Integrability as well as (5.262) follow from (5.265), (5.268) and Remark 5.140.

By (5.118) and (5.265),

\[ \psi_{ba} = \psi^+ + \sum_{k=1}^{\infty} \frac{1}{2^k} (\mathcal{H} \circ \tau \circ \tau \circ \cdots \circ \tau)(p) \]

so that

\[
(\psi_{ba}^\ast \psi_{ba})(p) = \left( \psi^+(p) + \sum_{k=1}^{\infty} \frac{1}{2^k} (\mathcal{H}(p-k)\psi^+(p-k)) \right)^2 = \\
(\psi^+ \ast \psi^+)(p) + \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{m=0}^{k} (\mathcal{H} \psi^+_m(p-k) \ast (\mathcal{H} \psi^+_{k-m})(p-k+m)) = (\psi^2)^{ba}
\]

(5.271)

where we used (5.270) (see also the note on p. 197).

To finish the proof of Theorem 5.120, note that on any finite interval the sum in (5.118) has only a finite number of terms and by (5.271) balanced averaging commutes with any finite sum of the type

\[
\sum_{k_1, \ldots, k_n} c_{k_1, \ldots, k_n} f_{k_1} \ast \cdots \ast f_{k_n}
\]

and then, by continuity, with any sum of the form (5.272), with a finite or infinite number of terms, provided it converges in \( L^1_{loc} \). Averaging thus commutes with all the operations involved in the equations (5.222). By uniqueness therefore, if \( Y_0 = Y^{ba} \) then \( Y_k = Y^{ba}_k \) for all \( k \). Preservation of reality is immediate since (5.89), (5.92) are real if (5.51) is real, therefore \( Y^{ba}_k \) is real-valued on \( \mathbb{R}^+ \setminus \mathbb{N} \) (since it is real-valued on \([0, 1) \cup (1, 2)\)) and so are, inductively, all \( Y_k \).

5.11 Appendix

5.11a AC(\( f \ast g \)) versus AC(\( f \ast AC(g) \))

Typically, the analytic continuation along curve in \( \mathcal{W}_1 \) which is not homotopic to a straight line does not commute with convolution.