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Exponential Asymptotics and the Theory of Analyzable Functions

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Contents

1. Review of some results in classical asymptotics

1.1 Asymptotic expansions and asymptotic power series

Classical asymptotics typically deals with the qualitative and quantitative description of the behavior of a function (in some direction) near a point, usually a singularity of the function. This description is usually provided in the form of an *asymptotic expansion*, a formal series (that is, there are no convergence requirements) of simpler functions $\tilde{f}_k$,

$$f \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k(t) \qquad (\text{as } t \to t_0) \tag{1.1}$$

in which each successive term is much smaller than its predecessors, written

$$\tilde{f}_{k+1}(t) = o(\tilde{f}_k(t)) \quad \text{or} \quad \tilde{f}_{k+1}(t) \ll \tilde{f}_k(t)$$

denoting

$$\lim_{t \to t_0} \tilde{f}_{k+1}(t)/\tilde{f}_k(t) = 0 \tag{1.2}$$

Functions asymptotic to a series. The relation $f \sim \tilde{f}$ between an actual function and a formal expansion is defined as a sequence of limits, the Poincaré definition of asymptoticity

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = f(t) - \tilde{f}^{[N]}(t) = o(\tilde{f}_N(t)) \quad (\forall N \in \mathbb{N}) \tag{1.3}$$

Condition (1.3) can then be also written as

$$f(t) - \sum_{k=0}^{N} \tilde{f}_k(t) = O(\tilde{f}_{N+1}(t)) \quad (\forall N \in \mathbb{N}) \tag{1.4}$$

where $g(t) = O(h(t))$ means $\limsup_{t \to t_0} |g(t)/h(t)| < \infty$

Note. It is often important to use one-sided limits or more generally to restrict the limiting process to special directions or curves in the complex plane ending at t_0. We allow for this case, and only mention the direction or curve when it matters for the result.

Especially in this case there are some technical advantages in changing over to $t_0 = +\infty$; in this case we shall usually denote by x the variable. We ordinarily use z for variable when the limiting point is zero.

1.1a Asymptotic power series

A special role is played by power series, which are series of the form

$$\tilde{S} = \sum_{k=0}^{\infty} c_k z^k \quad (z \to 0) \tag{1.5}$$

Remark. The prevailing convention allows for some (or even all) of the c_k's to be zero to ensure better algebraic properties. If a c_k is zero then (1.2) fails trivially in which case (1.5) is not, strictly speaking, an asymptotic series.

A function has a given asymptotic *power* series iff (1.3) by

$$f(z) - \sum_{k=0}^{N} c_k z^k = O(z^{N+1}) \quad (\forall N \in \mathbb{N}) \tag{1.6}$$

In this sense the power series at zero of e^{-1/x^2} is the zero series. It is certainly incorrect to conclude that the asymptotic behavior of e^{-1/x^2} is zero. We use the boldface notation $\boldsymbol{\sim}$ for the stronger asymptoticity condition in (1.3).

Asymptotic power series form an algebra; addition of asymptotic power series is defined in the usual way:

$$A\sum_{k=0}^{\infty} c_k z^k + B\sum_{k=0}^{\infty} c'_k z^k = \sum_{k=0}^{\infty}(Ac_k + Bc'_k) z^k$$

while multiplication is defined as in the convergent case

$$\left(\sum_{k=0}^{\infty} c_k z^k\right)\left(\sum_{k=0}^{\infty} c'_k z^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} c_j c'_{k-j}\right) z^k$$

Remark 1.7 *If the series $\tilde{f}$ is convergent and f is its sum (note the ambiguity of the "sum" notation) $f = \sum_{k=0}^{\infty} c_k z^k$ then $f \sim \tilde{f}$.*

The proof of this remark follows directly from the definition of convergence.

Lemma 1.8 *(Uniqueness of the asymptotic series to a function) If $f(z) \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k z^k$ as $z \to 0$ then the $\tilde{f}_k$ are unique.*

Proof. Assume that we also have $f(z) \sim \tilde{F} = \sum_{k=0}^{\infty} \tilde{F}_k z^k$. We then have (cf. (1.3))

$$\tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = o(z^N)$$

which is impossible unless $g_N(z) = \tilde{F}^{[N]}(z) - \tilde{f}^{[N]}(z) = 0$, as it can be easily checked, since g_N is a polynomial of degree N in z.

The proof of the following lemma is immediate:

Lemma 1.9 *(Algebraic properties of asymptoticity to a power series) If* $f \sim \tilde{f} = \sum_{k=0}^{\infty} c_k z^k$ *and* $g \sim \tilde{g} = \sum_{k=0}^{\infty} d_k z^k$ *then*
(i) $Af + Bg \sim A\tilde{f} + B\tilde{g}$
(ii) $fg \sim \tilde{f}\tilde{g}$

Sometimes it is convenient to check an (apparently) weaker condition of asymptoticity:

Lemma 1.10 *There exists a sequence* $p_n \to \infty$ *such that for any* n *there is a* $p(n)$ *such that*

$$f(z) - \tilde{f}^{[p_n]}(z) = o(z^n) \quad as \quad z \to 0$$

then $f \sim \tilde{f}$.

Proof. If $p_n \leq n$ for all n there is nothing to show, otherwise, without loss of generality we may assume that $p_n \geq n$ (indeed, otherwise we extract such a subsequence). We then have

$$f(z) - \tilde{f}^{[n]} = (f(z) - \tilde{f}^{[p_n]}) + (\tilde{f}^{[p_n]} - \tilde{f}^{[n]}) = o(z^n) \ (z \to 0)$$

as it can be easily since $z^{-n-1}(\tilde{f}^{[p_n]} - \tilde{f}^{[n]})$ is a polynomial.

We now show that any asymptotic series is asymptotic to some function. A sharper version of the proposition below, the Borel-Ritt lemma, will be proved later.

Proposition 1.11 *Let* $\tilde{f} = \sum_{k=0}^{\infty} a_k z^k$ *be a power series. There exists a function* $f(z)$ *such that* $f(z) \sim \tilde{f}$ *as* $z \to 0$.

Proof. The following elementary proof has some ideas in common with optimal truncation of series, a powerful numerical technique in asymptotics.

By Remark 1.7 we can assume, without loss of generality, that the series has zero radius of convergence.

For every z, we will add “sufficiently many but not too many” terms of the series $\tilde{f}$.

For every z let $N(z)$ be the largest N such that $|a_n| \leq 2^{-n}|z|^{-n/2}$ for all $n \leq N$. ($N(z) < \infty$, otherwise the series would have nonzero radius of convergence.) It is also easy to see that $N(z)$ is increasing as $|z|$ decreases and that $N(z) \to \infty$ as $z \to 0$. Consider

$$f(z) = \sum_{j=0}^{N(z)} a_n z^n$$

Let N be given and choose z_0 such that $N(z_0) \geq N$. For $|z| < |z_0|$ we have

$$\left| f(z) - \sum_{n=0}^{N} a_n z^n \right| = \left| \sum_{n=N+1}^{N(z)} a_n z^n \right| \leq \sum_{j=N+1}^{N(z)} |z^{j/2}| 2^{-j} \leq |z|^{N/2+1/2}$$

Using now Lemma 1.10, the proof follows.

There is certainly no uniqueness in this generality. Given a power series there are many functions asymptotic to it. Indeed there are many functions asymptotic to the (identically) zero power series at zero, in any sectorial punctured neighborhood of zero in the complex plane, and even on the Riemann surface of the log on $\mathbb{C} \setminus \{0\}$, e.g. $e^{-x^{-1/n}}$ has this property in a sector of width $2n\pi$.

1.1b Integration and differentiation of asymptotic power series.

While asymptotic power series can be safely integrated term by term as the next proposition will show, differentiation is more delicate. We will much later see that this asymmetry is largely in suitable spaces of functions and expansions. But for the moment note that the function $e^{-1/z^2} \sin(e^{1/z^4})$ is asymptotic to the zero power series as $z \to 0$ with z real although the derivative is unbounded and thus not asymptotic to the zero series.

Proposition 1.12 *Assume $f(x)$ is integrable near $x = 0$ and that*

$$f(z) \sim \tilde{f} = \sum_{k=0}^{\infty} \tilde{f}_k z^k$$

Then

$$\int_0^z f(s) ds \sim \int \tilde{f} := \sum_{k=0}^{\infty} \frac{\tilde{f}_k}{k+1} z^{k+1}$$

Proof. This follows from the fact that $\int_0^z o(s^n) ds = o(z^{n+1})$ as can be seen by immediate estimates.

Sectorial asymptotic power series of analytic function can be differentiated:

Proposition 1.13 *Assume $f(x)$ is analytic in the strip $S_a = \{x : |x| > R, |\Im(x)| < a\}$. Let $\alpha < a$ and and $S_\alpha = \{x : |x| > R, |\Im(x)| < \alpha\}$ and assume that*

$$f(x) \sim \tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k} \quad (|x| \to \infty, x \in S_\alpha)$$

Then, for $\alpha' < \alpha$ we have

$$f'(x) \sim \tilde{f}'(x) := \sum_{k=0}^{\infty} -\frac{k c_k}{x^{k+1}} \quad (|x| \to \infty, x \in S_{\alpha'})$$

Proof. We have $f(x) = \tilde{f}^{[N]}(x) + g_N(x)$ where clearly g is analytic in S_a and $|g(x)| \le Const.|x|^{-N-1}$ in S_α. But then, for $x \in S_{\alpha'}$ and $\delta = \frac{1}{2}(\alpha - \alpha')$ we get for some $C > 0$ which depends on δ but not on x,

$$|g_N'(x)| = \frac{1}{2\pi}\left|\oint_{|x-s|=\delta} \frac{g(s)ds}{(s-x)^2}\right| \le \frac{C}{|x|^{N+1}} \quad (|x| \to \infty,\ x \in S_{\alpha'})$$

By Lemma 1.10, the proof follows.

In many instances the functions (scales) f_k are combinations of exponentials, powers of x, and logarithms. This is not simply a matter of choice or an accident, but reflects some important fact about the relation between asymptotic expansions and functions which will be clarified shortly.

1.1c Asymptotics of integrals: first results

Example: Integration by parts and elementary truncation to the least term. A solution of the differential equation

$$f' - 2xf = 1 \tag{1.14}$$

is the complementary error function

$$I(x) = e^{x^2} \int_x^\infty e^{-s^2} ds \tag{1.15}$$

Let us find the asymptotic behavior of $I(x)$ for $x \to \infty$. One very simple technique is integration by parts, done in a way that the successive terms decrease rapidly. We have

$$I(x) = \frac{1}{2x} - \frac{e^{x^2}}{2}\int_x^\infty \frac{e^{-s^2}}{s^2}ds = \frac{1}{2x} - \frac{1}{4x^2} + \frac{3e^{x^2}}{4}\int_x^\infty \frac{e^{-s^2}}{s^4}ds = ...$$
$$= \sum_{k=0}^{m-1} \frac{(-1)^k \Gamma(k+\frac{1}{2})}{2\sqrt{\pi}x^{2k+1}} + \frac{(-1)^m e^{x^2}\Gamma(m+\frac{1}{2})}{2\sqrt{\pi}}\int_x^\infty \frac{e^{-s^2}}{s^{2m+1}}ds \quad (1.16)$$

where is is easy to see that the series generated in this way is an *alternating* series and by looking at the remainder term we see that the actual value of $I(x)$ is always contained between two successive truncations of the power series obtained, for instance

$$\frac{1}{2x} - \frac{1}{4x^3} \leq I(x) \leq \frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} \quad (1.17)$$

$$\frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} - \frac{5}{16x^7} \leq I(x) \leq \frac{1}{2x} - \frac{1}{4x^3} + \frac{3}{8x^5} - \frac{5}{16x^7} + \frac{105}{32x^9} \quad (1.18)$$

Thus the error term when truncating the series is always of the order of magnitude of the first discarded term. The series (1.16) has zero radius of convergence, and thus for large x, although the terms start by decreasing rapidly, they ultimately increase again and tend to infinity. The best approximation based on (1.16) is thus obtained by *optimal truncation*, at the ($x-$dependent) order where the discarded term is minimal. This procedure is called *truncation at the least term* and in an alternating series example like the present one is immediately justified; more analysis is required in general. The least term in our example is of order 10^{-12} when $x = 5$ and of order 10^{-45} when $x = 10$ (!) Although ultimately divergent the series provides very accurate information about the function represented.

*

Often solutions of differential or difference equations are presented in the form

$$F(x) = \int_a^b e^{xg(s)} f(s)ds \quad (1.19)$$

and the behavior as $x \to \infty$ of $F(x)$ is needed. Three particular cases are more important: (1) The case where all parameters are real (dealt with by the so-called Laplace method); (2) the case where everything is real except x which is taken to be purely imaginary (stationary phase method) and (3) Everything is complex and f and g are analytic (Steepest descent method). In this latter case, the integral may also come as a contour integral along some path.

(1) The Laplace method. Even when very little regularity can be assumed about the functions, we can still infer something about the large x behavior of (1.19).

Proposition 1.20 *If $g(s) \in L^\infty([a,b])$ then*

$$\lim_{x\to\infty}\left(\int_a^b e^{xg(s)}ds\right)^{1/x} = e^{\|g\|_\infty}$$

Proof. This is simply the fact that $\|f\|_n \to \|f\|_\infty$.

Note that this does not give the asymptotic expansion of (1.19) for large x in the sense of (1.3). For that, more regularity needs to be assumed.

Proposition 1.21 *(g is maximum at one endpoint) Assume f is continuous on $[a,b]$, $f(a) \neq 0$, g is in $C^1[a,b]$ and $g' < -\alpha < 0$ on $[a,b]$. Then*

$$J_x := \int_a^b f(s)e^{xg(s)}ds = \frac{f(a)e^{xg(a)}}{x|g'(a)|}(1+o(1)) \quad (x\to+\infty) \tag{1.22}$$

Note: The derivative of g enters in the final result, so regularity is needed.

Proof. Without loss of generality, we may assume $a=0, b=1$. Let ϵ be small enough and choose δ such that if $x<\delta$ we have $|f(x)-f(0)|<\epsilon$ and $|g'(x)-g'(0)|<\epsilon$.
We write

$$\int_0^1 f(s)e^{xg(s)}ds = \int_0^\delta f(s)e^{xg(s)}ds + \int_\delta^1 f(s)e^{xg(s)}ds \tag{1.23}$$

the last integral in (1.23) is bounded by

$$\int_\delta^1 f(s)e^{xg(s)}ds \le \|f\|_\infty e^{xg(0)}e^{x(g(\delta)-g(0))} \tag{1.24}$$

For the middle integral in (1.23) we have

$$\int_0^\delta f(s)e^{xg(s)}ds \le (f(0)+\epsilon)\int_0^\delta e^{x\left[g(0)+(g'(0)+\epsilon)s\right]}ds$$
$$\le -\frac{e^{xg(0)}}{x}\frac{f(0)+\epsilon}{g'(0)+\epsilon}\left[1-e^{x\delta(g'(0)+\epsilon)}\right] \tag{1.25}$$

Combining these estimates, as $x\to\infty$ we thus obtain

$$\limsup_{x\to\infty} xe^{-xg(0)}\int_0^1 f(s)e^{xg(s)}ds \le -\frac{f(0)+\epsilon}{g'(0)+\epsilon} \tag{1.26}$$

A lower bound is obtained in a similar way. Since ϵ is arbitrary, the result follows.

When the maximum is reached inside the interval of integration, a similar analysis requires more regularity.

Proposition 1.27 *(Interior maximum) Assume $f \in C[-1,1]$, $g \in C^2[-1,1]$ has a unique absolute maximum, at $x = 0$, and that $f(0) \neq 0$ and $g''(0) < 0$. Then*

$$\int_{-1}^{1} f(s)e^{xg(s)}ds = \sqrt{\frac{2\pi}{x|g''(0)|}} f(0)e^{xg(0)}(1 + o(1)) \quad (x \to +\infty) \tag{1.28}$$

Proof. The proof is similar to the previous one. Let ϵ be small enough and let δ be such that $|s| < \delta$ implies $|g''(s) - g''(0)| < \epsilon$ and also $|f(s) - f(0)| < \epsilon$. We write

$$\int_{-1}^{1} e^{xg(s)}f(s)ds = \int_{-\delta}^{\delta} e^{xg(s)}f(s)ds + \int_{|s|\geq\delta} e^{xg(s)}f(s)ds \tag{1.29}$$

The last term will not contribute in the limit since by assumption for some $\alpha > 0$ and $|s| > \delta$ we have $g(s) - g(0) < -\alpha < 0$ and thus

$$e^{-xg(0)}\sqrt{x}\int_{|s|\geq\delta} e^{xg(s)}f(s)ds \leq 2\sqrt{x}\|f\|_\infty e^{-x\alpha} \to 0 \text{ as } x \to \infty \tag{1.30}$$

On the other hand,

$$\int_{-\delta}^{\delta} e^{xg(s)}f(s)ds \leq (f(0)+\epsilon)\int_{-\delta}^{\delta} e^{xg(0)+\frac{x}{2}(g''(0)+\epsilon)s^2}ds$$
$$\leq (f(0)+\epsilon)e^{xg(0)}\int_{-\infty}^{\infty} e^{xg(0)+\frac{x}{2}(g''(0)+\epsilon)s^2}ds = \sqrt{\frac{2\pi}{|g''(0)-\epsilon|}}(f(0)+\epsilon)e^{xg(0)} \tag{1.31}$$

An inequality in the opposite direction is obtained in the same way by noting that

$$\frac{\int_{-a}^{a} e^{-xs^2}ds}{\int_{-\infty}^{\infty} e^{-xs^2}ds} \to 1 \quad \text{as } x \to \infty \tag{1.32}$$

as can be seen by changing variables to $u = sx^{-\frac{1}{2}}$.

With appropriate decay conditions, the interval of integration does not have to be compact. For instance, let $J \subset \mathbb{R}$ be an interval (finite or not) and $[a,b] \subset J$.

Proposition 1.33 *(Interior maximum, noncompact interval) Assume* $f \in C[a,b] \cap L^\infty(J)$, $g \in C^2[a,b]$ *has a unique absolute maximum at* $x = c$ *and that* $f(c) \neq 0$ *and* $g''(c) < 0$.

Assume further that g *is measurable in* J *and* $g(c) - g(s) = \alpha + h(s)$ *where* $\alpha > 0$, $h(s) > 0$ *on* $J \setminus [a,b]$ *and* $e^{-h(s)} \in L^1(J)$. *Then,*

$$\int_A^B f(s) e^{xg(s)ds} = \sqrt{\frac{2\pi}{x|g''(c)|}} f(c) e^{xg(c)} (1 + o(1)) \quad (x \to +\infty) \tag{1.34}$$

Proof. This case reduces to the compact interval case by noting that

$$\left| \sqrt{x} e^{-xg(c)} \int_{J\setminus[a,b]} e^{xg(s)} f(s) ds \right| \le \sqrt{x} \|f\|_\infty e^{-x\alpha} \int_J e^{-xh(s)} ds$$
$$\le Const. \sqrt{x} e^{-x\alpha} \to 0 \text{ as } x \to \infty \tag{1.35}$$

Example. We see that the last proposition applies to the Γ function by writing

$$n! = \int_0^\infty e^{-t} t^n dt = n^{n+1} \int_0^\infty e^{n(-t+\ln t)} dt \tag{1.36}$$

whence we get Stirling's formula

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + o(1))$$

1.1d Watson's Lemma

We note that in many instances integral representations of functions are amenable to Laplace transforms

$$\mathcal{L}F := \int_0^\infty e^{-xp} F(p) dp \tag{1.37}$$

The behavior of $\mathcal{L}F$ for large x relates to the behavior for small p of F.

It will be shown in the sequel that solutions of generic analytic differential equations, under mild assumptions can be conveniently expressed in terms of Laplace transforms.

For the error function note that

$$\int_x^\infty e^{-s^2} ds = x \int_1^\infty e^{-x^2u^2} du = \frac{x}{2} \int_0^\infty \frac{e^{-x^2 p}}{\sqrt{p+1}} dp$$

For the Γ function, writing $\int_0^\infty = \int_0^1 + \int_1^\infty$ in (1.36) we can make the substitution $t - \ln t = p$ and obtain (see §1.1e)

$$n! = \int_0^\infty e^{-np} W(p)dp$$

Furthermore, note that the integral in Proposition 1.21 can be brought to the form (1.37) by extending f by zero to the whole line and changing variable to $g(t+a) = g(a) + u$. Similarly $u = g(\text{sign}(s)\sqrt{s}) - g(0)$ in Proposition 1.27 brings it to a problem of the form (1.37).

Lemma 1.38 *Let* $F \in L^1(\mathbb{R}^+)$ $x = \rho e^{i\phi}, \phi \in (-\pi/2, \pi/2)$ *and assume*

$$F(p) \sim p^\beta$$

with $\Re(\beta) > -1$ *as* $p \to 0^+$. *Then*

$$\int_0^\infty F(p)e^{-px}dp \sim \Gamma(\beta+1)x^{-\beta-1} \quad (\rho \to \infty)$$

Proof. If $U(p) = p^{-\beta}F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let χ_A be the characteristic function of the set A and $\phi = \arg(x)$. We choose $C, a > 0$ such that $|F(p)| < C|p^\beta|$ on $[0, a]$. Since

$$\left| \int_a^\infty F(p)e^{-px}\mathrm{d}p \right| \le e^{-xa}\|F\|_1 \tag{1.39}$$

we have, and after the change of variable $s = p/|x|$,

$$x^{\beta+1}\int_0^\infty F(p)e^{-px}\mathrm{d}p = e^{i\phi(\beta+1)}\int_0^\infty s^\beta U(s/|x|)\chi_{[0,a]}(s/|x|)e^{-se^{i\phi}}\mathrm{d}s$$
$$+ O(|x|^{\beta+1}e^{-xa}) \to \Gamma(\beta+1) \quad (|x| \to \infty) \tag{1.40}$$

Watson's Lemma, presented below, states that the asymptotic series at infinity of $(\mathcal{L}F)(x)$ is obtained by formal term-by-term integration of the asymptotic series of $F(p)$ for small p, provided F has such a series.

Lemma 1.41 *Let* $F \in L^1(\mathbb{R}^+)$ *and assume* $F(p) \sim \sum_{k=0}^\infty c_k p^{k\beta_1+\beta_2-1}$ *as* $p \to 0^+$ *for some constants* β_i *with* $\Re(\beta_i) > 0$, $i = 1, 2$. *Then*

$$\mathcal{L}F \sim \sum_{k=0}^\infty c_k \Gamma(k\beta_1 + \beta_2)x^{-k\beta_1-\beta_2}$$

along any ray ρ *in the open right half plane* H.

Proof. Induction, using Lemma 2.67. □

1.1e Example: Gamma function

We start from the representation

$$n! = \int_0^\infty t^n e^{-t} dt = n^{n+1} \int_0^\infty e^{-n(s-\ln s)} ds$$
$$= n^{n+1} \int_0^1 e^{-n(s-\ln s)} ds + n^{n+1} \int_1^\infty e^{-n(s-\ln s)} ds \quad (1.42)$$

On $(0,1)$ and $(1,\infty)$ separately, the function $s - \ln(s)$ is monotonic and we may write, after inverting $s - \ln(s) = t$ on the two intervals to get $s_{1,2} = s_{1,2}(t)$,

$$n! = n^{n+1} \int_1^\infty e^{-nt} (s_2'(t) - s_1'(t)) dt = n^{n+1} e^{-n} \int_0^\infty e^{-np} G'(p) dp \quad (1.43)$$

where $G(p) = s_2(1+p) - s_1(1+p))$. In order to determine the asymptotic behavior of $n!$ we need to determine the small p behavior of the function $G'(p)$

Remark 1.44 *The function $G(p)$ is an analytic function in $\sqrt{p}$ and thus $G'(p)$ has a convergent Puiseux series*

$$\sum_{k=-1}^{\infty} c_k p^{k/2} = \sqrt{2} p^{-1/2} + \frac{\sqrt{2}}{6} p^{1/2} + \frac{\sqrt{2}}{216} p^{3/2} - \frac{139\sqrt{2}}{97200} p^{5/2} + ...$$

Thus, by Watson's Lemma, for large n we have

$$n! \sim \sqrt{2\pi n} n^n e^{-n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + ...\right) \quad (1.45)$$

Proof. We write $s = 1 + S$ and $t = 1 + p$ and the equation $s - \ln(s) = t$ becomes $S - \ln(1+S) = p$. Note that $S - \ln(1+S) = S^2 U(S)/2$ where $U(0) = 1$ and $U(S)$ is analytic for small S; with the natural branch of the square root, $\sqrt{U(S)} = H(S)$ is also analytic. We rewrite $S - \ln(1+S) = p$ as $SH(S) = \pm\sqrt{2}\sigma$ where $\sigma^2 = p$. Since $(SH(S))'_{|S=0} = 1$ the implicit function theorem ensures the existence of two functions $S_{1,2}(\sigma)$ (corresponding to the two choices of sign) which are analytic in σ. The concrete expansion may be gotten by implicit differentiation in $SH(S) = \pm\sqrt{2}\sigma$, for instance.

1.1f Borel-Ritt Lemma: end of proof in a half plane

Proposition 1.46 *Given a formal power series $\tilde{f} = \sum_{k=0}^{\infty} \frac{c_k}{x^{k+1}}$ there exists an entire function $f(x)$, of exponential order one, which is asymptotic to $\tilde{f}$ in the right half plane, i.e., if $\phi \in (-\pi/2, \pi/2)$ then*

$$f(x) \sim \tilde{f} \text{ as } x = \rho e^{i\phi}, \quad \rho \to +\infty$$

Proof. Let $\tilde{F} = \sum_{k=0}^{\infty} \frac{c_k}{k!} p^k$, let $F(p)$ be a function asymptotic to $\tilde{F}$ as in Proposition 1.11. Then clearly the function

$$f(x) = \int_0^1 e^{-xp} F(p) dp$$

is entire, bounded by $Const.e^{|x|}$, and, by Watson's Lemma has the desired properties.

Exercises.
(1) How can this method be modified to give a function analytic in a sector of opening $2\pi n$ for an arbitrary fixed n which is asymptotic to $\tilde{f}$?

(2) Assume F is bounded on $[0,1]$ and has an asymptotic expansion $F(t) \sim \sum_{k=0}^{\infty} c_k t^k$ as $t \to 0^+$. Let $f(x) = \int_0^1 e^{-xp} F(p) dp$

(a) Find necessary and sufficient conditions on F such that $\tilde{f}$, the asymptotic power series of f for large x, is a convergent series for $|x| > R > 0$.

(c) Show that in case (a) there is a convergent representation of f in the form $\tilde{f} + e^{-x} \tilde{f}_1$ where $\tilde{f}_1$ is also a convergent series for $|x| > R > 0$.

(b) Assume that $\tilde{f}$ *converges to* $f(x)$. Show that f is zero.

(3) The width of the sector in Proposition 1.46 cannot be extended to a more than a half plane: Show that if f is entire and bounded in a sector of opening exceeding π, and of exponential order one then it is constant. (This follows immediately from the Phragmen-Lindelöf principle; an alternative proof can be derived from elementary properties of Fourier transforms and contour deformation.) The exponential order has to play a role in the proof: check that the function $\int_0^{\infty} e^{-px-p^2} dp$ is bounded for $\arg(x) \in (-\frac{3\pi}{4}, \frac{3\pi}{4})$. How wide can such a sector be made?

1.1g Oscillatory integrals and the stationary phase method

In this setting, an integral of a function against a rapidly oscillating exponential becomes small as the frequency of oscillation increases. Again we first look at the case where there is minimal regularity; the following is a version of the Riemann–Lebesgue lemma.

Proposition 1.47 *Assume* $f \in L^1[0,1]$. *Then* $\int_0^{2\pi} e^{ixt} f(t) dt \to 0$ *as* $x \to \infty$.

It is enough to show the result on a set which is dense in L^1. Since trigonometric polynomials are dense in $C[0,2\pi]$ in the sup norm, and thus in $L^1[0,2\pi]$, it suffices to look at trigonometric polynomials, thus at e^{ikx} for fixed k, where the integral can be expressed explicitly and gives

$$\int_0^{2\pi} e^{ixs}e^{iks}ds = O(x^{-1}) \ \text{ for large } x. \qquad \square$$

No rate of decay follows without further knowledge about the regularity of f. We have the following characterization:

Proposition 1.48 *For $\eta \in (0,1]$ let the $H^\eta[0,1]$ be the Hölder continuous functions of order η on $[0,1]$, i.e., the functions with the property that there is some C such that for all $x, x' \in [0,1]$ we have $|f(x) - f(x')| \le C|x - x'|^\eta$.*

(i) We have $f \in H^\eta[0,1] \Rightarrow \int_0^1 f(s)e^{ixs}ds = O(x^{-\eta})$ as $x \to \infty$.

(ii) If $f \in L^1(\mathbb{R})$ and $|x|^\eta f(x) \in L^1(\mathbb{R})$ with $\eta \in (0,1]$, then its Fourier transform $\hat{f} = \int_{-\infty}^{\infty} f(x)e^{-ixs}ds$ is in $H^\eta(\mathbb{R})$.

(iii) Let $f \in L^1(\mathbb{R})$. If $x^n f \in L^1(\mathbb{R})$ with $n \in \mathbb{N}$ then $\hat{f} \in C^{[n]}(\mathbb{R})$; If $e^{|Ax|}f \in L^1(\mathbb{R})$ then $\hat{f}$ extends analytically in a strip of width $|A|$ centered on $\mathbb{R}$.

Note. The rate of decay may improve if the lack of regularity is due to behavior at isolated points for otherwise smoother functions. Such is for instance the function $f(x) = \sqrt{x}$, which is in $H^{1/2}[0,1]$ but not in $H^\eta[0,1]$ if $\eta > 1/2$, and yet $\int_0^1 e^{ixs}\sqrt{s}ds = O(x^{-1})$ as shown at the end of the proof of Proposition 1.56.

Proof. (i) We have as $x \to \infty$

$$\left|\int_0^1 f(s)e^{ixs}ds\right| =$$

$$\left|\sum_{j\in[0,\frac{x}{2\pi}-1)} \left(\int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} f(s)e^{ixs}ds + \int_{(2j+1)\pi x^{-1}}^{(2j+2)\pi x^{-1}} f(s)e^{ixs}ds\right)\right| + O(x^{-1})$$

$$= \left|\sum_{j\in[0,\frac{x}{2\pi}-1)} \int_{2j\pi x^{-1}}^{(2j+1)\pi x^{-1}} \big(f(s) - f(s+\pi/x)\big)e^{ixs}ds\right| + O(x^{-1})$$

$$\le \sum_{j\in[0,\frac{x}{2\pi}-1)} C\left(\frac{\pi}{x}\right)^\eta \frac{\pi}{x} \le Cx^{-\eta} + O(x^{-1}) \qquad (1.49)$$

(ii) We see that

$$\frac{\hat{f}(s) - \hat{f}(s')}{(s-s')^\eta} = \int_{-\infty}^{\infty} \frac{e^{ixs} - e^{ixs'}}{(xs - xs')^\eta} x^\eta f(x)dx$$

is bounded.

(iii) Follows in the same way as (ii), using dominated convergence.

Notes In part (i), compactness of the interval is crucial. Indeed, the function $f(x) = 1$ on the interval $[n, n + e^{-n^2}]$ for $n \in \mathbb{N}$ and zero otherwise is in $L^1(\mathbb{R})$ and further has the property that f and $e^{|Ax|}f \in L^1(\mathbb{R})$ for any A, and thus $\hat{f}$ is entire. Thus f is the Fourier transform of an entire function, $\hat{f}$, and nevertheless does not decay pointwise as $x \to \infty$.
(2) It is worth mentioning that in Laplace type integrals it suffices for a function to be continuous to ensure an $O(x^{-1})$ decay of the integral. This is for instance seen in Watson's Lemma when $\beta = 0$, but in Fourier-like integrals, continuity does not ensure $O(x^{-1})$ decay. When the conditions for the steepest descent method studied in the next section apply, a better control of decay of a Fourier type integral may be achieved by transforming it into a Laplace-like one.

Proposition 1.50 *Assume $f \in C^n[a,b]$. Then, if $m < n$ we have*

$$\int_a^b e^{ixt} f(t)dt = e^{ixa} \sum_{k=1}^{m} c_k x^{-k} + e^{ixb} \sum_{k=1}^{m} d_k x^{-k} + o(x^{-m})$$
$$= e^{ixt} \left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{m-1} \frac{f^{(m)}(t)}{(ix)^m} \right)\Bigg|_a^b + o(x^{-m}) \quad (1.51)$$

Proof. This follows by integration by parts since

$$\int_a^b e^{ixt} f(t)dt = e^{ixt} \left(\frac{f(t)}{ix} - \frac{f'(t)}{(ix)^2} + ... + (-1)^{m-1} \frac{f^{(m-1)}(t)}{(ix)^m} \right)\Bigg|_a^b$$
$$+ \frac{(-1)^m}{(ix)^m} \int_a^b f^{(m)}(t) e^{ixt} dt \quad (1.52)$$

Corollary 1.53 *(1) Assume $f \in C^\infty[a,b]$ is periodic with period $b-a$. Then $\int_a^b f(t)e^{int} = o(n^{-m})$ for any $m > 0$ as $n \to +\infty, n \in \mathbb{Z}$.*

(2) Assume $f \in C_0^\infty[a,b]$, a smooth function which vanishes with all derivatives at the endpoints; then $\hat{f}(x) = \int_a^b f(t)e^{ixt} = o(x^{-m})$ for any $m > 0$ as $x \to +\infty$.

Exercises. (a) Show that if is analytic in a neighborhood of $[a,b]$ but is not an entire function, then both series in (1.51) have zero radius of convergence.

(b) In Corollary 1.53 (2) show that $\limsup_{x\to\infty} e^{\epsilon|x|}|\hat{f}(x)| = \infty$ for any $\epsilon > 0$ unless $f = 0$.

Oscillatory integrals with monotonic phase.

Proposition 1.54 *Let the real valued functions $f \in C^m[a,b]$ and $g \in C^{m+1}[a,b]$ and assume $g' \neq 0$ on $[a,b]$. Then*

$$\int_a^b f(t)e^{ixg(t)}dt = e^{ixg(a)}\sum_{k=1}^{m} c_k x^{-k} + e^{ixg(b)}\sum_{k=1}^{m} d_k x^{-k} + o(x^{-m}) \quad (1.55)$$

as $x \to \infty$, where the coefficients c_k and d_k can be computed by Taylor expanding f and g at the endpoints of the interval of integration.

Proof. Since $g' \neq 0$ we may invert $g(t) = G$ in $C^{[m+1]}[a, b]$, change variables in the integral and write

$$\int_a^b f(t)e^{ixg(t)}dt = \int_{g(a)}^{g(b)} f(G(g))e^{ixg}G'(g)dg$$

and apply Proposition 1.50 to the latter integral. The computation of the coefficients c_k and d_k is straightforward.

Stationary phase method. We consider now the case when $g(s)$ has a stationary point inside the interval $[a, b]$. In this case the main contribution to the integral on the lhs of (1.55) comes from a neighborhood of the stationary point of g since around that point the oscillations that determine the integral to be small are less rapid. We have the following result:

Proposition 1.56 *Assume the real valued functions $f, g \in C^\infty[a, b]$ and that $g'(c) = 0$ $g''(x) \neq 0$ on $[a, b]$. Then for any $m \in \mathbb{N}$ we have*

$$J = \int_a^b f(s)e^{ixg(s)}ds = e^{ixg(c)}\sum_{k=1}^{2m} c_k x^{-k/2} + e^{ixg(a)}\sum_{k=1}^{m} d_k x^{-k} + e^{ixg(b)}\sum_{k=1}^{m} e_k x^{-k} + o(x^{-m}) \quad (1.57)$$

for large x, where the coefficients of the expansion can be calculated by Taylor expansion around a, b and c of the integrand. In particular, we have

$$c_1 = \sqrt{\frac{2\pi i}{g''(c)}}f(c)$$

Proof. It is convenient split the integral into $\int_a^c + \int_c^b$ and reduce to the case when the extremum is at one endpoint. By a change of variables we make $c = 0$ and $b = 1$, and by subtracting out $g(0)$ we arrange $g(0) = 0$. We analyze the case $g'' > 0$, the other case being similar. Let $g''(0) = 2\alpha$. On the interval $[0, 1]$ g is monotonic and we change variables to $g(s) = G$.

We can write

$$g(s) = \int_0^s (s-t)g''(t)dt = s^2\int_0(1-u)g''(su)du := s^2\alpha H(s)$$

with $H \in C^\infty$, $H > 0$ on $[0,1]$ and $H(0) = 1$. Thus $h(t) = \sqrt{H} \in C^\infty$, $h' \neq 0$ on $(0,1]$, and $(sh)'(0) = 1$ and $(sh)^{-1} \in C^\infty$ implying that the equation $g(s) = t$ has a solution of the form $G(\sqrt{t})$ with $G \in C^\infty$. With $b = g(1)$ we get

$$J = \int_0^b t^{-1/2} F(\sqrt{t}) e^{ixt} dt$$

with $F \in C^\infty$. As above, we can write $F(\sqrt{t}) = F(0) + \sqrt{t} F_1(\sqrt{t})$, $F_1 \in C^\infty$. Now

$$\int_0^b t^{-1/2} e^{ixt} dt = \int_0^\infty t^{-1/2} e^{ixt} dt - \int_b^\infty t^{-1/2} e^{ixt} dt$$
$$= \sqrt{\frac{\pi}{ix}} + \frac{i}{x\sqrt{b}} e^{ixb} - \frac{i}{2x} \int_b^\infty t^{-3/2} e^{ixt} \quad (1.58)$$

By integration by parts,

$$\int_0^b F_1(\sqrt{t}) e^{ixt} dt = \frac{1}{ix} F_1(\sqrt{t}) e^{ixt} \Big|_0^b - \frac{1}{ix} \int_0^b t^{-1/2} F_2(\sqrt{t}) e^{ixt} dt$$

with F_2 in C^∞. The proof is completed by induction.
Note It is easy to see that in the settings of Watson's Lemma and of Propositions 1.50, 1.54 and 1.56 the asymptotic expansions are differentiable, in the sense that the integral transforms are differentiable and their derivative is asymptotic to the formal derivative of the associated expansion.

1.1h Remarks about the form of asymptotic expansions

The asymptotic expansions seen in the previous examples have as a common feature that they are written in terms of powers of the variable, exponentials and logs, e.g.

$$\int_x^\infty e^{-s^2} ds \sim e^{-x^2} \left(1 + \frac{1}{2x} - \frac{1}{4x^2} + \frac{5}{8x^3} - ...\right) \quad (1.59)$$

$$n! \sim \sqrt{2\pi} e^{n \ln n - n - \frac{1}{2} \ln n} \left(1 + \frac{1}{12n} + ...\right) \quad (1.60)$$

$$\int_1^x \frac{e^t}{t} \sim e^x \left(\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x^3} + ...\right) \quad (1.61)$$

Hardy noted that "No function has yet presented itself whose asymptotic expansion cannot be expressed in terms of exponentials, power series and logs". The modern conjecture of Écalle states that functions of natural origin can be isomorphically represented by "transseries" in the same way as an analytic function is locally given by a convergent Taylor series. Transseries are *formal* combinations of exponentials, power series and logs which are asymptotic. It is convenient to take the limit setting $x \to +\infty$.

1.2 Steepest descent method

There are instances when there is further analytic structure in a problem involving oscillatory integrals with large parameter which can be used to get sharper estimates on the asymptotic behavior.

1.2a Examples

Example 1. Consider the problem of finding the asymptotic behavior of the integral

$$J(n) = \int_{-\pi}^{\pi} \frac{e^{-int}}{2 - e^{it}} dt := \int_{-\pi}^{\pi} F(t) dt$$

as $n \to \infty$. We see by corollary 1.53 that $J = o(x^{-m})$ for any $m \in \mathbb{N}$. In this case the stationary phase method fails to provide show what the leading asymptotic behavior of the integral is (it only shows what it *is not*). We are dealing here with an analytic periodic function, and the Fourier coefficients decay faster than power-like. We can use this analyticity information to understand in fact what the behavior is for large n. Note that F is analytic in $\mathbb{C} \setminus \{-i \ln 2 + 2k\pi i\}_{k \in \mathbb{Z}}$. and meromorphic in $\mathbb{C}$. Furthermore, as $N \to \infty$ we have $F(t - iN) \to 0$ exponentially fast. This allows us to push the contour of integration down, in the following way. Note that

$$\oint_C F(t) dt = -2\pi i \text{Res}\Big(F(t); t = -i \ln 2\Big)$$

where the contour C of integration is a clockwise rectangle with vertices $-\pi, \pi, -iN + \pi, -iN - \pi$ for any N sufficiently large. As $N \to \infty$ the integral over the segment $-iN + \pi, -iN - \pi$ goes to zero exponentially fast, and we find out that

$$\int_{-\pi}^{\pi} F(t) dt = \int_{-\pi}^{-\pi - i\infty} F(t) dt - \int_{\pi}^{\pi - i\infty} F(t) dt + \frac{\pi}{2} 2^{-n}$$

Note now that the two integrals cancel each-other completely because of periodicity of the integrand and we are left with

$$J(n) = \pi 2^{-n}$$

Note also that the same calculation works if we replace $n \in \mathbb{N}$ with $x \in \mathbb{R}^+$. In this case the integrals will not cancel each-other for all x and we end up with

$$J(x) = i(e^{ix\pi} - e^{-ix\pi}) \int_0^\infty \frac{e^{-xs}}{2+e^s} ds + \pi 2^{-x} = -2\sin \pi x \int_0^\infty \frac{e^{-xs}}{2+e^s} ds + \pi 2^{-x}$$

which is now in the form of a combination of integrals to which Watson's Lemma applies and gives a power series behavior and a small exponential. By applying Watson's lemma we get

$$J(x) - \sim 2\sin \pi x \left(\frac{1}{3x} - \frac{2}{9x^2} - \frac{4}{27x^3} + \frac{8}{27x^4} + \frac{80}{81x^5} - \frac{224}{243x^6} + ... \right) + \pi 2^{-x} \tag{1.62}$$

which is a simple example of a transseries. We shall see that (1.62) is the actual transseries of $J(x)$. However the power series diverges factorially and adding to it the exponentially small term makes classical sense only if n is an integer. The divergence follows from the fact that the term of order k of the series is by Watson's lemma $k!$ times the Taylor coefficient of the function $(2+e^s)^{-1}$ at $s = 0$ and this function is not entire. Thus its Taylor coefficients must grow faster than a^k for some a. Thus the power series part cannot be simply subtracted out of J to see "what is left" and on the other hand 2^{-x} is smaller for large enough x than any x^{-m} thus cannot be made part of the scales x^{-m}. In some sense we may say that Poincaré type asymptoticity is restricted to ordinal type ω and our example has higher ordinal length.

*

Example 2. The Bessel function $J_0(\xi)$ can be written as $\frac{1}{\pi}\mathrm{Re}\, I$, where

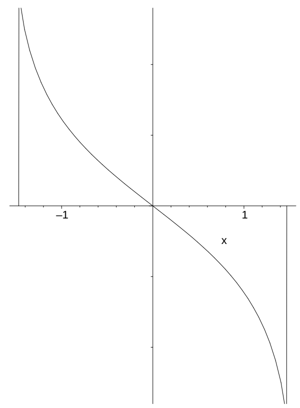
$$I = \int_{-\pi/2}^{\pi/2} e^{i\xi \cos t} dt \tag{1.63}$$

which in turn equals

$$I = \int_{-\pi/2}^{-\pi/2+i\infty} e^{i\xi \cos t} dt + \int_\gamma e^{i\xi \cos t} dt + \int_{\pi/2}^{\pi/2-i\infty} e^{i\xi \cos t} dt \tag{1.64}$$

as shown in the figure. All the curves involved in this decomposition of I are lines of constant imaginary part of the exponent, and the ordinary Laplace method can be applied to find their asymptotic behavior for $\xi \to +\infty$ (note also that the integral along the curve γ, called Sommerfeld contour, is the only one contributing to J_0).

*



Proposition 1.65 (Fourier coefficients of analytic functions) *Assume f is periodic of period 2π and is analytic in a strip of width $> R > 0$ sitting in the upper half plane over the real line. Then the Fourier coefficients $(2\pi)^{-1}\int_0^{2\pi} e^{int}f(t)dt$ are (at most) $O(e^{-nR})$ for large n.*

Proof. By analyticity, we have

$$\int_0^{2\pi} e^{int}f(t)dt = \int_0^{iR} e^{int}f(t)dt - \int_{2\pi}^{2\pi+iR} e^{int}f(t)dt + \int_{iR}^{2\pi+iR} e^{int}f(t)dt$$

The first two integrals on the rhs cancel by periodicity while the last one is manifestly $O(e^{-nR})$ for large n.

1.2b A more general discussion

Let

$$\int_C f(s)e^{xg(s)}ds \tag{1.66}$$

where g is analytic and f is meromorphic in a domain in the complex plane of the contour C and x is a large parameter.

The crucial idea is to use deformation of contours to bring the integral to one which is suitable to the application of the Laplace method. We can assume without loss of generality that x is real and positive.

(A) Let $g = u + iv$. and assume for C' is a curve such that $v = K$ is constant along C'. Then

$$\int_{C'} f(s)e^{xg(s)}ds = e^{xiK}\int_{C'} f(s)e^{xu(s)}ds = e^{xiK}\int_0^1 f(\gamma(t))e^{xu(s)}\gamma'(t)dt$$

is such that the Laplace method may apply. The method of steepest descent consists in using the meromorphicity of f, analyticity of g to deform the contour of integration such that modulo residues, the original integral can be written as a sum of integrals of the type C' mentioned. The name steepest descent comes from the following remark. As a consequence of the Cauchy-Riemann equations we have $\nabla u \cdot \nabla v = 0$ and the lines $v = K$ are lines of steepest variation of $|e^{xg(s)}|$, and that to best control the integral it is convenient to go along the descent direction. If we interpret $\nabla u \cdot \nabla v = 0$ as a PDE for v, $u_x v_x + u_y v_y = 0$, the characteristic curves of this equation solve the system of ODEs

$$\dot{x} = -u_x(x,y); \quad \dot{y} = -u_y(x,y) \tag{1.67}$$

Note that $(x(s), y(s))$; $s > 0$ are the curves of steepest descent of u. Assume for simplicity that g is entire and f is meromorphic. We can let the points on the curve $C = (x_0(\tau), y_0(\tau))$; $\tau \in [0,1]$ evolve with (1.67) keeping the endpoints fixed. More precisely, at time t consider the curve $(t) = C_1 \cup C_2 \cup C_3$ where $C_1 = (x(s, x_0(0)), y(s, y_0(0))$; $s \in [0,t)$, $C_2 = x(t, x_0(\tau)), y(t, y_0(\tau))$, $\tau \in [0,1]$) and $C_3 = (x(s, x_0(1)), y(s, y_0(1))$; $s \in [0,t)$ (see figure). Clearly,

$$\int_C f(s)e^{xg(s)}ds = \int_{C(t)} f(s)e^{xg(s)}ds \tag{1.68}$$

We can see that $z(t, x_0(\tau)) = (x(t, x_0(\tau)), y(t, x_0(\tau)))$ has a limit as $t \to \infty$ on the Riemann surface, since

$$\frac{d}{dt}u\Big(x(t), y(t)\Big) = -u_x^2 - u_y^2 \tag{1.69}$$

Define $\mathcal{S}$ as the smallest forward invariant set with respect to the evolution (1.67) which contains $(x_0(0), y_0(0))$, all the limits in $\mathbb{C}$ of $z(t, x_0(\tau))$ (by (1.69), these limits are saddle points of g, i.e. points where $g' = 0$) and the descent lines originating at these points. The set $\mathcal{S}$ is a union of steepest descent curves of u, $\mathcal{S} = \cup_n C_n$ and, if s_j are poles of f crossed by the curve $C(t)$ we have, under suitable convergence assumptions[1],

[1] which are required, as can be seen by applying the described procedure to very simple integral

$$\int_0^i e^{xe^{-z}}dz$$

The procedure described in (B) is better in many respects.

$$\int_C f(s)e^{xg(s)}ds = \sum_n \int_{C_n} f(s)e^{xg(s)}ds + 2\pi i \sum_j \mathrm{Res}(f(s)e^{xg(s)})_{s=s_j} \quad (1.70)$$

and the situation described in (A) above has been achieved.

One can allow for branch points of f, each of which adds a contributions of the form

$$\int_C \Delta f(s)e^{xg(s)}ds$$

where C is a cut starting at the branch point of f, along a line of steepest descent of g, and $\Delta f(s)$ is the jump across the cut of f.

(B) It is often more convenient to proceed as follows.

We may assume we are dealing with a simple smooth curve. We assume $g' \neq 0$ at the endpoints (the case of vanishing derivative is illustrated shortly on an example). Then, possibly after an appropriate small deformation of C we have $g' \neq 0$ along the path of integration C and g is invertible in a small enough neighborhood $\mathcal{D}$ of C. We make the change of variable $g(s) = -\zeta$ and note that the image of C is smooth and has at most finitely many self-intersections. We can break this curve into piecewise smooth, simple curves. Without loss of generality we then assume that the image , C' of C is simple and take the endpoints of C' to be 0 and i. We deform the contour of integration toward $+\infty$ and end up with a sum of integrals of the form

$$\sum_n \int_{c_n}^{c_n+\infty} f\left(s(\zeta)\right)e^{-x\zeta}\frac{ds}{d\zeta}d\zeta + 2\pi i \sum_j \mathrm{Res}\left(f((s(\zeta))e^{-x\zeta}\frac{ds}{d\zeta}\right)_{s=s_j} \\ + \sum_j \int_{d_j}^{d_j+\infty} \Delta\left[f(s(\zeta))\frac{ds}{d\zeta}\right]e^{-x\zeta}d\zeta \quad (1.71)$$

If more convenient, one can alternatively subdivide C such that g' is nonzero on the (open) subintervals.

Example In the integral (1.72) we have, using the substitution $\cos(t) = it$,

$$I = \int_{-\pi/2}^{\pi/2} e^{i\xi\cos t}dt = \int_{-\pi/2}^{0} e^{i\xi\cos t}dt + \int_0^{\pi/2} e^{i\xi\cos t}dt = 2\int_0^{\pi/2} e^{i\xi\cos t}dt \\ = \int_0^{\infty} \frac{e^{-\xi t}}{\sqrt{1+t^2}}dt + \int_i^{i+\infty} \frac{e^{-\xi t}}{\sqrt{1+t^2}}dt \quad (1.72)$$

which can be immediately brought to a combination of Laplace transforms of functions having convergent Puiseux series at the origin.

Other examples (1) To find the behavior of the integral

$$\int_{-1}^{1} \frac{e^{ixs}}{s^2+1} ds$$

for large positive x, we deform the contour of integration in the upper half plane toward $i\infty$ where lines of steepest descent "end," collect the residue at the pole and write

$$\int_{-1}^{1} \frac{e^{ixs}}{s^2+1} ds = -\int_{-1}^{-1+i\infty} \frac{e^{ixs}}{s^2+1} ds + \int_{1}^{1+i\infty} \frac{e^{ixs}}{s^2+1} ds + \pi e^{-x}$$
$$= -ie^{-ix} \int_0^\infty \frac{e^{-xt}}{1+(it-1)^2} dt + ie^{ix} \int_0^\infty \frac{e^{-xt}}{1+(it+1)^2} dt + \pi e^{-x}$$
$$\sim e^{-ix} \sum_{k=1}^{\infty} c_k x^{-k} + e^{ix} \sum_{k=1}^{\infty} d_k x^{-k} + \pi e^{-x} \quad (1.73)$$

by Watson's Lemma, where we kept the exponentially small term since it turns out that this is also the complete transseries of our function.

(2) Similarly, one can check that the contour in

$$\int_{-1}^{1} e^{ixs^2} ds$$

can be deformed in such a way that the integral becomes

$$2e^{ix} \int_0^\infty \frac{e^{-xu}}{2\sqrt{u+i}} du + \int_{-\sqrt{i}\infty}^{\sqrt{i}\infty} e^{ixs^2} ds = 2e^{ix} \int_0^\infty \frac{e^{-xu}}{2\sqrt{u+i}} du + \frac{\sqrt{2\pi i}}{\sqrt{x}}$$
$$\sim \frac{\sqrt{2\pi i}}{\sqrt{x}} + e^{ix} \sum_{k=1}^{\infty} c_k x^{-k} \quad (1.74)$$

(and where one of the integrals now passes through the saddle point $s = 0$); the last expression is the transseries of our integral.

1.2c Asymptotics of Taylor coefficients

There is dual relation between the (trans)asymptotic behavior of the Taylor coefficients of an analytic function and the structure of its singularities in the complex plane. We will study a few examples in which this relationship is exhibited.

Proposition 1.75 *Assume f is analytic in the open unit disk and on the boundary of the disk it has an isolated singularity, say at $z = 1$, in a neighborhood of which the function is described by a convergent Puiseux series. In* *other words f is analytic in a disk of radius* $1+\epsilon$ *with a cut on* $(1, 1+\epsilon)$ *and near* $x = 1$ *we have*

$$f(z) = (1-z)^{\beta_1} A_1(z) + \ldots + (1-z)^{\beta_m} A_m(z)$$

where $A_1, \ldots, A_m$ *are analytic in a neighborhood of* $z = 1$*. Then we have*

$$c_k \sim k^{-\beta_1 - 1} \sum_{j=0^\infty} \frac{c_{j;1}}{k^j} + \ldots + k^{-\beta_n - 1} \sum_{j=0^\infty} \frac{c_{j;m}}{k^j}$$

where the coefficients $c_{j;m}$ *can be calculated from the Taylor coefficients of the function* A_m*, and conversely, this asymptotic expansion determines the functions* A_m*. The theorem has a straightforward generalization to the case when there are finitely many isolated singularities on the unit circle.*

Proof. We have

$$c_{k-1} = \frac{1}{2\pi i} \oint \frac{f(s)}{s^k} ds$$

where the contour is a small circle around the origin. This contour can be deformed, by assumption, to the union between the $C_{1+\epsilon}$, the circle of radius $(1+\epsilon)$, an integral along and below the lower side of the cut, avoiding $z = 1$ and then moving forward along and above the upper side of the cut, and whose sum we denote as $\int_C$. The integral along $C_{1+\epsilon}$ can be estimated by

$$\frac{1}{2\pi} \left| \oint_{C_{1+\epsilon}} \frac{f(s)}{s^k} ds \right| \le \|f\|_\infty (1+\epsilon)^{-k} = O((1+\epsilon)^{-k})$$

and will not participate in the asymptotic expansion. By checking the branches of the roots we can write the integral along C as a sum of integrals of the form

$$\frac{1}{2\pi i} \int_C (1-s)^\beta A(s) s^{-k} ds \tag{1.76}$$

we can restrict ourselves to the case when β is not an integer, the other case being trivial. By performing an appropriate number of integrations we can make $\Re(\beta) > 0$. We then have

$$\begin{aligned} \frac{1}{2\pi i} \int_C (1-s)^\beta A(s) s^{-k} ds &= -\frac{\sin(\pi\beta)}{\pi} \int_1^{1+\epsilon} (s-1)^\beta A(s) s^{-k} ds \\ = -\frac{\sin(\pi\beta)}{\pi} \int_0^\epsilon t^\beta A(t)(1+t)^{-k} dt &= -\frac{\sin(\pi\beta)}{\pi} \int_0^\epsilon t^\beta A(t) e^{-k \ln(1+t)} dt \end{aligned} \tag{1.77}$$

where it is convenient to change variables to $t + 1 = e^u$. This is a regular change of variables, and noting that $e^u - 1 = u\phi(u)$ where $\phi(0) = 1$ we have

$$-\frac{\sin(\pi\beta)}{\pi}\int_0^{\epsilon} t^{\beta}A(t)e^{-k\ln(1+t)}dt = -\frac{\sin(\pi\beta)}{\pi}\int_0^{\ln(1+\epsilon)} u^{\beta}\phi(u)A(e^u)e^{-ku}du$$
$$= -\frac{\sin(\pi\beta)}{\pi}\int_0^{\ln(1+\epsilon)} u^{\beta}G(u)e^{-ku}du = \quad (1.78)$$

where the assumptions of Watson's lemma are satisfied and we thus have

$$c_{k-1} \sim k^{-\beta-1}\sum_{j=0}^{\infty}\frac{d_j}{k^j}$$

where the d_j can be calculated straightforwardly from the Taylor coefficients of A.

To generalize to the case when there are n isolated singularities on the unit circle note that the same path deformation argument would work, avoiding each isolated singularity separately, and we end up with a sum of integrals of the type studied. Each can be reduced to the $z = 1$ case by taking $g(z) = f(ze^{i\phi}$ and noting that $g^{(n)} = e^{in\phi}f^{(n)}$.

1.3 Singularities of differential equations

We will first review briefly some basic notions about singularities of linear differential equations.

1.3a Linear meromorphic differential equations. Regular and irregular singularities

A linear meromorphic m-th order differential equation has the canonical form

$$y^{(m)} + D_{n-1}(x)y^{(n-1)} + ... + D_0(x)y = D(x) \quad (1.79)$$

where the coefficients $D_j(x)$ are meromorphic near x_0. We note first that any equation of the form (1.79) can be brought to a homogeneous meromorphic of order $n = m + 1$

$$y^{(n)} + C_{n-1}(x)y^{(n-1)} + ... + C_0(x)y = 0 \quad (1.80)$$

by applying the operator $D(x)\frac{d}{dx}\frac{1}{D(x)}$ to (1.79). We want to look at the possible singularities of the solutions of this equation, $y(x)$. Note first that by the general theory of linear differential equations if all coefficients are analytic at a point x_0 then the solution is also analytic. Such a point is called regular point. In the linear case singularities can only arise because of singularities in the coefficients.

The main distinction is made with respect to the type of local solutions, whether they can be expressed as convergent asymptotic series (regular singularity) or not (irregular one).

Theorem 1.81 (Frobenius) *If near the point $x = x_0$ the coefficients C_{n-j}, $j = 1...n$ can be written as $(x - x_0)^{-j} A_{n-j}(x)$ where A_{n-j} are analytic, then there is a fundamental system of solutions in the form*

$$y_m(x) = (x - x_0)^{r_m} \sum_{j=0}^{N_m} (\ln(x - x_0))^j B_{j;m}(x) \tag{1.82}$$

where $B_{j;m}$ are analytic in an open disk centered at x_0 with radius equal to the distance from x_0 to the first singularity of A_j. The powers r_m are solutions of the indicial *equation*
$r(r-1)\cdots(r-n+1) + A_{n-1}(x_0)r(r-1)\cdots(r-n+2) + \ldots + A_0(x_0) = 0$
Furthermore, logs appear only in the resonant case, when two (or more) characteristic roots differ by an integer.

A straightforward way to prove the theorem is by induction on n. A transformation of the type $y = x^{r_m} z$ reduces the equation (1.80) to an equation of the same type, but where one of the characteristic roots is zero. One can show there is an analytic solution z_0 of this equation by inserting a power series, identifying the coefficients and estimating the growth of the coefficients. The substitution $z = z_0 \int g(s)ds$ gives an equation for g which is of the same type as (1.80) but of order $n - 1$. We will not go into the details of the general case but instead we will illustrate the approach on the simple equation

$$x(x-1)y'' + y = 0$$

around $x = 0$ whose indicial equation is $r(r-1) = 0$ (a resonant case). Substituting $y_0 = \sum_{k=0}^{\infty} c_k x^k$ in the equation and identifying the powers of x yields the recurrence

$$c_{k+1} = \frac{k^2 - k + 1}{k(k+1)} c_k$$

with $c_0 = 0$ and c_1 arbitrary. By linearity we may take $c_1 = 1$ and by induction we see that $c_k < 1$. Thus the power series has radius of convergence 1, and it converges up to the nearest singularity of the equation which is at $x = 1$. We let $y_0 = y_0 \int g(s)ds$ and get for h the equation

$$g' + c(x)g = 0$$

where $c(x) = 2\frac{y_0'}{y_0} = \frac{2}{x} + A(x)$ with $A(x)$ is analytic. The point $x = 0$ is a regular singular point and in fact we see that $g(x) = C_1 x^{-2} B(x)$ where C_1 is an arbitrary constant and $B(x)$ is some function analytic at $x = 0$. Thus $\int g(s)ds = C_1(\frac{a}{x} + b\ln(x) + A_1(x)) + C_2$ where $A_1(x)$ is analytic at $x = 0$.

Going back to the original variables we see that we have a fundamental set of solutions in the form $y_0(x), B_1(x) + \ln x B_2(x)$ where B_1 and B_2 are analytic.

A converse of this theorem also holds, namely

Theorem 1.83 (Fuchs) *If a meromorphic linear differential equation has, at $x = x_0$, a fundamental system of solutions in the form (1.82), then x_0 is a regular singular point of the equation.*

In fact, for irregular singularities the general formal solution of the equation may contain divergent power series and exponentially small (large) terms, which lead naturally to the concept of transseries, studied later.

Example. Consider the equation

$$y' + \frac{1}{x^2}y = 1 \tag{1.84}$$

which has an irregular singularity at $x = 0$ since the order of the pole in $C_0 = x^{-2}$ exceeds the order of the equation. Substituting $y = \sum_{k=0}^{\infty} c_k x^k$ we get $c_0 = c_1 = 0$, $c_2 = 1$ and in general the recurrence

$$c_{k+1} = -kc_k$$

whence $c_k = (-1)^k(k-1)!$ and the series has zero radius of convergence. The associated homogeneous equation $y' + \frac{1}{x^2}y = 0$ has the general solution $y = Ce^{1/x}$ with an exponential singularity at $x = 0$.

1.3b Singularities of nonlinear differential equations; formal Painlevé test

For nonlinear differential equations, the solutions may be singular at points x where the equation is regular. Indeed, for example, the equation

$$y' = y^2 + 1$$

has a one parameter family of solutions $y(x) = \tan(x + C)$; each solution has infinitely many poles. Since the location of these poles depends on C, thus on the solution itself, these singularities are called *movable* or *spontaneous*. Painlevé studied the problem of finding differential equations whose only movable singularities are poles. These equations were interpreted as giving "nice" functions, with reasonable behavior in the complex plane. We can think of this property as guaranteeing some form of integrability of the equation, in the following sense. If we denote by $Y(x; x_0; C_1, C_2)$ the solution of the differential equation $y'' = F(x, y, y')$ with initial conditions $y(x_0) = C_1, y'(x_0) = C_1$ we see that $y(x_1) = Y(x_1; x; y(x), y'(x))$ is formally constant along trajectories and so is $y'(x_1) = Y'(x; x_0; y(x), y'(x))$. This gives thus two "integrals of motion" in $\mathbb{C}$ provided the solution Y is well defined almost everywhere in $\mathbb{C}$, i.e., if Y is meromorphic.

On the contrary, movable branch-points, if bad enough, may make the inversion process badly multi-valued, and one may expect in such circumstances that any integral of motion, which is necessarily a function of the C_i, is badly behaved. Since the Painlevé test is not invariant under changes of coordinates, failure of the Painlevé test does not imply nonintegrability. M. Kruskal introduced a test of nonintegrability, the *poly-Painlevé test* which measures whether branching is "dense" in which case one does expect absence of integrals of motion.

Example Painlevé's equation P1. This equation is usually written in the form

$$g'' = 6g^2 + z \tag{1.85}$$

which, by the substitution $y(z) = \alpha y(\beta z)$, $\beta z = x$, $\beta = 6^{1/5}$, $\alpha = 6^{-4/5}$ becomes

$$y'' = y^2 + x \tag{1.86}$$

We will look at the local behavior of a solution that blows up, and will find solutions that are meromorphic but not analytic. In a neighborhood of a point where y is large the dominant equation is $y'' = y^2$ which can be integrated explicitly in terms of elliptic functions and its solutions have double poles. Alternatively, we could have looked for a power-like behavior

$$y \sim A(x - x_0)^p$$

where $p < 0$ and obtained, to leading order, the equation $Ap(p-1)x^{p-2} = A^2p^2$ which gives $p = -2$ and $A = 6$ (the solution $A = 0$ is inconsistent with our assumption). We are next looking for actual solutions which are of the form $6(x-x_0)^{-2}(1+o(1))$ and apply the "squeezing" method. Substituting $y(x) = 6(x-x_0)^{-2} + \delta(x)$ and taking $x = x_0 + z$ leads to the equation

$$\delta'' = \frac{12}{z^2}\delta + z + x_0 + \delta^2 \tag{1.87}$$

Note now that our assumption $\delta = o(z)^{-2}$ makes $\delta^2 = o(\frac{12}{z^2}\delta)$ which shows that the nonlinear term in (1.87) is negligible. (Thus, to leading order, our equation is linear! This is a general phenomenon: taking out more and more terms out of the local expansion, the correction becomes less and less important, and an equation becomes approximately linear with this procedure.) It is then natural to apply the general strategy in asymptotics, separating out the large terms from the small terms and setting an iteration scheme accordingly (or, equivalently, writing a fixed point equation for the solution based on this separation). We take $\delta(z)^2 = r(z)$ and solve the remaining (linear) equation as if r was known (by variation of constants), to get, after substituting r for its value an identity as an integral equation which by construction is supposed to be contractive.

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{D}{z^3} - \frac{1}{7z^3}\int_0^z s^4 r(s)ds + \frac{z^4}{7}\int_0^z s^{-3}r(s)ds$$
$$= -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + J(\delta) \quad (1.88)$$

the assumption that $\delta = o(z^{-2})$ forces $D = 0$. To find δ formally, we would simply iterate (1.88) in the following way: We take $r = 0$ first and obtain $\delta_0 = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4$. Then take $r = \delta_0^2$ and compute δ_1 from (1.88) and so on. This yields:

$$\delta = -\frac{1}{10}x_0 z^2 - \frac{1}{6}z^3 + Cz^4 + \frac{x_0^2}{1800}z^6 + \frac{x_0}{900}z^7 + ... \quad (1.89)$$

This series is actually convergent. To see that, we scale out the leading power of z in δ, z^2 and write $\delta = z^2 u$. The equation for u is

$$u = -\frac{x_0}{10} - \frac{z}{6} + Cz^2 - \frac{z^{-5}}{7}\int_0^z s^8 u^2(s)ds + \frac{z^2}{7}\int_0^z s u^2(s)ds$$
$$= -\frac{x_0}{10} - \frac{z}{6} + Cz^2 + J(u) \quad (1.90)$$

It is straightforward to check that, given C_1 large enough (compared to $x_0/10$ etc.) there is an ϵ such that this is a contractive equation for u in the ball $\|u\|_\infty < C_1$ in the space of analytic functions in the disk $|z| < \epsilon$. Our conclusion is that δ is analytic and that y is meromorphic near $x = x_0$.

Thus the equation P_I passes the local Painlevé test.

Note. The true Painlevé test requires that y is globally meromorphic, and we did *not* prove this. That indeed y is globally meromorphic is also true, but the proof is more delicate (see e.g. [1]).

An equation taken "at random" will generically fail even the local Painlevé test. For instance, for the simpler, autonomous, equation

$$f'' + f' + f^2 = 0 \quad (1.91)$$

the same analysis yields a local behavior starting with a double pole, $f \sim -6z^{-2}$. Taking $f = -6z^{-2} + \delta(z)$ with $\delta = o(z^{-2})$ again leads to a nearly linear equation for δ which can be solved by convergent iteration, using arguments similar to the ones above. The iteration is now (for some $a \neq 0$)

$$\delta = \frac{6}{5z} + Cz^4 - \frac{1}{7z^3}\int_0^z s^4 r(s)ds + \frac{z^4}{7}\int_a^z s^{-3}r(s)ds \quad (1.92)$$

but now the leading behavior of δ is larger, $\delta = \frac{6}{5z}$. Iterating in the same way as before, we see that this will eventually produce logs in the expansion for δ (it first appears in the second integral, thus in the form $z^4 \ln z$). We get

$$\delta = \frac{6}{5z} + \frac{1}{50} + \frac{z}{250} + \frac{7z^2}{5000} + \frac{79}{75000}z^3 - \frac{117}{2187500}z^4\ln(z) + Cz^4 + \ldots \quad (1.93)$$

where later terms will contain higher and higher powers of $\ln(z)$. This is effectively a series in powers of z and $\ln z$ a simple *transseries*, which here is classically convergent, as can be straightforwardly shown using the contractive mapping method, as above.

1.4 Singular perturbations

In problems depending analytically on a small parameter, internal or external, the dependence of the solution on this parameter may be analytic or not; the dependence on the parameter may be regular or singular. In differential equations, singular perturbations are usually those in which the small perturbation is such that the highest derivative does not participate in the dominant balance. An example is the Schrödinger equation

$$-\epsilon^2\psi'' + V(x)\psi - E\psi = 0 \quad (1.94)$$

for small ϵ, which will be studied in more detail later. Setting $\epsilon = 0$ removes the second derivative from the equation. Similarly, in the problem

$$x^2 f' + f = x^2 \quad (1.95)$$

the presence of x^2 in front of f' makes it subdominant if $f \sim x^p$ for some p. In this sense the Airy equation that we looked at in §2.109 is singularly perturbed at $x = \infty$, as can be seen by taking $x = 1/z$. It turns out that in many of these problems the behavior of solutions is exponential in the parameter, generically yielding level one transseries, of the form Qe^P where P and Q have algebraic behavior in the parameter. An exponential substitution of the form $f = e^w$ may be used in order to make the leading behavior algebraic. This is the first step in the method known as WKB.

We have already used this substitution in §2.109 to determine the asymptotic behavior of the Airy functions and of the factorial. We will first illustrate the idea in some further instances.

Consider the heat equation

$$\psi_t = \psi_{xx} \quad (1.96)$$

This is a degenerate (parabolic) PDE. The effect of this degeneracy is similar to that of a singular perturbation. If we attempt to solve the PDE in the spirit of Cauchy-Kowalewski's method by a power series

$$\psi = \sum_{k=0}^{\infty} t^k F_k(x) \tag{1.97}$$

this series will generically have zero radius of convergence. Indeed, introducing this expansion in the equation and identifying the powers of t we get a recurrence relation for the coefficients $F_k = F''_{k-1}/k$ whose solution, $F_k = F_0^{(2k)}/k!$ behaves like $F_k \sim k!$ for large k, if F is analytic but not entire.

If we take $\psi = e^w$ in (1.96) we get

$$w_t = w_x^2 + w_{xx} \tag{1.98}$$

where the assumption of algebraic behavior of w is expected to make $w_x^2 \gg w_{xx}$ and so the leading equation is approximately

$$w_t = w_x^2 \tag{1.99}$$

which can be solved by characteristics, e.g. in the following way. We take $w_x = u$ and get for u the quasilinear equation

$$u_t = 2uu_x \tag{1.100}$$

with a particular solution $u = -x/(2t)$, giving $w = -x^2/(4t)$. We thus take $w = -x^2/(4t) + \delta$ and get for δ the equation

$$\delta_t + \frac{x}{t}\delta_x + \frac{1}{2t} = \delta_x^2 + \delta_{xx} \tag{1.101}$$

where we have separated the relatively small terms to the rhs. We would normally solve the leading equation (the lhs of (1.101)) and continue the process, but for this equation we note that $\delta = -\frac{1}{2}\ln t$ solves not only the leading equation, but the full equation (1.101). Thus

$$w = -\frac{x^2}{4t} - \frac{1}{2}\ln t \tag{1.102}$$

which gives the classical heat kernel

$$\psi = \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}} \tag{1.103}$$

This exact solvability is of course rather accidental, but a perturbation approach formally works in a more PDE general context.

1.4a Singularly perturbed Schrödinger equation (1.94). Setting and heuristics

We look at (1.94) under the assumption that $V \in C^\infty(\mathbb{R})$ and would like to understand the behavior of solutions for small ϵ. Applying the WKB transformation $\psi = e^w$ we get

$$-\epsilon^2 w'^2 - \epsilon^2 w'' + V(x) - E = 0 \tag{1.104}$$

where, near an x_0 where $V(x_0) - E \neq 0$ the only consistent balance is between $-\epsilon^2 w'^2$ and $V(x) - E$ with $\epsilon^2 w''$ much smaller than both. We then write the equation in the iterative form

$$w'^2_{n+1} = \epsilon^{-2} U - w''_n \tag{1.105}$$

or

$$w' = \pm\sqrt{\epsilon^{-2} U - w''_n} = \pm\frac{\sqrt{U}}{\epsilon}\sqrt{1 - \frac{\epsilon^2 w''}{U}} \tag{1.106}$$

and solve it formally, taking first $w''_0 = 0$. To first order we thus have

$$w' = \pm\epsilon^{-1} U^{1/2} \tag{1.107}$$

Using this to approximate w'' we get

$$w' \approx \pm\epsilon^{-1} U^{1/2} - \frac{1}{4}\frac{U'}{U} \tag{1.108}$$

and thus

$$w \approx \pm\epsilon^{-1} \int U^{1/2}(s) ds - \frac{1}{4} \ln U \tag{1.109}$$

and thus

$$\psi \sim e^{\pm\epsilon^{-1} \int U^{1/2}(s) ds} U^{-1/4} \tag{1.110}$$

If we proceed formally we would get an expansion of the form

$$\psi \sim \exp\left(\pm\epsilon^{-1} \int U^{1/2}(s) ds\right) U^{-1/4} \left(1 + \epsilon F_1(x) + \epsilon^2 F_2(x) + ...\right) \tag{1.111}$$

There are two possibilities compatible with our assumption about x_0, namely $V(x_0) > E$ and $V(x_0) < E$. In the first case there is (formally) an exponentially small solution and an exponentially large one, in the latter two rapidly oscillating ones.

The points where $V(x_0) = E$ are special in this analysis and are called *turning points*. In applying our approximation procedure we needed the quantity $\epsilon^2 w'' U^{-1}$ to be small. To leading order, $w' = \epsilon^{-1} U^{1/2}$. The formal condition of validity of the iteration is then

$$\epsilon U' U^{-3/2} \ll 1 \tag{1.112}$$

which typically rules out small neighborhoods of points where $U = 0$. For instance if U has a simple root at $x = 0$, the only one that we will consider here (but multiple roots are not substantially more difficult) then condition

(1.112) reads $x \gg \epsilon^{2/3}$. The region where this condition holds is called *outer* region. In a small region where (1.112) fails, called *inner* region, a different approximation will be sought. We see that $V(x) - E = V'(0)x + x^2 h(x) =: \alpha x + x^2 h(x)$ where $h(x) \in C^\infty(\mathbb{R})$. We then write

$$-\epsilon^2 \psi'' + \alpha x = -x^2 h(x)\psi \tag{1.113}$$

and treat the rhs of (1.113) as a small perturbation. The substitution $x = \epsilon^{2/3} t$ makes the leading equation an Airy equation:

$$-\psi'' + \alpha t\psi = -\epsilon^{2/3} t^2 h(\epsilon^{2/3} t)\psi \tag{1.114}$$

which is a regularly perturbed equation! For a perturbation method to apply, we merely need that $x^2 h(x)\psi$ in (1.113) is much smaller than the lhs, roughly requiring $x \ll 1$. This shows that the inner and outer regions overlap, there is a subregion of both *the matching region* where both expansions apply, and where, by equating them, the free constants in each of them can be linked.

1.4b Outer region. Rigorous analysis

We first look at a region where $U(x)$ is bounded away from zero. We will write $U = F^2$.

Proposition 1.115 *Let $F \in C^\infty(\mathbb{R})$, $F^2 \in \mathbb{R}$, and assume $F(x) \neq 0$ in $[a, b]$. Then for small enough ϵ there exists a fundamental set of solutions of (1.94) in the form*

$$\psi_\pm = \Phi_\pm(x;\epsilon) \exp\left[\pm\epsilon^{-1} \int F(s)ds\right] \tag{1.116}$$

where $\Phi_\pm(x;\epsilon)$ are C^∞ in $\epsilon > 0$.

Proof. We show that there exists a fundamental set of solutions in the form

$$\psi_\pm = \exp\left[\pm\epsilon^{-1} R_\pm(x;\epsilon)\right] \tag{1.117}$$

where $R_\pm(x;\epsilon)$ are C^∞ in ϵ. The proof is by rigorous WKB.

Note first that linear independence is immediate, since for small enough ϵ the ratio of the two solutions cannot be a constant, given their ϵ behavior.

We take $\psi = e^{w/\epsilon}$ and get, as before, to leading order $w' = \pm F$. We look at the plus sign case, the other case being similar. It is then natural to substitute $w = F + \delta$; we get

$$\delta' + 2\epsilon^{-1} F\delta = -F' - \epsilon^{-1}\delta^2 \tag{1.118}$$

which we transform into an integral equation by treating the rhs as if it was known and integrating the resulting linear inhomogeneous differential equation. Setting $H = \int F$ the result is

$$\delta = -e^{-\frac{2H}{\epsilon}} \int_a^x F'(s) e^{\frac{2H(s)}{\epsilon}} ds - \frac{1}{\epsilon} e^{-\frac{2H}{\epsilon}} \int_a^x \delta^2(s) e^{\frac{2H(s)}{\epsilon}} ds =: J(\delta) =: \delta_0 + N(\delta) \tag{1.119}$$

We assume that $F > 0$ on (a, b), the case $F < 0$ being very similar. The case $F \in i\mathbb{R}$ is not too different either, as we will explain at the end.

Let now $\|F'\|_\infty = A$ in (a, b) and assume also that $\min_{s\in(a,b)} |U(s)| > B > 0$.

Lemma 1.120 *For small ϵ, the operator J is contractive in a ball $\mathcal{B} := \{\delta : \|\delta\|_\infty \leq 2AB^{-1}\epsilon\}$*

Proof. i) Preservation of $\mathcal{B}$. We have

$$|\delta_0(x)| \leq A e^{-\frac{2}{\epsilon}H(x)} \int_a^x e^{\frac{2}{\epsilon}H(s)} ds$$

By assumption, H is increasing on (a, b) and $H' \neq 0$ and thus, by the Laplace method, cf. Proposition 1.21, for small ϵ we have

$$|\delta_0(x)| \leq 2A e^{-\frac{2}{\epsilon}H(s)} \frac{e^{\frac{2}{\epsilon}H}}{\frac{2}{\epsilon}H'} \leq \epsilon AB^{-1}$$

Note We need this type of estimates to be uniform in $x \in [a, b]$ as $\epsilon \to 0$. To see that this is the case, we write

$$\int_a^x e^{\frac{2}{\epsilon}H(s)} ds = \int_a^x e^{\frac{2}{\epsilon}H(s)} \frac{2F(s)}{\epsilon} \frac{\epsilon}{2F(s)} ds$$

$$\leq \frac{\epsilon}{2B} e^{\frac{2}{\epsilon}H(s)} \Big|_a^x \leq \frac{\epsilon}{2B} e^{\frac{2}{\epsilon}H(x)} \tag{1.121}$$

Similarly,

$$\left| \frac{1}{\epsilon} e^{-\frac{2H}{\epsilon}} \int_a^x \delta^2(s) e^{\frac{2H(s)}{\epsilon}} ds \right| \leq 2\epsilon^2 A^2 B^{-3}$$

and thus, for small ϵ and $\delta \in \mathcal{B}$ we have

$$J(\delta) \leq \epsilon^{-1} AB^{-1} + 2\epsilon^2 A^2 B^{-3} \leq 2\epsilon AB^{-1}$$

ii) *Contractivity.* We have, with $\delta_1, \delta_2 \in \mathcal{B}$, using similarly Laplace's method,

$$|J(\delta_2) - J(\delta_1)| \le \frac{1}{\epsilon} e^{-\frac{2H}{\epsilon}} \int_a^x |\delta_2(s) - \delta_1(s)||\delta_2(s) + \delta_1(s)| e^{\frac{2H(s)}{\epsilon}} ds$$
$$\le \frac{\epsilon^2 A^2}{B^3} \|\delta_2 - \delta_1\| \quad (1.122)$$

and thus the map is contractive for small enough ϵ.

Note. We see that the conditions of preservation of $\mathcal{B}$ and contractivity allow for a dependence of (a,b) on ϵ. Assume for instance $a, b > 0$, $V(x) = E$ has no root in $[a, b+\gamma)$ with $\gamma > 0$, and that a is small. Assume further that $V(0) = E$ is a simple root, $|V'(0)| = m \neq 0$. Then for some $C > 0$ we have $B \ge Cm^2a^2$ and the condition of contractivity reads

$$\frac{\epsilon^2 m}{m^3 a^3} < 1$$

i.e. $a > \epsilon^{2/3}$ and for small enough ϵ this is also enough to ensure preservation of $\mathcal{B}$ which allows for matching with the inner region expansions.
We thus find that the equation $\delta = J(\delta)$ has a unique solution and that, furthermore, $\|\delta\| \le const.\epsilon$. Using this information and (1.122) which implies

$$\|J(\delta)\| \le \frac{\epsilon A}{B^2} 2AB^{-1}\epsilon$$

we easily get that, for some constants $C_i > 0$ independent on ϵ,

$$|\delta - \delta_0| \le C_1\epsilon|\delta| \le C_1\epsilon|\delta_0| + C_1\epsilon|\delta - \delta_0|$$

and thus

$$|\delta - \delta_0| \le C_2\epsilon|\delta_0|$$

and thus, applying again Laplace's method we get

$$\delta \sim \frac{-\epsilon F'}{2F} \quad (1.123)$$

which gives

$$\psi \sim \exp\left(\pm\epsilon^{-1} \int U^{1/2}(s) ds\right) U^{-1/4}$$

The proof of the C^∞ dependence on ϵ can be done by induction, using (1.123) to estimate δ^2 in the fixed point equation, to get an improved estimate on δ, etc.
In the case $F \in i\mathbb{R}$, the proof is the same, by using the Stationary Phase method instead of the Laplace Method.

1.4c Inner region. Rigorous analysis

By rescaling the independent variable we may assume without loss of generality that $\alpha = 1$ in (1.114) which we rewrite as

$$-\psi'' + t\psi = -\epsilon^{2/3}t^2 h_1(\epsilon^{2/3}t)\psi := f(t) \tag{1.124}$$

which can be transformed in an integral equation in the usual way,

$$\psi(t) = -\mathrm{Ai}(t)\int^t f(s)\mathrm{Bi}(s)ds + \mathrm{Bi}(t)\int^t f(s)\mathrm{Ai}(s)ds + C_1\mathrm{Ai}(t) + C_2\mathrm{Bi}(t) \tag{1.125}$$

where Ai, Bi are the usual Airy functions, with the asymptotic behavior

$$\mathrm{Ai}(t) \sim \frac{1}{\sqrt{\pi}}t^{-1/4}e^{-\frac{2}{3}t^{\frac{3}{2}}};\quad \mathrm{Bi}(t) \sim \frac{1}{\sqrt{\pi}}t^{-1/4}e^{\frac{2}{3}t^{\frac{3}{2}}} \tag{1.126}$$

and

$$|t^{-1/4}\mathrm{Ai}(t)| < const.,\quad |t^{-1/4}\mathrm{Bi}(t)| < const. \tag{1.127}$$

as $t \to -\infty$. In view of (1.126) we must be careful in choosing the limits of integration in (1.125). It is convenient to ensure that the second term does not have a fast growth as $t \to \infty$, and for this purpose we need to integrate from t toward infinity in the associated integral. The rule of thumb is to ensure that the maximum of the integrand is achieved near the endpoint of integration. We choose to look at an interval in the original variable $x \in I_M = [-M, M]$ where we shall allow for ϵ-dependence of M. We then write the integral equation with concrete limits in the form below, which we analyze in I_M.

$$\psi(t) = -\mathrm{Ai}(t)\int_0^t f(s)\mathrm{Bi}(s)ds + \\ \mathrm{Bi}(t)\int_M^t f(s)\mathrm{Ai}(s)ds + C_1\mathrm{Ai}(t) + C_2\mathrm{Bi}(t) = J\psi + \psi_0 \tag{1.128}$$

Proposition 1.129 *For some positive const., if ϵ is small enough (1.128) is contractive in the sup norm if $M \le const.\epsilon^{2/5}$.*

Proof. Using the Laplace method we see that for $t > 0$ we have

$$t^{-1/4}e^{-\frac{2}{3}t^{\frac{3}{2}}}\int_0^t s^{-1/4}e^{\frac{2}{3}s^{\frac{3}{2}}}ds \le const.(|t|+1)^{-1}$$

and also

$$t^{-1/4} e^{\frac{2}{3} t^{\frac{3}{2}}} \int_t^M s^{-1/4} e^{-\frac{2}{3} s^{\frac{3}{2}}} ds \le t^{-1/4} e^{\frac{2}{3} t^{\frac{3}{2}}} \int_t^\infty s^{-1/4} e^{-\frac{2}{3} s^{\frac{3}{2}}} ds \le const.(|t|+1)^{-1} \quad (1.130)$$

and thus for a constant independent of ϵ, using (1.126) we get

$$|J\psi(t)| \le const.\epsilon^{2/3}(|t|+1)^{-1} \sup_{s\in[0,t]} |\psi(s)|$$

for $t > 0$. For $t < 0$ we use (1.127) and get

$$\left| \mathrm{Ai}(t) \int_M^t f(s)\mathrm{Bi}(s) ds \right| \le (1+|t|)^{-1/4} \sup_{s\in[-t,0]} |f(s)|(const. + \int_t^0 s^{-1/4} ds$$

and get for a constant independent of ϵ

$$|J\psi(t)| \le const.\epsilon^{2/3}(1+|t|)^{5/2} \le const.\epsilon^{2/3}(\epsilon^{-2/3}M)^{5/2} < 1$$

We see that for small enough ϵ, the regions where the outer and inner equations are contractive overlap. This allows for performing asymptotic matching in order to relate these two solutions. For instance, from the contractivity argument it follows that

$$\psi = (1-J)^{-1}\psi_0 = \sum_{k=0}^{\infty} J^k \psi_0$$

giving a power series asymptotics in powers of $\epsilon^{2/3}$ for ψ.

1.4d Matching

We may choose for instance $x = const.\epsilon^{1/2}$ for which the inner expansion (in powers of $\epsilon^{2/3}$) and the outer expansion (in powers of ϵ) are valid at the same time. We assume that x lies in the oscillatory region for the Airy functions (the other case is slightly more complicated).

We note that in this region of x the coefficient of ϵ^k of the outer expansion will be large, of order $(U'U^{-3/2})^k \sim \epsilon^{-3k/4}$. A similar estimate holds for the terms of the inner expansion. Both expansions will thus effectively be expansions in $\epsilon^{-1/4}$. Since they represent the same solution, they must agree and thus the coefficients of the two expansions are linked. This enables fixing the constants C_1 and C_2 once the outer solution is prescribed.

2. Introduction to transseries and analyzable function

Classical asymptotics typically deals with the qualitative and quantitative description of the behavior of a function (in some direction) near a point, usually a singularity of the function. This description is usually provided in the form of an asymptotic expansion (1.1)in which each successive term is much smaller than its predecessors. To ensure the non-triviality of the representation, it is normally understood in asymptotics that if f is not identically zero, then the scales of the expansion are chosen so that at least one of the $\tilde{f}_k,\ k \in \mathbb{N}$, is nonzero. Thus, although $e^{-1/z}$ is, asymptotic for $z \downarrow 0$ to its Maclaurin series $\sum_{k=0}^{\infty} 0\, z^k$ (in the sense that $e^{-1/z} = o(z^m)$ for any m in the given limit) the series is not considered to contain the *asymptotic behavior* of $e^{-1/z}$ for $z \downarrow 0$ (such a description would be too uninformative, many functions are asymptotic to the zero series). Rather, one takes $e^{-1/z}$ to be its own asymptotic representation.

This leads to an important distinction between asymptotic expansions and Taylor series. While the operator $f \mapsto \mathcal{T}(f)$ which associates to f its Taylor series is clearly linear, the asymptotic expansion of f, $\mathcal{E}(f)$, is *not* linear, and in fact has very poor algebraic properties. Indeed, it is not difficult to see that

$$\mathrm{Ei}(x; C) = \mathrm{P}\int_C^x \mathrm{d}t \frac{\mathrm{e}^t}{t} \sim \sum_{k=0}^{\infty} \frac{k!\mathrm{e}^x}{x^{k+1}} = \tilde{f}(x) \quad (x \to +\infty) \tag{2.1}$$

for any starting point $C \in \mathbb{C}$, while if we assumed linearity we would get from (2.1) and (**??**) the unacceptable conclusion Const $= \mathrm{P}\int_{-\infty}^{C} t^{-1} e^t \mathrm{d}t = \mathrm{Ei}(x) - \mathrm{Ei}(x; C) \sim 0$. Although many obstacles to freely manipulating expansions arise in this simple way, there is no simple way out. Also because of the nontriviality of the kernel of $\mathcal{T}$, the description provided by classical asymptotics is fundamentally *incomplete*. There is no unambiguous way to determine f from its classical asymptotic expansion alone.

The previous discussion suggests trying a representation of the form

$$\mathrm{Ei}(x; C) \sim \sum_{k=0}^{\infty} \frac{k!\mathrm{e}^x}{x^{k+1}} + C_1 \quad (x \to +\infty) \tag{2.2}$$

but, because of the divergence of the series, C_1 cannot be defined straightforwardly as the expansion of $\mathrm{Ei}(x;C) - \sum_{k=0}^{\infty} k!\mathrm{e}^x x^{-k-1}$. Furthermore, (2.2) is clearly meaningless *relative* to definition (1.3).[1] Indeed, the limiting values implicit in the o symbol in (1.3) will be the same whether we take $f_k = \mathrm{e}^x x^{-k-1}$ or $\mathrm{e}^x x^{-k-1} + C_1$. In some sense C_1 is in the kernel of $\mathcal{E}$ (except of course, that $\mathcal{E}$ is not linear). Although $\mathrm{Ei}(x;C)$ and $\mathrm{Ei}(x;C')$ differ by a constant, this constant is **beyond all orders** of their *common* expansion (2.1), that is, the constant is **exponentially small** relative to the terms of the expansion.

It is however very important to note that convergence or divergence are *relative to a topology*. The series $\sum_{k=0}^{\infty} k!\mathrm{e}^x x^{-k-1}$ is divergent relative to the usual topology. It is natural to ask the question whether there are other topologies in suitable function spaces that would make this series convergent, and which have good properties.

2.1 Analytic function theory as a toy model of the theory of analyzable functions

One can meaningfully trace the source of these complications to the nontrivial nature of the singularity of $\mathrm{Ei}(x)$ at $x = \infty$. There is a sharp contrast between the limited properties of asymptotic expansions in general, with which very few operations are allowed, and the rich properties of $\mathcal{T}$ acting on the space of germs of analytic functions. In the latter case, which is a prototype in constructing a comprehensive theory of analyzable functions, the correspondence between functions and their expansions, Taylor expansion is a faithful isomorphism, linear and multiplicative and commuting with differentiation, restricted composition and integration.

Let A denote the set of germs of analytic functions at $z = 0$, let $\mathbb{C}[[z]]$ be the space of formal series in z with complex coefficients, of the form $\sum_{k=0}^{\infty} c_k z^k$, and define $\mathbb{C}_c[[z]]$ as the subspace of series with nonzero radius of convergence.

The Maclaurin series of a function in A is also its asymptotic series at zero. Moreover, the map $\mathcal{M} : A \mapsto \mathbb{C}_c[[z]]$ is an isomorphism and its inverse $\mathcal{M}^{-1} = \mathcal{S}$ is simply the operator of summation of series in $\mathbb{C}_c[[z]]$. $\mathcal{M}$ and $\mathcal{S}$ commute with all usual (and many unusual) function operations which are defined on $\mathcal{A}$, in particular we have, with $\tilde{f}, \tilde{f}_1$ and $\tilde{f}_2$ in $\mathbb{C}_c[[z]]$

[1] Until recently, for reasons not entirely clear, the common assumption was that expansions such as (2.2) must be meaningless altogether, i.e. relative to any *possible* definition of asymptotic expansions.

$$\begin{aligned}
&(1) && \mathcal{S}\{\alpha\tilde{f}_1+\beta\tilde{f}_2\}=\alpha\mathcal{S}\tilde{f}_1+\beta\mathcal{S}\tilde{f}_2\\
&(2) && \mathcal{S}\{\tilde{f}_1\tilde{f}_2\}=\mathcal{S}\tilde{f}_1\mathcal{S}\tilde{f}_2\\
&(3) && \mathcal{S}\{\tilde{f}^{\star}\}=\left(\mathcal{S}\tilde{f}\right)^{\star}\\
&(4) && \mathcal{S}\{\tilde{f}\}'=\left(\mathcal{S}\tilde{f}\right)';\quad \mathcal{S}\left\{\int_0^x\tilde{f}\right\}=\int_0^x\mathcal{S}\tilde{f}\\
&(5) && \mathcal{S}\{\tilde{f}_1\circ\tilde{f}_2\}=\mathcal{S}\tilde{f}_1\circ\mathcal{S}\tilde{f}_2\\
&(6) && \mathcal{S}1=1
\end{aligned} \tag{2.3}$$

$\tilde{f}^{\star}(z)=\overline{\tilde{f}}(\overline{z})$. In fact $\mathcal{M}$ is such a good isomorphism between A and $\mathbb{C}_c[[z]]$, that usually no distinction is made between formal (albeit convergent) expansions and their sums which are actual functions.

As a consequence, whenever a problem of an analytical nature can be solved in $\mathbb{C}_c[[z]]$, where the solution procedure is often algorithmic and of an algebraic nature, $\mathcal{S}$ provides an actual solution of the same problem. For example, if $\tilde{y}$ is a formal solution of the equation

$$\tilde{y}'=\tilde{y}^2+z \tag{2.4}$$

as a series in powers of z, with nonzero radius of convergence, and we let $y=\mathcal{S}y$ we may write, using (2.3),

$$\left(\tilde{y}'=\tilde{y}^2+z\right)\Leftrightarrow\left(\mathcal{S}\left\{\tilde{y}'\right\}=\mathcal{S}\left\{\ \tilde{y}^2\right\}+z\right)\Leftrightarrow\left(y'=y^2+z\right)$$

i.e. $\tilde{y}$ is a formal solution of (2.4) iff y is an actual solution. The same reasoning would work in most natural problems with analytic coefficients for which solutions $\tilde{y}\in C_C[[z]]$ can be found.

In contrast, asymptotic expansions in general are compatible with virtually no function operations, and solving formally a problem in a space of non-convergent expansions, within a classical setting and without further analysis contains little rigorously usable information. Rigor aside, in many problems it appears that formal expansions, even together with the terms beyond all orders, do provide nevertheless very reliable information about the properties of actual solutions.

It is the task of the theory of analyzable functions to transform these formal approaches into a rigorous method and make asymptotic analysis into a natural extension of analytic function theory in which an isomorphism like (2.3) holds in much wider generality.

The ideas of the theory of analyzable functions can be traced back to Euler, and were developed in the work of Cauchy, who discovered and rigorously applied optimal truncation techniques, by Stokes who used optimal truncation successfully in the study of solutions of differential systems and discovered what is now called Stokes' phenomenon, by Borel who found the first powerful technique to deal with divergent expansions, and by Dingle and

Berry who substantially extended optimal truncation methods. In the early 80's exponential asymptotics became a field of its own, with the theory of towers and nice functions introduced by Martin Kruskal and the theory of transseries and analyzable functions of Jean Écalle, who also proposed a very comprehensive generalization of the technique of Borel summation.

Setting of the problem. One operation is clearly missing from both A and $\mathbb{C}_c[[z]]$ namely division, and this severely limits the range of problems that can be solved in either A or $\mathbb{C}_c[[z]]$. The question is then which spaces $A_1 \supset A$ and $S_1 \supset \mathbb{C}_c[[z]]$ are closed under all function operations, including division, and are such that an extension of $\mathcal{M}$ is an isomorphism between them? (Because of the existence of an isomorphism between A_1 and the formal expansions S_1 the functions in A_1 will be called called, in agreement with Écalle, *formalizable*). Exploring the limits of formalizability is at the core of the modern theory of analyzable functions.

In addition to the obvious theoretical interest, there are many important practical applications. One application of such a theory, for instance for some generic classes of differential systems where it has been worked out, is the possibility of solving problems starting from formal expansions, which are easy to produce (in an algebraic and algorithmic way), and from which the isomorphism produces, constructively, actual solutions.

In general, the question of formalizability is delicate. Adjoining division to the set of operations opens the way to a slew of more sophisticated functions and representations. For instance, a short chain of simple operations leads from the identity, through log branched functions, to the exponential integral and its factorially divergent expansion:

$$x \underset{\div}{\longrightarrow} x^{-1} \underset{\int}{\longrightarrow} \ln x \underset{\div}{\longrightarrow} \frac{1}{\ln x} \underset{\int}{\longrightarrow} \int_0^x \frac{\mathrm{d}t}{\ln t} = \int_{-\infty}^{\ln x} \mathrm{d}t \frac{e^t}{t}$$

On the other hand, if a space $A_1 \supset A$ is only closed under simple function operations (algebraic ones, including division and complex conjugation ; differentiation and integration; composition and function inversion) this *may not* even suffice to solve differential equations. Indeed, A_1 would only contain functions that are expressible through elementary functions and quadratures thereof, too restricted to contain solutions of general differential systems. On the other hand, a too comprehensive enlargement of A is likely to have poor analytic properties.

While enlargements A_1 have been found for some specific classes of problems, it is still an open problem what space A_1 would be suitable for a general theory of analyzable functions.

In constructing the correspondence (isomorphism) between A_1 and S_1 one of the most powerful tools is Borel summation.

2.2 Formal asymptotic power series (APS)

Definition 2.5 *For $x \to \infty$, an APS is a formal structure of the type*

$$\sum_{i=1}^{\infty} \frac{c_i}{x^{k_i}} \tag{2.6}$$

where $k_1 < k_2 < ... < k_n < ....$

where $M \in \mathbb{Z}$ can be negative.

Examples. (1) Integer power series, i.e. series of the form

$$\sum_{k=M}^{\infty} \frac{c_k}{x^k} \tag{2.7}$$

(2) An important instance are the finitely generated power series, of the form

$$\sum_{k_i \geq M} \frac{c_{k_1,k_2,...,k_n}}{x^{\alpha_1 k_1 + ... + \alpha_n k_n}} \tag{2.8}$$

where $\alpha_1 > 0, ..., \alpha_n > 0$.

Proposition 2.9 *A series of the form (2.8) is (can be rearranged as) an APS.*

Proof. For the proof we note that for any $L \in \mathbb{Z}$, the set

$$\{(k_1, k_2, ..., k_n) \in \mathbb{Z}^n : k_i \geq M \text{ for } 1 \leq i \leq n \text{ and } L \geq \sum_{i=1}^{n} \alpha_i k_i\}$$

is finite. Indeed, k_i are bounded below, $\alpha_i > 0$ and $\sum_{i=1}^{n} \alpha_i k_i \to \infty$ if one or more k_i is unbounded.

Thus (2.8) can be written in the form (2.6). In particular we can define the dominance of a series in the following way:

Definition 2.10 *If S is a nonzero APS of the type (2.6) we define $\mathrm{Dom}(S)$ to be $c_{i_1} x^{-k_{i_1}}$ where i_1 is the first i in (2.6) for which $c_i \neq 0$.*

Operations with APS. The following operations are defined in a natural way and have the usual properties: $+, -, \times, /$ differentiation and composition $S_1 \circ S_2$ where S_2 is a series such that $k_1 < 0$.

Asymptotic order relation. We naturally write

$$Cx^p \ll Dx^q \quad \text{iff} \quad p < q$$

Definition 2.11 *For two nonzero APSs S_1, s_2 we write $S_1 \gg S_2$ iff $Dom(S_1) \gg Dom(S_2)$.*

Proposition 2.12 *$Dom(S_1 S_2) = Dom(S_1)Dom(S_2)$, and if $Dom(S) \neq const$ then $Dom(S') = Dom(S)'$.*

Proof. Straightforward.

Thus we have

Proposition 2.13 *(i) $S_1 \ll T$ and $S_2 \ll T$ imply $S_1 + S_2 \ll T$.*
(ii) $S_1 \gg T_1$ and $S_2 \gg T_2$ imply $S_1 S_2 \ll T_1 T_2$.
(iii) $S \ll T$ implies $\frac{1}{S} \gg \frac{1}{T}$.
(iv) $S \ll T \ll 1$ implies $S' \ll T' \ll 1$ and $1 \ll S \ll T$ implies $S' \ll T'$.
(v) There is the following trichotomy for two nonzero APSs : $S \ll T$ or $S \gg T$ or else $\frac{S}{T} - C \ll 1$ for some constant C.

Proof. Straightforward.

Proposition 2.14 *Any nonzero APS S can be uniquely decomposed in the following way*

$$S = L + C + s$$

where C is a constant and L and s are APS, with the property that L has nonzero coefficients only for positive *powers of x (L is purely large) and s has nonzero coefficients only for* negative *powers of x (s is purely small).*

Proof. Straightforward.

Example.

Proposition 2.15 *The differential equation*

$$y' + y = \frac{1}{x} + y^3 \tag{2.16}$$

has a unique solution as an APS which is purely small.

Proof. For the existence part, note that direct substitution of a formal integer power series $y_0 = \sum_{k=1}^{\infty} c_k x^{-k}$ leads to the recurrence relation $c_1 = 1$ and for $k \geq 2$,

$$c_k = (k-1)c_{k-1} + \sum_{k_1+k_2+k_3=k;k_i\geq 1} c_{k_1} c_{k_2} c_{k_3}$$

for which direct induction shows the existence of a solution, and we have

$$y_0 = \frac{1}{x} + \frac{1}{x^2} + \frac{3}{x^3} + \frac{12}{x^4} + \frac{60}{x^5} + \cdots$$

For uniqueness assume y_0 and y_1 are APS solutions and let $\delta = y_1 - y_0$. Then δ satisfies

$$\delta' + \delta = 3y_0^2\delta + 3y_0\delta^2 + \delta^3 \tag{2.17}$$

Since by assumption $\delta \ll 1$ we have $\mathrm{Dom}(\delta') \ll \mathrm{Dom}(\delta)$ and similarly $\mathrm{Dom}(3y_0^2\delta + 3y_0\delta^2 + \delta^3) \ll \mathrm{Dom}(\delta)$. But this implies $\mathrm{Dom}(\delta) = 0$ and thus $\delta = 0$.

The exponential.

Proposition 2.18 *The differential equations $f' \pm f = 0$ have no nontrivial solution as an APS.*

Proof. If $\mathrm{Dom}(f) = 0$ then $f = 1+s$ where s is purely small and thus $s' \ll s$. But then the equation $s'+s+1 = 0$ is contradictory. Similarly if $f = s$ where s is purely small then $s' + s = 0$ implies $Dom(s) = 0$. If instead $f \gg 1$ then $\mathrm{Dom}(f) \gg \mathrm{Dom}(f')$ which is a contradiction.

Thus the exponential, if we need to introduce it, is a new element. We would like its introduction to preserve the basic structures of APS and the asymptotic ordering. Then $e^x \gg 1$ or $e^x \ll 1$ or finally $e^x - c \ll 1$. The last inequality would give, after differentiation, $e^x \ll 1$ which contradicts it, so we only have two choices. Both are consistent, and correspond to choosing a sign in $x \to \pm\infty$. We choose $x \to +\infty$ and impose $e^x \gg 1$.

Proposition 2.19 *If s is a purely small series then the equation $y' = s'y$ (corresponding intuitively to $y = e^s$) has APS solutions of the form $C + s_1$ where s_1 is small. If we choose $C = 1$ then $s_1 = s_{1;1}$ is uniquely defined.*

Proof. Straightforward.

Definition 2.20 *We define $e^s = 1 + s_{1;1}$, and in general if $S = L+C+s$ we write $e^S = C(1+s_{1;1})e^L$ where e^L is to be thought of as a primary symbol, subject to the further definitions $e^{L_1+L_2} = e^{L_1}e^{L_2}$ and $(e^L)' = L'e^L$.*

Exponential power series (EPS). A simple example of EPS is a formal series of the form

$$\sum_{i,j=1}^{\infty} \frac{c_{ij}}{e^{k_i x} x^{k_j}} \tag{2.21}$$

where k_i are increasing in i and k_j are increasing in j. Again the usual operations are well defined on EPS (composition is nor defined on (2.21) but would be if more general terms of the form $C(1+s_{1;1})e^L$ are allowed; we will postpone this until the formal introduction of transseries).

The order relation will be defined by $e^{k_1 x}x^{k_2} \gg e^{k_3 x}x^{k_4}$ iff $k_1 > k_3$ or if $k_1 = k_3$ and $k_2 > k_4$. Then we can still define the dominance of a structure of the form (2.21).

The question is what is the general *formal* solution of (**??**)? For this we have to assume we have a space A of formal objects in which all operations involved in (**??**) make sense and have the usual properties. A would be some differential algebra. We would like this space to contain x^{-1} so that the equation makes sense, and then A will contain all inverse powers of x, including constants (power zero), and in fact to be large enough to contain our formal solution (**??**). It is natural to assume that $f' = 0$ has the general solution in A, $f = C$ for some constant C.

Then if $\tilde{y}$ is any solution of (**??**) then $\tilde{f} = \tilde{y} - \tilde{y}_0$ will satisfy the homogeneous equation $f' + f = 0$. To proceed, we may include solutions of this homogeneous equation (or postulate there are no nontrivial ones; in this latter case, we settle for with a differential algebra of formal power series). If we call this solution e^{-x} and the solution of the related equation $f' - f = 0$ by e^x we see that $(e^{-x}e^x)' = 0$ thus $e^{-x}e^x = C$ for some C and we can normalize our choice of e^x to make $C = 1$. Then the general solution of $y' + y = 0$ is Ce^{-x}. Indeed, we may multiply by e^x and get $(ye^x)' = 0$, i.e. $ye^x = C$ or $y = Ce^{-x}$.

Exponential power series solutions for (2.16). We now look again at the prototypical nonlinear equation (2.16) and we look for further small solutions. We take $y = y_0 + \delta$. The equation for δ is (2.17) where we search for solutions $\delta \ll 1$, in which assumption the terms on the right hand side of the equation are subdominant. We have $\delta' + \delta(1 + o(1)) = 0$ thus $\delta = Ce^{-x+o(x)}$and this suggests the substitution $\delta = e^w$. We get

$$w' + 1 = 3y_0^2 + 3y_0 e^w + e^{2w}$$

and since $e^w = \delta \ll 1$ the dominant balance is between the terms on the left hand side, thus $w = -x + C + w_1$ and we get

$$w_1' = 3y_0^2 + 3y_0 e^{-x} e^{w_1} + e^{-2x+2w_1}$$

We have $y_0 e^{-x} e^{w_1} = y_0 \delta = y_0 e^{-x+o(x)}$. Since $-x + o(x) \gg n \ln(x)$ we have $y_0 e^{-x} e^{w_1} \ll x^{-n}$ for any n and thus $w_1' = O(x^{-2})$ then $w_1 = O(x^{-1})$. Thus, $e^{w_1} = 1 + w_1 + w_1^2/2 + \ldots$ and consequently $3y_0 e^{-x} e^{w_1} + e^{-2x+2w_1}$ is negligible with respect to y_0^2. Again by dominant balance, to leading order, $w_1' = 3y_0^2$ and thus $w_1 = \int 3y_0^2 + w_2 := \phi_1 + w_2$ (ϕ_1 is a formal power series). It follows that, to leading order, we have

$$w_2' = 3y_0 e^{-x}$$

and thus $w_2 = \phi_2 e^{-x}$ where ϕ_2 is a power series. Continuing this process of iteration, we can see inductively that w must be of the form

$$w = -x + \sum_{k=0}^{\infty} \phi_k e^{-kx}$$

where ϕ_k are formal power series, which means

$$y = \sum_{k=0}^{\infty} e^{-kx} y_k \tag{2.22}$$

where y_k are also formal power series. Having this knowledge, it is more convenient to plug in(2.22) directly in the equation and solve for the unknown series y_k. We obtain the system

$$\begin{gathered}
y_0' + y_0 = x^{-1} + y_0^3 \\
y_1' = 3y_0^2 y_1 \\
\cdots \\
y_k' - ky_k - 3y_0^2 y_k = 3y_0 \sum_{k_1+k_2=k;k_i\ge 1} y_{k_1} y_{k_2} + \sum_{k_1+k_2+k_3=k;k_i\ge 1} y_{k_1} y_{k_2} y_{k_3} \\
\cdots
\end{gathered} \tag{2.23}$$

We can easily see by induction that this system of equations does admit a solution where y_k are power series. Furthermore, y_1 is defined up to an arbitrary multiplicative constant, and there is no further freedom in y_k, whose equation can be solved by our usual iteration procedure, after placing the subdominant term y_k' on the RHS.

Choosing then y_0 in such a way that $y_1^{[1]} = 1 + ax^{-1} + ...$ we have $y_1 = Cy_1^{[1]}$. By the special structure of the RHS of the general equation in (2.23) we see that if $y_k^{[1]}$ is the solution with the choice $y_1 = y_1^{[1]}$ we see, by induction, that the solution when $y_1 = Cy_1^{[1]}$ is $C^k y_k^{[1]}$. Thus the general formal solution of (**??**) in our setting should be

$$\sum_{k=0}^{\infty} C^k y_k^{[1]} e^{-kx}$$

where $y_0^{[1]} = y_0$.

2.3 Preview of general properties of transseries

Transseries will be studied in more carefully later.

1. Transseries have an exponential level (height) which is the highest order of composition of the exponential, and similarly a logarithmic depth; both of these are finite; $\exp(\exp(x^2)) + \ln x$ has height 2 and depth 1.
   It is convenient to first construct transseries without logs and then define the general ones by composition to the right with an iterated log.

2. Transseries of level zero are simply finitely generated *asymptotic* power series. That is, given $\alpha_1, ..., \alpha_n$ with $\Re(\alpha_i) > 0$ a level zero transseries is a sum of the form

$$S = \sum_{k_i \ge M_i} c_{k_1,...,k_n} x^{-\alpha_1 k_1 - ... - \alpha_n k_n} \tag{2.24}$$

where $M_1, ..., M_n$ are *integers*, positive or negative; the terms of S are therefore nonincreasing in k_i and bounded *above* by $O(x^{-\alpha_1 M_1 - ... - \alpha_n M_n})$.

3. A term of the form $m = x^{-\alpha_1 k_1 - ... - \alpha_n k_n}$ is a level zero (trans)monomial.
4. The lower bound for k_i easily implies that there are only finitely many terms with the same monomial. Indeed, the equation $\alpha_1 k_1 + ... + \alpha_n k_n = p$ does not have solutions if $\Re(\alpha_i) k_i > |p| + \sum_{j \ne i} |\alpha_j||M_j|$.
5. A transmonomial is small $m = o(1)$ and large if $1/m$ is small. m is neither large nor small iff $m = 1$ i.e., $-\alpha_1 k_1 - ... - \alpha_n k_n = 0$; this is a degenerate case and for some purposes it is not considered a monomial.
6. A level zero transseries can be decomposed as $L + \mathit{const} + s$ where L, which could be zero, is the purely large part in the sense that it contains only large monomials and s is small.

   Assuming the coefficient of $x^{-\alpha_1 M_1 - ... - \alpha_n M_n}$ is nonzero, we can write

$$S = \mathit{const}\, x^{-\alpha_1 M_1 - ... - \alpha_n M_n}(1 + s)$$

   where s is small.

7. Operations are defined on level zero transseries in the natural way. The product of level zero transseries is a level zero transseries where as in 4 above the lower bound for k_i entails that there are only finitely many terms with the same monomial in the product.
8. It is easy to see that the expression $(1 - s)^{-1} := 1 - s + s^2 - ...$ is well defined and this allows definition of division via

$$1/S = \mathit{const}^{-1} x^{\alpha_1 M_1 + ... + \alpha_n M_n}(1 - s)^{-1}$$

9. $x^{\alpha_1 M_1 + ... + \alpha_n M_n}$ is the leading order and *const* is the leading constant.
10. It can be checked that level zero transseries form a differential field. Composition $S(s)$ is also well defined whenever s is a *large* transseries.
11. Level one. The exponential e^x has no asymptotic *power* series at infinity (in particular, its power series about zero is not of the form (2.24) and e^x is taken to be its own expansion. It is a new element.
12. A level one transmonomial is of the form $\mu = m e^L$ where m is a level zero transmonomial and L is a purely large level zero transseries. μ is *large* if the leading constant of L is positive and small otherwise. If L is large and positive then e^L is, by definition, much larger than any monomial of level zero. We define naturally $e^{L_1} e^{L_2} = e^{L_1 + L_2}$.

13. A level one transseries is of the form

$$S = \sum_{k_i \ge M_i} c_{k_1,\dots,k_n} \mu_1^{-k_1} \cdots \mu_n^{-k_n} := \sum_{\mathbf{k} \ge \mathbf{M}} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} \tag{2.25}$$

where μ_i are *large* level one transmonomials.
With the operations defined naturally as above, level one transseries form a differential field.

14. We define, for a small transseries, $e^s = \sum_{k=0}^{\infty} s^k/k!$. If s is of level zero, then e^s is of level zero too.
15. The construction proceeds similarly, by induction and a general exponential-free transseries is one obtained at *some level* of the induction. They form a differential field.
16. It can be shown, by induction, that $S' = 0$ *iff* $S = const$.
17. *Dominance*: If $S \ne 0$ then there is a largest transmonomial $\mu_1^{-k_1} \cdots \mu_n^{-k_n}$ in S, with nonzero coefficient, C. Then $\mathrm{Dom}(S) = C\mu_1^{-k_1} \cdots \mu_n^{-k_n}$. If S is a nonzero transseries, then $S = \mathrm{Dom}(S)(1+s)$ where s is purely small, i.e., all the transmonomials in s are small. It can be shown (the construction will be given later) that a base of monomials can then be chosen such that all M_i in s are positive.
18. Topology.
    a) If $\tilde{S}$ is the space of transseries generated by the monomials $\mu_1, \dots, \mu_n$ then, by definition, the sequence $S^{[j]}$ converges to S given in (2.25) if for any $\mathbf{k}$ there is a $j_0 = j_0(\mathbf{k})$ such that $c_{\mathbf{k}}^{[j]} = c_{\mathbf{k}}$ for all $j \ge j_0$.
    b) This topology is metrizable, see the discussion after Definition 3.17.
    c) In this topology, addition and multiplication are continuous, but multiplication by scalars is not.
    d) It is easy to check that any Cauchy sequence is convergent and transseries form a complete linear metric space.
    e) Contractive mappings: A function (operator) $\mathcal{A} : \tilde{S} \to \tilde{S}$ is contractive if for some $\alpha < 1$ and any $S_1, S_2 \in \tilde{S}$ we have $d(A(S_1) - A(S_2)) \le \alpha\, d(S_1 - S_2)$.
    f) *Fixed point theorem*. It can be proved in the usual way that if $\mathcal{A}$ is contractive, then the equation $S = S_0 + \mathcal{A}(S)$ has a unique fixed point.
    Examples –This is a convenient way to show the existence of multiplicative inverses. It is enough to invert $1 + s$ with s small. We choose a basis such that all M_i in s are positive. Then the equation $y = -s - sy$ is contractive.
    –The equation $y = 1/x - y'$ is contractive within level zero transseries; It has a unique solution.
19. If $L_n = \log(\log(\dots\log(x)))$ n times, and T is an exponential-free transseries then $T(L_n)$ is a general transseries. They form a differential field, closed under integration, composition to the right with large transseries, and

many other operations; this closure is proved as part of the general induction.

20. The theory of differential equations in transseries has many similarities with the usual theory. For instance it is easy to show, using an integrating factor and 16 above that the equation $y' = y$ has the general solution Ce^x and that the equation $y" = xy$ has at most two linearly independent solutions. We will find two such solutions in the examples below.

*

The type of exponential growth is related to the factorial power of divergence of the power series. For illustration we take

$$g'' + 2z^{-1}g' - z^{-5}g = 1 \tag{2.26}$$

The presence of a pole of higher order than the equation makes the power series expansion $\sum_k c_k z^k$ of a solution diverge ($c_k \propto k^p,\ p > 0$), since at the level of the recurrence for the c_k it implies that coefficients with larger k are given in terms of earlier ones multiplied by powers of n. In our specific case we get

$$c_{n+3} = n(n+1)c_n$$

with the solution

$$c_{3k} = const.\Gamma(k + 1/3)\Gamma(k)$$

roughly,

$$c_k \propto (k!)^{2/3} \tag{2.27}$$

2.3a Representability in terms of Laplace transforms

We divide by the exponential and change variable $\frac{2}{3}x^{3/2} = s$ to linearize the exponent and ensure that the transformed function has an asymptotic series with factorial divergence. Such a series can be obtained by Watson's lemma from a convergent series. Inverse Laplace transform in then likely to regularize the equation.

Taking $f(x) = e^{\frac{2}{3}x^{3/2}}h(\frac{2}{3}x^{3/2})$ we get

$$h'' + \left(2 + \frac{1}{3s}\right)h' + \frac{1}{3s}h = 0 \tag{2.28}$$

and with $H = \mathcal{L}^{-1}(h)$ we get

$$p(p-2)H' = \frac{5}{3}(1-p)H$$

which indeed has a regular singularity at $p = 0$. The solution is

$$H = Cp^{-5/6}(2-p)^{-5/6}$$

and it can be easily checked that any integral of the form

$$h = \int_0^{\infty e^{i\phi}} e^{-ps} H(p)dp$$

for $\phi \neq 0$ is a solution of (2.28) yielding the expression

$$f = e^{\frac{2}{3}x^{3/2}} \int_0^{\infty e^{i\phi}} e^{-\frac{2}{3}x^{3/2}p} p^{-5/6}(2-p)^{-5/6}dp \tag{2.29}$$

for a solution of the Airy equation. A second solution can be obtained in a similar way, replacing $e^{\frac{2}{3}x^{3/2}}$ by $e^{-\frac{2}{3}x^{3/2}}$, or by taking the difference between two integrals of the form (2.29).

Example 2. By a similar method, we can find a formal solution for the Gamma function $a_{n+1} = na_n$. We look directly for transseries of level at least one, $a_n = e^{f_n}$ and thus $f_{n+1} = \ln n + f_n$. It is clear that $f_{n+1} - f_n \ll f_n$; this suggests to write $f_{n+1} = f_n + f'_n + \frac{1}{2}f''_n + ...$ and, taking $f' = h$ we get the equation

$$h_n = \ln n - \frac{1}{2}h'_n - \frac{1}{6}h''_n - ... \tag{2.30}$$

which is contractive in transseries of zero height. We get

$$h = \ln n - \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4}...$$

and thus

$$f_n = n\ln n - n - \frac{1}{2}\ln n + \frac{1}{12n} - \frac{1}{360n^3}... + C$$

Expression as Laplace transform. The procedure in (2.30) indicates factorial divergence and suggests taking inverse Laplace transform of $g_n = f_n - (n\ln n - n + \frac{1}{2}\ln n)$.

Inverse Laplace transform is given by the *Bromwich integral* along a vertical contour in the right half plane:

$$(\mathcal{L}^{-1}F)(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xp}F(p)dp$$

The recurrence satisfied by g is

$$g_{n+1} - g_n = q_n = 1 - \left(\frac{1}{2} + n\right)\ln\left(1 + \frac{1}{n}\right) = \frac{1}{12n^2} - \frac{1}{12n^3} + ...$$

First note that $\mathcal{L}^{-1}q = p^{-2}\mathcal{L}^{-1}q''$ which can be easily evaluated by residues since

$$q'' = \frac{1}{n} - \frac{1}{n+1} - \frac{1}{2}\Big(\frac{1}{(n+1)^2} + \frac{1}{n^2}\Big)$$

Thus, with $\mathcal{L}^{-1}g_n := G$ we get

$$(e^{-p}-1)G(p) = \frac{1 - \frac{p}{2} - \left(\frac{p}{2}+1\right)e^{-p}}{p^2}$$

$$g_n = \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2}+1\right)e^{-p}}{p^2} e^{-np} dp$$

(It is easy to check that the integrand is analytic at zero; its Taylor series is $\frac{1}{12} - \frac{1}{720}p^2 + O(p^3)$.)

The integral is well defined, and it easily follows that

$$f_n = C + n(\ln n - 1) - \frac{1}{2}\ln n + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2}+1\right)e^{-p}}{p^2} e^{-np} dp$$

solves our recurrence. The constant $C = \frac{1}{2}\ln(2\pi)$ is most easily obtained by comparing with Stirling's series (1.45) and we thus get the identity

$$\ln \Gamma(n) = n(\ln n - 1) - \frac{1}{2}\ln n + \frac{1}{2}\ln(2\pi) + \int_0^\infty \frac{1 - \frac{p}{2} - \left(\frac{p}{2}+1\right)e^{-p}}{p^2} e^{-np} dp \tag{2.31}$$

which holds with n replaced by $z \in \mathbb{C}$ as well.

Other recurrences can be dealt with in the same way. One can calculate $\sum_{j=1}^n j^{-1}$ as a solution of the recurrence

$$s_{n+1} - s_n = \frac{1}{n}$$

Proceeding as in the Gamma function example, we have $f' - \frac{1}{n} = O(n^{-2})$ and the substitution $s_n = \ln n + g_n$ yields

$$g_{n+1} - g_n = \frac{1}{n} + \ln\Big(\frac{n}{n+1}\Big)$$

and in the same way we get

$$f_n = C + \ln n + \int_0^\infty e^{-np}\Big(\frac{1}{p} - \frac{1}{1-e^{-p}}\Big)dp$$

where the constant can be obtained from the initial condition, $f_1 = 0$,

$$C = -\int_0^{\infty} e^{-p}\Big(\frac{1}{p} - \frac{1}{1-e^{-p}}\Big)dp$$

which, by comparison with the usual asymptotic expansion of the harmonic sum also gives

$$\gamma = \int_0^{\infty} e^{-p}\Big(\frac{1}{1-e^{-p}} - \frac{1}{p}\Big)dp$$

Comparison with (2.31) gives

$$\sum_{j=0}^{n-1}\frac{1}{j} - \gamma = \ln n + \int_0^{\infty} e^{-np}\Big(\frac{1}{p} - \frac{1}{1-e^{-p}}\Big)dp = \frac{\Gamma'(n)}{\Gamma(n)} \tag{2.32}$$

Exercise: Zeta function. Use the same strategy to show that

$$n!\zeta(n) = \int_0^{\infty} p^{n-1}\frac{e^{-p}}{1-e^{-p}}dp = \int_0^1 \frac{\ln^{n-1} s}{1-s}ds \tag{2.33}$$

2.3b What is special about Borel summation

The Laplace Transform is defined on integrable functions of at most exponential growth by

$$\mathcal{L}\{F\}(x) := \int_0^{\infty} e^{-px}F(p)\mathrm{d}p \quad (\Re(x) > x_0)$$

When dealing with functions defined in the complex domain it is useful to allow for different contours of integration; $\mathcal{L}_\phi$ denotes the Laplace Transform in the direction ϕ:

$$\mathcal{L}_\phi\{F\}(x) := \int_0^{\infty e^{i\phi}} e^{-px}F(p)\mathrm{d}p \quad (\Re(xe^{-i\phi}) > x_0)$$

The formal Laplace Transform, still denoted $\mathcal{L} : \mathbb{C}[[p]] \mapsto \mathbb{C}[[x^{-1}]]$ is defined by

$$\mathcal{L}\{s\} = \mathcal{L}\left\{\sum_{k=0}^{\infty} c_k p^k\right\} = \sum_{k=0}^{\infty} c_k \mathcal{L}\{p^k\} = \sum_{k=0}^{\infty} c_k k! x^{-k-1} \tag{2.34}$$

(with $\mathcal{L}\{p^{\alpha-1}\} = \Gamma(\alpha)x^{-\alpha}$ the definition extends straightforwardly to non-integer power series). The **Borel Transform**, $\mathcal{B} : \mathbb{C}[[x^{-1}]] \mapsto \mathbb{C}[[p]]$ is the (formal) inverse of the operator $\mathcal{L}$ in (2.34).

We note that

$$\mathcal{B}\{x^{-1}\} = 1$$

whereas

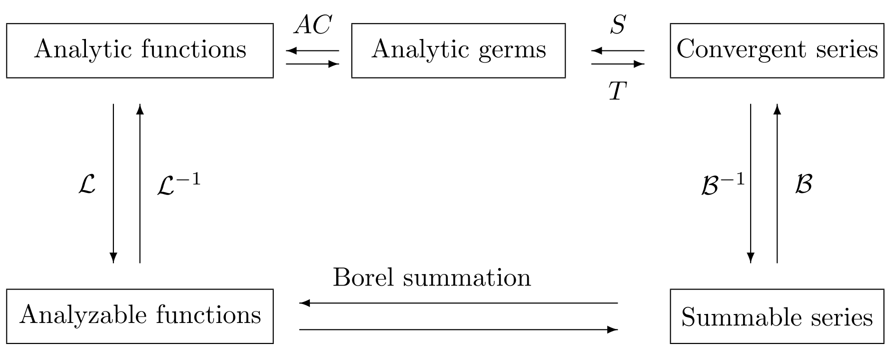
$$\mathcal{L}^{-1}\{x^{-1}\} = \begin{cases} 1 & \Re(p) > 0 \\ 0 & \Re(p) < 0 \end{cases}$$

This difference is related to allowing for directional Laplace Transforms. Otherwise, if $\tilde{f}$ is convergent, the functions $\mathcal{B}\{\tilde{f}\}$ and $\mathcal{L}^{-1}\{\tilde{f}\}$ coincide in the right half plane (cf. also Remark 2.42 below).

Because the $k-th$ coefficient of $\mathcal{B}\{\tilde{f}\}$ is smaller by a factor $k!$ than the corresponding coefficient of $\tilde{f}$, $\mathcal{B}\{\tilde{f}\}$ may converge even if $\tilde{f}$ does not. Since factorial divergence is commonplace in analytic problems (for reasons that will become clear in the sequel) this convergence-improving property of $\mathcal{B}$ is very useful.

Also important is that the combination $\mathcal{LB}$ is, formally, **the identity** operator, and must thus have, when properly interpreted, good commutation properties with function operations.

These two facts account for the central role played by $\mathcal{LB}$, the operator of Borel summation in the theory of analyzable functions.



Definition of Borel summation and basic properties.

Series of the form $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-\beta_1 k_1 - \ldots - \beta_m k_m - r}$ with $\Re(\beta_j) > 0$ frequently arise as formal solutions of differential systems. We will first analyze the case $m = 1, r = 1, \beta = 1$ but the theory extends without difficulty to more general series.

Borel summation is relative to a direction, see Remark 2.42. The same formal series $\tilde{f}$ may yield different functions by Borel summation in different directions.

Borel summation along $\mathbb{R}^+$ consists in three operations:

1. Borel Transform, $\tilde{f} \mapsto \mathcal{B}\{\tilde{f}\}$. *Assuming* that $\mathcal{B}\{\tilde{f}\}$ is convergent and extends analytically in a neighborhood of $\mathbb{R}^+$, step 2 is possible:

2. Summation and analytic continuation, $\mathcal{B}\{\tilde{f}\} \mapsto AC \circ \sum\{\mathcal{B}\{\tilde{f}\}\} =: F(p)$, with F real analytic on $\mathbb{R}^+$. The further assumption that $F(p)$ grows at most exponentially makes the last step possible:

3. Laplace Transform, $F \mapsto \int_0^\infty F(p)e^{-px}\mathrm{d}p =: \mathcal{LB}\{\tilde{f}\}$, defined in some half plane $\Re(x) > x_0$.

The *domain* of Borel summation is the subspace $S_{\mathcal{B}}$ of series for which the conditions for the steps 1-3 above are met. For 3 we can require that for some constants C_F, ν_F we have $|F(p)| \le C_F e^{\nu_F p}$.

Note 2.35 *Equivalently we can say that the series $\tilde{f}$ is Borel summable if it is the asymptotic series as $x \to +\infty$ of $\mathcal{L}F$ with F analytic in a neighborhood $\mathcal{D}_{\mathbb{R}^+}$ of $\mathbb{R}^+$ (in particular, we say such a function is* real-analytic on $[0,+\infty)$*) and exponentially bounded at infinity. The domain $\mathcal{D}_{\mathbb{R}^+}$ as well as the bounds may depend on F. The definition is unambiguous since on the one hand the asymptotic series of a function is unique, and, by Watson's Lemma, if the asymptotic series of $\mathcal{L}F$ is zero, then the Taylor series of F at $p = 0$ is zero as well, and then $F \equiv 0$.*

Definition 2.36 (Inverse Laplace space convolution) *If $f, g \in L^1_{loc}$ then*

$$(f * g)(p) := \int_0^p f(s)g(p-s)ds \tag{2.37}$$

Lemma 2.38 *The space of functions which are in $L^1[0,\epsilon)$ for some $\epsilon > 0$ and real-analytic on $(0,\infty)$ is closed under convolution. If F and G are exponentially bounded then so is $F * G$.*

Proof. The statement about L^1 is standard. Analyticity follows by writing

$$\int_0^p f_1(s)f_2(p-s)ds = p\int_0^1 f_1(pt)f_2(p(1-t))dt \tag{2.39}$$

which is manifestly analytic in p. Clearly,

$$|F * G| \le C_F C_G p\, e^{(\nu_F+\nu_G)p} \le C_F C_G\, e^{(\nu_F+\nu_G+1)p}$$

Proposition 2.40 *(i) $S_{\mathcal{B}}$ is a differential algebra,*[2] *and $\mathcal{LB} : S_{\mathcal{B}} \mapsto \mathcal{LB}S_{\mathcal{B}}$ is a differential algebra isomorphism.*[3]

(ii) If $S_c \subset S_{\mathcal{B}}$ denotes the differential algebra of convergent power series, and we identify a convergent power series with its sum, then $\mathcal{LB}$ is the identity on S_c.

(iii) In addition, for $\tilde{f} \in S_B$, $\mathcal{LB}\{\tilde{f}\} \sim \tilde{f}$ as $|x| \to \infty$, $\Re(x) > 0$.

Proof. (i) Clearly $S_{\mathcal{B}}$ is a linear space; furthermore, $\tilde{f} = 0 \iff \mathcal{B}\tilde{f} = 0 \iff \mathcal{LB}\{\tilde{f}\} = 0$ (the last step follows from the injectivity of $\mathcal{L}$ which, in our case also follows from Watson's Lemma as in Note 2.35 above.)

To show multiplicativity, we use Note 2.35. Analyticity and exponential bounds of $|F * G|$ follow from Lemma2.38. Consequently, $F * G$ is Laplace

[2] with respect to formal addition, multiplication, and differentiation of power series.

[3] See also § 2.3c for a summary of the properties of $\mathcal{L}$.

transformable, and by elementary properties of Laplace transforms (or by performing a simple change of variables in a double integral) we see that

$$\mathcal{L}(F * G) = \mathcal{L}F\,\mathcal{L}G$$

It remains to show that the asymptotic expansion of $\mathcal{L}(F * G)$ is indeed the product of the asymptotic series of $\mathcal{L}F$ and $\mathcal{L}G$, which is a consequence of the more general fact that the asymptotic series of a product is the product of the corresponding asymptotic series.

(ii) Since $\tilde{f}_1 = \tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k-1}$ is convergent, then $|c_k| \le CR^k$ for some C, R and $F(p) = \sum_{k=0}^{\infty} c_k p^k/k!$ is entire, $|F(p)| \le \sum_{k=0}^{\infty} CR^k p^k/k! = Ce^{Rp}$ and thus F is Laplace transformable for $|x| > R$. By dominated convergence we have for $|x| > R$,

$$\mathcal{L}\Big\{\sum_{k=0}^{\infty} c_k p^k/k!\Big\} = \lim_{N\to\infty} \mathcal{L}\Big\{\sum_{k=0}^{N} c_k p^k/k!\Big\} = \sum_{k=0}^{\infty} c_k x^{-k-1} = f(x)$$

(iii) This part follows simply from Watson's lemma, cf. § 2.3c . □

Remark 2.41 *The results above can be rephrased for more general series of the form* $\sum_{k=0}^{\infty} c_k x^{-k-r}$ *by noting that for* $\Re(\rho) > -1$ *we have*

$$\mathcal{L}p^{\rho} = x^{-\rho-1}\Gamma(\rho+1)$$

and thus

$$\mathcal{B}\left(\sum_{k=0}^{\infty} c_k x^{-k-r}\right) = c_0 \frac{p^{r-1}}{\Gamma(r)} + \frac{p^{r-1}}{\Gamma(r)} * \mathcal{B}\left(\sum_{k=1}^{\infty} c_k x^{-k}\right)$$

Furthermore, Borel summation summation naturally extends to to series of the form

$$\sum_{k=-M}^{\infty} c_k x^{-k-r}$$

where $M \in \mathbb{N}$ *by defining*

$$\mathcal{L}\mathcal{B}\left(\sum_{k=-M}^{\infty} c_k x^{-k-r}\right) = \sum_{k=-M}^{0} c_k x^{-k-r} + \mathcal{L}\mathcal{B}\left(\sum_{k=0}^{\infty} c_k x^{-k-r}\right)$$

and more general powers can be allowed, replacing analyticity in p *with analyticity in* $p^{\beta_1}, ..., p^{\beta_m}$.

Remark 2.42 We note that in the last step in Borel summation we may take the integral in p along a different half-line in $\mathbb{C}$, as long as $\Re(xp) > 0$, and the algebraic properties are preserved. But it is easy to check that the path matters, in general. For instance, if $x \in \mathbb{R}^+$ and $\mathcal{B}\tilde{f} = (1-p)^{-1}$, the half line can be any ray in the open right half plane, other than $\mathbb{R}^+$. But

$$\int_0^{\infty e^{i0+}} \frac{e^{-xp}}{1-p} dp - \int_0^{\infty e^{i0-}} \frac{e^{-xp}}{1-p} dp = 2\pi i e^{-x}$$

thus a convention for a choice of ray is needed.
(i) For the ray $e^{i\phi}\mathbb{R}^+$, we take $F = AC\sigma\mathcal{B}\tilde{f}$ and define

$$(\mathcal{LB})_\phi \tilde{f} = \int_0^{\infty e^{-i\phi}} e^{-px} F(p) dp = \mathcal{L}_{-\phi} F = \mathcal{L}F(\cdot e^{-i\phi}) \tag{2.43}$$

i.e., by convention, Laplace Transform is taken in the direction that ensures $xp \in \mathbb{R}^+$. We can also say that Borel summation of $\tilde{f}$ along the ray $\arg(x) = \phi$ is defined as the (real) Borel summation of $\tilde{f}(xe^{i\phi})$.

(ii) Related to (i), control over the analytic properties of $\mathcal{B}\tilde{f}$ near $p = 0$ is essential to Borel summability (classical or generalized). Indeed, by a result due to Borel and Ritt, for any power series $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k}$ and *any* sector S there exist (many) functions f analytic in S and asymptotic in S to $\tilde{f}$ [**?**]. Now, choosing $\delta > 0$, a sector S of angle larger than $\pi + \delta$, and any f such that $f \sim \tilde{f}$ in S, and denoting $f_1 = f$, then Proposition 2.52 below shows that $f - c_0 - c_1 x^{-1} = \mathcal{L}\{F_1\}$ with F_1 analytic in a sector of angle δ; in addition, by Watson's lemma (see Lemma 2.70), $\mathcal{L}\{F_1\} \sim \tilde{f}_1$ in S. *Any* series would thus be "summable" (very non-uniquely) in this weak sense. Summable series $\tilde{f}$ are distinguished by the analytic properties of F_1 at $p = 0$.

(iii) Since in most cases of interest $\mathcal{B}\tilde{f}$ has singularities in the complex plane, different functions $\mathcal{LB}_\phi \tilde{f}$ are obtained for different ϕ. For example, we have

$$\mathcal{LB}_\phi \sum_{k=0}^{\infty} \frac{k!}{x^{k+1}} = \mathcal{L}_{-\phi}\{(1-p)^{-1}\} = \begin{cases} \mathrm{e}^{-x}\mathrm{Ei}(x) - \pi i & \text{for } \phi \in (-\pi, 0) \\ \mathrm{e}^{-x}\mathrm{Ei}(x) + \pi i & \text{for } \phi \in (0, \pi) \end{cases} \tag{2.44}$$

(iv) On the other hand it can be seen by deforming the contour in $\mathcal{L}$ that if $\mathcal{B}\tilde{f}$ is analytic and has uniform exponential bounds at infinity for $\arg(p) \in (-\delta_1, \delta_2)$, then the function $\mathcal{LB}_\phi \tilde{f}$ is the same for all $\arg(x) \in (-\delta_2, \delta_1)$, in contrast to (2.44).

Recovering exact solutions from formal series. If a differential equation has a formal solution $\tilde{f} \in S_{\mathcal{B}}$ then $\mathcal{LB}\tilde{f}$ is an actual solution of the same equation. For example

$$f' - f = x^{-1} \tag{2.45}$$

for $x \to \infty$ has the series solution $\tilde{f} = \sum_{k=0}^{\infty}(-x)^{-k-1}k!$ and $\mathcal{B}\{\tilde{f}\} = \sum_{k=0}^{\infty}(-p)^k$ sums to the Laplace transformable function $(1+p)^{-1}$. Now, for any $\tilde{f} \in S_{\mathcal{B}}$ and $f \in \mathcal{LB}(S_{\mathcal{B}})$ we have

$$\tilde{f}' - \tilde{f} - x^{-1} = 0 \iff \mathcal{LB}\big(\tilde{f}' - \tilde{f} - x^{-1}\big) = 0 \tag{2.46}$$

$$\iff \big(\mathcal{LB}\{\tilde{f}\}\big)' - \mathcal{LB}\{\tilde{f}\} - x^{-1} = 0 \tag{2.47}$$

In particular,

$$\mathcal{LB}\{\tilde{f}\} = \int_0^{\infty} \frac{e^{-px}\mathrm{d}p}{1+p} = f \tag{2.48}$$

is an actual solution of (2.45). Solving the analytic problem (2.45) in $\mathcal{LB}(S_{\mathcal{B}})$ has reduced thus to an essentially *algebraic* question, that of finding $\tilde{f}$.

Some of the difficulties of Borel summation. A serious problem with classical Borel summation is that its domain of definition is not large enough. First, $\mathcal{LB}$ only applies to power series, while for instance the general solution of (2.45) is $\mathcal{LB}\{\tilde{f}\} + Ce^x$. This deficiency could, however, be corrected by naturally extending $\mathcal{LB}$ to the exponential by the formula $\mathcal{LB}\exp(ax) = \exp(ax)$.

There is a subtler and much more severe difficulty, however. The change of variable $x \mapsto (-x)$ in (2.45) leads to the equation $f' + f = 1/x$, with formal solution $\tilde{f} = \sum_{k=0}^{\infty} k!x^{-k-1}$. We now get $\sum \mathcal{B}\tilde{f} = (1-p)^{-1}$ which is *not* Laplace transformable, because of the nonintegrable singularity at $p = 1$. Although one can avoid the singularity by shifting the contour of $\mathcal{L}$ in the complex plane, there is **no systematic way** to define the shift to allow for arbitrary location of the (isolated) singularities of $\mathcal{B}\tilde{f}$, and if the contour of integration has to depend on $\tilde{f}$, then the linearity of $\mathcal{LB}$ is lost. (Commutation with complex conjugation is also lost if the contour of integration is not fixed.) Restricting however the *location* of singularities would make Borel summability incompatible with the trivial change of variable $x \mapsto \text{Const}.x$.

Finally, $\mathcal{LB}$ cannot be usefully restricted to those $\tilde{f} \in S_{\mathcal{B}}$ for which $F_1 = \mathcal{B}\{x^{\tilde{r}}f\}$ is entire and $|F_1(p)| \le C_1 e^{C_2|p|}$ in $\mathbb{C}$, because this simply entails the convergence of $\tilde{f}$. Indeed, by shifting the contour of integration in $\int_0^{\infty} e^{-px}F_1(p)dp$ and rotating x simultaneously to keep xp real and positive, we see that $(\mathcal{L}F_1)(x)$ is single-valued near infinity. By Proposition 2.40 $\mathcal{L}F_1 \sim \tilde{f}_1$ as $|x| \to \infty$, $x \in \mathbb{C}$, therefore ∞ is a removable singularity of $\mathcal{L}F_1$ and the series $\tilde{f}_1$ converges.

Incidentally, the example just given shows an important feature of divergent expansions. A single-valued function f cannot be asymptotic to the same divergent expansion in every direction in the complex plane: the asymptotic behavior of f must therefore vary with the direction at infinity. This is a manifestation of **Stokes' phenomenon**.

Since the restrictions needed for classical Borel summation to apply do not allow it to define a sufficiently general isomorphism, one looks instead at extensions of $\mathcal{LB}$, as an **operator**.

2.3c Appendix. Properties of the Laplace transform

Proposition 2.49 *If $F \in L^1(\mathbb{R}^+)$ then $\mathcal{L}F$ is analytic in the right half plane H and continuous on the imaginary axis ∂H, and $\mathcal{L}\{F\}(x) \to 0$ as $x \to \infty$ in H.*

Proof. Continuity and analyticity are preserved by integration against a finite measure ($F(p)dp$). Equivalently, they follow by dominated convergence, as $\epsilon \to 0$, of $\int_0^\infty e^{-isp}(e^{-ip\epsilon} - 1)F(p)dp$ and $\int_0^\infty e^{-xp}(e^{-p\epsilon} - 1)\epsilon^{-1}F(p)dp$ respectively, the last integral for $\Re(x) > 0$. The stated limit also follows easily from dominated convergence, if $|\arg(x) \pm \pi/2| > \delta$; the general case follows from the case $|\arg(x)| = \pi/2$ which is a consequence of the Riemann-Lebesgue lemma. $\square$

First inversion formula. Let $\mathcal{H}$ denote the space of analytic functions in H.

Proposition 2.50 *(i) $\mathcal{L} : L^1(\mathbb{R}^+) \mapsto \mathcal{H}$ and $\|\mathcal{L}\{F\}\|_\infty \le \|F\|_1$.*

(ii) $\mathcal{L} : L^1 \mapsto \mathcal{L}(L^1)$ is invertible, and the inverse is given by

$$F(x) = \hat{\mathcal{F}}^{-1}\{\mathcal{L}\{F\}(it)\}(x) \tag{2.51}$$

for ($x \in \mathbb{R}^+$) where $\hat{\mathcal{F}}$ is the Fourier transform.

Proof. Part (i) is immediate, since $|e^{-xp}| \le 1$. (ii) Extending F on $\mathbb{R}^-$ by zero we have $\mathcal{L}\{F\}(it) = \int_{-\infty}^\infty e^{-ipt}F(p)\mathrm{d}p = \hat{\mathcal{F}}F$. $\square$

Second inversion formula. Laplace transform is not surjective from L^1 to $\mathcal{H}$ but functions in $\mathcal{H}$ with sufficient decay do belong to $\mathcal{L}(L^1)$.

Proposition 2.52 *(i) Assume $\delta \ge 0$ and f is analytic in a sector of angle more than π, $H_\delta := \{x : |\arg(x)| < \pi + \delta\}$ and continuous on ∂H_δ, and that for some $K > 0$ and any $x \in \overline{H_\delta}$*

$$|f(x)| \le K(|x|^2 + 1)^{-1} \tag{2.53}$$

Then $\mathcal{L}^{-1}f$ is well defined by

$$F = \mathcal{L}^{-1}f = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \mathrm{d}t\, e^{pt}f(t) \tag{2.54}$$

and

$$\int_0^{\infty} \mathrm{d}p\, e^{-px} F(p) = \mathcal{L}\mathcal{L}^{-1} f = f(x)$$

and in addition $\|\mathcal{L}^{-1}\{f\}\|_{\infty} \le K\pi$ *and* $\mathcal{L}^{-1}\{f\} \to 0$ *as* $p \to \infty$.

(ii) If $\delta > 0$ *then* $F = \mathcal{L}^{-1} f$ *is analytic in the sector* $S = \{p \ne 0 : |\arg(p)| < \delta\}$. *In addition,* $\sup_S |F| \le K\pi$ *and* $F(p) \to 0$ *as* $p \to \infty$ *in* S.

Proof. (i) We have

$$\int_0^{\infty} \mathrm{d}p\, e^{-px} \int_{-\infty}^{\infty} \mathrm{d}s\, e^{ips} i f(is) = \int_{-\infty}^{\infty} \mathrm{d}t\, f(is) \int_0^{\infty} \mathrm{d}p\, e^{-px} e^{ips} \tag{2.55}$$

$$= \int_{-i\infty}^{i\infty} f(z)(x-z)^{-1} \mathrm{d}z = 2\pi i f(x) \tag{2.56}$$

where we applied Fubini's theorem and then pushed the contour of integration past x to infinity. The norm is obtained by majorizing $|f e^{ips}|$ by $K(|x^2|+1)^{-1}$.

(ii) We have for any $\delta' < \delta$, by (2.53),

$$\int_{-i\infty}^{i\infty} \mathrm{d}s\, e^{ps} f(s) = \left(\int_{-i\infty}^{0} + \int_0^{i\infty} \right) \mathrm{d}s\, e^{ps} f(s)$$

$$= \left(\int_{-i\infty e^{-i\delta'}}^{0} + \int_0^{i\infty e^{i\delta'}} \right) \mathrm{d}s\, e^{ps} f(s) \tag{2.57}$$

and analyticity is clear in (2.57).

For (ii) we note that (i) applies in $\bigcup_{|\delta'|<\delta} e^{i\delta'} H_0$. □

Proposition 2.58 *Let* F *be analytic in the open sector* $S_p = e^{i\phi}\mathbb{R}^+$ *with* $\phi \in (-\delta, \delta)$ *be such that* $|F(|x|e^{i\phi})| \le g(|x|)$ *for some* $g \in L^1[0, \epsilon)$ *bounded as* $x \to \infty$. *Then* $f = \mathcal{L}F$ *is analytic in the sector* $S_x = \{x : |\arg(x)| < \pi/2 + \delta\}$ *and* $f(x) \to 0$ *as* $|x| \to \infty, \arg(x) = \theta \in (-\pi/2 - \delta, \pi/2 + \delta)$.

Proof. Because of the analyticity of F and the decay conditions for large p, the path of Laplace integration can be rotated by any angle $\phi \in (-\delta, \delta)$ without changing $(\mathcal{L}F)(x)$ (see also the next example). This means Proposition 2.49 applies in $\cup_{|\phi|<\delta} e^{i\phi} H$.

Note that without further assumptions on $\mathcal{L}F$, F is *not* necessarily analytic at $p = 0$.

Corollary 2.59 The kernel of $\mathcal{L}$ is trivial: *if* $F \in L^1(\mathbb{R}^+)$ *and* $\mathcal{L}F = 0$ *then* $F = 0$.

Proof. An immediate consequence of the first inversion formula. □

Example Rotation of the contour of integration in the complex plane is a convenient way to calculate the change in asymptotic behavior with respect to the sector of analysis. We illustrate this on a simple case:

$$y(x) := \int_0^{\infty} \frac{e^{-px}}{1+p} dp \tag{2.60}$$

and we would like, say, to find the asymptotic behavior in the complex plane of the analytic continuation of this integral with respect to x after one anticlockwise loop around infinity. Analytic continuation (if it exists) is unique, so we proceed in a natural way, by deforming the contour of integration. We note that for $x \in \mathbb{R}^+$ $y(x)$ also equals

$$y(x) = \int_0^{\infty e^{-i\pi/4}} \frac{e^{-px}}{1+p} dp \tag{2.61}$$

but this integral is manifestly analytic for $\arg(x) \in (-\frac{\pi}{4}, \frac{3\pi}{4})$. Thus the analytic continuation $y(xe^{i\pi/2})$, $x \in \mathbb{R}^+$, is given by

$$y(xe^{i\pi/2}) = \int_0^{\infty e^{-i\pi/4}} \frac{e^{-pxe^{i\pi/2}}}{1+p} dp \tag{2.62}$$

which is also equal to

$$y(xe^{i\pi/2}) = \int_0^{\infty e^{-3i\pi/4}} \frac{e^{-pxe^{i\pi/2}}}{1+p} dp \tag{2.63}$$

which is now manifestly analytic for $\arg(x) \in (\pi/4, 5\pi/4)$. We have, in particular, the analytic continuation

$$y(xe^{i\pi}) = \int_0^{\infty e^{-3i\pi/4}} \frac{e^{-pxe^{i\pi}}}{1+p} dp \tag{2.64}$$

in which it would be convenient to rotate again the contour of integration clockwise, by $\pi/2$. This time however we cross a pole of the integrand and collect a residue, $-2\pi i e^x$ in the process.

$$y(xe^{i\pi}) = \int_0^{\infty e^{-5i\pi/4}} \frac{e^{-pxe^{i\pi}}}{1+p} dp - 2\pi i e^x \tag{2.65}$$

Now the process can be continued in the same way, and we get

$$y(xe^{2\pi i}) = \int_0^{\infty} \frac{e^{-px}}{1+p} dp - 2\pi i e^x \tag{2.66}$$

Remark. It is useful to note that by continuity and analyticity, it is enough to have $\mathcal{L}F(x) = 0$ on any set with an accumulation point in the right half plane to ensure $F \equiv 0$.

Asymptotic properties Laplace transforms. The asymptotic behavior of Laplace integrals is particularly important given that every analyzable function should be convergently expressed by generalized Borel summation of a transseries.

Lemma 2.67 *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim p^\beta$ with $\Re(\beta) > -1$ as $p \to 0^+$. Then $\mathcal{L}F \sim \Gamma(\beta+1)x^{-\beta-1}$ along any ray ρ in the open right half plane H°.*

Proof. If $U(p) = p^{-\beta}F(p)$ we have $\lim_{p\to 0} U(p) = 1$. Let χ_A be the characteristic function of the set A and $\phi = \arg(x)$. We choose $C, a > 0$ such that $|F(p)| < C|p^\beta|$ on $[0,a]$. Since

$$\left|\int_a^\infty F(p)e^{-px}\mathrm{d}p\right| \le e^{-xa}\|F\|_1 \tag{2.68}$$

we have, and after the change of variable $s = p/|x|$,

$$x^{\beta+1}\int_0^\infty F(p)e^{-px}\mathrm{d}p = e^{i\phi(\beta+1)}\int_0^\infty s^\beta U(s/|x|)\chi_{[0,a]}(s/|x|)e^{-se^{i\phi}}\mathrm{d}s$$
$$+ O(|x|^{\beta+1}e^{-xa}) \to \Gamma(\beta+1) \quad (|x|\to\infty) \tag{2.69}$$

Watson's Lemma, presented below, states that the asymptotic series at infinity of $(\mathcal{L}F)(x)$ is obtained by formal term-by-term integration of the asymptotic series of $F(p)$ for small p, provided F has such a series.

Lemma 2.70 *Let $F \in L^1(\mathbb{R}^+)$ and assume $F(p) \sim \sum_{k=0}^\infty c_k p^{k\beta_1+\beta_2-1}$ as $p \to 0^+$ for some constants β_i with $\Re(\beta_i) > 0$, $i = 1, 2$. Then*

$$\mathcal{L}F \sim \sum_{k=0}^\infty c_k\Gamma(k\beta_1+\beta_2)x^{-k\beta_1-\beta_2}$$

along any ray ρ in the open right half plane H°.

Proof Induction, using Lemma 2.67. □

2.4 Gevrey classes, least term truncation, and Borel summation

In the simple example of $\mathrm{Ei}(x)$, factorial divergence is associated with the possible presence of exponentially small terms, terms beyond all orders. This and the power-of-factorial-like divergence of formal asymptotic series of solutions of differential equations are quite general phenomena, as will become clear in the following chapters.

Let now $\tilde{f} = \sum_{k=0}^{\infty} c_k x^{-k}$ be a formal power series and f a function asymptotic to it. The definition (1.3) provides for large x estimates of the value of $f(x)$ within $o(x^{-N})$, $N \in \mathbb{N}$, which are, as we have seen, insufficient to pin down a unique f associated to $\tilde{f}$. Simply widening the sector in which (1.3) is required cannot change this situation since, for instance, $\exp(-x^{1/m})$ is beyond all orders of $\tilde{f}$ in a sector of angle almost $m\pi$.

It seems then reasonable to attempt to (a) lower the errors in the approximation of f by the truncates of $\tilde{f}$ to less than $O(e^{-\text{Const.}|x|})$, to roughly match the "natural" indeterminacy of f, and then (b) look for estimates in a wide enough sector in the hope of ruling out any possible terms beyond all orders, in this way restoring uniqueness of the association between f and $\tilde{f}$. In some cases this strategy is successful. One important technique in this class is due to *Gevrey* (see e.g. [**?**]).

The formal series

$$\tilde{f}(x) = \sum_{k=0}^{\infty} c_k x^{-k}$$

is Gevrey of order $1/m$, or Gevrey-$(1/m)$ if $|c_k| \le C_1 C_2^k (k!)^m$ for some C_1, C_2. Taking $x = y^m$ and $\tilde{g}(y) = \tilde{f}(x)$, then $\tilde{g}$ is Gevrey-1 (albeit not necessarily an integer power series, but noninteger power series can be treated very similarly) and we will focus on this case.

Remarks 2.71 (a) The Gevrey order of the series $\sum_k (k!)^r x^k$ $r > 0$, is the same as that of $\sum_k (rk)! x^k$. Indeed, if $\epsilon > 0$ we have, by Stirling's formula,

$$Const\ (1+\epsilon)^{-k} \le (rk)!/(k!)^r \sim Const\ k^{\frac{1}{2}-r} \le Const\ (1+\epsilon)^k$$

(b) There is a simple connection between the Gevrey order of formal power series solutions of a differential equation at an irregular singular point and the type of exponentials of the associated homogeneous equation. For illustration consider the example of the equation $x^{q+1}y' - ay = 1$ in a neighborhood of zero, with $q \in \mathbb{N}$. The coefficients c_k of a formal power series solution $\tilde{y} = \sum_{k \ge 0} c_k x^k$ satisfy the recurrence $a_0 = 0$ and $(k-q)c_{k-q} + ac_k = 0$ if $k - q > 0$. If $q \ge 1$ we get $c_{jq+q} = a^j j!$, the series diverges and $x = 0$ is an irregular singularity. Using part (a) above we see that the series is Gevrey-q. On the other hand, the solution of the homogeneous equation $x^{q+1}y' - ay = 0$ is $C \exp\left(-\frac{a}{q} x^{-q}\right)$. Precise asymptotic control of the coefficients of formal power series solutions can be obtained for quite general differential systems, see e.g. [**?**].

Exercise. Formulate and prove a more general result in the spirit of Remark 2.71 (b) for n-th order linear differential equations.

*

Let $\tilde{f}$ be Gevrey-1. A function f is *Gevrey*-1 *asymptotic* to $\tilde{f}$ as $x \to \infty$ in a sector S if for some C_3, C_4, C_5, and all $x \in S$ with $|x| > C_5$ and all N we have

$$|f(x) - \tilde{f}^{[N]}| \le C_1 C_2^{N+1} |x|^{-N-1} (N+1)! \tag{2.72}$$

i.e. the error $f - \tilde{f}^{[N]}$ is of the same form as the first omitted term in $\tilde{f}$.

Remark 2.73 *If $\tilde{f}$ is Gevrey-1 and f is Gevrey-1 asymptotic to $\tilde{f}$ then f can be approximated by $\tilde{f}$ with exponential precision in the following sense. Let $N = \lfloor |x/C_2| \rfloor$, the integer part of $|x|$; then for any $C > C_2$ we have*

$$f(x) - \tilde{f}^{[N]}(x) = o(e^{-|x|/C}) \ |x| \text{ large}) \tag{2.74}$$

Indeed, applying Stirling's formula we have

$$N! N^{-N} C_2^N |x|^{-N} = O(\sqrt{N} e^{-|x|/C_2})$$

□

Notes. (a) A heuristic discussion about the strategy may be helpful now; rigorous statements will follow.

Usually the imprecision implied by (2.74) is larger than the potential terms beyond a Gevrey-1 series $\tilde{f}$, at least in *some* directions.

However, if the estimate (2.74) holds for f in a sector $S_{\pi+}$ of opening more than π, then it is easy to see that (2.74) cannot hold at the same time for f and for $f + C' e^{-C'' x^p} x^{-m}$, no matter what C'', m, p are, unless $C' = 0$. Since terms beyond all orders, if present, are expected to be some combinations of powers, exponentials and logs, these and similar attempts suggest that if f satisfies (2.74) in $S_{\pi+}$, then f is unique. Theorem 2.75 below shows that this is true.

(b) It is also interesting that when there is a unique f in $S_{\pi+}$ with the property (2.74), then $\tilde{f}$ is Borel summable, and f is *precisely the Borel sum of $\tilde{f}$* (Theorem 2.75 below).[4]

(c) However the same theorem suggests that unless the series $\tilde{f}$ is trivial, there must exist *some* $S_{\pi+}$ in which *no* f is Gevrey−1-asymptotic to $\tilde{f}$ and where this method of associating an f to $\tilde{f}$ fails. In addition we note that there is no entire function of exponential order one at infinity (i.e., $f(x) \le C_1 \exp(C_2|x|)$) which is Gevrey−1 asymptotic to a divergent series in more than a half plane. Indeed if there was such a function f then the Phragmén-Lindelöf principle applied in $\mathbb{C} \backslash S_{\pi+}$ would imply that f is bounded at infinity, thus f and $\tilde{f}$ would be constant.

(d) *Summation to the least term* as will be detailed in the Chapter 4, is in a sense a refined version of Gevrey asymptotics. It requires *optimal constants* in addition to an improved form of Rel. (2.72). In this way the

[4] Borel summability is clearly not ensured by the Gevrey character of $\tilde{f}$ alone, since such estimates give no information about $\sum \mathcal{B}\tilde{f}$ beyond the implied disk of convergence.

imprecision of approximation of f by $\tilde{f}$ turns out to be smaller than the largest exponentially small term beyond all orders, and thus the cases in which uniqueness is ensured are more numerous.

Connection between Gevrey asymptotics and Borel summation.

Theorem 2.75 *Let* $\tilde{f} = \sum_{k=2}^{\infty} c_k x^{-k}$ *be a Gevrey-*1 *series and assume the function* f *is analytic for large* x *in* $S_{\pi+} = \{x : |\arg(x)| < \pi/2 + \delta\}$ *for some* $\delta > 0$ *and Gevrey-*1 *asymptotic to* $\tilde{f}$ *in* $S_{\pi+}$. *Then*

(i) f *is unique.*

(ii) $\tilde{f}$ *is Borel summable in any direction* $e^{i\theta}\mathbb{R}^+$ *with* $|\theta| < \delta$ *and* $f = \mathcal{L}\mathcal{B}_\theta \tilde{f}$.

(iii) $\mathcal{B}(\tilde{f})$ *is analytic (at* $p = 0$ *and) in the sector* $S_\delta = \{p : \arg(p) \in (-\delta, \delta)\}$, *and uniformly bounded in any closed subsector.*

(iv) Conversely, if $\tilde{f}$ *is Borel summable along any ray in the sector* S_δ *given by* $|\arg(x)| < \delta$, *and uniformly bounded in any closed subsector of* S_δ, *then* f *is Gevrey-1 with respect to its asymptotic series* $\tilde{f}$ *in the sector* $|\arg(x)| \le \pi/2 + \delta$.

Notes. (i) In particular, when the assumptions of the theorem are met, Borel summability follows using only *asymptotic estimates.*

(ii) We also see that the cases described in Theorem 2.75 in which Gevrey estimates ensure uniqueness of the association between $\tilde{f}$ and f are *less general* than those in which $\tilde{f}$ is Borel summable.

Proof of Theorem 2.75. (i) If f_1 and f_2 satisfy the assumption of the theorem, then by Proposition 2.73, for some constants C_1, C_2 we have

$$|f_1(x) - f_2(x)| < C_1 e^{-C_2|x|} \tag{2.76}$$

in a sector of opening more then π. Since f_1 and f_2 are analytic, Phragmén-Lindelöf's principle gives $f_1 - f_2 = 0$. Alternatively, we could note that by Proposition 2.52 $\mathcal{L}^{-1}\{f_1 - f_2\}$ exists and is analytic for $\arg(p) \in (-\delta, \delta)$ and that, by (2.76), for $|p| < C_2$ the contour of integration in (2.54) can be pushed to infinity implying that $\mathcal{L}^{-1}\{f_1 - f_2\} = 0$ on the interval $(0, C_2)$. By analyticity $\mathcal{L}^{-1}\{f_1 - f_2\} \equiv 0$ and the inversion formula gives $f_1 - f_2 = 0$.

(ii) By a simple change of variables we arrange $C_1 = C_2 = 1$. The series $\tilde{F}_1 = \mathcal{B}\tilde{f}$ is convergent for $|p| < 1$ and defines an analytic function, F_1. By Proposition 2.52, the function $F = \mathcal{L}^{-1} f$ is analytic for $|p| > 0, |\arg(p)| < \delta$, and $F(p)$ is analytic and uniformly bounded if $|\arg(p)| < \delta_1 < \delta$. We now show that F is analytic for $|p| < 1$. Taking p real, $p \in [0, 1)$ we obtain in view of (2.72) that

$$\begin{aligned} |F(p) - \tilde{F}^{[N-1]}(p)| &\le \int_{-i\infty+N}^{i\infty+N} d|s| \left| f(s) - \tilde{f}^{[N-1]}(s) \right| e^{\Re(ps)} \\ &\le N! e^{pN} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{|x + iN|^N} = N! e^{pN} \int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2 + N^2)^{N/2}} \end{aligned} \tag{2.77}$$

where we take $x = N\tan t$ and get the estimate for the last term

$$\frac{N!e^{pN}}{N^{N-1}}\int_{-\pi/2}^{\pi/2}\cos^{N-2}(t)\mathrm{d}t \sim 2\pi N e^{(p-1)N} \quad (N\to\infty) \tag{2.78}$$

(using, for instance, by the Laplace method). Since the RHS in (2.78) vanishes in the limit $N\to\infty$ for $p\in[0,1)$, this implies $F = F_1$ for $p\in[0,1)$, thus $F = F_1$ for any p with $|p|<1$ and also for any p with $|\arg(p)|<\delta$.

Since $\sum\mathcal{B}\tilde{f} = \mathcal{L}^{-1}f$, (iii) follows now from Proposition 2.52.

(iv) Let $|\phi|<\delta$. We have, by integration by parts,

$$f(x) - \tilde{f}^{[N-1]}(x) = x^{-N}\mathcal{L}\frac{d^N}{dp^N}F \tag{2.79}$$

On the other hand, F is analytic in S_a, some $a = a(\phi)-$neighborhood of the sector $\{p : |\arg(p)| < |\phi|\}$. Estimating Cauchy's formula on an $a-$circle around the point p with $|\arg(p)|<|\phi|$ we get

$$|F^{(n)}(p)| \le N!a(\phi)^{-N}\|F(p)\|_{\infty;S_a}$$

Thus, by (2.79), with $|\theta|\le|\phi|$ chosen so that $\gamma = \cos(\theta - \arg(x))$ is maximal we have

$$\begin{aligned}\left|f(x) - \tilde{f}^{[N]}\right| &= \left|x^{-N}\int_0^{\infty\exp(-i\theta)}F^{(N)}(p)e^{-px}dp\right| \\ &\le N!a^{-N}|x|^{-N}\|F\|_{\infty;S_a}\int_0^\infty e^{-px\gamma}dp = N!a^{-N}\gamma^{-1}|x|^{-N-1}\|F\|_{\infty;S_a}\end{aligned} \tag{2.80}$$

□

Stokes lines. Theorem 2.75 and the discussion in §2.3b show that for a non-convergent Gevrey-1 series $\tilde{f}$ there must exist sectors of opening more than π where no f is Gevrey-1 asymptotic to $\tilde{f}$. These "singular" directions reflect the presence of the local Stokes phenomenon.

Definition. *Let $\tilde{f}$ be Gevrey-1.*

We say that $\tilde{f}$ is **Gevrey-1 asymptotic in** $S(\phi;\epsilon;R)$ *where*

$$S(\phi;\epsilon;R) = \{x : |x|>R, |\arg(x)-\phi| < \pi/2+\epsilon\}$$

if there exists f analytic $S(\phi;\epsilon;R)$ such that $f \overset{G_1}{\sim} \tilde{f}$ in $S(\phi;\epsilon;R)$ (then this f is unique, by Theorem 2.75).

If ϕ is such that $\tilde{f}$ is not Gevrey-1 asymptotic in $S(\phi;\epsilon;R)$, we say that $d_\phi = \{x : \arg(x) = \phi\}$ is a **Stokes ray** *for $\tilde{f}$.*

Proposition 2.81 *Let $\tilde{f}$ be Gevrey-1. Then $\tilde{f}$ is divergent* **iff** *it has at least a Stokes ray.*

Proof. This property of $\tilde{f}$ is clearly independent of any finite number of terms in $\tilde{f}$ so we may assume $\tilde{f} = \sum_{k=2}^{\infty} f_k x^{-k}$. If $\tilde{f}$ is convergent then clearly it has no Stokes directions. For the converse, we assume that $\tilde{f}$ has no Stokes directions and for $\phi \in [0, 2\pi + \delta]$ we let $\epsilon_\phi > 0, R_\phi, f_\phi$ be such that $f_\phi \overset{G_1}{\sim} \tilde{f}$ in $S(\phi; \epsilon_\phi; R_\phi)$. If $E(\phi)$ is the sup of ϵ_ϕ such that $\tilde{f}$ is Gevrey-1 asymptotic in $S(\phi; \epsilon_\phi; R_\phi)$ for some R_ϕ then it is easy to check that $E(\phi)$ is continuous in ϕ and then, for some $N \in \mathbb{N}$ we have $\inf_{\phi \in [0, 2\pi+\delta]} E(\phi) > (2/N) > 0$. In all sectors $S_j = S(j/N; \epsilon_{j/N}; R_{j/N})$ with $0 \leq j/N < 2\pi + \delta$ the series $\tilde{f}$ is Gevrey-1 asymptotic, and since $S_j \cap S_{j+1}$ is wider than π we have by Theorem 2.75 that $f_{(j+1)/N} = f_{j/N}$ if $0 \leq j/N < 2\pi + \delta$. Thus $f_{j/N} = f$ is independent of j and in particular f is single-valued at infinity. Thus, by Liouville's theorem f is analytic at infinity and $\tilde{f}$ is convergent.

2.4a Strategies of Borel summation of formal power series solutions: an introduction

Assume we intend to solve using Borel summability techniques an ODE, say

$$y' + y = x^{-2} + y^3 \tag{2.82}$$

To find a formal power series solution we proceed as usual, separating out the dominant terms, in this case y and x^{-2}. We get the iterations scheme

$$y_{[n]}(x) - x^{-2} = y_{[n-1]}^3 - y_{[n]}' \tag{2.83}$$

with $y_{[0]} = 0$. After a few iterations we get

$$\tilde{y}(x) = x^{-2} + 2\,x^{-3} + 6\,x^{-4} + 24\,x^{-5} + 121\,x^{-6} + 732\,x^{-7} + 5154\,x^{-8} + \cdots \tag{2.84}$$

For differential equations of this kind there exist results in great generality as to the Borel summability of formal transseries solutions, and we shall see a few of these in the sequel. The purpose now is to illustrate a strategy of proof that is convenient and which applies to a reasonably large class of settings.

It would be technically awkward to prove that after Borel transform the series is convergent, extends analytically along the real line and better approach is has the required exponential bounds towards infinity.

A better approach is to get a direct grip on the Borel transform of $\tilde{y}$ via the equation it satisfies. This equation is the formal inverse Laplace transform of (2.83), namely, setting $Y = \mathcal{B}\tilde{y}$

$$-pY + Y = p + Y * Y * Y := p + Y^{*3} \tag{2.85}$$

We then show that the equation (2.85) has a (unique) solution which is analytic in a neighborhood of the origin together with a sector centered on $\mathbb{R}^+$ in which this solution has exponential bounds. Thus Y is Laplace transformable, and immediate verification shows that $y = \mathcal{L}Y$ satisfies (2.82).

Furthermore, since the Maclaurin series $S(Y)$ formally satisfies (2.85) then the formal Laplace (inverse Borel) transform $\mathcal{B}^{-1}SY$ is a *formal* solution of (2.82), and thus equals $\tilde{y}$ since this solution, as we proved in many similar settings is unique. But since then $SY = \mathcal{B}\tilde{y}$ it follows that $\tilde{y}$ is Borel summable, and the Borel sum solves (2.82).

The transformed equations are expected to have analytic solutions, therefore to be more regular than the original ones.

Regularizing the heat equation.

$$f_{xx} - f_t = 0 \tag{2.86}$$

Since (2.86) is parabolic, power series solutions

$$f = \sum_{k=0}^{\infty} t^k F_k(x) = \sum_{k=0}^{\infty} \frac{F_0^{(2k)}}{k!} t^k \tag{2.87}$$

are divergent even if F_0 is analytic (but not entire). Nevertheless, under suitable assumptions, Borel summability results of such formal solutions have been shown by Lutz, Miyake, and Schäfke [**?**] and more general results of multisummability of linear PDEs have been obtained by Balser [**?**].

The heat equation can be regularized by a suitable Borel summation. The divergence implied, under analyticity assumptions, by (2.87) is $F_k = O(k!)$ which indicates Borel summation with respect to t^{-1}. Indeed, the substitution

$$t = 1/\tau; \quad f(t,x) = t^{-1/2} g(\tau, x) \tag{2.88}$$

yields

$$g_{\tau\tau} + \tau^2 g_\tau + \frac{1}{2}\tau g = 0$$

which becomes after formal inverse Laplace transform (Borel transform) in τ,

$$p\hat{g}_{pp} + \frac{3}{2}\hat{g}_p - \hat{g}_{xx} = 0 \tag{2.89}$$

which is brought, by the substitution $\hat{g}(p,x) = p^{-\frac{1}{2}} u(x, 2p^{\frac{1}{2}})$; $y = 2p^{\frac{1}{2}}$, to the wave equation, which is hyperbolic, thus *regular*

$$u_{xx} - u_{yy} = 0. \tag{2.90}$$

Existence and uniqueness of solutions to regular equations is guaranteed by Cauchy-Kowalevsky theory. For this simple equation the general solution is certainly available in explicit form: $u = f(x-y) + g(x+y)$ with f, g arbitrary twice differentiable functions. Since the solution of (2.90) is related to a solution of (2.86) through (2.88), to ensure that we do get a solution it is easy to check that we need to choose $f = g$ (up to an irrelevant additive constant which can be absorbed into f) which yields,

$$f(t,x) = t^{-\frac{1}{2}} \int_0^\infty y^{-\frac{1}{2}} \left[u\left(x + 2\,y^{\frac{1}{2}}\right) + u\left(x - 2\,y^{\frac{1}{2}}\right)\right] \exp\left(-\frac{y}{t}\right) dy \quad (2.91)$$

which, after splitting the integral and making the substitutions $x \pm 2\,y^{\frac{1}{2}} = s$ is transformed into the usual Heat kernel solution,

$$f(t,x) = t^{-\frac{1}{2}} \int_{-\infty}^\infty u(s) \exp\left(-\frac{(x-s)^2}{4t}\right) ds \quad (2.92)$$

*

2.4b Convolutions: elementary properties

The transformed equation (2.85) is a convolution equation and it is useful to list first some elementary properties of convolutions. Some spaces are well suited for the study of convolution algebras.

(1) Let $\nu \in \mathbb{R}^+$ and define $L^1_\nu := \{f : \mathbb{R}^+ : f(p)e^{-\nu p} \in L^1(\mathbb{R}^+)\}$; then the norm $\|f\|_\nu$ is defined as $\|f(p)e^{-\nu p}\|_1$ where $\|\cdot\|_1$ denotes the L^1 norm.

Proposition 2.93 *L^1_ν is a Banach algebra with respect to convolution.*

Proof. Note first that if $f \in L^1_\nu$ then the Laplace transform of f exists for $Re(x) \ge \nu$ and $f, g \in L^1_\nu$ implies

$$\begin{aligned}\|f * g\|_\nu &= \int_0^\infty e^{-\nu p} \left| \int_0^p f(s) g(p-s) ds \right| dp \\ &= \int_0^\infty \left| \int_0^p f(s)e^{-\nu s} g(p-s) e^{-\nu(p-s)} ds \right| dp \\ &\le \int_0^\infty \int_0^p \left|f(s)e^{-\nu s}\right| \left|g(p-s)e^{-\nu(p-s)}\right| ds dp \\ &= \int_0^\infty \left|f(s)e^{-\nu s}\right| \int_0^\infty ds \left|g(s)e^{-\nu s}\right| ds = \|f\|_\nu \|g\|_\nu \end{aligned} \quad (2.94)$$

In particular, convolution is well defined in L^1_ν and we have, by a very similar calculation,

$$\mathcal{L}\left[f * g\right] = (\mathcal{L}f)(\mathcal{L}g) \quad (2.95)$$

Furthermore,

$$\mathcal{L}\left[f * (g * h)\right] = \mathcal{L}[f]\mathcal{L}\left[g * h\right] = \mathcal{L}[f]\mathcal{L}[g]\mathcal{L}[h] = \mathcal{L}\left[(f * g) * h\right] \quad (2.96)$$

and since the Laplace transform is injective, we get

$$f * (g * h) = (f * g) * h \tag{2.97}$$

and convolution is associative. Similarly, it is easy to see that

$$f * g = g * f, \qquad f * (g + h) = f * g + f * h \tag{2.98}$$

(2) Another important space is $\mathcal{A}_{K;\nu}$, the space of analytic functions analytic in a star-shaped neighborhood $\mathcal{N} \in \mathbb{C}$ of the interval $[0, K]$ in the norm ($\nu \in \mathbb{R}^+$)

$$\|f\| = K \sup_{p \in \mathcal{N}} \left| e^{-\nu|p|} f(p) \right|$$

Note This norm is equivalent with the sup norm, but is useful in controlling exponential growth.

Proposition 2.99 *The space $\mathcal{A}_{K;\nu}$ is a Banach algebra with respect to convolution.*

Proof. Analyticity of convolution is proved in the same way as Lemma 2.38. Associativity and commutativity of convolution are shown either by a strategy similar to the one in the previous proposition, or by direct verification.

To show continuity of convolution we let $|p| = P$, $p = Pe^{i\phi}$ and note that

$$\left| Ke^{-\nu P} \int_0^p f(s) g(p-s) ds \right| = \left| Ke^{-\nu P} \int_0^P f(te^{i\phi}) g((P-t)e^{i\phi}) dt \right|$$
$$= \left| K^{-1} \int_0^P Kf(te^{i\phi}) e^{-\nu t} Kg((P-t)e^{i\phi}) e^{-\nu(P-t)} dt \right|$$
$$\leq K^{-1} \|f\| \|g\| \int_0^P d|t| = \|f\| \|g\| \tag{2.100}$$

(3) Finally, we note that the space $\mathcal{A}_{K,\nu;0} = \{f \in \mathcal{A}_{K,\nu} : f(0) = 0\}$ is a closed subalgebra of $\mathcal{A}_{K,\nu}$.

Focusing spaces and algebras. An important property of the norms introduced, on the spaces L^1_ν and $\mathcal{A}_{K,\nu;0}$ is that for any f in these spaces $\|f\| \to 0$ as $\nu \to \infty$. In the case L^1_ν this is an immediate consequence of dominated convergence.

More generally, we say that a family of norms $\|\|_\nu$ depending on a parameter $\nu \in \mathbb{R}^+$ is **focusing** if for any f with $\|f\|_{\nu_0} < \infty$

$$\|f\|_\nu \downarrow 0 \text{ as } \nu \uparrow \infty \tag{2.101}$$

Let $\mathcal{E}$ be a linear space and $\{\|\|_\nu\}$ a family of norms satisfying (2.101). For each ν we define a Banach space $\mathcal{B}_\nu$ as the completion of $\{f \in \mathcal{E} : \|f\|_\nu < \infty\}$. Enlarging $\mathcal{E}$ if needed, we may assume that $\mathcal{B}_\nu \subset \mathcal{E}$. For $\alpha < \beta$, (2.101) shows that the identity is an embedding of $\mathcal{B}_\alpha$ in $\mathcal{B}_\beta$. Let $\mathcal{F} \subset \mathcal{E}$ be the projective limit of the $\mathcal{B}_\nu$. That is to say

$$\mathcal{F} := \bigcup_{\nu>0} \mathcal{B}_\nu \tag{2.102}$$

is endowed with the topology in which a sequence is convergent if it converges in *some* $\mathcal{B}_\nu$. We call $\mathcal{F}$ a **focusing space**.

Consider now the case when $(\mathcal{B}_\nu, +, *, \|\|_\nu)$ are commutative Banach algebras. Then $\mathcal{F}$ inherits a structure of a commutative algebra, in which $*$ ("convolution") is continuous. We say that $(\mathcal{F}, *, \|\|_\nu)$ is a **focusing algebra**.

Examples The spaces $\bigcup_{\nu>0} L^1_\nu$ and $\bigcup_{\nu>0} \mathcal{A}_{K;\nu;0}$ are focusing algebras.

2.5 Borel summability of solutions of nonlinear equations: an introduction

We will analyze a simple example which will however illustrate many of the important technical points in Borel summation of nonlinear systems. Consider the equation:

$$y' - y = x^{-2} + y^3 \tag{2.103}$$

Formal inverse Laplace transform of (2.103) yields, with the notation $\mathcal{L}^{-1}y = Y$ and $Y^{*3} = Y * Y * Y$,

$$-pY - Y = p + Y^{*3} \tag{2.104}$$

Proposition 2.105 *(i) Assume $Y \in L^1_\nu$ is a solution of (2.104). Then $y = \mathcal{L}Y$ is a solution of (2.103).*

(ii) Assume Y is analytic at the origin then y has an asymptotic power series as $x \to \infty$ which is the formal Laplace transform of the Maclaurin series of Y.

(iii) In the assumption (iii), the formal power series solution of (2.103) is Borel summable, and it Borel sums to y.

Proof. (i) Taking $y = \mathcal{L}Y$ we get straightforwardly equation (2.103) by taking the Laplace transform of (2.104).

(ii) Watson's Lemma directly implies this conclusion.

(iii) Since the Maclaurin series of Y TY is a formal solution of (2.104) it follows easily that its formal Laplace transform $\mathcal{B}^{-1}TY$ is a formal solution of (2.103). But the uniqueness of the formal power series solution for (2.103) is shown as for (2.16).

We now show that the assumptions in (i) and (ii) in the previous Proposition hold.

Proposition 2.106 *For large enough ν, the equation*

$$Y = -\frac{p}{p+1} - \frac{1}{p+1}Y^{*3} = \mathcal{N}Y$$

is contractive in a small ball in L^1_ν and thus has a unique solution there.

Proof. Let ϵ be small, choose ν such that $\|p(p+1)^{-1}\|_\nu \le \epsilon/2$ (which is possible since L^1_ν is a focusing algebra) and define the ball $B_\epsilon = \{f : \|f\|_\nu \le \epsilon$. It is easy to see that $\mathcal{N}(B_\epsilon) \subset B_\epsilon$. Contractivity follows from

$$\|Y_1^*3 - Y_2^{*3}\| = \|(Y_1 - Y_2) * (Y_1^{*2} + Y_1 * Y_2 + Y_2^{*2})\| \le 3\epsilon^2\|(Y_1 - Y_2)\|$$

2.5a Preview of solution of differential equations by generalized Borel summation

The type of equations for which complete rigorous results exist are of the form

$$\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n, \quad x \in \mathbb{C} \tag{2.107}$$

where
(i) $\mathbf{f}$ is *analytic* in a neighborhood $\mathcal{V}_x \times \mathcal{V}_\mathbf{y}$ of $(0, \mathbf{0})$, under the genericity conditions that:
(ii) the eigenvalues λ_j of the matrix $\hat{\Lambda} = -\left\{\frac{\partial f_i}{\partial y_j}(0, \mathbf{0})\right\}_{i,j=1,2,\dots n}$ are linearly independent over $\mathbb{Z}$ (in particular $\lambda_j \neq 0$) and such that
(iii) $\arg \lambda_j$ are all different.

(In fact somewhat less restrictive conditions are used, namely those of [**?**] §1.1.2.)

By elementary changes of variables, the system (3.117) can be brought to the *normalized form* [**?**],

$$\mathbf{y}' = -\hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x^{-1}, \mathbf{y}) \tag{2.108}$$

where $\hat{\Lambda} = \operatorname{diag}\{\lambda_j\}$, $\hat{A} = \operatorname{diag}\{\alpha_j\}$ are constant matrices, $\mathbf{g}$ is analytic at $(0, \mathbf{0})$ and $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}) + O(|\mathbf{y}|^2)$ as $x \to \infty$ and $\mathbf{y} \to 0$. Performing a further transformation of the type $\mathbf{y} \mapsto \mathbf{y} - \sum_{k=1}^M \mathbf{a}_k x^{-k}$ (which takes out M terms of the formal asymptotic series solutions of the equation), makes

$$\mathbf{g}(|x|^{-1}, \mathbf{y}) = O(x^{-M-1}; |\mathbf{y}|^2; |x^{-2}\mathbf{y}|) \quad (x \to \infty;\ \mathbf{y} \to 0)$$

where

$$M \geq \max_j \Re(\alpha_j)$$

and $O(a;b;c)$ means (at most) of the order of the largest among a, b, c.

Our analysis applies to solutions $\mathbf{y}(x)$ such that $\mathbf{y}(x) \to 0$ as $x \to \infty$ along some arbitrary direction $d = \{x \in \mathbb{C} : \arg(x) = \phi\}$. We shall exemplify some of these transformations in the sequel.

An n-parameter formal solution of (3.115) (under the assumptions mentioned) as a combination of powers and exponentials is found in the form

$$\tilde{\mathbf{y}}(x) = \sum_{k_1,k_2,\ldots,k_n=0}^{\infty} \left\{ C_1^{k_1} C_2^{k_2} \cdots C_n^{k_n} e^{-(k_1\lambda_1+k_2\lambda_2+\ldots k_n\lambda_n)x} x^{k_1\alpha_1+k_2\alpha_2+\ldots k_n\alpha_n} \tilde{\mathbf{y}}_{k_1,k_2,\ldots,k_n} \right\} := \sum_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^n} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \tilde{\mathbf{y}}_{\mathbf{k}}(x) \quad (2.109)$$

where $\tilde{\mathbf{y}}_{\mathbf{k}}$ are (usually factorially divergent) formal power series and in general

$$\tilde{\mathbf{y}}_{\mathbf{k}}(x) = \sum_{r=0}^{\infty} \frac{\tilde{\mathbf{y}}_{\mathbf{k};r}}{x^r} \quad (2.110)$$

that can be determined by formal substitution of (2.109) in (3.115); $\mathbf{C} \in \mathbb{C}^n$ is a vector of parameters[5] (we use the notations $\mathbf{C}^{\mathbf{k}} = \prod_{j=1}^{n} C_j^{k_j}$, $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_n)$, $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_n)$, $|\mathbf{k}| = k_1 + ... + k_n$).

Given a direction d in the complex x-plane the *transseries* (on d), are, in our context, those exponential series (2.109) which are formally *asymptotic* on d, i.e. the terms $\mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} x^{-r}$ (with $\mathbf{k} \in (\mathbb{N}\cup\{0\})^n$, $r \in \mathbb{N}\cup\{0\}$) form a well ordered set with respect to $\gg$ on d (see also [**?**]). In other words, indices i for which the corresponding term $e^{-\lambda_i x}$ is not formally small in d may not appear, that is, must be associated with $C_i = 0$.

The results proven for this type of equations may be, informally, summarized in the following.

i) All $\tilde{\mathbf{y}}_{\mathbf{k}}$ are generalized Borel summable at the same time.

ii) The Borel summed series $\mathbf{y}_{\mathbf{k}} = \mathfrak{B}\tilde{\mathbf{y}}_{\mathbf{k}}$ exist in a half plane $H = \{x : \Re(x) > x_0\}$ for some x_0 independent of $\mathbf{k}$ and are analytic there.

[5] In the general case when some assumptions made here do not hold, the general formal solution may also involve compositions of exponentials, logs and powers [**?**].

iii) There exists a constant $\mathbf{c}$ *independent of* $\mathbf{k}$ *so that* $\sup_{x\in H} |\mathbf{y_k}| \le \mathbf{c^k}$. *Thus, the new series,*

$$\mathbf{y} = \sum_{\mathbf{k}\in(\mathbb{N}\cup\{0\})^n} \mathbf{C^k} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \mathbf{y_k}(x) \tag{2.111}$$

is convergent for any $\mathbf{C}$ *for which the corresponding expansion (2.109) is a transseries, in a region given by the condition* $|C_i e^{-\lambda_i x} x^{\alpha_i}| < c_i^{-1}$ *(remember that* C_i *is zero if* $|\mathrm{e}^{-\lambda_i x}|$ *is not small).*

iv) The function $\mathbf{y}$ *obtained in this way is a solution of the differential equation (3.117).*

v) Any solution of the differential equation (3.117) which tends to zero in some direction d *can be written in the form (2.111) for a unique* $\mathbf{C}$*, this constant depending usually on the sector where* d *is. This dependence is a manifestation of the Stokes phenomenon.*

vi) The Borel summation operator $\mathfrak{B}$ *is the usual Borel summation in any direction* d *of* x *which is not a Stokes direction. However* $\mathfrak{B}$ *is still an isomorphism, whether* d *is a Stokes direction or not.*

Some remarks about structure of singularities in Borel space and resurgence phenomena. Let us look at a very simple prototypical example

$$y'' + (2 + x^{-1})y' - (3 + x^{-1})y = x^{-1}y^2$$

We take $y_1 = y, y_2 = y'$ and get a system of equations of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} 0 \\ y_1^2 \end{pmatrix}$$

Diagonalization of the two 2-by-2 matrices on the right hand side is achieved easily my making a transformation of the dependent variable of the form $\mathbf{y} \mapsto (\hat{M}_1 + x^{-1}\hat{M})\mathbf{y}$ for suitably chosen $\hat{M}_i$ and the system that results is of the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} g_1(x^{-1}, y_1, y_2) \\ g_2(x^{-1}, y_1, y_2) \end{pmatrix}$$

satisfying our assumptions. In this particular example, the eigenvalues, though not linearly independent over $\mathbb{Z}$ still satisfy the weaker conditions in [**?**] and the general theory applies. If the direction of interest for the variable x is $\mathbb{R}^+$, then the only admissible exponential is e^{-x} as e^{2x} tends to infinity instead of being small. Thus there is in the direction of $\mathbb{R}^+$ a one-parameter only family of transseries, in the form

$$\sum_{k=0}^{n} C^k \mathrm{e}^{-kx} x^{k\alpha} \tilde{\mathbf{y}}_k(x)$$

Analytic continuations. The series $\tilde{\mathbf{y}}_k$ will be classically Borel summable in any direction other than $\mathbb{R}^+$ and $\mathbb{R}^-$. It turns out that along any Stokes direction, here $\mathbb{R}^+$ and $\mathbb{R}^-$, the Borel transforms $Y_k = \mathcal{B}\tilde{\mathbf{y}}_k$ develop arrays of singularities. These singularities are located at positive multiple integers of 1, and -2. It is proved that the functions $\mathbf{Y}_k$ can be continued analytically along any paths in the complex plane that go towards infinity (the modulus of p increases along the path) and cross between the singular points in the arrays at most once. Borel summability along the special directions of the singularities is ensured both in a sense of distributions, in which generalized Laplace transform is taken through the singular points, or, equivalently, as a specific average of analytic continuations along the paths mentioned above. The averaging formula is the same, irrespective of the differential equation.

Resurgence. This is another very important phenomenon that occurs in differential systems, in which the higher index series $\tilde{\mathbf{y}}_\mathbf{k}$ are related to $\tilde{\mathbf{y}}_\mathbf{0}$ in a way that does not depend on the differential equation and permits reconstruction of the $\tilde{\mathbf{y}}_\mathbf{k}$, thus of the general formal solution and ultimately of the whole differential equation from the mere knowledge of $\tilde{\mathbf{y}}_\mathbf{0}$. For instance under proper normalization, the $\mathbf{Y}_\mathbf{k}$ are related to differences in the analytic continuations of $\mathbf{Y}_0$ along the various paths between singularities.

Normalization procedures. Many equations which are not presented in the form (3.117) can be brought to this form by changes of variables. The key idea to do this in a systematic way is to calculate the transseries solutions of the equation, find the transformations which bring that to the normal form (2.109), and then apply these transformations to the original variables in the differential equation. The first part of the analysis need not be rigorous, as the conclusions are made rigorous in the sequel.

We illustrate this on a simple equation, as $t \to \infty$:

$$u' = u^3 - t \tag{2.112}$$

This is not of the form (3.117) due to the fact that $g(u,t) = u^3 - t$ is not analytic in t *at* $t = \infty$. This can be however remedied in the way we described before.

As we have already seen before, dominant balance for large t requires writing the equation (2.112) in the form

$$u = (t + u')^{1/3} \tag{2.113}$$

and we have $u' \ll t$. Three branches of the cubic root are possible and are investigated similarly, but we aim here merely at illustration and choose the simplest. Iterating (2.113) in the usual way, we are lead to a formal series solution in the form

$$\tilde{u} = t^{1/3} + \frac{1}{9}t^{-4/3} + ... = t^{1/3} \sum_{k=0}^{\infty} \frac{\tilde{u}_k}{t^{5k/3}} \tag{2.114}$$

To find the full transseries we now substitute $u = \tilde{u} + \delta$ in (2.112) and keep the dominant terms. We get

$$\frac{\delta'}{\delta} = \frac{9}{5}t^{2/3} + \frac{2}{3}\ln t$$

from which it follows that

$$\delta = Ct^{2/3}e^{\frac{9}{5}t^{5/3}} \tag{2.115}$$

Since the normalized transseries must have exponentials of the form e^{-x}, the adequate independent variable must then be $x = -\frac{9}{5}t^{5/3}$. In this variable, the formal power series (2.114) takes the form

$$\tilde{u} = x^{1/5} \sum_{k=0}^{\infty} \frac{\tilde{u}_k}{x^k} \tag{2.116}$$

But this should have been of the form $\sum_{k=0}^{\infty} \frac{\tilde{u}_k}{x^k}$. Thus the right dependent variable is $h = x^{1/5}u$. In this variable, we are led to the equation

$$h' + \frac{1}{5x}h + 3h^3 - \frac{1}{9} = 0 \tag{2.117}$$

where analyticity at infinity is now ensured! The only remaining transformation is to pull out a few terms out of h, to make the nonlinearity of the order $g = O(x^{-2}, h^2)$. This is done by calculating, again by dominant balance, the first two terms in the $1/x$ power expansion of h, namely $1/3 - x^{-1}/15$ and subtracting them out of h, i.e., changing to the new dependent variable $y = h - 1/3 + x^{-1}/15$. This yields

$$y' = -y + \frac{1}{5x}\,y + g(y, x^{-1}) \tag{2.118}$$

where

$$g(y, x^{-1}) = -3(y^2 + y^3) + \frac{3y^2}{5x} - \frac{1}{15x^2} - \frac{y}{25x^2} + \frac{1}{3^2 5^3 x^3} \tag{2.119}$$

We see that

$$\lambda = 1, \quad \alpha = 1/5 \tag{2.120}$$

3. Rigorous construction of transseries

Sections §2.109 and §2.3 are helpful to motivate the rigorous but relatively more abstract constructions of this chapter.

Transseries are comprehensive generalizations of power series, constructed to represent analytic functions not only at regular points but at complicated singularities as well.

Because of Stokes' phenomenon the same function may have different transseries in different directions at the singularity. Loosely speaking, transseries are *asymptotic, finitely generated* combinations powers, logarithms and exponentials, and represent a closure of usual power series under a wide variety of operations. Because of this, they are able to describe most functions of "natural origin", to use an expression of Écalle. Asymptotic means that the terms can be ordered decreasingly, with respect to the order relation $f_1(x) \ll f_2(x)$ if $f_1(x) = o(f_2(x))$ as $x \to x_0$. It is usually convened to place x_0 at ∞ and make the direction of the analysis that of $\mathbb{R}^+$. A simple example of a transseries of exponential level 1 with generators $1/x, e^{-x}$, as $x \to +\infty$, is

$$\sum_{k,m=0}^{\infty} c_{km} e^{-kx} x^{-m}$$

where $c_{km} \in \mathbb{C}$. An example of a transseries of exponential level 2, with level 0 generators x^{-1} and $x^{-\sqrt{2}}$, level 1 generator $\exp(x)$, and level 2 generators $\exp(\sum_{k=0}^{\infty} c_k e^x x^{-k})$ and $\exp(-e^x)$ is

$$e^{\sum_{k=0}^{\infty} c_k e^x x^{-k}} + \sum_{k=0}^{\infty} d_k x^{-k\sqrt{2}} + e^{-e^x}$$

Some examples of transseries-like expressions which are in fact *not* transseries as $x \to +\infty$ are $\sum_{k=0}^{\infty} x^k$ (it fails the asymptoticity condition) and $\sum_{k=0}^{\infty} e^{-e^{nx}}$ (it does not have finitely many generators, this property is described precisely in the sequel).

The underlying structure behind the condition of asymptoticity is that of *well ordering*. In order to formalize the notion of transseries and study their properties, it is useful to first introduce and study more general abstract expansions, over a well ordered set.

3.0b Totally ordered sets; well ordered sets

Let A be an ordered set, with respect to $\leq$. If $x \not\leq y$ we write $x > y$ or $y < x$. A is **totally ordered** if any two elements are comparable, i.e., if for any $x, y \in A$ we have $x \geq y$ or $y \geq x$. If A is not totally ordered, it is called **partially ordered**.

The set A is **well ordered** with respect to $>$ if every nonempty totally ordered subset (*chain*) of A has a minimal element, i.e.

$$A' \subset A \implies \exists M \in A' \text{ such that } \forall x \in A',\ M \leq x$$

If any nonempty totally ordered subset of A has a **maximal** element, we say that A is well ordered with respect to $<$.

3.0c Finite chain property

A has the **finite descending chain** property if there is no *infinite strictly decreasing* sequence in A, in other words if $f : \mathbb{N} \mapsto A$ is decreasing, then f is constant for large n.

Proposition 3.1 *A is well ordered with respect to $>$* iff *it has the finite descending chain property.*

Proof. A strictly decreasing infinite sequence is obviously totally ordered and has no minimal element. For the converse, if there exists $A' \subset A$ such that $\forall\, x \in A'\, \exists\, y =: f(x) \in A',\ f(x) < x$ then for $x_0 \in A'$, the sequence $\{f^{(n)}(x_0)\}_{n \in \mathbb{N}}$ is an infinite descending chain in A.

Example: multi-indices. $\mathbb{N}$ is well ordered with respect to $>$, and so is

$$\mathbb{N}^M - \mathbf{k}_0 := \{\mathbf{k} \in \mathbb{Z}^M : \mathbf{k} \geq -\mathbf{k}_0\}$$

with respect to the order relation $\mathbf{m} \geq \mathbf{n}$ iff $m_i \geq n_i\, \forall\, i \geq k$. Indeed an infinite descending sequence $\mathbf{n}_i$ would be infinitely descending on at least one component.

Proposition 3.2 *Let $\mathbf{k}_0 \in \mathbb{Z}^M$ be fixed. Any infinite set A in $\mathbb{N}^M - \mathbf{k}_0$ contains a* strictly increasing *(infinite) sequence.*

Proof. The set A is unbounded, thus there must exist at least one component $i \leq M$ so that the set $\{m_i : \mathbf{m} \in A\}$ is also unbounded; say $i = 1$. We can then choose a sequence $S = \{\mathbf{m}_n\}_{n \in \mathbb{N}}$ so that $(\mathbf{m}_n)_1$ is strictly increasing. If the set $\{(\mathbf{m}_n)_j : \mathbf{m} \in A;\ j > 1\}$ is bounded, then there is a subsequence S' of S so that $(m_2, ..., m_n)_{n'}$ is a constant vector. Then S' is a strictly increasing sequence. Otherwise, one component, say $(\mathbf{m}_{n'})_2$ is unbounded, and we can choose a subsequence S'' so that $(m_1, m_2)_{n''}$ is increasing. The argument continues in this fashion until in at most M steps an increasing sequence is constructed.

Corollary 3.3 *Any infinite set of multi-indices in $\mathbb{N}^M$ contains at least two comparable elements.*

Corollary 3.4 *Let A be a nonempty set of multi-indices in $\mathbb{N}^M - \mathbf{k}_0$. There exists a unique and* **finite** *minimizer set $\mathcal{M}_A$ such that none of its elements are comparable and for any $a' \in A$ there is an $a \in \mathcal{M}_A$ such that $a \le a'$.*

Proof. Consider the set C of all *maximal* totally ordered subsets of A (every chain is contained in a maximal chain; also, in view of countability, Zorn's lemma is not needed). Let $\mathcal{M}_A$ be the set of the least elements of these chains, i.e. $\mathcal{M}_A = \{\min c : c \in C\}$. Then $\mathcal{M}_A$ is finite. Indeed, otherwise, by Corollary 3.4 at least two elements in $\mathcal{M}_A$ such that $a'_1 < a'_2$. But this contradicts the maximality of the chain whose least element was a'_2. It is clear that if $\mathcal{M}'_A$ is a minimizer then $\mathcal{M}'_A \supset \mathcal{M}_A$. Conversely if $m \in \mathcal{M}'_A \setminus \mathcal{M}_A$ then $m \not< a$, $\forall a \in \mathcal{M}_A$ contradicting the definition of $\mathcal{M}_A$.

3.0d First step in formalizing transseries. Abstract series

If $\mathcal{G}$ is a commutative group with an order relation, we call it an **abelian ordered group** if the order relation is compatible with the group operation, i.e., $a \le A$ and $b \le B \implies ab \le AB$ (e.g. $\mathbb{R}$ or $\mathbb{Z}^M$ with addition). Let $\mathcal{G}$ be an abelian ordered group and let $\mu : \mathbb{Z}^M \mapsto \mathcal{G}$ be a *decreasing group morphism*, i.e.,

(1) $\mu_{\mathbf{0}} = 1$.

(2) $\mu_{\mathbf{k}_1+\mathbf{k}_2} = \mu_{\mathbf{k}_1}\mu_{\mathbf{k}_2}$.

(3) $\mathbf{k}_1 > \mathbf{k}_2 \implies \mu_{\mathbf{k}_1} < \mu_{\mathbf{k}_2}$.

Then $\mu(\mathbb{Z})$ is the subgroup finitely generated by $\mu_{\mathbf{e}_j}; j = 1, ..., M$ where $\mathbf{e}_j$ is the unit vector in the direction j in $\mathbb{Z}^M$, and since $\mathbf{e}_j > 0$ it follows that

$$\mu_{\mathbf{e}_j} < 1,\ j = 1, ..., M$$

In view of our final goal, a simple example to keep in mind is the multiplicative group of *monomials* $\mathcal{G}_1$, generated by the functions $x^{-1/2}$, $x^{-1/3}$ and e^{-x}, for large positive x. The order relation on $\mathcal{G}_1$ is $\mu_1 < \mu_2$ if $|\mu_1(x)| < |\mu_2(x)|$ for all large x. When $M = 3$ we choose $\mu(\mathbf{k}) = x^{-k_1/2-k_2/3}e^{-k_3 x}$. After Remark 3.21 we mainly focus on the case where $\mathcal{G}$ is totally ordered.

Remark 3.5 *The relation $\mu_{\mathbf{k}_1} = \mu_{\mathbf{k}_2}$ induces an equivalence relation on $\mathbb{Z}^r$; we denote it by $\equiv$.*

For instance in $\mathcal{G}_1$, since $1/2$ and $1/3$ are not rationally independent, there exist distinct $\mathbf{k}', \mathbf{k}$ so that $\mathbf{k}' \in \mathbb{N}^3 : \mu_{\mathbf{k}'} = \mu_{\mathbf{k}}$.

Remark 3.6 *Clearly any choice of μ_i with $\mu_i < 1$ for $i = 1, ..., M$ defines an order preserving morphism via*

$$\mu(\mathbf{k}) = \prod_{i=1}^{M} \mu_i^{k_i} := \boldsymbol{\mu}^{\mathbf{k}} \tag{3.7}$$

Ordered morphisms preserve well-ordering:

Proposition 3.8 *Let $P \subset \mathbb{Z}^M$ be* well ordered *(an important example for us is $P = \mathbb{N}^M - \mathbf{k}_0$) and μ an order preserving morphism. Then $\mu(P)$ is well ordered.*

Proof. Assuming the contrary, let $J = \{\mathbf{k}_n\}_{n\in\mathbb{N}}$ be such that $\mu_J := \{\mu(\mathbf{k}_n)\}_{n\in\mathbb{N}}$ is an infinite strictly *ascending* chain in $\mu(P)$. Then the index set J is clearly infinite, and then, by Proposition 3.2 it has a strictly increasing subsequence J'. Then $\boldsymbol{\mu}_{J'}$ is a *descending* subsequence of μ_J, which is a contradiction.

Corollary 3.9 *The following sets are finite:*

1. $\{\mathbf{k}' \in \mathbb{N}^M - \mathbf{k}_0 : \mu_{\mathbf{k}'} = \mu_{\mathbf{k}}\}$ *(for fixed $\mu_{\mathbf{k}}$).*
2. $\{\mathbf{k}, \mathbf{k}' \in \mathbb{N}^M - \mathbf{k}_0 : \mathbf{k} + \mathbf{k}' = \mathbf{k}''\}$ *(for fixed $\mathbf{k}''$)*
3. $\{\mathbf{k} \in \mathbb{N}^M - \mathbf{k}_0 : |\mathbf{k}| := \sum_{i=1}^M k_i < C\}$ *(for fixed C).*

Proof. By Proposition 3.2, if the set in 1. was infinite, there would exist a strictly increasing subsequence of $\mathbf{k}'$, for which then $\{\boldsymbol{\mu}^{\mathbf{k}'}\}$ would be strictly decreasing, contradiction. The second part follows if we take $\boldsymbol{\mu}^{\mathbf{k}} = \mathbf{k}$ and the proof of the last part is similar.

Definition 3.10 *Let $\mathbf{k}_0 \in \mathbb{Z}^M$. We let $\tilde{\mathcal{A}}_{\mathbf{k}_0}(\mu_1, ..., \mu_M) = \tilde{\mathcal{A}}_{\mathbf{k}_0}$ be the space of formal series in $\mu_1, ..., \mu_M$, where $\mu_j < 1$, (i.e. the space of real or complex* functions *on $\mu(\mathbb{N}^M - \mathbf{k}_0)$ with usual addition and convolution (3.16).) and we define $\tilde{\mathcal{A}} = \cup_{\mathbf{k}_0\in\mathbb{Z}}\tilde{\mathcal{A}}_{\mathbf{k}_0}$.*

Definition 3.11 *The sum*

$$\tilde{S}_c = \sum_{\mathbf{k}\geq\mathbf{k}_0} c_{\mathbf{k}}\boldsymbol{\mu}^{\mathbf{k}}$$

is in collected *form if, by definition, $c_{\mathbf{k}} \neq 0 \implies \mathbf{k} = \max\{\mathbf{k}' : \mathbf{k}' \equiv \mathbf{k}\}$, where the maximum is with respect to the lexicographic order. (In other words the coefficients are collected and assigned to the earliest μ in its equivalence class.)*

Remark 3.12 *By Proposition 3.14 every nonzero sum can be written in collected form.*

Definition 3.13 *Remark gives natural equivalence relation on $\tilde{\mathcal{A}}$, namely $\tilde{S}_1 \equiv \tilde{S}_2$ if the collected form of $\tilde{S}_1 - \tilde{S}_2$ is the zero series.*

Proposition 3.14 *$\tilde{\mathcal{A}}$ is an algebra with respect to componentwise multiplication by scalars, componentwise addition, and the inner multiplication*

$$\tilde{S}\tilde{S}' = \sum_{\mathbf{k}\geq\mathbf{k}_0}\sum_{\mathbf{k}'\geq\mathbf{k}_0'} c_{\mathbf{k}}c_{\mathbf{k}'}\boldsymbol{\mu}^{\mathbf{k}+\mathbf{k}'} = \sum_{\mathbf{k}''\geq\mathbf{k}_0+\mathbf{k}_0'} \boldsymbol{\mu}^{\mathbf{k}''}c_{\mathbf{k}''} \tag{3.15}$$

where

$$c_{\mathbf{k}''} = \sum_{\substack{\mathbf{k}\ge\mathbf{k}_0,\mathbf{k}'\ge\mathbf{k}_0\\ \mathbf{k}+\mathbf{k}'=\mathbf{k}''}} c_{\mathbf{k}}c_{\mathbf{k}'} \tag{3.16}$$

The same is true for $\tilde{\mathcal{A}}(\mu_1,...,\mu_M)/\equiv$. $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ *is an algebra if* $\mathbf{k}_0 \ge 0$.

Proof. Straightforward.

3.0e Pointwise-discrete topology on $\tilde{\mathcal{A}}$ (the asymptotic topology)

This topology is introduced in the following way:

Definition 3.17 The sequence $\tilde{S}^{(n)}$ in $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ converges in the asymptotic topology if for any $\mathbf{k}$, $c_{\mathbf{k}}^{(n)}$ becomes constant (with respect to n) eventually. *This induces a natural topology on* $\tilde{\mathcal{A}}_{\mathbf{k}_0}/\equiv$.

This topology is metrizable. We choose the translation-invariant distance

$$d(\tilde{S}^{(1)},\tilde{S}^{(2)}) := d(\tilde{S}^{(1)} - \tilde{S}^{(2)},0)$$

where

$$d(\tilde{S},0) := d(\tilde{S}) = \sup_{\mathbf{k}\ge\mathbf{k}_0;c_{\mathbf{k}}\ne 0} e^{-|\mathbf{k}|}$$

(the condition $c_{\mathbf{k}} \ne 0$ in the definition of the distance is important for our purpose; as usual, $|\mathbf{k}| = \sum_{i=1}^{M} k_i$.)

Notes 3.18

1. *We make the convention* $\sup\emptyset = 0$.
2. *The triangle inequality, follows from the fact that if* $\tilde{S}_1,\tilde{S}_2 \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$ *then*

$$\{\mathbf{k} : c_{\mathbf{k}}^{[1]} + c_{\mathbf{k}}^{[2]} \ne 0\} \subset \{\mathbf{k} : c_{\mathbf{k}}^{[1]} \ne 0\} \cup \{\mathbf{k} : c_{\mathbf{k}}^{[2]} \ne 0\} \tag{3.19}$$

3. *With this choice of distance we clearly have*

$$d(\boldsymbol{\mu}^{\mathbf{k}_1}\boldsymbol{\mu}^{\mathbf{k}_2}) = d(\boldsymbol{\mu}^{\mathbf{k}_1+\mathbf{k}_2}) = d(\boldsymbol{\mu}^{\mathbf{k}_1})d(\boldsymbol{\mu}^{\mathbf{k}_2}) \tag{3.20}$$

4. *It is however* not *true in general that* $\boldsymbol{\mu}^{\mathbf{k}_1} < \boldsymbol{\mu}^{\mathbf{k}_2}$ *implies* $d(\boldsymbol{\mu}^{\mathbf{k}_1}) < d(\boldsymbol{\mu}^{\mathbf{k}_2})$. *For instance in the example* $\mathcal{G}_1$ *given at the beginning of §3.0d* $e^{-x}x^{n/2} < x^{-n/2}$ *for any* $n \in \mathbb{N}$ *but* $d(e^{-x}x^{n/2}) = e^{n-1}$ *and* $d(x^{-n/2}) = e^{-n}$. *The absence of some "desired" properties of* d *reflects the fact that* d *is a homeomorphism between* $(\mathcal{G},\cdot)$ *and* $(\mathbb{R}^+,\cdot)$ *and* $(\mathcal{G},\cdot,<)$ *is (usually) not archimedian. See also Proposition 3.25 below and Note 3.39.*

Remark 3.21 *It is easy to check that for any in* $\tilde{S} \in \tilde{\mathcal{A}}$ *we have*

$$\tilde{S} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\boldsymbol{\mu}^{\mathbf{k}} = \lim_{M\to\infty} \sum_{\mathbf{k}\ge\mathbf{k}_0;|\mathbf{k}|\le M} c_{\mathbf{k}}\boldsymbol{\mu}^{\mathbf{k}}$$

From this point on, we assume $\mathcal{G}$ is a **totally ordered abelian group**. Let $\tilde{S} \in \tilde{\mathcal{A}}$.

Remark 3.22 *A subgroup of $\mathcal{G}$ generated by n elements $\mu_1 < 1, ..., \mu_n < 1$ is totally ordered and well ordered, and thus can be indexed by a set of ordinals Ω, in such a way that $\omega_1 < \omega_2$ implies $\mu_{\omega_1} > \mu_{\omega_2}$. A sum*

$$\tilde{S} = \sum_{\omega \in \Omega_{\tilde{S}}} c_\omega \mu_\omega \tag{3.23}$$

where we agree to omit from $\Omega_{\tilde{S}}$ all ordinals for which $c_\omega = 0$ is called the asymptotic *form of $\tilde{S}$.*

Definition 3.24 Dominant term, magnitude. *Assume $\tilde{S} = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} \in \mathcal{A}_{\mathbf{k}_0}$ is presented in collected form. The set $\boldsymbol{\mu}^{\mathbf{k}} : c_{\mathbf{k}} \neq 0$ is then totally ordered and must have a maximal element $\boldsymbol{\mu}^{\mathbf{k}_1}$, by Proposition 3.8. of $\tilde{S}$. We say that*

- $c_{\mathbf{k}_1} \boldsymbol{\mu}^{\mathbf{k}_1} =: \operatorname{dom}(\tilde{S})$ *is the dominant term of $\tilde{S}$ and*
- $\boldsymbol{\mu}^{\mathbf{k}_1} =: \operatorname{mag}(\tilde{S})$ *is the* (dominant) magnitude *of $\tilde{S}$*

(equivalently, $\operatorname{mag}(\tilde{S}) = \mu_{\min \Omega_{\tilde{S}}}$). We allow for $\operatorname{mag}(\tilde{S})$ to be zero, **iff** $\tilde{S} = 0$.

Proposition 3.25 *(i) We have $\tilde{S}^{[n]} \to 0 \Rightarrow \operatorname{mag} \tilde{S}^{[n]} \to 0$ and thus the magnitude is a continuous. Also, if $|\mathbf{k}_2| \geq 0$, then*

$$d(\boldsymbol{\mu}^{\mathbf{k}_1} \boldsymbol{\mu}^{\mathbf{k}_2}) = d(\boldsymbol{\mu}^{\mathbf{k}_1 + \mathbf{k}_2}) \leq d(\boldsymbol{\mu}^{\mathbf{k}_1}) d(\boldsymbol{\mu}^{\mathbf{k}_2}) \tag{3.26}$$

(The condition $|\mathbf{k}_2| \geq 0$ is needed, by Note 3.18.)

(ii) Addition is continuous in the asymptotic topology. Multiplication is continuous from $\tilde{\mathcal{A}}_{\mathbf{k}_0} \times \tilde{\mathcal{A}}_{\mathbf{k}_1} \to \tilde{\mathcal{A}}_{\mathbf{k}_0 + \mathbf{k}_1}$. A Cauchy sequence in $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ is clearly convergent, and in this sense $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ is a complete topological space.

Note that multiplication by scalars is **not continuous**.

Proof. (i) follows immediately from the definition of the topology and of $\operatorname{mag}(\cdot)$. Continuity of addition follows from the triangle inequality, cf. (3.19). For the product note that $\tilde{S}^{[m]} \to 0$ implies $d(\tilde{S}^{[m]}) \leq \epsilon < 1$ for $m > m_\epsilon$ which entails that $\epsilon \geq \sup\{d(\boldsymbol{\mu}^{\mathbf{k}}_{[m]}) : c^{[m]}_{\mathbf{k}} \neq 0$ for some $\mathbf{k}, m > m_\epsilon\}$, and continuity follows from (i) (3.26).

The following property is an immediate consequence of Corollary 3.13:

Remark 3.27 *For any nonzero $\tilde{S}$ we can write*

Remark 3.28 *If $\operatorname{mag}(S_1) > \operatorname{mag}(S_2)$ then $\operatorname{mag}(S_1 + S_2) = \operatorname{mag}(S_1)$ and if $S = \sum_j = 1^\infty S^[j]$ converges then $\operatorname{mag}(S) \leq \operatorname{mag} S^[j]$ for some j.*

$$\tilde{S} = c_{\mathbf{k}_1} \operatorname{mag}(\tilde{S})(1 + \sum c'_{\mathbf{k}'} \boldsymbol{\mu}^{\mathbf{k}'}) = \operatorname{dom}(\tilde{S})(1 + \tilde{S}_1)$$

Product form.

Proposition 3.29 *Any $\tilde S \in \tilde{\mathcal{A}}$ can be written in the form*

$$c\ \mathrm{mag}(\tilde S)\left(1+\sum_{\mathbf{k}>0} c_{\mathbf{k}}\boldsymbol{\mu}^{\mathbf{k}}\right) \tag{3.30}$$

i.e.,

$$c\ \mathrm{mag}(\tilde S)(1+\tilde S_1) \tag{3.31}$$

where $\tilde S_1 \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$ for some $\mathbf{k}_0 > 0$ (cf. also Remark 3.27) and $\mathrm{mag}(\tilde S_1) < 1$.

Proof. We have, by Remark 3.27,

$$\tilde S = c\ \mathrm{mag}(\tilde S)\left(1+\sum_{\mathbf{k}\ge\mathbf{k}_0} c'_{\mathbf{k}}\mu_1^{k_1}\cdots\mu_M^{k_M}\ \mathrm{mag}(\tilde S)^{-1}\right) \tag{3.32}$$

where all the elements in the last sum are less than one.

Let A be the set of multi-indices in the sum in (3.32) for which *some* $k_i < 0$ and let A' be its minimizer in the sense of Corollary 3.4, a finite set. We now consider the extended set of generators

$$\{\overline{\mu}_i : i \le M'\} := \{\nu\, \mathrm{mag}(\tilde S)^{-1} : \nu = \mu_i \text{ with } i \le M \text{ or } \nu = \boldsymbol{\mu}^{\mathbf{k}} \text{ with } \mathbf{k}\in A'\}$$

We clearly have $\overline{\mu}_i < 1$. By the definition of A', for any term in the sum in (3.32) either $\mathbf{k} > 0$ or else $\mathbf{k} = \mathbf{k}'+\mathbf{k}''$ with $\mathbf{k}' \in A'$ and $\mathbf{k}'' \ge 0$. In both cases we have $c_{\mathbf{k}}\boldsymbol{\mu}^{\mathbf{k}} = c_{\mathbf{k}'}\overline{\mu}_{\mathbf{k}'}$ with $\mathbf{k}' \in \mathbb{Z}^{M'}$ and $\mathbf{k}' > 0$. Thus $\tilde S$ can be rewritten in the form

$$c\ \mathrm{mag}(\tilde S)\left(1+\sum_{\mathbf{k}>0} c_{\mathbf{k}}\boldsymbol{\mu}^{\mathbf{k}}\right)\quad (\mathbf{k}\in\mathbb{N}^{M'})$$

where the assumptions of the Proposition are satisfied.

Remark 3.33 *The set $\{\,\mathrm{mag}(\tilde S) : \tilde S \in \tilde{\mathcal{A}}_{\mathbf{k}_0}\}$ is bounded (above) by a magnitude $\boldsymbol{\mu}_0 = \mu(\tilde{\mathcal{A}}_{\mathbf{k}_0}) \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$.*

Proof. This follows from Proposition 3.8.

The proposition below discusses the closure of $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ under restricted *infinite sums.*

Proposition 3.34 *Let $\mathbf{j}_0, \mathbf{k}_0, \mathbf{l}_0 \in \mathbb{Z}^M$ with $\mathbf{k}_0 + \mathbf{l}_0 = \mathbf{j}_0$ and consider the sequence in $\tilde{\mathcal{A}}_{\mathbf{k}_0}$*

$$\tilde S^{(\mathbf{m})} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}^{(\mathbf{m})}\boldsymbol{\mu}^{\mathbf{k}}$$

and a fixed $T \in \tilde{\mathcal{A}}_{\mathbf{l}_0}$,

$$T = \sum_{\mathbf{k} \geq \mathbf{l}_0} c'_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}}$$

Then **trans-composition**

$$T(\tilde{S}) := \sum_{\mathbf{k} \geq \mathbf{l}_0} c'_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} \cdot \tilde{S}^{(\mathbf{k})}$$

obtained by replacing each $\boldsymbol{\mu}^{\mathbf{k}}$ *in* T *by the product* $\boldsymbol{\mu}^{\mathbf{k}} \cdot \tilde{S}^{(\mathbf{k})}$ *is well defined in* $\tilde{\mathcal{A}}_{\mathbf{j}_0}$ *as the limit of* truncates

$$T^{[\mathbf{k}']}(\tilde{S}) = \sum_{\mathbf{l}_0 \leq \mathbf{k}; |\mathbf{k}| \leq M} c'_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} \cdot \tilde{S}^{(\mathbf{k})} \tag{3.35}$$

Proof. The proof now follows from Note 3.18 by checking that

$$d(\boldsymbol{\mu}^{\mathbf{k}} \tilde{S}^{(\mathbf{k})}) \leq d(\boldsymbol{\mu}^{\mathbf{k}+\mathbf{k}_0}) = d(\boldsymbol{\mu}^{\mathbf{k}}) d(\boldsymbol{\mu}^{\mathbf{k}_0})$$

Note 3.36 (Restricted composition) *Let* $\tilde{S}_1, ..., \tilde{S}_M \in \mathcal{A}$ *and assume*

$$\operatorname{mag}(\tilde{S}_i) = \nu_i < 1$$

Let $\tilde{S}_i = c_i \nu_i (1 + \tilde{T}_i)$ *where, using Proposition 3.29 we can assume that*

$$\tilde{T}_i = \sum_{\mathbf{k} > 0} c_{\mathbf{k};i} \boldsymbol{\mu}^{\mathbf{k}} \tag{3.37}$$

We enlarge $\mathcal{A}_{\mathbf{k}_0}$ *so that* $\nu_1, ... \nu_M \in \mathcal{A}_{\mathbf{k}_0}$. *If*

$$\tilde{S} = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} \tag{3.38}$$

then we define

$$\tilde{S} \circ \tilde{\mathbf{S}} = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} \boldsymbol{\nu}^{\mathbf{k}} \prod_{i=1}^{M} (1 + \tilde{T}_i)^{k_i}$$

Note 3.39 *The condition that* $\operatorname{mag}(\tilde{S})^{(m)}$ *decreases strictly in* m *does* **not** *suffice for* $\sum_{m \geq 0} c_m \tilde{S}^{(m)}$ *to be well defined. Indeed, the terms* $x^{-m} + e^{-x} =: \tilde{S}^{(m)}$ *have strictly decreasing magnitudes and yet* $d(\tilde{S}^{(m)}) = e^{-1} \not\to 0$ *so* $\sum_m \tilde{S}^{(m)}$ *does not converge; moreover, the formal expression* $\sum_{m \geq 0} (x^{-m} + e^{-x})$ *seems meaningless anyway.*

3.0f Contractive operators on $\tilde{\mathcal{A}}_{\mathbf{k}_0}(\boldsymbol{\mu}_1, ..., \boldsymbol{\mu}_M)$

Definition 3.40 *Let J be a linear operator from $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ or from one of its subspaces, to $\tilde{\mathcal{A}}_{\mathbf{k}_0}$,*

$$J\tilde{S} = J \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} \boldsymbol{\mu}^{\mathbf{k}} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} J \boldsymbol{\mu}^{\mathbf{k}} \tag{3.41}$$

Then J is called asymptotically contractive on $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ if for some $c < 1$ we have

$$d(J\boldsymbol{\mu}^{\mathbf{j}}) \le c\, d(\boldsymbol{\mu}^{j}) \tag{3.42}$$

Equivalently,

$$J\boldsymbol{\mu}^{\mathbf{j}} = \sum_{\mathbf{p}>0} c_{\mathbf{j};\mathbf{p}} \boldsymbol{\mu}^{\mathbf{j}+\mathbf{p}}$$

We note that by (3.42) and Proposition 3.34, J is well defined.

Definition 3.43 *The linear or nonlinear operator J is (asymptotically) contractive in the set $A \subset \mathcal{A}_{\mathbf{k}_0}$ if $J : A \mapsto A$ and the following condition holds. There is a $c < 1$ such that for any f_1 and f_2 in A*

$$d(J(f_1) - J(f_2)) \le c\ d(f_1 - f_2) \tag{3.44}$$

Remark 3.45 *The sum of asymptotically contractive operators is contractive; the composition of contractive operators, whenever defined, is contractive.*

Proposition 3.46 *(i) If J is linear and contractive on $\tilde{\mathcal{A}}_{\mathbf{k}_0}$ then for any $\tilde{S}_0 \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$ the fixed point equation $\tilde{S} = J\tilde{S} + \tilde{S}_0$ has a unique solution $\tilde{S} \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$.*

(ii) In general, if $A \subset \mathcal{A}_{\mathbf{k}_0}$ is closed and $J : A \mapsto A$ is a (linear or nonlinear) contractive operator on A, then $\tilde{S} = J(\tilde{S})$ has a unique solution is A.

Proof. For existence, we note the convergence of the recurrence $f_{n+1} = J(f_n)$:

$$d(f_{n+1} - f_n) = d(J(f_n) - J(f_{n-1})) \le c\, d(f_n - f_{n-1}) \le const\, c^n$$

For uniqueness, if $\tilde{S}_1$ and $\tilde{S}_2$ are solutions we have $d(J(\tilde{S}_1 - \tilde{S}_2)) = d(\tilde{S}_1 - \tilde{S}_2)$ which implies $\tilde{S}_1 = \tilde{S}_2$.

Corollary 3.47 *Let $\tilde{S} \in \mathcal{A}_0$ be arbitrary and $\tilde{S}_n = \sum_{\mathbf{k}>0} c_{\mathbf{k};n} \boldsymbol{\mu}^{\mathbf{k}} \in \mathcal{A}_0$ for $n \in \mathbb{N}$. Then the operator defined by*

$$J(y) = \tilde{S} + \sum_{n\ge2} \tilde{S}_{n-2} y^n$$

is contractive in the set $\{y : \mathrm{mag}(y) < 1\}$.

Proof. We have

$$J(y+\delta) - J(y) = \delta \sum_{n\geq 2} \tilde{S}_{n-2} \left(\sum_{j=1}^{n-1} y^j \delta^{n-j} \right)$$

3.0g The field of finitely generated formal series

Let $\mathcal{G}$ be a totally ordered abelian group. We now define the algebra:

$$\tilde{\tilde{S}} = \bigcup_{\substack{\mathbf{k}_0 \in \mathbb{Z}^M \\ \mu_1<1,\ldots,\mu_M<1 \\ M\in\mathbb{N}}} \tilde{\mathcal{A}}_{\mathbf{k}_0}(\mu_1, ..., \mu_M) \tag{3.48}$$

modulo the obvious inclusions, and with the induced topology (convergence in $\tilde{\tilde{S}}$ means convergence in one of the $\tilde{\mathcal{A}}_{\mathbf{k}_0}(\mu_1, ..., \mu_M)$).

Proposition 3.49 *$\tilde{\tilde{S}}$ is a field.*

Proof. The only property that needs verification is the existence of a reciprocal for any nonzero $\tilde{S}$. Using Proposition 3.29 we only need to consider the case when

$$\tilde{S} = 1 + \sum_{\mathbf{k}>0} c_{\mathbf{k}} \mu^{\mathbf{k}}$$

Since multiplication by t is manifestly contractive (see § 3.0f), $\tilde{S}^{-1}$ is the solution (unique by Proposition 3.46) of

$$\tilde{S}^{-1} = 1 - t\tilde{S}^{-1}$$

Closure under infinite sums.

Corollary 3.50 *(i) Let $\mathbf{k}_0 > 0$ and $\tilde{S} \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$ and $\{c_n\}_{n\in\mathbb{N}} \in \mathbb{C}$ be any sequence. Then*

$$\sum_{n=0}^{\infty} c_n \tilde{S}^n \in \tilde{\mathcal{A}}_{\mathbf{k}_0}$$

(ii) More generally, if $\tilde{S}_{01}, ..., \tilde{S}_{0M}$ are of the form $\tilde{S}_0$ and $\{c_{\mathbf{k}}\}_{\mathbf{k}\in\mathbb{N}^M}$ is a multi-sequence of constants, then $\sum_{\mathbf{k}\geq 0} c_{\mathbf{k}} \tilde{S}_0^{\mathbf{k}} = \sum_{\mathbf{k}\geq 0} c_{\mathbf{k}} \tilde{S}_{01}^{k_1} \cdots \tilde{S}_{0M}^{k_M}$ is well defined.

Proof. See Note 3.36.

Formal series with real coefficients. Order relation. Let $\tilde{\tilde{S}}_{\mathbb{R}}$ the subfield consisting in $\tilde{S} \in \tilde{\tilde{S}}_{\mathbb{R}}$ which have real coefficients. We say that $\tilde{\tilde{S}}_{\mathbb{R}} \ni \tilde{S} > 0$ if $\mathrm{dom}(\tilde{S})/\operatorname{mag}(\tilde{S}) > 0$. Then every nonzero $\tilde{S} \in \tilde{\tilde{S}}_{\mathbb{R}}$ is either positive or else $-\tilde{S}$ is positive. This induces a *total* order relation on $\tilde{\tilde{S}}_{\mathbb{R}}$, by writing $\tilde{S}_1 > \tilde{S}_2$ if $\tilde{S}_1 - \tilde{S}_2 > 0$.

3.1 Transseries

In constructing a space of transseries , one aims at constructing a differential field containing x^{-1}, which is closed under a "all" important operations. A smaller closure usually imparts better properties of composing elements.

The construction presented differs in many technical respects from the one of Écalle, and the transseries space is smaller than his.

Still some of the construction steps and the structure of the final object are similar enough to Écalle's, to justify using the terminology and same notations.

3.1a Notations

- $\sqcup$ —small transmonomial.
- $\sqcap$ —large transmonomial.
- $\square$ —any transmonomial, large or small.
- $\sqcup\!\sqcup$ —small transseries .
- $\sqcap\!\sqcap$ —large transseries .
- $\square\!\square$ —any transseries , small or large.

3.2 Inductive construction of logarithm-free transseries

3.2a Level 0: power series

Let $x \in \mathbb{R}^+$ be a large variable and let $\mathcal{G}$ be the multiplicative group $(x^\sigma, \cdot, \ll), \sigma \in \mathbb{R}$, with the order relation $x^{\sigma_1} \ll x^{\sigma_2}$ if $x^{\sigma_1} = o(x^{\sigma_2})$ as $x \to \infty$, i.e., if $\sigma_1 < \sigma_2$.

It is easy to see that $\mathcal{G}$ is **totally ordered** with respect to $\ll$.

The space of level zero log-free transseries is by definition $\tilde{\mathcal{T}}^{[0]} = \tilde{\tilde{S}}\,(\mathcal{G})$. By Proposition 3.49, $\tilde{\mathcal{T}}^{[0]}$ is a field.

If $\tilde{T} \in \tilde{\mathcal{T}}^{[0]}$, then $\tilde{T} = \square$ iff $\tilde{T} = x^\sigma$ for some $\sigma \neq 0$, $\tilde{T} = \sqcap$ if $\sigma > 0$ and $\tilde{T} = \sqcup$ if $\sigma < 0$.

The general element of $\tilde{\mathcal{T}}^{[0]}$ is a level zero transseries , $\square\!\square^{[0]}$ or $\square\!\square$ in short. We have

$$\square\!\square = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} \sqcup^{\mathbf{k}} \tag{3.51}$$

There are two order relations: $<$ and $\ll$ on $\tilde{T} \in \tilde{\mathcal{T}}^{[0]}$.

- We have $\square\!\square_1 \ll \square\!\square_2$ iff $\mathrm{mag}(\square\!\square_1) \ll \mathrm{mag}(\square\!\square_2)$ (note it is only the magnitude and not constant in front of it, positive or negative, that matters);
- $\square\!\square > 0$ if $\square\!\square \neq 0$ and the real number $\mathrm{dom}(\square\!\square)/\,\mathrm{mag}(\square\!\square)$ is positive.

Definition 3.52 A transseries is **small**, i.e. $\boxbar = \sqcup\!\sqcup$ iff in (3.51) we have $c_{\mathbf{k}} = 0$ whenever $\sqcup^{\mathbf{k}} \not\ll 1$. Correspondingly, transseries is **large**, i.e. $\boxbar = \sqcap\!\sqcap$ iff in (3.51) we have $c_{\mathbf{k}} = 0$ whenever $\sqcup^{\mathbf{k}} \not\gg 1$. We note that $\boxbar = \sqcup\!\sqcup$ iff $\operatorname{mag}(\boxbar) \ll 1$ (there is an asymmetry: the condition $\operatorname{mag}(\boxbar) \gg 1$ does *not* imply $\boxbar = \sqcap\!\sqcap$, since it does not prevent the presence of small terms in $\boxbar$). *Any transseries can then be written uniquely as*

$$\boxbar = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} \sqcup^{\mathbf{k}} = \sum_{\mathbf{k}\ge\mathbf{k}_0;\ \sqcup^{\mathbf{k}}>1} c_{\mathbf{k}} \sqcup^{\mathbf{k}} + const + \sum_{\mathbf{k}\ge\mathbf{k}_0;\ \sqcup^{\mathbf{k}}<1} c_{\mathbf{k}} \sqcup^{\mathbf{k}}$$
$$= \sqcap\!\sqcap + const + \sqcup\!\sqcup := L(\boxbar) + C(\boxbar) + s(\boxbar) \quad (3.53)$$

3.2b Level 1: Exponential power series

The set $\mathcal{G}^{[1]}$ of transmonomials of exponentiality one consists by definition in the formal expressions

$$\square^{[1]} = \square^{[0]} \exp(\sqcap\!\sqcap^{[0]}), \quad \text{where } \square^{[0]} \text{ and } \sqcap\!\sqcap^{[0]} \in \tilde{\mathcal{T}}^{[0]}$$

where we allow for $\sqcap\!\sqcap^{[0]} = 0$ and set $\exp(0) = 1$.

Multiplication is *defined* by

$$\square_1^{[0]} \exp(\sqcap\!\sqcap_1^{[0]}) \square_2^{[0]} \exp(\sqcap\!\sqcap_2^{[0]}) = (\square_1^{[0]} \square_2^{[0]}) \exp(\sqcap\!\sqcap_1^{[0]} + \sqcap\!\sqcap_2^{[0]})$$

and we see that $(\mathcal{G}^{[1]}, \cdot)$ is a commutative group, .

The order relations are introduced in the following way. First, we say that

$$\square^{[1]} = \square^{[0]} \exp(\sqcap\!\sqcap^{[0]}) \gg 1 \text{ means}$$
$$(\sqcap\!\sqcap^{[0]} > 0) \text{ or } (\sqcap\!\sqcap^{[0]} = 0 \text{ and } \square^{[0]} \gg 1) \quad (3.54)$$

In particular, by construction, if $\sqcap\!\sqcap^{[0]} > 0$ then $\exp(\sqcap\!\sqcap^{[0]})$ exceeds any monomial of level zero $\square^{[0]}$.

It is easy to see that

$$\square_1^{[1]} \gg 1 \ \text{ and } \ \square_2^{[1]} \ \text{ imply } \ \square_1^{[1]} \square_2^{[1]} \gg 1 \quad (3.55)$$

We can thus define $\gg$ in $\mathcal{G}^{[1]}$ by

$$\square_1^{[1]} \gg \square_2^{[1]} \text{ if } \square_1^{[1]} / \square_2^{[1]} \gg 1 \quad (3.56)$$

It can be easily checked that $(\mathcal{G}^{[1]}, \cdot, \ll)$ is a totally ordered abelian group. The abelian ordered group of zero level monomials, $(\mathcal{G}^{[0]}, \cdot, \gg)$, is naturally identified with the set of transmonomials for which $\sqcap\!\sqcap^{[0]} = 0$.

Definition 3.57 *The space $\tilde{\mathcal{T}}^{[1]}$ of level one transseries is by definition $\tilde{\tilde{\tilde{S}}}$ $(\mathcal{G}^{[1]})$.*

By Proposition 3.49, $\tilde{\mathcal{T}}^{[1]}$ is a field. By construction, the space $\tilde{\mathcal{T}}^{[0]}$ is embedded in $\tilde{\mathcal{T}}^{[1]}$. Formula (3.51) is the general expression of a level one transseries, where now $\sqcup$ is a transmonomial of level one.

The second order relation, $>$, is defined by

$$\square\!\square > 0 \text{ means } \operatorname{dom}(\square\!\square)/\operatorname{mag}(\square\!\square) > 0 \tag{3.58}$$

3.2c Induction step: level n transseries

Assuming the transseries of level $\leq n-1$ are constructed, transseries of level n together with the order relation, are constructed exactly as in § 3.2b , replacing [0] by $[n-1]$ and [1] by $[n]$. The group $\mathcal{G}^{[1]}$ of transmonomials of order at most n consists in expressions of the form

$$\square^{[n]} = x^{\sigma} \exp(\sqcap\!\sqcap^{[n-1]}) \tag{3.59}$$

where $\sqcap\!\sqcap^{[n-1]}$ is either zero or a large transseries of level $n-1$ with the multiplication:

$$x^{\sigma_1} \exp(\sqcap\!\sqcap_1^{[n-1]}) x^{\sigma_2} \exp(\sqcap\!\sqcap_2^{[n-1]}) = x^{\sigma_1+\sigma_2} \exp(\sqcap\!\sqcap_1^{[n-1]} + \sqcap\!\sqcap_2^{[n-1]}) \tag{3.60}$$

The order relation is given by

$$x^{\sigma_1} \exp(\sqcap\!\sqcap_1^{[n-1]}) \gg x^{\sigma_2} \exp(\sqcap\!\sqcap_2^{[n-1]}) \iff \tag{3.61}$$

$$\left(\sqcap\!\sqcap_1^{[n-1]} > \sqcap\!\sqcap_2^{[n-1]}\right) \text{ or } \left(\sqcap\!\sqcap_1^{[n-1]} = \sqcap\!\sqcap_2^{[n-1]} \text{ and } \sigma_1 > \sigma_2\right) \tag{3.62}$$

$$\square\!\square^{[n]} = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} (\sqcup^{[n]})^{\mathbf{k}} \tag{3.63}$$

As in § 3.2b , $\tilde{\mathcal{T}}^{[n-1]}$ is naturally embedded in $\tilde{\mathcal{T}}^{[n]}$.

3.2d General log-free transseries , $\tilde{\tilde{\tilde{\mathcal{T}}}}$

This is the space of arbitrary level transseries , the inductive limit of the finite level spaces of transseries :

$$\tilde{\tilde{\tilde{\mathcal{T}}}} = \bigcup_{n=0}^{\infty} \tilde{\mathcal{T}}^{[n]}$$

Clearly $\tilde{\tilde{\tilde{\mathcal{T}}}}$ is a field. The order relation is the one inherited from $\tilde{\mathcal{T}}^{[n]}$. The topology is also that of an inductive limit, namely a sequence converges iff it converges in $\tilde{\mathcal{T}}^{[n]}$ for *some* n.

3.2e Further properties of transseries

Definition. The level $l(\square\!\square)$ of $\square\!\square$ is n if $\square\!\square \in \tilde{\mathcal{T}}^{[n]}$ and $\square\!\square \notin \tilde{\mathcal{T}}^{[n-1]}$.

Proposition 3.64 *If $n = l(\sqcap\!\sqcap_1) > l(\sqcap\!\sqcap_2)$ then $\sqcap\!\sqcap_1 \gg \sqcap\!\sqcap_2$.*

Proof. We may clearly take $n \ge 1$. Since (by definition) $\sqcap\!\sqcap \gg 1$ we must have, in particular, $\mathrm{dom}(\sqcap\!\sqcap) = cx^{\sigma}\exp(\sqcap\!\sqcap^{-})$ with $\sqcap\!\sqcap^{-} \ge 0$. By induction, and the assumption $l(\sqcap\!\sqcap_1) = n$ we must have $\sqcap\!\sqcap_1^{-} > 0$ and $l(\sqcap\!\sqcap_1^{-}) = n-1$. The proposition follows since, by again by the induction step, $\sqcap\!\sqcap_1^{-} \gg \sqcap\!\sqcap_2^{-}$.

Corollary 3.65 *If $\square$ is of level no less than 1, then either $\square$ is large, and then $\square \gg x^{\alpha}$, $\forall \alpha \in \mathbb{R}$ or else $\square$ is small, and then $\square \ll x^{-\alpha}$, $\forall \alpha \in \mathbb{R}$.*

Remark 3.66 *We can define* generating monomials *of $0 \ne \square\!\square \in \tilde{\mathcal{T}}^{[n]}$ a minimal subgroup $\mathcal{G} = \mathcal{G}(\square\!\square)$ of $\mathcal{G}^{[n]}$ with the following properties:*

- $\square\!\square \in \overset{\approx}{\mathcal{S}}(\mathcal{G})$;
- $x_1^{\sigma}\exp(\sqcap\!\sqcap_1) \in \mathcal{G}$ *implies* $x_1^{\sigma} \in \mathcal{G}$ *and, if* $\sqcap\!\sqcap_1 \ne 0$, *then* $\mathcal{G} \supset \mathcal{G}(\sqcap\!\sqcap_1)$.

By induction we see that $\mathcal{G}(\square\!\square)$ is finitely generated for any $\square\!\square \in \mathcal{T}^{[n]}$.

3.2f Closure of $\overset{\approx}{\mathcal{T}}$ under restricted composition and differentiation

Composition was discussed in general in Note 3.36. In the context of transseries, we take into account the fact that everything is ultimately generated from one variable, x; this leads to a refinement of composition, see §3.2h .

Proposition 3.67 *$\overset{\approx}{\mathcal{T}}$ and $\mathcal{T}^{[n]}$; $n \in \mathbb{N}$ are differential fields.*

Proof. Differentiation $\mathcal{D} = \frac{\mathrm{d}}{\mathrm{d}x}$ is introduced inductively on $\overset{\approx}{\mathcal{T}}$, as term by term differentiation, in the following way. Differentiation in $\tilde{\mathcal{T}}^{[0]}$ is defined as:

$$\mathcal{D}\square\!\square = \sum_{\mathbf{k} \ge \mathbf{k}_0} c_{\mathbf{k}} \mathcal{D} \sqcup^{\mathbf{k}} \tag{3.68}$$

where, as mentioned in §3.2a we have $\sqcup = x^{-\sigma}$ for some $\sigma \in \mathbb{R}^+$ and, in a natural way, we set $\mathcal{D}x^{-\sigma} = -\sigma x^{-\sigma-1}$. This makes $\mathcal{D}\square\!\square \in \tilde{\mathcal{T}}^{[0]}$, and the generating transmonomials of $\mathcal{D}\square\!\square$ are those of $\square\!\square$ together with x^{-1}.

We assume by induction that differentiation $\mathcal{D} : \tilde{\mathcal{T}}^{[n-1]} \mapsto \tilde{\mathcal{T}}^{[n-1]}$ has been defined for all transseries of level at most $n-1$. (In particular, $\mathcal{D}\square\!\square$ is finitely generated.) We define

$$\mathcal{D}\left(\square^{[n]}\right) = \mathcal{D}\left(x^{\sigma}\exp(\sqcap\!\sqcap^{[n-1]})\right) = \sigma x^{\sigma-1}\exp(\sqcap\!\sqcap^{[n-1]}) \\ + x^{\sigma}\mathcal{D}\sqcap\!\sqcap^{[n-1]}\exp(\sqcap\!\sqcap^{[n-1]}) \quad (3.69)$$

A level n transseries is

$$\square\!\square = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\sqcup^{\mathbf{k}} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\prod_{j=1}^{M}\sqcup_j^{k_j} \quad (3.70)$$

and we write in a natural way

$$\mathcal{D}\square\!\square^{[n]} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}\sum_{m=1}^{M} k_m\sqcup_m^{k_m-1}\mathcal{D}\sqcup_m\prod_{m\ne j=1}^{M}\sqcup_j^{k_j} \\ = \sum_{m=1}^{M}\sqcup_m^{-1}\sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}k_m\sqcup_m^{k_m-1}\mathcal{D}\sqcup_m\prod_{m\ne j=1}^{M}\sqcup_j^{k_j} \quad (3.71)$$

and the result follows from the induction hypothesis, since

$$\sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}k_m\sqcup^{\mathbf{k}}\frac{\mathcal{D}\sqcup_m}{\sqcup_m} = \frac{\mathcal{D}\sqcup_m}{\sqcup_m}\sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}k_m\sqcup^{\mathbf{k}} \in \tilde{\mathcal{T}}^{[n]} \quad (3.72)$$

Corollary 3.73 *If $\mathcal{G}_{\square\!\square}$ is the group (finitely) generated by all generators in any of the levels of $\square\!\square$, then $\mathcal{D}\square\!\square$ is generated by the transmonomials of $\mathcal{G}_{\square\!\square}$ together possibly with x^{-1}. If $\square\!\square \ne \text{Const.}$ then $l(\square\!\square) = l(\square\!\square')$.*

Proof. Immediate induction; cf. also the beginning of the proof of Proposition 3.67.

The properties of differentiation are the usual ones:

Proposition 3.74 *$\mathcal{D}(fg) = g\mathcal{D}f + f\mathcal{D}g$, $\mathcal{D}const = 0$ and $\mathcal{D}(f\circ g) = \mathcal{D}g(\mathcal{D}f\circ g)$ (for composition, see § 3.2h).*

Proof. The proof is straightforward induction.

In the space of transseries, differentiation which is henceforth denoted by $''\prime''$, is also compatible with the order relation (a property which is of course not true in general function spaces).

Proposition 3.75 *For any $\sqcap\!\sqcap_i, i = 1, 2$ and $\sqcup\!\sqcup_i, i = 1, 2$ we have*

1. $\sqcap\!\sqcap_1 \gg \sqcap\!\sqcap_2 \Leftrightarrow \sqcap\!\sqcap_1' \gg \sqcap\!\sqcap_2'$
2. $\sqcap\!\sqcap > 0 \Rightarrow \sqcap\!\sqcap' > 0$
3. $\sqcup\!\sqcup_1 \gg \sqcup\!\sqcup_2 \Leftrightarrow \sqcup\!\sqcup_1' \gg \sqcup\!\sqcup_2'$
4. $\sqcap\!\sqcap_1' \gg \sqcup\!\sqcup_1'$

Proof. We first prove by induction 1. and 2. The other properties are shown very similarly. They hold for level zero transseries . Assume they are true for transseries of level $\le n-1$; we first show that for $\sqcap^{[n]} = x^\sigma e^{\sqcap\!\!\sqcap}$, a large transmonomial of level n, we have $(\sqcap^{[n]})' \gg 1$ and $(\sqcap^{[n]})' > 0$. Indeed this follows immediately from the induction hypothesis and Remark 3.28 since $l(\sqcap\!\!\sqcap) < n$ and

$$(\sqcap^{[n]})' = \sqcap^{[n]}(\sqcap\!\!\sqcap' + \sigma x^{-1})$$

Positivity follows immediately from the fact that $\sqcap\!\!\sqcap' > 0$ and $\sqcap\!\!\sqcap \gg 1 \gg x^{-1}$. Now if $\sqcap_1^{[n]} \gg \sqcap_2^{[n]}$ then $\sqcap^{[n]} := \sqcap_1^{[n]}/\sqcap_2^{[n]} \gg 1$ and, by differentiation, it follows from the previous conclusion that

$$(\sqcap_1^{[n]})' > \frac{\sqcap_1^{[n]}}{\sqcap_2^{[n]}}(\sqcap_1^{[n]})' \gg (\sqcap_1^{[n]})'$$

and thus, again by Remark 3.28 we have both $(\sqcap\!\!\sqcap_1^{[n]})' > 0$ and

$$\sqcap\!\!\sqcap_1^{[n]} \gg \sqcap\!\!\sqcap_2^{[n]} \Rightarrow (\sqcap\!\!\sqcap_1^{[n]})' \gg (\sqcap\!\!\sqcap_2^{[n]})'$$

Corollary 3.76 *We have* $\mathcal{D}\Box\!\!\Box = 0 \iff \Box\!\!\Box = \mathrm{Const}.$

Proof. We have to show that if $\Box\!\!\Box = \sqcap\!\!\sqcap + \sqcup\!\!\sqcup \ne 0$ then $\Box\!\!\Box' \ne 0$. If $\sqcap\!\!\sqcap \ne 0$ then (for instance) $\sqcap\!\!\sqcap + \sqcup\!\!\sqcup \gg x^{-1} = \sqcup\!\!\sqcup$ and then $\sqcap\!\!\sqcap' + \sqcup\!\!\sqcup' \gg x^{-2} \ne 0$. If instead $\sqcap\!\!\sqcap = 0$ then $(1/\Box\!\!\Box) = \sqcap\!\!\sqcap_1 + \sqcup\!\!\sqcup_1 + c$ and we see that $(\sqcap\!\!\sqcap_1 + \sqcup\!\!\sqcup_1)' = 0$ which, by the above, implies $\sqcap\!\!\sqcap_1 = 0$ which gives $1/\sqcup\!\!\sqcup = \sqcup\!\!\sqcup_1$, a contradiction.

Proposition 3.77 *Assume* $\Box\!\!\Box = \sqcap\!\!\sqcap$ *or* $\Box\!\!\Box = \sqcup\!\!\sqcup$*. Then:*
(i) If $l(\mathrm{mag}(\Box\!\!\Box)) \ge 1$ *then* $l(\mathrm{mag}(\Box\!\!\Box^{-1}\Box\!\!\Box')) < l(\mathrm{mag}(\Box\!\!\Box))$.
(ii) $\mathrm{dom}(\Box\!\!\Box') = \mathrm{dom}(\Box\!\!\Box)'(1 + \sqcup\!\!\sqcup)$.

Proof. Straightforward induction.

3.2g Transseries with complex coefficients

Complex transseries $\mathcal{T}_{\mathbb{C}}$ are constructed in a similar way as real transseries, replacing everywhere $\sqcap\!\!\sqcap_1 > \sqcap\!\!\sqcap_2$ by $\Re\sqcap\!\!\sqcap_1 > \Re\sqcap\!\!\sqcap_2$. Thus there is only one order relation in $\mathcal{T}_{\mathbb{C}}$, $\gg$. Difficulties arise when exponentiating transseries whose dominant term is imaginary. Operations with complex transseries are then limited. We will only use complex transseries in contexts that will prevent these difficulties.

3.2h Restricted composition

The right composition $\square\square_1 \circ \square\square_2$ is defined on $\tilde{\tilde{\tilde{\mathcal{T}}}}$, if $\mathrm{mag}(\square\square_2) \gg 1$ and $\mathrm{dom}(\square\square_2) > 0$. The definition is inductive.

We first define the power and the exponential of a transseries . Assume powers and exponentials have been defined for all transseries of level $\le n-1$. Let $\square\square = c\,\mathrm{mag}(\square\square)(1+\sqcup\sqcup) \in \tilde{\mathcal{T}}^{[n]}$ be any transseries such that $c>0$, cf. Proposition 3.29. By the definition of $\mathrm{mag}(\cdot)$ and (3.63), $\mathrm{mag}(\square\square)$ is a transmonomial, $\mathrm{mag}(\square\square) = \sqcap^{[n-1]}\exp(\sqcap\sqcap^{[n-1]})$. We let

$$\begin{aligned}\square\square^{\sigma} &= c^{\sigma}\left(\sqcap^{[n-1]}\right)^{\sigma}\exp(\sigma\sqcap\sqcap^{[n-1]})(1+\sqcup\sqcup)^{\sigma}\\ &= c^{\sigma}\left(\sqcap^{[n-1]}\right)^{\sigma}\exp(\sigma\sqcap\sqcap^{[n-1]})(1+\sqcup\sqcup)^{\sigma}\\ &= c^{\sigma}\left(\sqcap^{[n-1]}\right)^{\sigma}\exp(\sigma\sqcap\sqcap^{[n-1]})\sum_{k=0}^{\infty}\binom{n}{\sigma}\sqcup\sqcup^{n}\end{aligned} \tag{3.78}$$

where $\binom{n}{\sigma}$ are the generalized binomial coefficients, the infinite sum is well defined, by Corollary 3.50, and thus $\square\square^{\sigma}$ is well defined as well. Then, if $\boldsymbol{\sigma} \in (\mathbb{R}^+)^M$ and if $\square\square^{[0]} = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} x^{-\boldsymbol{\sigma}\cdot\mathbf{k}}$ is a level zero transseries , we write

$$\square\square^{[0]}\circ\square\square = \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}}(\square\square^{-1})^{\boldsymbol{\sigma}\cdot\mathbf{k}}$$

which is well defined by Corollary 3.50 (ii) and Proposition 3.29. We note that, under our assumptions for $\square\square$, $\square\square^{[0]}\circ\square\square > 0$ is positive iff $\square\square^{[0]} > 0$

Similarly, we write cf. Definition 3.52

$$\exp(\square\square) = e^{\sqcap\sqcap + C + \sqcup\sqcup} = Ae^{\sqcap\sqcap}\sum_{k=0}^{\infty}\frac{\sqcup\sqcup^{n}}{n!} = C'\,\square^{[n+1]}\square\square^{[n]} \tag{3.79}$$

well defined by the definition of a transmonomial, Corollary 3.50 (ii) and Proposition 3.29. Now the definition of general composition is straightforward induction. We assume that composition is defined at all $\le n-1$ levels, and that in addition $\square\square^{[n-1]}\circ\square\square > 0$ if $\square\square^{[n-1]} > 0$. Then, with $\mathbf{L}, \mathbf{S}$ denoting the large part and small part respectively of a transseries,we have

$$\begin{aligned}
\square\!\square^{[n]} \circ \square\!\square &= \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} (\sqcup^{[n]} \circ \square\!\square)^{\mathbf{k}} \\
&= \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} (\square^{[n-1]} \circ \square\!\square)^{\mathbf{k}} \exp(-\sqcap\!\sqcap^{[n-1]} \circ \square\!\square) \\
&= \sum_{\mathbf{k}\ge\mathbf{k}_0} c_{\mathbf{k}} (\square^{[n-1]} \circ \square\!\square)^{\mathbf{k}} \Big[\exp(-\mathbf{L}(\sqcap\!\sqcap^{[n-1]} \circ \square\!\square) C \exp(-\mathbf{S}(\sqcap\!\sqcap^{[n-1]} \circ \square\!\square)\Big] \\
&= \sum_{\mathbf{k}\ge\mathbf{k}_0} c'_{\mathbf{k}} (\square^{[n-1]} \circ \square\!\square)^{\mathbf{k}} \sqcup^{[n]} \exp(-\mathbf{S}(\sqcap\!\sqcap^{[n-1]} \circ \square\!\square) \qquad (3.80)
\end{aligned}$$

and the last sum exists by the induction hypothesis and Proposition 3.34.

Proposition 3.81 (restricted logarithm) *If $\sqcup\!\sqcup$ is a small transseries of level n, then there exists a unique transseries $\sqcup\!\sqcup_e$, of level n such that*

$$e^{\sqcup\!\sqcup_e} = 1 + \sqcup\!\sqcup$$

Proof. Uniqueness is immediate. For existence, note that the equation below is contractive in the space of small transseries with the same generators as $\sqcup\!\sqcup$:

$$\sqcup\!\sqcup_e = \sqcup\!\sqcup - \sum_{k=2}^{\infty} \frac{\sqcup\!\sqcup_e^k}{k!}$$

Corollary 3.82 *If $\sqcap\!\sqcap$ is a large transseries of level n then there exists a transseries $\square\!\square_e = \sqcap\!\sqcap_e + c + \sqcup\!\sqcup_e$ where $\sqcap\!\sqcap_e$ has level n such that*

$$\sqcap\!\sqcap \circ \exp(x) = e^{\square\!\square_e}$$

Proof. We have, by the previous Proposition,

$$\sqcap\!\sqcap = x^{\sigma} e^{\sqcap\!\sqcap_1}(1 + \sqcup\!\sqcup) \Rightarrow \sqcap\!\sqcap(e^x) = e^{\sigma x + \sqcap\!\sqcap_1(e^x) + \sqcup\!\sqcup_e(e^x)} \qquad (3.83)$$

A first result about integration.

Proposition 3.84 *Let T be a log-free transseries and $\tau = T \circ \exp(x)$. Then the equation $\square\!\square' = \tau$ has a unique solution $\square\!\square = \sqcap\!\sqcap_0 + C + \sqcup\!\sqcup$. We choose the antiderivative $\mathcal{P}$ by $\mathcal{P}\tau = \sqcap\!\sqcap_0 + \sqcup\!\sqcup_0$.*

Proof. Uniqueness follows Corollary 3.76. By linearity of differentiation, it is enough to show the result when T, and thus τ is a nonzero transmonomial. The case $l(T) = 1$ is immediate; we then assume $\tau = \sqcap = e^{\sigma x + \sqcap\!\sqcap_1(e^x)}$, the other case being similar. We look for $\mathcal{P}\tau$ in the form $g\tau$. Then g satisfies the equation

$$g = \frac{\tau}{\tau'} - \frac{g'\tau}{\tau'} \qquad (3.85)$$

We have $\tau/\tau' = e^{-e^x \sqcap\!\sqcap_1'(e^x) - \sigma} = e^{\square\!\square}$. We look for g in the form $g = e^{\square\!\square_g}$ with $g \ll \tau$. Since $1/\tau' \ll 1$, τ/τ' is of this form. Furthermore, $g \ll \tau$ implies $\square\!\square_g \ll \sqcap\!\sqcap_1(e^x)$ thus $\square\!\square_g' \ll (\sqcap\!\sqcap_1(e^x))'$ which implies $g'/g \ll \tau'/\tau$ or $g'\tau/\tau' \ll g$ and (3.85) is contractive in this space.

3.3 General transseries

We define

$$L_n(x) = \underbrace{\log\log\ldots\log(x)}_{n\ times} \tag{3.86}$$

$$E_n(x) = \underbrace{\exp\exp\ldots\exp(x)}_{n\ times} \tag{3.87}$$

$$\tag{3.88}$$

with the convention $E_0(x) = L_0(x) = x$.
We write $\exp(\ln x) = x$ and then any log-free transseries can be written as $\square\square(x) = \square\square \circ E_n(L_n(x))$. This defines right composition with L_n in this trivial case, as $\square\square_1 \circ L_n(x)) = (\square\square \circ E_n) \circ L_n(x) := \square\square(x)$.

More generally, we define $\mathcal{T}$, the space of general transseries , as a set of formal compositions

$$\mathcal{T} = \{\square\square \circ L_n : \square\square \in \overset{\approx}{\mathcal{T}}\}$$

with the algebraic operations (symbolized below by $\star$) inherited from $\overset{\approx}{\mathcal{T}}$ by

$$(\square\square_1 \circ L_n) \star (\square\square_2 \circ L_{n+k}) = [(\square\square_1 \circ E_k) \star \square\square_2] \circ L_{n+k} \tag{3.89}$$

and using (3.89), differentiation is defined by

$$\mathcal{D}(\square\square \circ L_n) = \left[(\prod_{k=0}^{n-1} L_k)^{-1} \right] (\mathcal{D}\square\square) \circ L_n$$

Proposition 3.90 *$\mathcal{T}$ is an ordered differential field, closed under restricted composition.*

Proof. The proof is straightforward, by substitution from the results in § 3.2.

We will denote generically the elements of $\mathcal{T}$ with the same symbols that we used for $\overset{\approx}{\mathcal{T}}$.

Proposition 3.91 *$\mathcal{T}$ is closed under integration.*

Proof. The idea behind the construction of $\mathcal{D}^{-1}$ is the following: we first find an invertible operator J which is *to leading order* $\mathcal{D}^{-1}$; then the equation for the correction will be contractive. Let $\square\square = \sum_{\mathbf{k} \ge \mathbf{k}_0} \sqcup^{\mathbf{k}} \circ L_n$. To unify the treatment, it is convenient to use the identity

$$\int_x \square\square(s)ds = \int_{L_{n+2}(x)} (\square\square \circ E_{n+2})(t) \prod_{j \le n+1} E_j(t)\mathrm{d}t = \int_{L_{n+2}(x)} \square\square_1(t)\mathrm{d}t$$

where the last integrand, $\Box_1(t) = \Box_2(e^t)$ with $\Box_2$ a log-free transseries. The result now follows from Proposition 3.84.

In the following we also use the notation $\mathcal{D}\Box = \Box'$ and we write $\mathcal{P}$ for the antiderivative $\mathcal{D}^{-1}$ constructed above.

Definition 3.92 (Level and depth) *By construction, for any element* $\Box$ *of* $\mathcal{T}$ *there is a minimal* $m \geq 0$ *such that* $\Box \circ E_m \in \widetilde{\widetilde{\mathcal{T}}}$*; assume the level of* $\Box \circ E_m$ *in* $\widetilde{\widetilde{\mathcal{T}}}$ *is* n*. We then say that* $\Box$ *has level* $n - m$ *and depth* m*.*

Note 3.93 $\mathcal{P}$ *is an antiderivative without constant terms, i.e,*

$$\mathcal{P}\Box = \sqcap + \sqcup$$

Proof. This follows from Proposition 3.84

Proposition 3.94 *We have*

$$\begin{aligned} \mathcal{P}(\Box_1 + \Box_2) &= \mathcal{P}\Box_1 + \mathcal{P}\Box_2 \\ (\mathcal{P}\Box)' = \Box;\ \mathcal{P}\Box' &= \Box(0) \\ \mathcal{P}(\Box_1 \Box_2') &= \Box_1 \Box_2 - \mathcal{P}(\Box_1' \Box_2) \\ \Box_1 \gg \Box_2 &\implies \mathcal{P}\Box_1 \gg \mathcal{P}\Box_2 \\ \Box > 0 &\implies \mathcal{P}\Box > 0 \end{aligned} \tag{3.95}$$

where

$$\Box = \sum_{\mathbf{k} \geq \mathbf{k}_0} c_{\mathbf{k}} \sqcup^{\mathbf{k}} \implies \Box(0) = \sum_{\mathbf{k} \geq \mathbf{k}_0; \mathbf{k} \neq 0} c_{\mathbf{k}} \sqcup^{\mathbf{k}}$$

Proof. All the properties are straightforward; preservation of inequalities uses Proposition 3.75.

Remark 3.96 *Let* $\sqcup\!\sqcup_0 \in \mathcal{T}$*. The operators defined by*

$$J_1(\Box) = \mathcal{P}(e^{-x}(\text{Const.} + \sqcup\!\sqcup_0)\Box(x)) \tag{3.97}$$

$$J_2(\Box) = e^{\pm x} x^{\sigma} \mathcal{P}(x^{-2} x^{-\sigma} e^{\mp x}(\text{Const.} + \sqcup\!\sqcup_0)\Box(x)) \tag{3.98}$$

are contractive on $\mathcal{T}$*.*

Proof. For (3.97) it is enough to show contractivity of $\mathcal{P}(e^{-x}\cdot)$. This is a straightforward calculation similar to the proof of Proposition 3.91. We have for some n $\Box(x) = \sum_{\mathbf{k} \geq \mathbf{k}_0} \sqcup^{\mathbf{k}}(L_n(x))$ where $\sqcup_j \in \widetilde{\widetilde{\mathcal{T}}}$.

$$\mathcal{P}e^{-x}(\Box \circ L_n) = \mathcal{P}\left(e^{-E_{n+2}} \prod_{1\le j\le n+2} E_j \exp(\sqcap\!\sqcap \circ E_2)\right) \circ L_{n+2}$$
$$= \left[\frac{e^{-E_{n+2}} \prod_{1\le j\le n+2} E_j \exp(\sqcap\!\sqcap \circ E_2)}{-E'_{n+2} + \sum_{0\le j\le n+1} E'_j + \sqcap\!\sqcap' \circ E'_2}(1 + \sqcup\!\sqcup)\right] \circ L_{n+2}$$
$$\ll \prod_{1\le j\le n+2} E_j \exp(\sqcap\!\sqcap \circ E_2) \quad (3.99)$$

The proof of (ii) is similar.

3.4 Differential equations

3.4a Examples: simple differential systems in $\mathcal{T}$

The theory of differential equations in $\mathcal{T}$ is similar to the corresponding theory for functions.

Example 1. The general solution of the differential equation

$$f' + f = 1/x \quad (3.100)$$

in $\mathcal{T}$ (for $x \to +\infty$) is $\Box\!\Box(x;C) = \sum_{k=0}^{\infty} k!x^{-k} + Ce^{-x} = \Box\!\Box(x;0) + Ce^{-x}$.

Indeed, the fact that $\Box\!\Box(x;C)$ is a solution follows immediately from the definition of the operations in $\mathcal{T}$. To show uniqueness, assume $\Box\!\Box_1$ satisfies (3.100). Then $\Box\!\Box_2 = \Box\!\Box_1 - \Box\!\Box(x;0)$ is a solution of $\mathcal{D}\Box\!\Box + \Box\!\Box = 0$. Then $\Box\!\Box_2 = e^x\Box\!\Box$ satisfies $\mathcal{D}\Box\!\Box_2 = 0$ i.e., $\Box\!\Box_2 = \text{Const}$.

The particular solution $\Box\!\Box(x;0)$ is the unique solution of the equation $f = 1/x - \mathcal{D}f$ which is manifestly contractive in the space of level zero transseries (cf. § 3.0f). However this same equation is not contractive for transseries of positive level, (because e.g. $\mathcal{D}e^x = e^x$) as expected, since the solution is not unique in $\mathcal{T}$

Second order linear equations.

Note 3.101 *If an equation of the form*

$$y'' + p(x)y' + q(x) = 0$$

has one nonzero solution in $\mathcal{T}$, then the space of solutions is two-dimensional. (There are equations with no nonzero solutions in $\mathcal{T}$, e.g. $y'' = -y$; in practice, complex trransseries are used to deal with these cases.)

Proof. Assume y_1 is a solution; we let $f = y/y_1$. Then $y_1 f'' + (2y'_1 + py_1)f' = 0$ from which the claim follows easily.

Example 2. The Airy equation

$$y'' = xy \tag{3.102}$$

In $\tilde{\mathcal{T}}^{[0]}$ there is only the null series solution (since the equation $y = y''/x$ is contractive there). We then look for solutions in the form e^W W is large, of level at least zero; the large part should contain some (positive) power of x (otherwise the order of the large part can be lowered by substitution). The equation becomes

$$W'' + W'^2 = x$$

Note 3.103 *In a WKB-type substitution $y = e^W$ where $l(W) \ge 0$ and $W \gg x^\sigma$ with $\sigma > 0$ we have $W'' \ll W'^2$.*

Proof. By assumption, $W = \sqcap\!\sqcap + C + \sqcup\!\sqcup$ where $\sqcap\!\sqcap \gg x^p$ for some $p > 0$; we have, applying Proposition 3.75,

$$\sqcap\!\sqcap' \gg x^{p-1} \Rightarrow \frac{1}{\sqcap\!\sqcap'} \ll x^{1-p} \Rightarrow \frac{\sqcap\!\sqcap''}{\sqcap\!\sqcap'^2} \ll x^{-p} \ll 1 \quad \square$$

By Note 3.103 we denote $W' = f$ and write

$$f = \pm\sqrt{x}\sqrt{1 - f'/x} = \pm\sqrt{x}\Big(1 - \frac{1}{2}\frac{f'}{x} - \frac{1}{8}\frac{f'^2}{x^2}\Big) \tag{3.104}$$

It is easy to check that (3.104) is contractive in the space $\tilde{\mathcal{T}}^{[0]}$. We get

$$W_\pm = \pm\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{4}\ln x + \sqcup\!\sqcup_\pm$$

where

$$\sqcup\!\sqcup_\pm = \pm\frac{5}{48}x^{-\frac{3}{2}} + \frac{5}{64x^3}\cdots$$

which gives

$$y_\pm = x^{-\frac{1}{4}}e^{\pm\frac{2}{3}x^{\frac{3}{2}}}\left(1 + \sqcup\!\sqcup_\pm\right)$$

where now

$$\sqcup\!\sqcup_\pm = \pm\frac{5}{48}x^{-\frac{3}{2}} \pm \frac{295}{2304}x^{-\frac{9}{2}}\cdots$$

Note 3.101 shows that the space of solutions of (3.102) in $\mathcal{T}$ is generated by $y_\pm$.

Simple nonlinear equations. Consider the first order differential equation:

$$f' = J_1(f) = F_0(x^{-1}) - f - \frac{\beta}{x}f - g(x^{-1}, f) \tag{3.105}$$

where

$$F_0(x^{-1}) = \sum_{k\geq 2} \frac{F_{0k}}{x^k}$$
$$g(x^{-1}, f) = \sum_{k\geq 0;\ l\geq 1} g_{kl} x^{-k} y^l \tag{3.106}$$

where $g_{01} = g_{11} = 0$.
We see that J_1 is well defined if $f = \sqcup\!\sqcup \in \mathcal{T}$ (cf. Proposition 3.34), and it is under this assumption that we study J_1.[1]

(1). Solutions of (3.106) in $\tilde{\mathcal{T}}^{[0]}$. The equation

$$f = J_2(f) = -f' + F_0(x^{-1}) - \frac{\beta}{x} f - g(x^{-1}, f) \tag{3.107}$$

is contractive in $\tilde{\mathcal{T}}^{[0]}$ (this follows immediately from §3.0f). Thus there exists in $\tilde{\mathcal{T}}^{[0]}$ a unique solution $\tilde{f}_0$. Since (3.107) is also contractive in the subspace of $\tilde{\mathcal{T}}^{[0]}$ of series of the form $\sum_{k=2}^{\infty} \frac{c_k}{x^k}$ we have

$$\tilde{f}_0 = \sum_{k=2}^{\infty} \frac{c_k}{x^k} \tag{3.108}$$

Note. The iteration $f_{n+1} = J_1 f_n,\ f_1 = x^{-1}$ is convergent in $\mathcal{T}$ and, if $f_i = \sum_{k=2}^{i} c_k^{[i]} x^{-k}$ then $c_k^{[i]} = c_k$ for $k \leq i$, and this is a practically very convenient way to calculate the coefficients c_i.

(2) Let now $\delta = f - \tilde{f}_0$. Then

$$\begin{aligned}\delta' &= -\delta - \frac{\beta}{x}\delta - g(x^{-1}, \tilde{f}_0 + \delta) + g(x^{-1}, \tilde{f}_0 + \delta) \\ &= -\delta - \frac{\beta}{x}\delta + \sum_{k\geq 0;\ l\geq 1} c_{kl} x^{-k} \delta^l \end{aligned} \tag{3.109}$$

with

$$c_{01} = c_{11} = 0 \tag{3.110}$$

or

$$\frac{\delta'}{\delta} = -1 - \frac{\beta}{x} - \sum_{k\geq 2} \frac{c_{k1}}{x^k} + \sum_{k\geq 0;\ l\geq 1} c_{k;l+1} x^{-k} \delta^l \tag{3.111}$$

Since by assumption $\delta \ll 1$ we have

[1] Except when there are only finitely many nonzero terms in the sum in (3.106), J_1 is not in $\mathcal{T}$ if $f \gg 1$ ($\mathrm{mag}(f^n)$ would be unbounded).

$$\ln \delta = C - x + \beta \ln x + \sum_{k\ge 1} \frac{c_{k+1;1}}{kx^k} + x\sqcup\!\sqcup(x)$$

and thus $\delta \ll \exp(-cx)$ for any $c < 1$ so that

$$\ln \delta = C - x + \beta \ln x + \sum_{k\ge 1} \frac{c_{k+1;1}}{kx^k} + \exp(-cx)\sqcup\!\sqcup(x)$$

whence, by composition with exp we get

$$\delta = Cx^{\beta}e^{-x} \sum_{k\ge 1} \frac{d_{k+1;1}}{kx^k} + \exp(-cx)\sqcup\!\sqcup(x)$$

Equation (3.111) implies

$$\delta = C'x^{\beta}e^{-x}\tilde{y}_0 \exp\left(\int \sum_{k\ge 0;\ l\ge 1} c_{k;l+1}x^{-k}\delta^l\right); \quad \left(\tilde{y}_0 = \sum_{k\ge 0} \frac{d_{k+1;1}}{kx^k}\right) \tag{3.112}$$

and (3.112) is contractive by Remark 3.45 and Remark 3.96. In particular, for every C there is a unique $\delta(x;C)$ satisfying (3.112).

Remark 3.113 *We have*

$$\delta = \sum_{k=1}^{\infty} C^k x^{\beta k} e^{-kx} \tilde{f}_k(x) \tag{3.114}$$

where $\tilde{f}_k \in \tilde{\mathcal{T}}^{[0]}$ *and*

$$\tilde{f}_k(x) = \sum_{j=0}^{\infty} \frac{f_{k;j}}{x^j}$$

Proof. This is a straightforward consequence of (3.112).

Formal linearization. Let $z = Cx^{\beta}e^{-x}$. We have $C(x,\delta) = x^{-\beta}e^{x}\sum_{k\ge 1}\delta^k \tilde{g}_k(x)$. A direct calculation shows that $C' = C_x + C_\delta \delta' = 0$. The transformation $(x \mapsto x; y \mapsto C(x, y - f_0))$ formally linearizes (3.105).

3.5 Higher dimensional systems of ODEs

The analysis in the previous section generalizes rather straightforwardly to analytic differential systems in $\mathbb{C}^n$.

3.6 Transseries solutions at irregular singularities

Consider the differential system

$$\mathbf{y}' = \mathbf{f}(x^{-1}, \mathbf{y}) \qquad \mathbf{y} \in \mathbb{C}^n \tag{3.115}$$

We look at solutions $\mathbf{y}$ such that $\mathbf{y}(x) \to 0$ as $x \to \infty$ along some direction $d = \{x \in \mathbb{C} : \arg(x) = \phi\}$. The following conditions are assumed
(a1) The function $\mathbf{f}$ is analytic at $(0, 0)$.
(a2) Nonresonance: the eigenvalues λ_i of the linearization

$$\hat{\Lambda} := -\left(\frac{\partial f_i}{\partial y_j}(0,0)\right)_{i,j=1,2,\ldots n} \tag{3.116}$$

are linearly independent over $\mathbb{Z}$ (in particular nonzero) and such that $\arg \lambda_i$ are different from each other (i.e., the Stokes lines are distinct; we will require somewhat less restrictive conditions, see § 3.1).

The system (3.115) can then be brought to the form

$$\mathbf{y}' = -\hat{\Lambda}\mathbf{y} + \frac{1}{x}\hat{A}\mathbf{y} + \mathbf{g}(x^{-1}, \mathbf{y}) \tag{3.117}$$

where $\hat{\Lambda} = \operatorname{diag}\{\lambda_i\}$, $\hat{A} = \operatorname{diag}\{\alpha_i\}$ are constant matrices, $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}, \mathbf{y}^2)$, $(x \to \infty, \mathbf{y} \to 0)$. (Note: with respect to [**?**] we have $\hat{A} = -\hat{B}$, $\mathbf{f}_0$ was incorporated in $\mathbf{g}$ and we omitted the normalization making $\Re(\alpha_i) > 0$). This is straightforward algebra [**?**] whose details we omit (see however § 3.2 where all this is exemplified in the two-dimensional case).

The general solution of (3.117) in $\mathcal{T}_{\mathbb{C}}$ is an $n_1 \le n$ parameter transseries. Let d be a ray in $\mathbb{C}$ and

$$\tilde{\mathbf{y}}(x) = \sum_{\mathbf{k} \ge 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\boldsymbol{\alpha}\cdot\mathbf{k}} \tilde{\mathbf{s}}_{\mathbf{k}}(x) = \sum_{\mathbf{k} \ge 0} \mathbf{C}^{\mathbf{k}} \mathrm{e}^{-\boldsymbol{\lambda}\cdot\mathbf{k}x} x^{\mathbf{m}_0\cdot\mathbf{k}} \tilde{\mathbf{y}}_{\mathbf{k}}(x) \tag{3.118}$$

Then $\tilde{\mathbf{y}}$ is a transseries on d iff $C_i = 0$ for all i so that $\mathrm{e}^{-\lambda_i x} \not\to 0$ as $x \to \infty$ in d.

Remark 3.119 *(i) If* $\mathbf{g}(x^{-1}, \mathbf{y}) \equiv 0$ *the (now linear) system (3.117) has the general solution*

$$\mathbf{y} = \mathrm{e}^{-x\hat{\Lambda}} \mathbf{C} x^{\hat{A}}$$

(ii) More generally, if $\mathbf{g}(x^{-1}, \mathbf{y}) = \mathbf{G}(x^{-1})$ *is a transseries, then the general solution of (3.117) is*

$$\mathbf{y} = \mathrm{e}^{-x\hat{\Lambda}} x^{\hat{A}} \mathbf{C} + \mathrm{e}^{-x\hat{\Lambda}} x^{\hat{A}} \mathcal{P}\left(\mathrm{e}^{x\hat{\Lambda}} x^{-\hat{A}} \mathbf{g}\right) \tag{3.120}$$

Proof. In both cases the system is diagonal and the result follows immediately from the case when $n = 1$, which we now assume. (i) follows from §**??**. With $C = 0$, (3.120) is a solution of (3.117), so (ii) reduces to (i).

Proposition 3.121 *The general solution of (3.117) in* $\mathcal{T}_{\mathbb{C}}$ *with the restriction* $\mathbf{y} \ll 1$ *is of the form (3.118).*

Proof. If $\mathbf{y}$ is a solution of (3.117) then we have, by Remark 3.119

$$\mathbf{y} = \mathrm{e}^{-x\hat{A}} x^{\hat{A}} \mathbf{C} + \mathrm{e}^{-x\hat{A}} x^{\hat{A}} \mathcal{P}\Big(\mathrm{e}^{x\hat{A}} x^{-\hat{A}} \mathbf{g}(x^{-1}, \mathbf{y})\Big) \tag{3.122}$$

for some $\mathbf{C}$. Since $\mathbf{y} \ll 1$ we have $\mathbf{g}(x^{-1}, \mathbf{y}) \ll 1$ and thus

$$\mathcal{P}\Big(\mathrm{e}^{x\hat{A}} x^{-\hat{A}} \mathbf{g}(x^{-1}, \mathbf{y})\Big) \ll \mathrm{e}^{x\hat{A}} x^{-\hat{A}}$$

Again since $\mathbf{y} \ll 1$, we then have $C_i = 0$ for all i for which $e^{-\lambda_i x} \not\ll 1$.
Note. With the condition $\mathbf{y} \ll 1$, eq. (3.122) has a unique solution.

Indeed, the difference of two solutions $\mathbf{y}_1 - \mathbf{y}_2$ satisfies the equation

$$\mathbf{y}_1 - \mathbf{y}_2 = \mathrm{e}^{-x\hat{A}} x^{\hat{A}} \mathcal{P}\Big(\mathrm{e}^{x\hat{A}} x^{-\hat{A}} \big[\mathbf{g}(x^{-1}, \mathbf{y}_1) - \mathbf{g}(x^{-1}, \mathbf{y}_2)\big]\Big) \tag{3.123}$$

Since $\mathbf{g}(x^{-1}, \mathbf{y}) = O(x^{-2}, \mathbf{y}^2)$ we have

$$\mathbf{g}(x^{-1}, \mathbf{y}_1) - \mathbf{g}(x^{-1}, \mathbf{y}_2) = O(x^{-2}\boldsymbol{\delta}, |\mathbf{y}||\boldsymbol{\delta}|)$$

which by Proposition 3.94 implies $\boldsymbol{\delta} = o(\boldsymbol{\delta})$, i.e., $\delta = 0$.

Using Remark 3.96 it is easy to check that (3.123) is an asymptotically contractive equation, in the space of $\mathbf{y}$ which are $\ll x^{-2}$ thus it has a solution $\mathbf{y}^{[0]}$ with this property. Since the previous note shows the solution of (3.117) with $\mathbf{y} \ll 1$ is unique, we have $\mathbf{y} = \mathbf{y}^{[0]}$. Formula (3.118) is obtained by straightforward iteration of (3.123).

Normalization example: the Panlevé equation $\mathbf{P_I}$ (1.85) when $z \to +\infty$. Finding a normalized form of the equation is done through simple changes of variables. These in turn are derived from the the transseries solution (we allow for complex ones) of the equation that we compute next. The change of variables is one that make all arguments of the exponentials linear in x and the dominance of the transseries solutions $o(x^{-M})$ for an appropriate M.

Formal solutions are derived in a straightforward way. In $\tilde{\mathcal{T}}^{[0]}$ (cf. §3.2) it is easy to check that the dominance Ax^p of any nontrivial series must have $p < 1$. Then, in $\tilde{\mathcal{T}}^{[0]}$ we have $g'' \ll g^2$ and solutions can be found from the contractive equations

$$g = \pm 6^{-1/2} i \sqrt{z - g''} \tag{3.124}$$

By contractivity, the two choices of sign give raise to solutions, they are distinct, and again by contractivity and the previous discussion, there are exactly two solutions in $\tilde{\mathcal{T}}^{[0]}$. We look at the + choice only, the other case being very similar. The power series solution can be generated by iterating (3.124), which yields

$$\tilde{g}_0 = 6^{-1/2} i z^{1/2} + \frac{1}{48z^2} - \frac{47i}{4608 z^{9/2}} + \cdots \qquad (3.125)$$

It is easy to check there are no large solutions of P_I with dominance of level higher than zero. We thus look for other solutions in the form $\tilde{g}_0 + \delta$ where δ must be small, and of level at least one, i.e. $\delta = e^{-W}$ with $\Re(W) > 0$. The equation for δ is

$$\delta'' - \left(2i\sqrt{6}\sqrt{z} + \frac{1}{4z^2} + \cdots\right)\delta = 6\delta^2 \qquad (3.126)$$

in which, taking $\delta = e^{-W}$ and noting that ${W'}^2 \gg W''$, one solves for W' with the information $\Re(W) > 0$ and integrate once. The equation for W,

$$W = C - \mathcal{P}\left[\sqrt{2i\sqrt{6}z^{1/2} + \frac{1}{4z^2} + \cdots + W'' + 6e^{-W}}\right] \qquad (3.127)$$

is contractive for any C. This gives which, upon iteration gives

$$\begin{aligned} W = &-\frac{4}{5}6^{1/4}(1+i)z^{5/4} + \frac{6^{3/4}(1-i)}{120}z^{-5/4} + \cdots \\ &- \frac{(1+i)6^{3/4}e^C}{2i} z^{-1/4} \exp\left(-\frac{4}{5}6^{1/4}(1+i)z^{5/4}\right)(1+\cdots) + \cdots \end{aligned} \qquad (3.128)$$

To normalize the transseries, the natural variable is $z^{5/4}$. Returning to (1.85) we take

$$x = \frac{(-24z)^{5/4}}{30}, y = \sqrt{-\frac{z}{6}}y_1(x) \qquad (3.129)$$

and obtain

$$y_1'' - \frac{1}{2}y_1^2 + \frac{1}{2} + \frac{1}{x}y' - \frac{4}{25x^2}y_1 = 0$$

where, as a last step we make a transformation that ensures that the leading behavior of the unknown function is $o(x^{-2})$ namely $h = y_1 - 1 + \frac{4}{25x^2}$ which yields

$$h'' + \frac{1}{x}h' - h = \frac{1}{2}h^2 + \frac{392}{625x^4} \qquad (3.130)$$

Written as a system, (3.130) satisfies the desired assumptions. A simple way to derive the normal form is to note that the first two terms on the rhs of (3.130) contribute to $\mathbf{g}$ and the last one to $\mathbf{F}_0$. The left hand side has solutions of the form $h = C_1 x^{-1/2} e^{-x} + C_2 x^{-1/2} e^{-x}$. To get a diagonal Λ we choose combinations of h and h' which have only one exponential present. These are $u = h + h'$ and $v = h - h'$ respectively. In terms of u and v we have

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \frac{1}{2x} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + O\left(\frac{1}{x^2}, x^{-1/2}(|u|+|v|)\right)$$
$$= \Lambda \begin{pmatrix} u \\ v \end{pmatrix} + \frac{1}{2x} B_1 \begin{pmatrix} u \\ v \end{pmatrix} + O\left(\frac{1}{x^2}, x^{-1/2}(|u|+|v|)\right) \quad (3.131)$$

To obtain a system whose x^{-1} term is also diagonal we take

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(I + \frac{1}{x} N\right) \begin{pmatrix} u \\ v \end{pmatrix} \quad (3.132)$$

and note that, if

$$N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$N\Lambda - \Lambda N = \begin{pmatrix} 0 & -2b \\ 2c & 0 \end{pmatrix} \quad (3.133)$$

and thus we choose $b = 1/2$, $c = -1/2$ to get

$$\mathbf{y}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y} - \frac{1}{2x} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y} + O\left(\mathbf{y}^2, \mathbf{y} x^{-2}\right) \quad (3.134)$$