(ii) If \( f \in L^1(\mathbb{R}) \) and \( |x|^\eta f(x) \in L^1(\mathbb{R}) \) with \( \eta \in (0, 1] \), then its Fourier transform \( \hat{f} = \int_{-\infty}^{\infty} f(s) e^{-ixs} ds \) is in \( C^0(\mathbb{R}) \).

(iii) Let \( f \in L^1(\mathbb{R}) \). If \( x^n f \in L^1(\mathbb{R}) \) with \( n - 1 \in \mathbb{N} \) then \( \hat{f} \) is \( n \) times differentiable, with the \( n - 1 \)th derivative Lipschitz continuous. If \( e^{Ax} f \in L^1(\mathbb{R}) \) then \( \hat{f} \) extends analytically in a strip of width \( |A| \) centered on \( \mathbb{R} \).

PROOF  
(i) We have as \( x \to \infty \) (\( \lfloor \cdot \rfloor \) denotes the integer part)

\[
\left| \int_0^1 f(s) e^{ixs} ds \right| = \left| \sum_{j=0}^{\lfloor \frac{x}{2\pi} \rfloor - 1} \left( \int_{(2j+1)\pi x}^{(2j+2)\pi x} f(s) e^{ixs} ds \right) + O(x^{-1}) \right|
\]

\[
= \left| \sum_{j=0}^{\lfloor \frac{x}{2\pi} \rfloor - 1} \int_{2j\pi x}^{(2j+1)\pi x} \left( f(s) - f(\pi/s) \right) e^{ixs} ds \right| + O(x^{-1})
\]

\[
\leq \sum_{j=0}^{\lfloor \frac{x}{2\pi} \rfloor - 1} a \left( \frac{\pi}{x} \right)^\eta \frac{\pi}{x} \leq \frac{1}{2} a \pi^\eta x^{-\eta} + O(x^{-1}) \quad (3.58)
\]

(ii) We see that

\[
\left| \frac{\hat{f}(s) - \hat{f}(s')}{|s - s'|^\eta} \right| = \left| \int_{-\infty}^{\infty} \frac{e^{-ixs} - e^{-ixs'}}{|s - s'|^\eta} x^n f(x) dx \right| \leq \int_{-\infty}^{\infty} \left| \frac{e^{-ixs} - e^{-ixs'}}{(x - x')^\eta} \right| |x^n f(x)| dx
\]

(3.59)

is bounded. Indeed, by elementary geometry we see that for \( \phi_1 - \phi_2 < 1 \) we have

\[
|\exp(i\phi_1) - \exp(i\phi_2)| \leq |\phi_1 - \phi_2| \leq |\phi_1 - \phi_2|\eta \quad (3.60)
\]

while for \( |\phi_1 - \phi_2| \geq 1 \) we see that

\[
|\exp(i\phi_1) - \exp(i\phi_2)| \leq 2 \leq 2|\phi_1 - \phi_2|\eta
\]

(iii) Follows in the same way as (ii), using dominated convergence. \( \square \)

Exercise 3.61 Complete the details of this proof. Show that for any \( \eta \in (0, 1] \) and all \( \phi_{1,2} \in \mathbb{R} \) we have \( |\exp(i\phi_1) - \exp(i\phi_2)| \leq 2|\phi_1 - \phi_2|\eta \).

Note. In Laplace type integrals Watson’s lemma implies that it suffices for a function to be continuous to ensure an \( O(x^{-1}) \) decay of the integral, whereas in Fourier-like integrals, the considerably weaker decay (3.57) is optimal as seen in the exercise below.