Fields Retrospective Workshop: Tropical Geometry of genus 2 curves. Fields 2018

Joint with Hannah Markwig (arXiv: 1801.00378)

Suppl. Material: people.math.osu.edu /cueto.5/ tropicalGeometryGenusTwoCurves

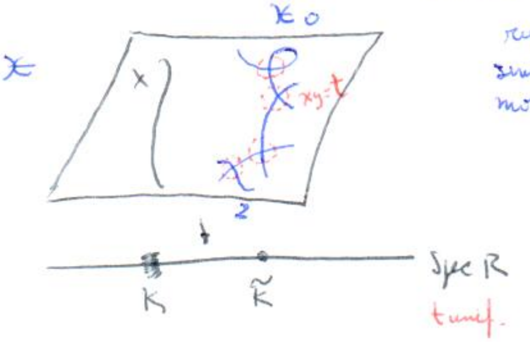
Guiding Q: Given $X/\mathbb{C}$ sm projective & a family over Δ^* = punctured disc with fibers iso to X, can we compute its limit at 0 (the topology at) using trop geometry? TODAY: A is yes for genus 2.

[Baker, Bolognese, Madeline Brandt, Lynn Chua: "From Curves To Trop Jacobians & Back" (Field volume)]

§0: Setup

X/K sm. proj curve.

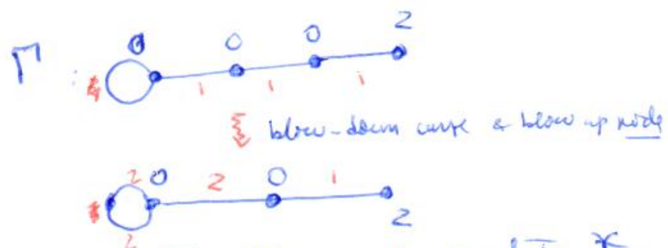
$K=\overline{K}$ (complete) non-Arch valued field, valuation nontrivial, R = valuation ring $\supset m$ max'l ideal, $\tilde{K}=R/m$ res field $\boxed{(\mathrm{char}\,\tilde{K}\neq 2,3)}$



regular semistable model of X/R

$\mathfrak{X}$ = reg surface, proper & flat $/R$

$\mathfrak{X}_0$ = reduced curve $/\tilde{K}$ with at worst nodal double pts



genus $X=3$

$X \longmapsto \mathfrak{X}_0 \rightsquigarrow \boxed{\text{Skeleton of } X =: \Gamma}$ = weighted metric graph dual to $\mathfrak{X}_0$

(Abstract Tropicalization of X)

$\text{genus}(\Gamma)=b_1(\Gamma)+\sum_{v\in V(\Gamma)} g(v)$. If X has a divisor $\rightsquigarrow$ add markings to Γ (legs w/ ∞ length) to vertices (divisor supp & mult)

How? Semistable reduction: Construct $\mathfrak{X}$ by blow-ups, ramified covers $t\mapsto t^{1/n}$, normalizations

[BBC] $\rightsquigarrow$ compute Jacobians via Tropicalization / period matrices projection

AIM Predict Γ by finding nice models for X & tropicalizing in coordinates.

Tropical strategy? (COORD. DEPENDENT!)

- $X \xhookrightarrow[cl]{i} (K^*)^n$ [$\neq Y_\Sigma$] $\rightsquigarrow \mathrm{Trop}(X,i)=\text{closure}\{(-\mathrm{val}(x_1),\dots,-\mathrm{val}(x_n)) : \underline{x}\in X(K)\}\subseteq \mathbb{R}^n$
  irred curve; $(K^*)^n = \mathrm{Spec}\, K[x_1^{\pm},\dots,x_n^{\pm}]$
  is a rat'l (weighted polyhedral complex, balanced at vertices ($\rightsquigarrow$ embedded weighted metric graph w/ legs)
- $\mathrm{Trop}(X,i)=\mathrm{trop}(X^{an})$
- Baker-Payne-Rabinoff ['14] • $\Gamma \xrightarrow{\mathrm{trop}} \mathrm{Trop}(X,i)$ is PL w/ $\mathbb{Z}$-slopes (stretching factors)
  - $\exists$ sufficient conditions (all trop smooth + trop mult = 1) to see an isometric copy of Γ in $\mathrm{Trop}(X,i)$
  [for whole $\mathrm{Trop}(X,i)$, we need markings on Γ]
  - $\exists$ re-embedding (non-effective) $X \xhookrightarrow{i} (K^*)^n$, $j: X \to (K^*)^m$ where (trop, j) gives the isometry

[CM] Find good embeddings / repair bad ones without knowing Γ.

§1 Motivation: genus 1

- $E = V(f) \subseteq \mathbb{P}^2$ sm cubic $/K$ Iso classes of $E \longleftrightarrow j_E = j$-inv of K

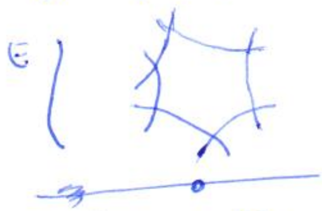
conductor discr. formula

① = deg 0 rat'l function in coeffs of $f = \dfrac{A}{\mathrm{Disc}(f)}$

② = $\begin{matrix} E \\ \downarrow 2:1 \\ \mathbb{P}^1 \end{matrix}$ = deg 0 rat'l funct in 4 branched pts.

- 2 types of graphs

(ENRICA'S TALK)



BAD reduction

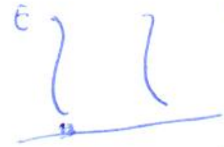
$\hookrightarrow \Gamma =$ 

[length $= -\mathrm{val}(j_E) > 0$]

(trop j-invariant)

[KMM, BPR]

or

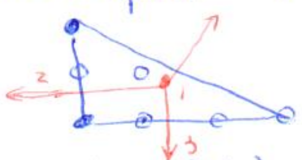


GOOD red.

$\hookrightarrow \Gamma = \bullet 1$ $(-\mathrm{val}(j_E) < 0)$

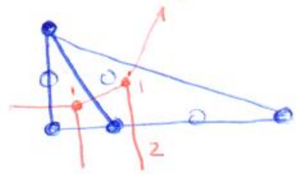
- Best f: Weierstrass form $f = y^2 - (x^3 + ax + b)$ $a, b \in K$

Trop(f) is a tree ($\Rightarrow$ No genus!)



$(2\,\mathrm{val}\,b \leq 3\,\mathrm{val}\,a)$

or



$(2\,\mathrm{val}\,b > 3\,\mathrm{val}\,a)$

Q: Where is Γ?

- Chan-Sturmfels '12: Can re-embed in $(K^\times)^2$ in honeycomb form (solve deg 6 eqn.)

$\subseteq \mathbb{R}^2$

$\hookrightarrow$ Z-length $= -\mathrm{val}(j_E)$

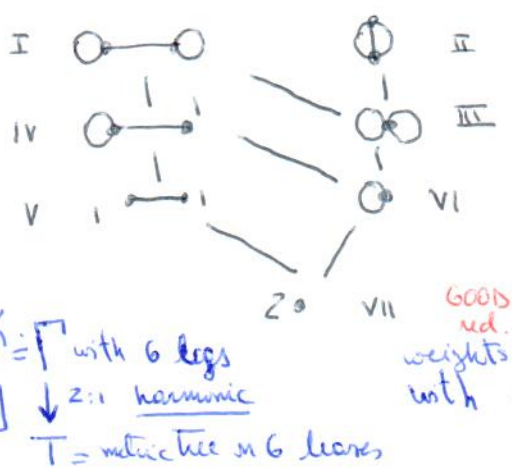
- C-Markwig '16: Combinatorial way of linearly repairing short loops / unfolding (bounded) double edges in dim ≤ 4.

§2 Genus 2 char $\neq 2, 3$

Key: sm genus 2 are hyperelliptic $\begin{matrix} X \\ \downarrow 2:1 \\ \mathbb{P}^1 \end{matrix}$ branched at 6 pts

$\rightsquigarrow$ hyperelliptic eqn: $f = y^2 - u \prod_{i=1}^{6} (x - \alpha_i)$ $\alpha_i \neq \alpha_j$ (in $K^\times$) $u \in K^\times$

- Poset of skeleta: (genus 2)



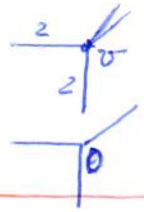
- Tropical analog: $\tilde{\Gamma} = \Gamma$ with 6 legs $\downarrow$ 2:1 harmonic T = metric tree on 6 leaves [Baker-Norine, Chan]

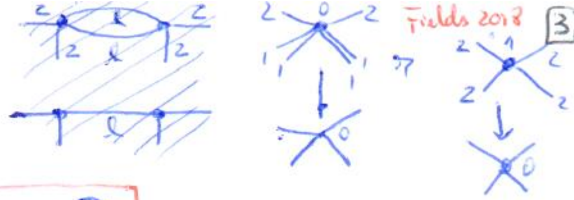
weights = stretching factors = 1 or 2.
with simple branching at its 6 ends

Trop Riemann-Hurwitz (Local) $2 - 2g(v) = 2\,d_v - \#\{e \ni v : w(e) = 2\}$ $\forall\, v \in V(\Gamma)$

$\hookrightarrow$ local degree: • 2 at vertices & interior of wt 2 edges
• 1 at interior of weight 1 edges

$R = \sum_{q \in X} \underbrace{(v_{(q)} - 1)}_{\text{ramif index}}\, q$ ram divisor. RH: $2g - 2 = 2(0-2) + \deg R$

Eg  $\boxed{d_v = 2}$ $\delta_v = 0$ (only option!) [check on any edge in the star of $\mathbf{0}$]



$\boxed{\text{Q1: Can we predict tree \& skeleta from } \alpha_1, \ldots, \alpha_6 \text{?}}$

THM1 [CM] (1) Each cover $\tilde{\Gamma} \xrightarrow{2:1} T$ is uniquely determined by the relative order of $w_i = -val(\alpha_i)$ $i = 1 \ldots 6$ & $d_{ij} = -val(\alpha_i - \alpha_j)$ (if $w_i = w_j$). [BBC]

(2) $\exists$ 7 witness regions (Table) & an algorithm to move any 6-tuple to one of these regions with an explicit $\varphi \in Aut(\mathbb{P}^1)$ (combine separation of pts $z \to z - \ldots$ with inversions $z \to \frac{1}{z}$)

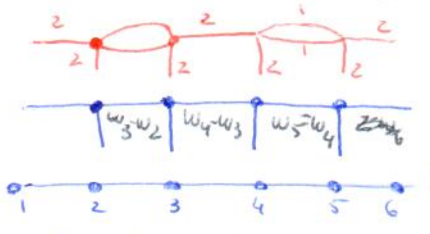
Proof idea: Line w/ 6 markings $\leftrightarrow$ pt in $Gr_0(2,6)/(K^\times)^6 = \begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1 & \cdots & \alpha_6 \end{pmatrix} \rightsquigarrow \Phi(\alpha) = \left((\alpha_i - \alpha_j) \right)_{i,j} / (K$

- Trop Plücker relns: $\max\{x_{ij} + x_{kl}, x_{ik} + x_{jl}, x_{il} + x_{jk}\}$ $\rightsquigarrow$ metric tree & weights from linear algebra. Leaves $\rightsquigarrow$ 6 legs with ∞ length.

$\boxed{\text{Trop } \Phi(\alpha) = (\underbrace{-val(\alpha_i - \alpha_j)}_{= x_{ij}})_{i,j}}$ $\frac{1}{2}$ distance $\;/\mathbb{R}^6$ lattice of index

- Trop RH recovers $\tilde{\Gamma}$ & $\tilde{\Gamma} \xrightarrow{2:1} T$.

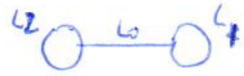
Eg (I): $w_1 < \cdots < w_6$ ($val\,\alpha_1 > \cdots > val\,\alpha_6$) $\rightsquigarrow \frac{d}{2} = (\overset{12}{w_2}, \overset{13}{w_3}, \overset{14}{w_4}, \overset{15}{w_5}, \overset{16}{w_6}, \overset{23}{w_3}, w_4, w_5, w_6, \overset{26}{w_4}, \overset{34}{w_5}, \overset{35}{\ldots}$ $w_5, w_6, w_6)$.



$L_0 = \frac{1}{2}(w_4 - w_3)$

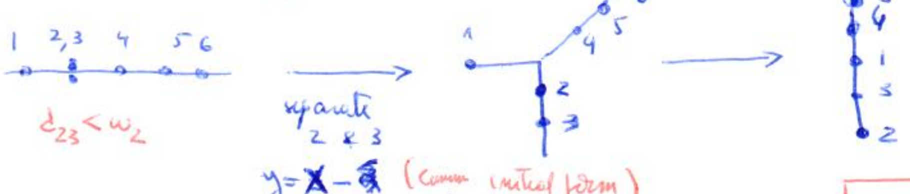
$L_1 = 2(w_5 - w_4)$

$L_2 = 2(w_3 - w_2)$



In general: • Know how to do local lift at star trees w/ ≤ 6 legs & $g_v = 0, 1, 2$.
• Glue local lifts together.

Note: Specializations are compatible with graphs & trees

Algorithm: 

$y = x - \xi$ (common initial form)
$val\,\xi = val\,\alpha_2$

$\boxed{f: y^2 = x \prod_{i=2}^{5} (x - \alpha_i)}$

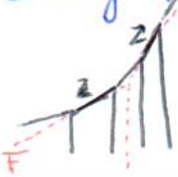
$\boxed{\text{Q2: What happens with } \Gamma \xrightarrow{(\text{Trop}, f)} \text{Trop } f \subseteq \mathbb{R}^2}$ TREE!

Assume some 7 regions & $\alpha_6 = 0$, $\alpha_1 = \infty$

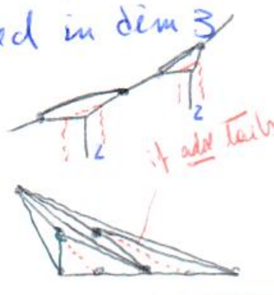
THM 2 [CM] On the 7 witness regions, naive Trop can be repaired in dim 3 via $I = \langle f, z - y + \sqrt{\alpha_5 \alpha_4 \alpha_3}\, x - \sqrt{\alpha_5}\, x^2 \rangle$

[$z - h$ with trop $h = F$]
$F = \max\{y, A + x, B + 2x\}$

Eg (I) 



Alg. Jn. of $in_w f$ hit at "where to repair"



Proof remarks: • All except Type II can be seen in ~~XZ~~ projection.

• In Type II, need all 3 proj (elimination of x var is "HARD")

• Gröbner cones of the coords of relevant ~~coefficients~~ of 3 projections (in $\mathbb{R}^4_{(-\mathrm{val}\,\alpha_i)_{i=2}^5}$) subdivide each region $\Rightarrow$ get all $\neq$ cont types of 3 projections

• Reverse engineering (+ refine se-ent to get control of cancellations of expected initial terms of the coefficients) to get cont types in $\mathbb{R}^3$.

HANDOUT: can you ~~guess~~ types / repairing loci from the Newton subdivisions

→ $\boxed{\text{Q3. Analogs of trop j-invariants?}}$ Algebraically: 3 IGUSA invariants j_1, j_2, j_3 $\Leftrightarrow$ 3 h. pts + 0, 1, ∞.

• They are rat'l functions in $\mathbb{Q}(\alpha, \ldots, \alpha_6)$

• Mestre: Given $j_1, j_2, j_3 \in K$ ~> build X/L with $j_i(X) = j_i$ in some field extn. $L|K$.

Write: $\Delta_{ij} := (\alpha_i - \alpha_j)^2 \quad 1 \le i, j \le 6 \quad i \ne j$ $\qquad C: y^2 = u \prod_{i=1}^{6}(x - \alpha_i)$

Def: $j_1(X) = \dfrac{A^5}{D}, \quad j_2(X) = \dfrac{A^3 B}{D}, \quad j_3(X) = \dfrac{A^2 C}{D}$ where

$A = u^2 \sum_{(ij)(kl)(m,n)} \Delta_{ij}\Delta_{kl}\Delta_{mn}$ 15 tripartitions of [6] $\quad$ deg = 6 # terms = 141

$B = u^4 \sum_{ijk|lmn} \Delta_{ij}\Delta_{kl}\Delta_{mn}\Delta_{lm}\Delta_{mn}\Delta_{nl}$ $\quad$ 10 (3,3) partitions of [6] $\quad$ deg = 12 #terms = 1531

$C = u^6 \sum_{\substack{ijk|lmn \\ (il, jm, kn)}}$ ———— $\Delta_{il}\Delta_{jm}\Delta_{kn}$ $\quad$ (60 terms) deg = 18 #terms = 8531

$D = \mathrm{Disc}^2 = u^{10} \prod_{1 \le i < j \le 6} \Delta_{ij}$ $\quad$ deg = 30 #terms = 56183

Def: $j_i^{trop}(\Gamma) := -\mathrm{val}(j_i(X))^{i=1,2,3}$ for a generic X with abstract trop realization Γ

Tropical Igusa invariants • Trop Igusa inv. are continuous PL functions on M_2^{trop} & linear on each cell.

THM 3 [CM] (I) $j_1^{trop}{}_{\text{o-o}} = L_1 + 12L_0 + L_2$, $j_2^{trop}{}_{\text{o-o}} = j_3^{trop}{}_{\text{o-o}} = L_1 + 8L_0 + L_2$

(II) $j_1^{trop}{}_{\oplus} = j_2^{trop}{}_{\oplus} = j_3^{trop}{}_{\oplus} = L_1 + L_0 + L_2$ $\quad$ symmetries of the graph are respected

Rmk: • Formulas specialize to lower dim'l cells

• Genericity: expected valn of A, B, C = wt $\mathrm{in}_{\underline{w}} A$, ... for $\underline{w}$ giving each Trop ...

~~For (II): replace w_4 by $\alpha_{34} = -\mathrm{val}(\alpha_3 - \alpha_4)$ $[\alpha_4 = \alpha_3 - \alpha_{34}]$~~ | ~~Eg (I) $\mathrm{in}_{\underline{w}} A = 6\alpha_4^2\alpha_5^2\alpha_6^2$~~, in $\mathrm{in}_{\underline{w}}\Delta_{ij} = \prod_{i<j}(\mathrm{in}_{\underline{w}}(\alpha_i - \alpha_j))^2$

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Eg (I): $\mathrm{in}_{\underline{w}} A = 6\, \alpha_4^2 \alpha_5^2 \alpha_6^2$
[M2]
$\mathrm{in}_{\underline{w}} B = 4\, \alpha_3^2 \alpha_4^2 \alpha_5^4 \alpha_6^4$
$\mathrm{in}_{\underline{w}} C = 8\, \alpha_3^2 \alpha_4^4 \alpha_5^6 \alpha_6^6$
$\mathrm{in}_{w} D = \alpha_2^2 \alpha_3^4 \alpha_4^6 \alpha_5^8 \alpha_6^{10}$
[MAX weight]

(III) $\mathrm{in}_{w'} A' = 8\, \alpha_3^2 \alpha_5^2 \alpha_6^2$
$\mathrm{in}_{w'} B' = 4\, \alpha_3^4 \alpha_5^4 \alpha_6^4$
$\mathrm{in}_{w'} C' = 8\, \alpha_3^6 \alpha_5^6 \alpha_6^6$
$\mathrm{in}_{w'} D' = \alpha_2^2 \alpha_3^8 \alpha_{34}^2 \alpha_5^8 \alpha_6^{10}$
Replace $\alpha_4 = \alpha_3 - \underline{\alpha_{34}}$
$w_4 \rightsquigarrow d_{34} = -\mathrm{val}(\alpha_{34})$ } $w \rightsquigarrow w'$

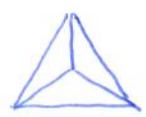
Lower cells: Initials are NOT monomials, so we want to ensure NO cancellations to control exp val $\overset{?}{=}$ val.

Rmk: (I). $\mathrm{in}_w A\ \mathrm{in}_w B = 3\, \mathrm{in}_w C$ (II) $\mathrm{in}_{w'} A'\ \mathrm{in}_{w'} B' = 4\, \mathrm{in}_{w'} C'$

$\rightsquigarrow$ $\boxed{j_4 = j_2 - 4 j_3 \quad \text{new Invariant}}$ $\rightsquigarrow j_4^{trop} = -\mathrm{val}(j_4(X))$ for generic X given Trop X.

THM 3 (addendum) j_4^{trop} is PL & cont on M_2^{trop} with
(I) $j_4^{trop}{}_{\text{o-o}} = j_3^{trop}{}_{\text{o-o}}$ (II) $j_4^{trop}{}_{\theta} = L_0 + L_1 + L_2 - \min\{L_0, L_1, L_2\}$

Why? $Q_4 = AB - C$ & use Gröbner fan to subdivide θ cone into 3 pieces. (leading term changes!)



Char 3 Issue: Only for A & Types (I), (IV) & (V) $A = 4A_4 + \underline{6A_6 + 12A_{12} + 120A_{120}}$
all with coeff ± 1 & disjoint support.

$\mathrm{in}_{\text{o-o}} 4A_4 = -4\, \alpha_3 \alpha_4 \alpha_5^2 \alpha_6^2$
$\mathrm{in}_{\text{o-o}} Q = 6\, \alpha_4^2 \alpha_5^2 \alpha_6^2$ } Ties only if $w_3 = w_4 - 1$.

- If Q wins $\rightsquigarrow j_4^{trop}{}_{\text{o-o}} = j_3^{trop}{}_{\text{o-o}}$,
- If A_4 — $\rightsquigarrow j_1^{trop} = L_1 + 2L_0 + L_2 = j_2^{trop}$
$j_3^{trop} = L_1 + 4L_0 + L_2$
- If ties & cancellation $\rightsquigarrow$???

Tropical Geometry of Genus Two Curves

Maria Angelica Cueto and Hannah Markwig

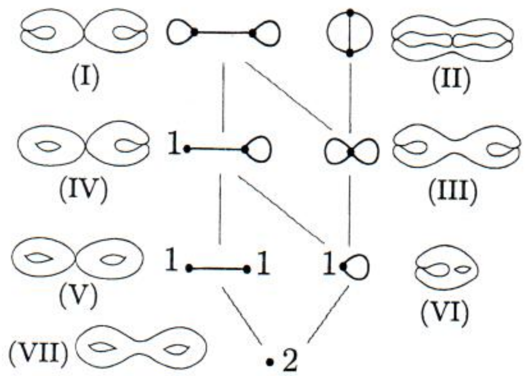


Fig. 1: Poset of weighted dual graphs of genus 2 curves.

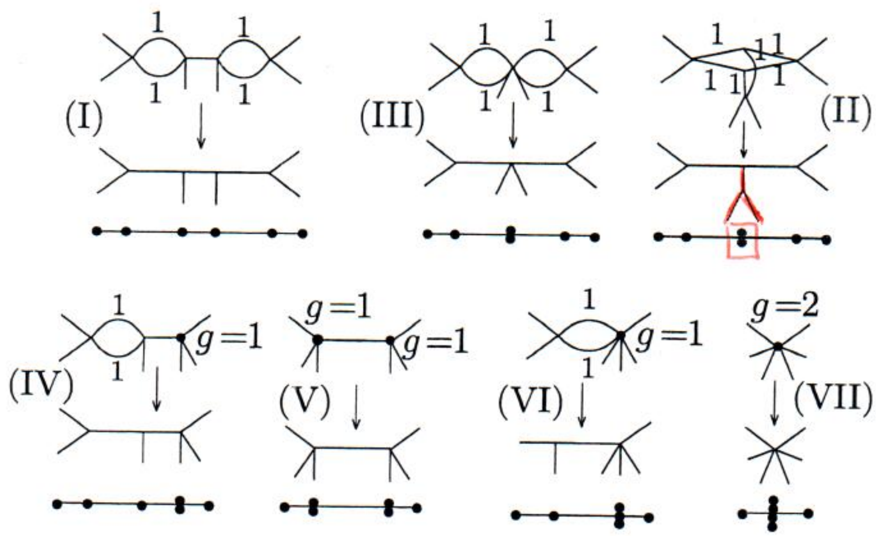


Fig. 2: Harmonic 2–to–1 covers of all possible metric trees on 6 leaves by abstract genus 2 trop. curves with 6 leaves, and the ordering of the valuations of the 6 branch points. All weights of the edges in the source curve equal 2 or 1.

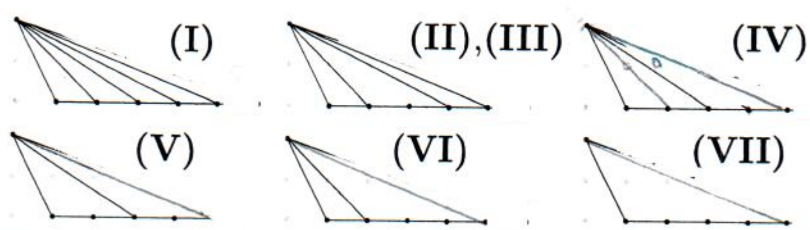


Fig. 3: Newton subdivisions induced by the hyperelliptic equation

$$y^2 = \cancel{x}\prod_{i=1}^{6}(x-\alpha_i).$$

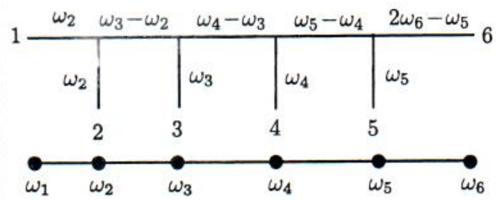
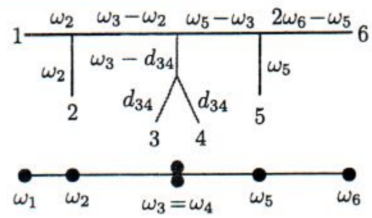
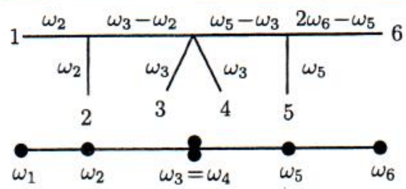
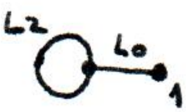
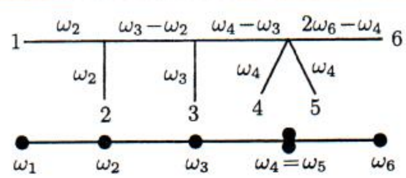
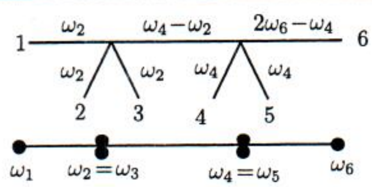
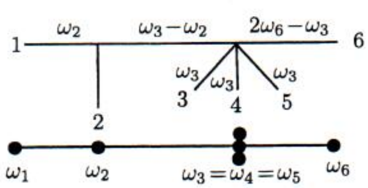
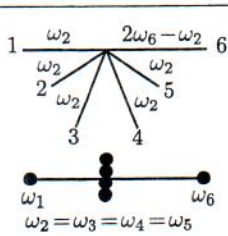
Type	Cover with lengths on $M_{0,6}^{\mathrm{trop}}$	Defining conditions	Lengths on M_2^{trop}
 (I)		$\omega_1<\omega_2<\omega_3<\omega_4<\omega_5<\omega_6$	$L_0=(\omega_4-\omega_3)/2$ $L_1=2(\omega_5-\omega_4)$ $L_2=2(\omega_3-\omega_2)$
 (II)		$\omega_1<\omega_2<\omega_3<\omega_5<\omega_6$ $\omega_3=\omega_4$ $\mathrm{in}(\alpha_3)=\mathrm{in}(\alpha_4)$	$L_0=2(\omega_3-d_{34})$ $L_1=2(\omega_5-\omega_3)$ $L_2=2(\omega_3-\omega_2)$
 (III)		$\omega_1<\omega_2<\omega_4<\omega_5<\omega_6$ $\omega_3=\omega_4$ $\mathrm{in}(\alpha_3)\neq\mathrm{in}(\alpha_4)$	$L_0=0$ $L_1=2(\omega_5-\omega_3)$ $L_2=2(\omega_3-\omega_2)$
 (IV)		$\omega_1<\omega_2<\omega_3<\omega_4<\omega_6$ $\omega_4=\omega_5$ $\mathrm{in}(\alpha_4)\neq\mathrm{in}(\alpha_5)$	$L_0=(\omega_4-\omega_3)/2$ $L_1=0$ $L_2=2(\omega_3-\omega_2)$
 (V)		$\omega_1<\omega_2<\omega_4<\omega_6$ $\omega_2=\omega_3$, $\omega_4=\omega_5$ $\mathrm{in}(\alpha_2)\neq\mathrm{in}(\alpha_3)$ $\mathrm{in}(\alpha_4)\neq\mathrm{in}(\alpha_5)$	$L_0=(\omega_4-\omega_2)/2$ $L_1=0$ $L_2=0$
 (VI)		$\omega_1<\omega_2<\omega_3<\omega_6$ $\omega_3=\omega_4=\omega_5$ $\mathrm{in}(\alpha_3)\neq\mathrm{in}(\alpha_4)$ $\mathrm{in}(\alpha_3)\neq\mathrm{in}(\alpha_5)$ $\mathrm{in}(\alpha_4)\neq\mathrm{in}(\alpha_5)$	$L_0=0$ $L_1=0$ $L_2=2(\omega_3-\omega_2)$
 (VII)		$\omega_1<\omega_2<\omega_6$ $\omega_2=\omega_3=\omega_4=\omega_5$ $\mathrm{in}(\alpha_i)\neq\mathrm{in}(\alpha_j)$ for $1<i<j<6$	$L_0=0$ $L_1=0$ $L_2=0$

Table: Comb. types, valuations and lengths for $\omega_i=-\,\mathrm{val}(\alpha_i)$, $d_{34}=-\,\mathrm{val}(\alpha_3-\alpha_4)$.