Asymptotic and transasymptotic matching: formation of singularities

Proposition 6.19  Take \( \lambda_1 = 1 \), \( n = 1 \) in (5.51) and let \( \xi = x^\alpha e^{-x} \). Assume that the corresponding \( F_0 \) in (6.13) is not entire (this is generic\(^1\)). Let \( \mathbb{D}_r \) be the maximal disk where \( F_0 \) is analytic\(^2\). Assume \( \xi_0 \in \partial \mathbb{D}_r \) is a singular point of \( F_0 \) such that \( F_0 \) admits analytic continuation in a ramified neighborhood of \( \xi_0 \). Then \( y \) is singular at infinitely many points, asymptotically given by

\[
x_n = 2n\pi i + \alpha_1 \ln(2n\pi i) + \ln C - \ln \xi_0 + o(1) \ (n \to \infty)
\]

(6.20)

(recall that \( C = C_+ \)).

Remark 6.21  We note that asymptotically \( y \) is a function of \( C e^{-x} x^\alpha \). This means that there are infinitely many singular points, nearly periodic, since the points \( x \) so that \( C e^{-x} x^\alpha = \xi_0 \) are nearly periodic.

We need the following result which is in some sense a converse of Morera’s theorem.

Lemma 6.22  Assume that \( f(\xi) \) is analytic on the universal covering of \( \mathbb{D}_r \setminus \{0\} \). Assume further that for any circle around zero \( C \subset \mathbb{D}_r \setminus \{0\} \) and any \( g(\xi) \) analytic in \( \mathbb{D}_r \) we have \( \int_C f(\xi)g(\xi)d\xi = 0 \). Then \( f \) is in fact analytic in \( \mathbb{D}_r \).

PROOF  Let \( a \in \mathbb{D}_r \setminus \{0\} \). It follows that \( \int_a^\xi f(s)ds \) is single-valued in \( \mathbb{D}_r \setminus \{0\} \). Thus \( f \) is single-valued and, by Morera’s theorem, analytic in \( \mathbb{D}_r \setminus \{0\} \). Since by assumption \( \int_C f(\xi)\xi^n d\xi = 0 \) for all \( n \geq 0 \), there are no negative powers of \( \xi \) in the Laurent series of \( f(\xi) \) about zero: \( f \) extends as an analytic function at zero.

PROOF of Proposition 6.19  By Lemma 6.22 there is a circle \( \mathcal{C} \) around \( \xi_s \) and a function \( g(\xi) \) analytic in \( \mathbb{D}_r(\xi - \xi_s) \) so that \( \int_{\mathcal{C}} F_0(\xi)g(\xi)d\xi = 1 \). In a neighborhood of \( x_n \in X \) the function \( f(x) = e^{-x} x^\alpha \) is conformal and for large \( x_n \)

\[
- \oint_{f^{-1}(\mathcal{C})} y(x)g(f(x))f'(x)dx = \oint_{\mathcal{C}} (F_0(\xi) + O(x_n^{-1}))g(\xi)d\xi = 1 + O(x_n^{-1}) \neq 0 \quad (6.23)
\]

It follows that for large enough \( x_n \), \( y(x) \) is not analytic inside \( \mathcal{C} \) either. Since the radius of \( \mathcal{C} \) can be taken \( o(1) \) the result follows.

\(^1\) After suitable changes of variables; see comments after Theorem 6.57.
\(^2\) By Theorem 6.57 \( F_0 \) is always analytic at zero.