psi)^{+} + \sum_{k=1}^{\infty} \left( \mathcal{H} \sum_{m=0}^{k} \psi_{m}^{+} \ast \psi_{k-m}^{+} \right) \circ \tau^{-k} \]  

(5.267)

Note. Check that, if \( f, g \) are 0 on \( \mathbb{R}^- \) and \( k \geq m \), then

\[
\int_{0}^{p} f(s-m)g(p-[s-(k-m)])ds = \int_{0}^{p-k} f(t)g(p-k-t)dt \]  

(5.265)

Now the analyticity of \( \psi \ast \psi \) in \( \mathcal{R}_1 \) follows: on the interval \( p \in (j, j+1) \) we have from (5.265)

\[
(\psi \ast \psi)^{-}(p) = (\psi \ast \psi)^{-}(p) = (\psi^{*2})^{+}(p) + \sum_{k=1}^{j} \sum_{m=0}^{k} \left( \psi_{m}^{+} \ast \psi_{k-m}^{+} \right) \circ \tau^{-k} \]  

(5.268)

Again, formula (5.268) is useful for analytically continuing \( (\psi \ast \psi)^{-} \) along a path as the one depicted in Fig. 5.1. By dominated convergence, \( (\psi \ast \psi)^{\pm} \in \mathcal{Q}_{\gamma(t, \infty)}(5.260) \). By (5.265), \( \psi_{m}^{+} \) are analytic in \( \mathcal{R}_{1}^{+} := \mathcal{R}_{1} \cap \{ p : \text{Im}(p) > 0 \} \) and thus by (5.266) the right side of (5.268) can be continued analytically in \( \mathcal{R}_{1}^{+} \). The same is then true for \( (\psi \ast \psi)^{-} \). The function \( (\psi \ast \psi) \) can be extended analytically along paths that cross the real line from below. Likewise, \( (\psi \ast \psi)^{+} \) can be continued analytically in the lower half-plane so that \( (\psi \ast \psi) \) is analytic in \( \mathcal{R}_{1}^{+} \).

Combining (5.268), (5.266) and (5.263) we get a similar formula for the analytic continuation of the convolution product of two functions, \( f, g \) satisfying the assumptions of Proposition 5.259

\[
(f \ast g)^{-} = f^{+} \ast g^{+} + \sum_{k=1}^{j} \left( \mathcal{H} \sum_{m=0}^{k} f_{m}^{+} \ast g_{k-m}^{+} \right) \circ \tau^{-k} \]  

(5.269)

Note that (5.269) corresponds to (5.264) and in those notations we have:

\[
(f \ast g)_{k} = \sum_{m=0}^{k} f_{m} \ast g_{k-m} \]  

(5.270)