Some results on inhomogeneous discriminants

Abstract

We study generalized Horn-Kapranov rational parametrizations of inhomogeneous sparse discriminants from both a theoretical and an algorithmic perspective. In particular, we focus on the computation of inhomogeneous discriminants related to integer matrices B as an implicitization problem with base points corresponding to a H-K parametrization ψ_B. We show that all these parametrizations are birational refining a previous result by Kapranov and prove some results on the corresponding implicit equations. We also propose a combinatorial algorithm to compute the degree of inhomogeneous discriminantal surfaces associated to uniform matrices.

If time permits, we will present some examples that show numerous difficulties for computing the (Hilbert-Samuel) multiplicities of these points. Joint work with Alicia Dickenstein.

Reference: $http://arxiv.org/abs/math/0610031$

1 Motivation to study the problem

1.1 A-discriminants

- $A = \{a_1, \ldots, a_n\} \subset \mathbb{Z}^{d-1}$ (n lattice points) ($n > d-1$)
- $F_A = \sum_{a \in A} x_a t^a$ (generic polynomial in $d-1$ variables $(t_1, \ldots, t_{d-1})$ with exponents in A).

[GKZ] + (certain hypothesis) $\Rightarrow$ exists an **irred.** poly $D_A \in \mathbb{Z}[x_a : a \in A]$ (*unique* up to sign) s.t.

$$D_A(c) = 0 \iff \exists t = (t_1, \ldots, t_{d-1}) \in (\mathbb{C}^*)^{d-1} \; s.t. \; F_A(c)(t) = \frac{\partial F_A(c)}{\partial t_i}(t) = 0 \; \forall \, i.$$

Definition 1.1 *$D_A :=$ A-discriminant (affine invariant of A).*

Remark 1.2 *Key properties:*

- *D_A is an affine invariant;*
- *certain homogeneity condition.*

Next step: built a matrix (also called A) in $\mathbb{Z}^{d\times n}$ with columns $(1, a)$ (for all $a \in A$). Assume A is of maximal rank d.

Built $B \in \mathbb{Z}^{n\times(n-d)}$ with columns a $\mathbb{Z}$ basis of $\ker A$.

(Idea: B expresses affine dependencies of the initial config. of pts.)

Definition 1.3 $B :=$ *Gale dual of* A

Remark 1.4 *Properties of* B*:*

1. B *is of full rank* $m = n - d = n - rk(A)$*;*
2. $\sum$ *rows of* $B = (0, \ldots, 0)$ *(because 1st. row of* A *is* $(1, \ldots, 1)$*).*
3. $g_B =$ *(gcd of mxl minors of* B*)* $= 1$.

Definition 1.5 *Under conditions (1) and (2),* B *is called* ***regular****.*

Remark 1.6 D_A *is* A*-homogeneous, i.e.* $\exists\, v \in \mathbb{Z}^d$ *s.t. all monomials* c^ν *in* $D_A = \sum_\nu d_\nu\, x^\nu$ *satisfy* $A \cdot \nu = v$*. (i.e. quasi-homogenous relative to the weight defined by any vector in the row span of* A*).*

Want: poly with A-homogeneity $= 0$.

"Take our this homogeneities" $\Rightarrow \forall\, \nu_0$ s.t. $A \cdot \nu = v$:

$$D_A(x) \;=\; x^{\nu_0} \sum_\nu d_\nu\, x^{\nu-\nu_0},$$

where $d_\nu \in \mathbb{Z} \setminus \{0\}$ and $\nu - \nu_0 \in \ker_{\mathbb{Z}}(A) = \langle \text{cols.}B \rangle$.

- Write each $\nu - \nu_0$ as a $\mathbb{Z}$- linear combination of the columns $v^{(1)}, \ldots, v^{(m)}$ of B. Call $y_i = v^{(i)}$ (new indeterminates).
- $\Rightarrow \exists$ Laurent polynomial $\Delta_B(y)$ in m variables such that up to a monomial (x^{ν_0}), $\Delta_B(x^{v^{(1)}}, \ldots, x^{v^{(m)}})$ equals $D_A(x)$.

Remark 1.7 Δ_B *has the* ***same*** *number of monomials and the same coefficients as* D_A*.*

GOAL: Compute Δ_B and describe de hypersurface $(\Delta_B = 0) \subset \mathbb{C}^m$ ($\Rightarrow$ get D_A!!!!).

1.2 Horn-Kapranov rational parametrization

- Nice way of constructing the hypersurface $(\Delta_B = 0)$.
- Need only *some* prop. of B.

Setting: $C \in \mathbb{Z}^{n\times m}$ of mxl rank m ($m \geq n$) s.t. $\sum$ rows of $C = 0$ (i.e. C regular), and has no *zero* rows.

- Call $C_1, \ldots, C_n \in \mathbb{Z}^m$ rows of C and define:

$$l_k(u_1, \ldots, u_m) := \langle C_k \, , \, (u_1, \ldots, u_m) \rangle \quad \forall \, k = 1, \ldots, n. \tag{1}$$

- Construct rational map:

$$\psi_C : \mathbb{C}^m \dashrightarrow \mathbb{C}^m \qquad (u_1, \ldots, u_m) \mapsto (y_1, \ldots, y_m) \, ,$$

where

$$y_k \; = \; \prod_{i=1}^{n} l_i(u_1, \ldots, u_m)^{c_{i,k}} \qquad \forall k = 1, \ldots, m. \tag{2}$$

and set $S_C = \overline{im \, \psi_C} \subset \mathbb{C}^m$.

Remark 1.8 *C regular $\Rightarrow$ y_k have degree* 0.

- Better presentation: define

$$f_0 \; = \; \prod_{i=1}^{n} l_i^{-min\{0, \, c_{i,k} \, : \, k=1,\ldots,m\}}$$

(i.e. the least common denominator of all the y_k's) and write

$$y_k \; = \; \frac{f_k}{f_0}, \quad k = 1, \ldots m. \tag{3}$$

Remark 1.9 *By Remark 1.8,* $\deg f_0 = \ldots = \deg f_m = d_C$ *(can be read from matrix C)*

$$d_C \; = \; -\sum_{i=1}^{n} \min\{0, c_{i,k} \; : \; k = 1, \ldots, m\}. \tag{4}$$

(i.e. pick most negative entry in each row and change sign)

Corollary 1.10 *Can define*

$$\psi_C : \mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^m, \tag{5}$$

where $\psi_C = (f_0 : f_1 : \cdots : f_m)$ is defined ***outside*** *the* base point locus $\mathcal{Z} = V(f_0, \ldots, f_m)$.

Definition 1.11 *$S_C :=$ proj. variety defined by ψ_C ($\subset \mathbb{P}^m$).*

Question 1.12
- *Describe $\mathcal{Z}$ in terms of matrix C.*
- *When does S_C be a hypersurface? In this case: $S_C = (\Delta_C = 0)$!!!*

Example 1.13

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \quad \Rightarrow \quad B = \begin{pmatrix} 1 & 2 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e. the discriminant $D_A(x_0, x_1, x_2, x_3)$ of the generic polynomial $F_A = x_0 + x_1 t + x_2 t^2 + x_3 t^3$ equals

$$D_A(x) \;=\; -27x_3^2x_0^2 + 18x_3x_0x_2x_1 + x_2^2x_1^2 - 4x_2^3x_0 - 4x_3x_1^3.$$

Note that $g_B = 1$, so that the columns of B are a basis of the integer kernel of A. Calling $l_0(u,v) := u + 2v$, $l_1(u,v) := -2u - 3v$, $l_2(u,v) := u$ and $l_3(u,v) := v$, the parametrization ψ_B equals

$$\begin{cases} y_1 & := \frac{l_0 l_2}{l_1^2} \\ y_2 & := \frac{l_0^2 l_3}{l_1^3} \end{cases}$$

Its closed image is the hypersurface $S_B = \{\Delta_B = 0\}$, where

$$\Delta_B(y_1, y_2) = -4y_2 - 27y_2^2 + y_1^2 + 18y_2y_1 - 4y_1^3,$$

$$\Delta_B(y_1, y_2) = D_A(1, 1, y_1, y_2).$$

Conversely, up to a monomial D_A equals $\Delta_B(x_0x_2/x_1^2, x_0^2x_3/x_1^3)$.

2 Base point locus

Remark 2.1 *l_k give hyperplane arrangement in $\mathbb{C}^m$ (and $\mathbb{P}^{m-1}$).*

Definition 2.2
- *$\mathcal{F}$:= flat in this arrangement = $V(l_{i_1}, \ldots, l_{i_r})$ (linear space).*
- *$\mathcal{L}(\mathcal{F})$:= {all linear forms l_j vanishing on $\mathcal{F}$}, i.e. all linear forms $l_j \in \mathbb{Q}\langle l_{i_1}, \ldots, l_{i_r}\rangle$.*
- Basic flat *:= $\mathcal{F}$ s.t. all of $f_0, \ldots, f_m$ vanish on $\mathcal{F}$.*

Remark 2.3 *f_k vanishes on $\mathcal{F}$ $\iff$ if it contains a linear factor from $\mathcal{L}(\mathcal{F})$.*

Lemma 2.4 *The base point locus $\mathcal{Z}$ equals the union of all basic flats.*

Remark 2.5 (KEY) *Eliminate all common factors from $f_0, \ldots, f_m$ (eventually modifying certain rows from C) $\Rightarrow$ can assume codim($\mathcal{Z}$) ≥ 2.*

Corollary 2.6 *If $m = 3$, $\mathcal{Z}$ is finite.*
(However: complicated structure!!)
If $m > 3$ in general $\dim(\mathcal{Z}) > 0$ *(some components)*

Definition 2.7 *If $codim(S_C) = 1$, call C non-defective.*

Lemma 2.8 *Let $C \in \mathbb{Z}^{n\times m}$ be a regular matrix of rank $r < m$. Then $\operatorname{codim}(S_C) > m - r$, so C is defective.*

Proof. All y_i are homog rational fnc of fixed rows $l_1, \ldots, l_r$ of deg. $0 \Rightarrow$ rat. fnc. of $r-1$ variables $l_1/l_r, \ldots, l_{r-1}/l_r$. $\Rightarrow codim(S_C) \geq m - (r-1)$. □

Remark 2.9 *Converse is not true. Pick $n = 2n'$ and $C := \left(\frac{M}{-M}\right)$ where $M \in \mathbb{Z}^{n'\times m}$ has rank $m \leq n'$. In this case ψ_C is const. map.*

Remark 2.10 *$\exists$ algorithms from checking defectiveness.*

1. *Compute generic rank of Jacobian matrix $J(\psi_C)$:*

$$rk(J(\psi_C)(u)) = m - 1 \text{ for gen.} u \iff C \text{ non-defective}$$

2. *Tropical approach in [3].*

3 Birrationality of ψ_C

Key-definition: *Gauss map* of a hypersurface $S \subset G$ (G algebraic group).

Definition 3.1 *Our case: $G = (\mathbb{C}^*)^m$ and $S = (\Delta = 0)$. Then the (logarithmic) Gauss map is*

$$\gamma(y) = (y_1 \frac{\partial \Delta}{\partial y_1}(y) : \ldots : y_m \frac{\partial \Delta}{\partial y_m}(y)) \,, \tag{6}$$

mapping a regular (smooth) point $y \in S$ to a projective point in $\mathbb{P}^{m-1}$.

Assumption: $S^* = S \cap (\mathbb{C}^*)^m \neq \emptyset$.

Remark 3.2 *(Geometric interpretation)*
$\gamma_S \longleftrightarrow$ looking at image of $log(y) = \log(y) = (\log(y_1), \ldots, \log(y_m))$ for $y \in S^*$ and consider Gauss map of $\log(S^*)$.

Theorem 3.3 (K; _ , D.) *Let $S \subset \mathbb{C}^m$ be an algebraic irreducible hypersurface.*

The Gauss map $\gamma_S : S \dashrightarrow \mathbb{P}^{m-1}$ is birational if and only if there exist a non-defective and regular integer matrix $C \in \mathbb{Z}^{n\times m}$ of full rank, and a constant $\lambda \in (\mathbb{C}^*)^m$ such that $S = \lambda \cdot S_C$, i.e. S is a torus translate by λ of a generalized inhomogeneous discriminant hypersurface.

Moreover, in this case, $\lambda \cdot \psi_C$ is birational and the logarithmic Gauss map γ_S is its inverse.

Comment 3.4

- ***Our contribution:*** *removing the incorrect hypothesis about the gcd $g_C = 1$. Explanation for this "mistake": Remark 4.4.*

- *Proofs: exactly the same as Kapranov's original ones. "If" direction: condition $g_C = 1$ is superfluous. "Only if" direction: the last statement is false.*

- ***Key tools** for (⇐): $J(log(\lambda\psi_C))$ is symmetric. $J(\log(\lambda\psi_C)) = J(\lambda\psi_C){\cdot}D$ D =diag. matrix with multipl. inverse of coordinates of $\lambda\psi_C$ as entries. $rkJ(\lambda\psi_C) = m-1$ (C non defective). Use implicit partial differentiation of $\Delta(\lambda\psi_C) = 0$ ($S = (\Delta = 0)$).*

Proof.[Proof of the "if" part in Theorem 3.3] Let C be a regular non-defective $n \times m$ integer matrix, a point $\lambda \in (\mathbb{C}^*)^m$ in the torus, and consider the map $\psi'_C := \lambda\psi_C$. We need to show that the logarithmic Gauss map is its birational inverse. Denote by Δ an irreducible equation of its closed image. The principal observation is that the Jacobian matrix of $\log(\psi'_C)$ is symmetric since

$$\frac{\partial}{\partial u_k}\log((\psi'_C)_j) \;=\; \sum_{i=1}^{n}\frac{c_{i,k}\;c_{i,j}}{l_i(u)}.$$

Moveover, a straightforward computation shows that for any point u in the preimage of the torus, the Jacobian matrices $J(\psi'_C)$ and $J(\log(\psi'_C))$ have the same rank since $J(\log(\psi'_C)) = J(\psi'_C)\cdot D$, where D is the diagonal matrix with diagonal entries the multiplicative inverses of the coordinates of ψ'_C. This rank is equal to $m-1$ by our hyphotesis that C is non-defective. Now, on one side, implicit partial differentiation of the equality $\Delta(\psi'_C(u)) = 0$ implies that the vector $\gamma_C(y)$ lies in the kernel of the transposed Jacobian matrix $J(\log(\psi'_C))^t$ for any y in the image of ψ'_C. On the other side, since the coordinates of ψ'_C are homogeneous forms of degree 0, it follows from Euler's formula applied to the coordinates of $\log(\psi'_C))$ that any point u in the preimage of the torus lies in the kernel of $J(\log(\psi'_C)(u)$. Then, u is proportional to $\gamma_C(\psi'_C(u))$, when this vector is non zero. □

4 Monomial changes of coordinates and factorizations

GOAL: Analyze choice of C s.t. $A \cdot C = 0$ and relate different Δ_C's obtained.

4.1 Some definitions

Setting: $C \in \mathbb{Z}^{n\times m}$ regular and non-defective (⇒ mxl rk, equiv $g_C \neq 0$).

Reduction: Replace all row vectors in C lying in the same one-dim flat $\mathcal{F}$ by their sum, without essentially changing the coordinates of the parametrization ψ_C except for constants (if the sum gives the zero vector, we keep the constants but we don't keep a zero row).

Warning: We may have changed g_C!! ⇒ work with general $g_C(\neq 0)$. Convention: B has $g_B = 1$.

Setting: Matrices $C_1, C_2 \in \mathbb{Z}^{n\times m}$ s.t. Cols (C_1) $\mathbb{Z}$-span Cols (C_2). Equiv. $\exists\, M \in \mathbb{Z}^{m\times m}$ s.t. $C_1 = C_2 \cdot M$.

Remark 4.1 *Suppose $g_{C_2} = 1$ (ie. $\mathbb{Z}C_2$ satturated lattice of $\mathbb{Z}C_1$). The lattice ideal $I(\mathbb{Z}C) = \langle x^u - x^v : u, v \in \mathbb{N}^n, u - v \in \mathbb{Z}C\rangle$ (in n variables) is radical with $|g_C|$ primary components, which correspond to torus translates of the toric variety defined by the lattice ideal $I(\mathbb{Z}B)$ [4]. We will see in Theorem 4.7 how this is reflected in the precise relation between the irreducible m-variate polynomials Δ_B and Δ_C.*

Definition 4.2

- *Linear map $\Lambda_M : \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$ $\Lambda_M(u) = M \cdot u^t$.*
- *Denote $Col(M) = \{M^{(1)}, \ldots, M^{(m)}\}$. Define the (multiplicative) monomial map $\alpha_M : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^m$:*

$$\alpha_M(y) = (\prod_{i=1}^{m} y_i^{M_{i,1}}, \ldots, \prod_{i=1}^{m} y_i^{M_{i,m}}) = (y^{M^{(1)}}, \ldots, y^{M^{(m)}}) \,. \qquad (7)$$

(Note: $\alpha_{M_1\cdot M_2} = \alpha_{M_2} \circ \alpha_{M_1}$ and α_M is a $|\det M| - 1$ mapping.)

Lemma 4.3 (C,C) *(in [1])*

$$\begin{array}{ccc} \mathbb{P}^{m-1} & \xrightarrow{\Lambda_M} & \mathbb{P}^{m-1} \\ \psi_{C_1} \downarrow & \equiv & \downarrow \psi_{C_2} \\ (\mathbb{C}^*)^m & \xleftarrow[\alpha_M]{} & (\mathbb{C}^*)^m \end{array}$$

Remark 4.4 *Assume $g_{C_2} = 1$, and call $C = C_1, B = C_2$. Then, $|\det(M)| = g_C$. Suppose that we didn't know Thm 3.3 but instead we suspected (or proved) that ψ_B is birational. From the equality $\psi_C = \alpha_M \circ \psi_B \circ \Lambda_M$, where Λ_M is birational and α_M is a g_C to 1 mapping, one is tempted to deduce that ψ_C is also a g_C to 1 mapping. But indeed, we have already proved that it is birational.*

Explanation:

Lemma 4.5 *Supp. $C_1 = C_2 \cdot M$ and C_1, C_2 are non-defective regular integer matrices. Then*

$$\tilde{\alpha}_M = \alpha_{M|_{(\Delta_{C_2}=0)}} : (\Delta_{C_2} = 0) \dashrightarrow (\Delta_{C_1} = 0),$$

is a birational map.

Proof. (Sketch) Restrict α_M to image of ψ_{C_2} and use density and Lemma 4.3. Thm 3.3 $\Rightarrow$ $\tilde{\alpha}_M$ birational. ☐

4.2 Factorization Theorem

Convention: $\Delta_{C_i} \in \mathbb{Z}[y_1, \ldots, y_m]$ with cont 1, defined up to sign.
(*Reason:* H-K param. is given by rational forms with rational coeff.)

Recall: α_M is a $|\det M| - 1$ mult. map.

Definition 4.6

$$G_M := \ker \alpha_M = \{\varepsilon \in (\mathbb{C}^*)^m \,:\, \alpha_M(\varepsilon) = (1, \ldots, 1)\} \tag{8}$$

group with induced coordinatewise mult.

Theorem 4.7 *[_ , D] (Factorization Thm)*
Let C_1, C_2 are non-defective $n \times m$ regular integer matrices such that $C_1 = C_2 \cdot M$. There exists v in the lattice $\mathbb{Z}M$ (gen by cols of M) (or equiv., s.t. $\varepsilon^v = 1$ for all $\varepsilon \in G_M$) such that

$$\Delta_{C_1} \circ \alpha_M(y) = y^v \prod_{\varepsilon \in G_M} \Delta_{C_2}(\varepsilon \cdot y)\ . \tag{9}$$

Proof.(Sketch)

1. Density and properness arguments:
$$(\Delta_{C_1} \circ \alpha_M(y) = 0) \cap (\mathbb{C}^*)^m \ = \bigcup_{\varepsilon \in G_m} (\Delta_{C_2}(\varepsilon \cdot y) = 0) \cap (\mathbb{C}^*)^m.$$

2. $\Delta_{C_2}(\varepsilon \cdot y) \mid \Delta_{C_1} \circ \alpha_M(y)$ for all $\varepsilon \in G_M$ as Laurent poly: clear by irred of (LHS).

3. Factors $\Delta_{C_2}(\varepsilon \cdot y)$ are pairwise coprime: $\tilde{\alpha}_M$ is birational.

4. Nullstellensatz $\Rightarrow$
$$\Delta_{C_1} \circ \alpha_M(y) = q y^v \prod_{\varepsilon \in G_M} \Delta_{C_2}(\varepsilon \cdot y)^{n_\varepsilon}, \quad q \in \mathbb{C}^*. \tag{*}$$

5. Show all n_ε are equal ($:= N$): substitute $y \mapsto \delta \cdot y$ for $\delta \in G_M$ and use unique factorization.

6. $v \in \mathbb{Z}M \iff \varepsilon^v = 1$ for all $\varepsilon \in G_M$ (conseq of prev item):use Smith Normal Form of M.

7. $N = 1$: Supp $N > 1$. Diff eq (*) + $\tilde{\alpha}_M$ birat. $\Rightarrow J(\Delta_{C_1})(y) = 0$ on $(\Delta_{C_1} = 0)$, Contr! (Δ_{C_1} irred.)

8. $q = 1$: use content 1 arguments (i.e. both sides are integer poly and cont 1). Reduce to case of $M = diag(1, \ldots, p, \ldots, 1)$, p prime and use Field extension + Arithmetic arguments.

□

Corollary 4.8

- *How to eliminate* α_M*? Replace:* $y \to \alpha_{Adj(M)}(y) \Rightarrow$ *obtained* $\Delta_{C_1}(y_1^g, \ldots, y_n^g)$ ($g = \det M$).
- *Replace* B *by* B' *given by reduced basis of* $\mathbb{Z}B$ *(LLL-algorithm):*

$$\Delta_B(y) \;=\; y^v \Delta_{B'}(\alpha_{M^{-1}}(y)) \;,$$

Advantage: coeff of $\psi_{B'}$ *are smaller* $\Rightarrow$ *obtain* $\Delta_{B'}$ *via elimination techniques (standard basis of* $\langle y_i f_0 - f_i, f_0 t - 1 : i = 1, \ldots, m\rangle$ *to eliminate variables* u*'s,* t*).*

5 The degree of Δ_C and the computation of local multiplicities in $m = 3$ case

Assume: $C \in \mathbb{Z}^{n\times m}$ regular, non-defective matrix with no *zero* rows. AND: finite number of base points.

Well-known formula:

Proposition 5.1 *(Intersection formula, [5])*

$$d_C^2 = \underbrace{deg(\psi_C)}_{=1!!!}\; deg(S_C) + \sum_{\mathcal{F}\ \text{basic}} e_{\mathcal{F}}, \tag{10}$$

where $e_{\mathcal{F}}$ *denotes the Hilbert-Samuel multiplicity of* $p_{\mathcal{F}}$ *[7, 8, 9].*

Problem: Computing $e_{\mathcal{F}}$ is hard.

Some steps forward:

- Probabilistic algorithm (reduce to local complete intersections)
- Monomial case (ex. all mxl minors $\neq 0$) $\to$ combinatorial algorithm [11].

Reduction: $m = 3$. Easy generalization for any m.

Definition 5.2 $p = p_{\mathcal{F}} \in \mathcal{Z}$*, pick Noeth. local ring* $A_p := \mathcal{O}_{\mathbb{P}^2,p}$ *and the localized base point locus ideal* $I_p := \langle f_0, f_1, f_2, f_3\rangle A_p$*.*

Define Samuel function A_p *w.r.t.* I_p *as:*

$$\chi_{A_p}^{I_p}(r) = l(A_p/I_p^{r+1}) = \dim_{\mathbb{C}}(A_p/I_p^{r+1}) \quad \textit{for all } r \in \mathbb{N}\;,$$

($l(\_) =$ *length as* A_p*-module).*

Fact: This function is *pseudo-polynomial*, i.e. $\exists\, PS_{A_p}^{I_p}(X)$ in $\mathbb{Q}[X]$ (which takes integer values over $\mathbb{Z}$) such that we have $PS_{A_p}^{I_p}(r) = \chi_{A_p}^{I_p}(r)$ for $r \gg 0$.

Remark 5.3 $\deg PS_{A_p}^{I_p}(X) = 2$ *and Lead Coeff* $= e/2!$ *with* $e \in \mathbb{N}_0$*.*

Definition 5.4 $e_{\mathcal{F}} := e$ *(*$= 2! \cdot LC(PS_{A_p}^{I_p}(X))$*).*

5.1 Probabilistic approach

Proposition 5.5 *If $p = p_{\mathcal{F}}$ determines a loc. complete intersection (i.e. I_P admits 2 generators after picking an affine patch), then $e_{\mathcal{F}} = \dim A_p/I_p$.*

(Algorithm: pick standard basis of I_p w.r.t. local order $\prec$. And count # of monomials not *in $in_{\prec}(I_p)$).*

Recall: $I_p = \langle f_0, \ldots, f_3 \rangle A_p$

- Pick 2 generic $\mathbb{C}$-lin. comb of the 4 generators:

$$J_p := < v_0^0 f_0 + v_1^0 f_1 + v_2^0 f_2 + v_3^0 f_3 \ , \ v_0^1 f_0 + v_1^1 f_1 + v_2^1 f_2 + v_3^1 f_3 > ,$$

- J_p is generically a complete intersection inside I_p and a *reduction ideal* of I_p (i.e. same Hilb-Samuel fnc.)
- $\Rightarrow e_{\mathcal{F}} := \dim_{\mathbb{C}}(A_p/J_p)$ with probability 1.

Corollary 5.6 $e_{\mathcal{F}} \geq \dim_{\mathbb{C}} (A_p/I_p)$ *(since $J_p \subset I_p$), so:*

$$deg(\Delta_C) \ \leq \ d_C^2 - \sum_{\mathcal{F} \text{ basic}} \dim_{\mathbb{C}}(A_{p_{\mathcal{F}}}/I_{p_{\mathcal{F}}}).$$

5.2 Monomial case

Suppose $p = (1:0:0)$ is a base point (after translation) and that I_p is monomial.

Algorithm 5.7 *Computation of Hilbert-Samuel Multiplicities for the monomial case and $m = 3$.*

- *Set $x_0 = 1$ and let $\tilde{I}_p$ be the specialization of the ideal I_p.*
- *Compute the convex hull $\mathcal{C}$ of the exponents of the bivariate monomials in $\tilde{I}_p$.*
- *Then: $e_p = 2!{\cdot}Vol(\mathbb{N}_0^2 \smallsetminus \mathcal{C})$ equals the normalized volume of the complement $\mathcal{K}$ of $\mathcal{C}$ in the first orthant.*

Example 5.8 I_p monomial AND complete intersection: $I_p = \langle x_1^{m_1}, x_2^{m_2} \rangle$, so $e_p = m_1 m_2$, as asserted in both situations.

Remark 5.9 *Algorithm DOES NOT work for general ideals: $e(in_{\prec}(I_0)) \neq e(I_0)$. Moreover ([11]):*

$$e(I_0) \leq e(in_{\prec}(I_0)) \leq 2!\, e(I_0).$$

Comment 5.10

- *p is zero of* exactly 2 *rows $\Rightarrow$ monomial case.*
- *Some cases: reduce to monomial case via change of coordinates*
- *This doesn't solve the whole problem ($\exists$ examples where no change of coordinates works)*
- *Base points can be really nasty $\rightarrow$ no clear computational method for general case. Explicit computations in the paper.*

References

[1] R. Curran. *Toric ideals and discriminants in codimensions greater than two*. Doctoral dissertation. University of Massachusetts, Amherst, 2005.

[2] D. Delfino, A. Taylor, W. V. Vasconcelos, N. Weininger and R.H. Villarreal. *Monomials ideals and the computation of multiplicities*, in: Commutative Ring Theory and Applications (M. Fontana, S.-E. Kabbaj and S. Wiegandl, Eds.), Lectures Notes in Pure and Applied Math. 231, Marcel Dekker, New York, 2002, 87-107.

[3] A. Dickenstein, E. M. Feichtner and B. Sturmfels. *Tropical Discriminants*. math.AG/0510126.

[4] D. Eisenbud and B. Sturmfels, *Binomial ideals*. Duke Math. J. **84** (1996), no. 1, 1–45.

[5] W. Fulton. *Intersection Theory*. Springer-Verlag, New York, 1984.

[6] I. M. Gel'fand, M. Kapranov and A. V. Zelevinsky . *Discriminants, Resultants and Multidimensional Determinants*. Birkhauser, Boston–Basel–Berlin, 1994.

[7] H. Matsumura. *Commutative Ring Theory*. Cambridge studies in advanced mathematics, 8. Cambridge University Press, 1989.

[8] W. Bruns and J. Herzog. *Cohen-Macaulay rings*. Cambridge studies in advanced mathematics, 39. Cambridge University Press, 1993.

[9] D. Cox. *What is the multiplicity of a base point?*. Slides from the expository talk given at XIV CLA, La Falda, Argentina, 2001. Available at http://www.amherst.edu/˜dacox/.

[10] M. Kapranov. *A characterization of A-discriminantal hypersurfaces in terms of the logarithmic Gauss map*. Mathematische Annalen, 290, 1991, 277–285.

[11] D. Delfino, A. Taylor, W. V. Vasconcelos, N. Weininger and R.H. Villarreal. *Monomials ideals and the computation of multiplicities*. Commutative Ring Theory and Applications (M. Fontana, S.-E. Kabbaj and S. Wiegandl, Eds.), Lectures Notes in Pure and Applied Math. 231, Marcel Dekker, New York, 2002, 87-107.