FRANKFURT ROADMAP

L1
Valuation Rings
Valuations on fields
Ordered ab. gps
EXAMPLES

L2
Chevalley's Thm

L2
Rank of a valuation
Rank vs dim.
Fundamental Ineq

L2
Prime Valuations ⇝ 1st & 2nd kind centers
divisorial

L2
Main Thm for 1st kind
Rank : Normal V ⇒ |S| = 1 [(*)]

L3
Main Thm for 2nd kind

L2
COMM ALG results
{ Noether N/M
{ Going Up/Down
Primary Decomp
L3

L1
- p-adic valuations
- $\mathfrak{P}_0$ min'l prime in [illegible]

} discrete rank 1

- $\mathbb{C}\{\{t\}\}$. rank 1 not discrete
- $v = (v_1, \ldots, v_r)$ higher rank (w/ lex order)
- Anticipate : order of vanishing

L1
Val ⇒ Normal

L2
Integral closure $= \bigcap_v \mathcal{O}_v$

For valuation rings
Noetherian ⟺ rank=1, discrete
L1S

L3
* Order of vanishing
* Weil Divisors
* Projective Varieties
* Chains of subvarieties (L1)

Valuative Criteria for projective vars.

L3
Conjugations under Gal(L|K) & valuation extensions

L2S L3
Extending valuations over alg extensions

L1
R normal domain, Noetherian, 1-dim'l, local
⇓
R is DVR rank 1
(*)

Frankfurt School (07/04–08/16)

References: • "Valued fields" (Engler–Prestel)
• "Commutative Algebra vol II" (Zariski–Samuel)

Lecture 1 : Introduction to Valuations

Def: Fix K a field, $\mathcal{O} \subset K$ subring is a valuation ring of K if for all $x \in K^\times$, either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$.

Remark: $\mathcal{O}$ is a local ring with ! maximal ideal $\mathfrak{m} = \mathcal{O} \setminus \mathcal{O}^\times$ ($\mathcal{O}^\times$: units in $\mathcal{O}$)

Pf/ • $x \in \mathfrak{m}$, $a \in \mathcal{O} \Rightarrow ax \in \mathfrak{m}$ ✓
• $0 \in \mathfrak{m}$ ✓
• $x, y \in \mathfrak{m} \overset{?}{\Rightarrow} x+y \in \mathfrak{m}$. Enough to show: $x \in \mathfrak{m} \Rightarrow 1-x \notin \mathfrak{m}$ $\left[\frac{x}{x+y} + \frac{y}{x+y} = 1\right]$

If $1-x \in \mathfrak{m} \Rightarrow (1-x)^{-1} \notin \mathcal{O}$. Then: • $\frac{1-x}{x} = x^{-1} - 1 \notin \mathcal{O}$ (b/c $x^{-1} \notin \mathcal{O}$)
• $\frac{x}{1-x} \notin \mathcal{O}$ [if in $\mathcal{O}$, then $\frac{1-x+1}{1-x} = 1 + \frac{1}{1-x} \in \mathcal{O}$
$\Rightarrow (1-x)^{-1} \in \mathcal{O}$ Contr!]

$\Rightarrow \mathcal{O}$ not valn ring. Contr!

Lemma: Valuation rings are normal, ie integrally closed (in their quotient field)

Pf/ $x \in \overline{\mathcal{O}} \setminus \mathcal{O} \Rightarrow x^{-1} \in \mathcal{O}$ & $x^n + \sum_{i=0}^{n-1} a_i x^i = 0$ for some $a_i \in \mathcal{O}$

Multiply by $x^{-n} \in \mathcal{O} \Rightarrow 0 = x^{-n}\left(x^n + \sum_{i=0}^{n-1} a_i x^i\right) = 1 + \sum_{i=1}^{n} a_{n-i} x^{-i}$

so $1 = \sum_{i=1}^{n} (-a_{n-i}) x^{-i} = x^{-1} \underbrace{\left(\sum_{i=0}^{n-1} (-a_{n-i-1}) x^{-i}\right)}_{\in \mathcal{O} \ \& \ = x}$ Contr!

Examples: ① $K = \mathbb{Q}$, p prime; $\mathcal{O} = \{\frac{a}{b} : (a,b)=1, p \nmid b\}$, $\mathfrak{m} = \{p\frac{a}{b} : (a,b)=1, p \nmid b\}$

② $K = \mathbb{C}(x)$, $\mathcal{O} = \{\frac{a}{b} : (a,b)=1, x \nmid b\}$, $\mathfrak{m} = \{x\frac{a}{b} : (a,b)=1, x \nmid b\}$, $\mathcal{O}/\mathfrak{m} \simeq \mathbb{C}$
(function that are regular at 0) (regular functions vanishing at 0) $\frac{a(x)}{b(x)} \longrightarrow \frac{a(0)}{b(0)}$

Note: In ① $\mathcal{O} = \mathbb{Z}_{(p)}$ (localization away from p)
$\mathfrak{m} = p\mathbb{Z}_{(p)}$ $\Big\}$ $\mathcal{O}/\mathfrak{m} = \frac{\mathbb{Z}_{(p)}}{p\mathbb{Z}_{(p)}} \simeq \left(\mathbb{Z}/p\mathbb{Z}\right)_{(\bar{p})} \cong \mathbb{F}_p$ ($\bar{p}=0$)
($\mathbb{Z}_p = \mathbb{F}_p$!)

① Order of vanishing? $v_p : K \longrightarrow \mathbb{Z}$ $\quad \frac{a}{b} = p^s \frac{a'}{b'} \longmapsto s$ $\quad (p \nmid a', p \nmid b')$
② $v : K \longrightarrow \mathbb{Z}$ "order of vanishing" is similar $\quad \mathcal{O} = \{v_p(x) \geq 0\}$, $\mathfrak{m} = \{v_p(x) > 0\}$
$\mathbb{Z}$ = ab gp w/ order <

Def: $(\Gamma, +, 0)$ abelian group is ordered if $\leq$ is an order on Γ satisfying
$\delta \leq \gamma \Rightarrow \delta + \lambda \leq \gamma + \lambda \ \forall \lambda \in \Gamma$ (ie addition is monotone)

Examples: $\Gamma' = \mathbb{Z}$, $\mathbb{Z}^n_{lex}$ ($\underline{x} \leq_{lex} \underline{y}$ if first nonzero entry in $\underline{y}-\underline{x}$ is ≥ 0) ^{n} ②

$\rightsquigarrow \Gamma = \mathbb{R}^n_{lex}$

Thm (Hahn) These are essentially all, ie given Γ, we can find a set I and an inclusion

$\Gamma \overset{i}{\hookrightarrow} \mathbb{R}^{(I)}_{lex}$ order preserving homomorphism ($\rightsquigarrow$ injective!)

Def: $v: K^\times \twoheadrightarrow \Gamma$ is a valuation on K if Γ is an ordered abelian gp and

v satisfies: $\begin{cases} v(ab) = v(a)+v(b) & \forall a,b \\ v(a+b) \geq \min\{v(a), v(b)\} & \forall a,b, a+b \neq 0 \end{cases}$ ($\Rightarrow$ image $v(=\Gamma)$ is a group, called VALUATION GROUP)

Extend $v: K \twoheadrightarrow \Gamma \cup \{\infty\}$ by $v(0)=\infty$

NOTE $v(1) = v(1.1) = 2v(1) \Longrightarrow v(1)=0$

- $\mathcal{O}_v = \{x : v(x) \geq 0\}$ is its valuation ring ($x \notin \mathcal{O} \Rightarrow v(x) < 0 \Rightarrow -v(x) = v(x^{-1}) > 0 \Rightarrow x^{-1} \in \mathcal{O}$)
  $\cup$
- $m_v = \{x : v(x) > 0\}$ is the unique maximal ideal of $\mathcal{O}$ ($v(x)=0 \Rightarrow 0 = v(1) = v(xx^{-1}) = v(x^{-1}) \Rightarrow x^{-1} \in \mathcal{O}_v$)

Note: $v(1)=0 \rightsquigarrow v(\mathcal{O}^\times)=0$.

Remark: $v(a) < v(b) \Longrightarrow v(a+b) = v(a)$ (ie min is achieved!)

Pf/ $v(b) = v(-b) \Rightarrow \underline{v(a)} = v(a+b-b) \geq \min\{\underline{v(a+b)}, v(-b)\} \geq \min\{\underline{\min\{v(a), v(b)\}}, v(b)\} = \underline{v(a)}$

$\Rightarrow v(a+b) = v(a)$.

Proposition Let $\mathcal{O} \subseteq K$ be a valuation ring of K. Then, there exists a valuation v on K with $\mathcal{O} = \mathcal{O}_v$.

Pf/ $\Gamma := K^\times/\mathcal{O}^\times$ is an ab gp under multiplication with unit 1 $\rightsquigarrow x\mathcal{O}^\times + y\mathcal{O}^\times := xy\mathcal{O}^\times$

Define order $\leq$: $x\mathcal{O}^\times \leq y\mathcal{O}^\times$ iff $\frac{y}{x} \in \mathcal{O}$ well-defined ✓ [$x\mathcal{O}^\times \geq 1\mathcal{O}^\times \Leftrightarrow x \in \mathcal{O}$ ($x \neq 0$)]

- Total order b/c $\mathcal{O}$ is a valn ring
- $x\mathcal{O}^\times \leq y\mathcal{O}^\times \Rightarrow x\mathcal{O}^\times + z\mathcal{O}^\times = xz\mathcal{O}^\times \leq yz\mathcal{O}^\times = y\mathcal{O}^\times + z\mathcal{O}^\times \quad \forall z \in K^\times$ ✓

$\rightsquigarrow v: K^\times \to \Gamma$ $v(x) = x\mathcal{O}^\times$ is a valn w/ $\mathcal{O}_v = \mathcal{O}$.

- $v(xy) = xy\mathcal{O}^\times = x\mathcal{O}^\times + y\mathcal{O}^\times$ ✓ ; • $v(x+y) = (x+y)\mathcal{O}^\times$ If $\frac{y}{x} \in \mathcal{O} \Rightarrow \frac{x+y}{x} \in \mathcal{O}$
  $\Rightarrow v(x+y) \geq v(x) = \min\{v(x), v(y)\}$

Def: Two valuations v_1, v_2 on K are equivalent if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.

Proposition: $v_i: K \twoheadrightarrow \Gamma_i \cup \{\infty\}$ are equiv valns $\Longleftrightarrow \exists \rho: \Gamma_1 \to \Gamma_2$ order preserving iso st $\rho \circ v_1 = v_2$

Pf/ $\tau_i: K^\times/\mathcal{O}^\times_{v_i} \to \Gamma_i$ $\tau_i(x\mathcal{O}^\times_{v_i}) = v_i(x)$ is order preserving iso $\Rightarrow \rho = \tau_2 \circ \tau_1^{-1}$ works □

EXAMPLES: ⓪ trivial valn on any field $v(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$

① $v_p : \mathbb{Q}^\times \twoheadrightarrow \mathbb{Z}$ p-adic valuation (p prime) $\frac{a}{b} = p^s \frac{a'}{b'}$, $p \nmid a'$, $p \nmid b'$ ⇝ $v_p(\frac{a}{b}) = s$

⇝ p-adic norms on $\mathbb{Q}$ $|x| = p^{-v_p(x)}$ (only non trivial valns on $\mathbb{Q}$)

$\Rightarrow |\ | : K \longrightarrow \mathbb{R}_{\geq 0}$
$$\begin{cases} |x| = 0 \iff x = 0 \\ |xy| = |x|\,|y| \quad \text{(multiplicative)} \\ |x+y| \leq \max\{|x|, |y|\} \quad \text{(ULTRAMETRIC; non-Arch } \Delta\text{-ineq)} \end{cases}$$
$|n| = |\underbrace{1 + \dots + 1}_{n \text{ times}}| \leq 1$

$\mathrm{val} : K^\times \longrightarrow \mathbb{R}$ $\mathrm{val}(\cdot) = -\log|\cdot|$ (LECTURE IV)

⇝ Take completion $\hat{K}$ w.r.t. this topology $|\ |_p : \hat{K} \longrightarrow \mathbb{R}_{\geq 0}$ ($|\lim_\alpha x_\alpha| := \lim_\alpha |x_\alpha|$)
($\hat{K}$ field containing K) (x_α) Cauchy sequence

⇝ $v_p : \hat{\mathbb{Q}} := \mathbb{Q}_p \twoheadrightarrow \mathbb{Z}$ $v_p(x) = -\log_p |x|_p$.

② $f \in k[x]$ irreducible polynomial $\Rightarrow$ f-adic valuation on $k(x)$ $v(\frac{g}{h}) = v(f^s \frac{g_1}{h_1}) = s$, $f \nmid g_1$, $f \nmid h_1$.

⇝ • R = Unique Factorization Domain, $\mathfrak{p}_0$ minimal prime $\neq 0$ $\Rightarrow \mathfrak{p}_0 = (f)$ f irred.
• $\dim_{\text{Krull}} R < \infty$ ⇝ extends to a valn on $K(x)$: ($\mathfrak{p}_0$-adic!)

③ $v_\infty : K(x) \twoheadrightarrow \mathbb{Z}$ degree-valuation $v\left(\sum_{i=m}^{N} a_\alpha x^\alpha\right) = m$ ⇝ $v_\infty\left(\frac{f}{g}\right) = v_\infty(f) - v_\infty(g)$
($v_\infty(fg) = v_\infty(f) + v_\infty(g)$, $v_\infty(f+g) \geq \min\{v_\infty(f), v_\infty(g)\}$)

Note: ② & ③ are the only rank 1 valns on $K(x)$

④ $\mathbb{C}\{\{t\}\} = \left\{ \sum_{\alpha \geq k} a_\alpha t^{\alpha/n} : n \in \mathbb{N}, k \in \mathbb{Z} \right\}$ (Field of Puiseux Series)

val(f) = lowest exponent $\Gamma = \mathbb{Q}$ ⇝ key for Tropical Geometry
[non-discrete rank 1 valn]

⑤ Given $v_i : K^\times \twoheadrightarrow \Gamma_i$. ($i = 1, \dots, r$) valuations on K, we can construct
$\Gamma = \Gamma_1 \times \dots \times \Gamma_r$ ab gp w/ lex-order $\underline{a} = (a_1, \dots, a_r) \leq_{lex} (b_1, \dots, b_r) = \underline{b}$
if 1st non-zero entry in $\underline{b} - \underline{a}$ is $\geq_i 0$ in Γ_i.

$v = (v_1, \dots, v_r) : K^\times \twoheadrightarrow \Gamma$ is a (higher rank) valuation

Notion of rank = LECTURE 2.

Proposition: $(R, \mathfrak{m})$ local Noetherian domain, integrally closed, $\dim R = 1$
$\Rightarrow \mathfrak{m} = (\pi)$ for some $\pi \in R$ (uniformizer) & R is a DVR (wrt π-adic valuation)

Pf/ (1) dim 1 + Noetherian $\Rightarrow$ for any $a \notin R^\times \cup \{0\}$: $\exists\, b \in R \setminus (a)$ st
$\mathfrak{m} = \{x \in R \mid xb \in (a)\}$

[Why? For any $c \in R \setminus (a)$ define $(a) \subseteq I_c := \{x \in R \mid xc \in (a)\} \subsetneq R$ ideal
By Noetherianity, $\exists$ maximal one I_b.
Claim: I_b is a proper prime ideal $(rs \in I_b,\ s \notin I_b \Rightarrow I_b \subset \underset{r}{\underset{\in}{I_{sb}}} \Rightarrow r \in I_b) \Rightarrow I_b = \mathfrak{m}$ $\boxed{\dim = 1}$

- $K = \operatorname{Quot}(R) \rightsquigarrow \mathfrak{m}^{-1} := \{r \in K : r\mathfrak{m} \subseteq R\}$
- $R \subset \mathfrak{m}^{-1} \subset K \rightsquigarrow$ Claim: $\exists\, r \in \mathfrak{m}^{-1}$: $r\mathfrak{m} = R$

Otherwise: By local condition of R, then $r\mathfrak{m} \subset \mathfrak{m}$ $\forall r \in \mathfrak{m}^{-1}$ so $r : \mathfrak{m} \to \mathfrak{m}$ is R-linear map
But $\mathfrak{m}$ is f.g. (R Noeth.), say $\mathfrak{m} = (a_1, \dots, a_k)$ so $r(a_j) = \sum_{\ell=1}^{k} \underset{\in R}{c_{j\ell}}\, a_\ell \rightsquigarrow C \in R^{k\times k}$ matrix
By [HC]: $\det(r I_d - C) = 0$ gives a monic eqn for r w/ coeffs in R
Since R is int closed, we get $r \in R$.
Conclusion: $\mathfrak{m}^{-1} = R$ & $\underset{(b\mathfrak{m} \subseteq (a))}{\frac{b}{a} \in \mathfrak{m}^{-1} = R} \Rightarrow b \in (a)$ Contr!

Pick $r \in \mathfrak{m}^{-1}$ s.t $r\mathfrak{m} = R$; $1 = r\pi$ for some $\pi \in \mathfrak{m} \Rightarrow_{\mathfrak{m} \ni} x = x \cdot 1 = \underset{\in R}{(xr)}\pi$
$\therefore \mathfrak{m} = (\pi)$.

(2) $t \in R \rightsquigarrow t = u\pi^n$ $n \geq 0$, $u \in R^\times$ for a unique n
$\Rightarrow$ Define $v(t) = n$ can be extended to K as $v\left(\frac{t}{s}\right) = v(t) - v(s)$.
This is the π-adic valuation on K $(\mathcal{O}_v = R,\ \mathfrak{m}_v = \mathfrak{m},\ \Gamma = \mathbb{Z})$

Application: R integrally closed, Noetherian domain, $\mathfrak{P}_0$ = minimal prime
Then $R_{\mathfrak{P}_0}$ is ________________________, local, dim = 1 $\Rightarrow$ we can use Proposition to define the $\mathfrak{P}_0$-adic valuation, $v_{\mathfrak{P}_0}$ on $K = \operatorname{Quot}(R)$. Here

$$\boxed{\mathcal{O}_{v_{\mathfrak{P}_0}} = R_{\mathfrak{P}_0}, \quad \mathfrak{m}_{v_{\mathfrak{P}_0}} = \mathfrak{P}_0 R_{\mathfrak{P}_0} \quad ; \quad \Gamma = \mathbb{Z}}$$

Example: R coord ring of a normal variety, $\mathfrak{P}_0$ = ideal of codim 1 subvariety of R.

Lecture II: Extensions of Valuations & divisorial valuations

Theorem (Chevalley's Extension Thm) Fix K field, $R \subsetneq K$ subring, $\mathfrak{p} \in \operatorname{Spec} R$, $\mathfrak{p} \neq 0$.
Then: $\exists$ $\mathcal{O} \subseteq K$ valuation ring w/ mxl ideal $\mathfrak{m}$ s.t.

$$R \subset \mathcal{O} \quad \& \quad \mathfrak{m} \cap R = \mathfrak{p}.$$

We say the valuation assoc to $\mathcal{O}$ is non-neg on R & it's center in R is $\mathfrak{p}$.

P.f/ We define $\Sigma := \{ (A, I) \mid R_{\mathfrak{p}} \subseteq \underset{\text{ring}}{A} \subset K \ , \ \mathfrak{p}R_{\mathfrak{p}} \subseteq \underset{\text{ideal}}{I} \subsetneq A \}$

- $\Sigma \neq \emptyset$ $\ ((R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}}) \in \Sigma)$
- Define partial order $\leq$ on Σ: $(A_1, I_1) \leq (A_2, I_2) \iff A_1 \subseteq A_2, \ I_1 \subseteq I_2$

Each chain $\{(A_j, I_j)\}_{j \in J}$ in Σ is bounded above by $\left(\bigcup_{j \in J} A_j, \bigcup_{j \in J} I_j \right) \in \Sigma$.

$\Rightarrow$ Zorn's Lemma gives a mxl element $(\mathcal{O}, \mathfrak{m}) \in \Sigma$.

- Note: $R \subseteq R_{\mathfrak{p}} \subset \mathcal{O}$, $\mathfrak{p}R_{\mathfrak{p}} \subseteq \mathfrak{m}$ so $R_{\mathfrak{p}} \cap \mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$ ($R_{\mathfrak{p}}$ is local) $\Rightarrow \boxed{\mathfrak{m} \cap R = \mathfrak{p}}$
- $\mathcal{O}$ is a local ring w/ mxl ideal $\mathfrak{m}$. $((\mathcal{O}, \mathfrak{m}) \leq (\mathcal{O}_{\mathfrak{m}}, \mathfrak{m}\mathcal{O}_{\mathfrak{m}}) \in \Sigma$ so $\mathcal{O}_{\mathfrak{m}} = \mathcal{O} \Rightarrow \mathcal{O}$ is local!
  $\mathfrak{m} \neq \mathcal{O}$; $(\mathcal{O}, \mathfrak{m})$ mxl.
- $\mathcal{O}$ is a valuation ring:

Assume it's not & pick $x \in K^\times$ with $x, x^{-1} \notin \mathcal{O}$. $\Rightarrow \mathcal{O} \subsetneq \mathcal{O}[x], \mathcal{O}[x^{-1}] \subseteq K$ subrings.

By maximality of $(\mathcal{O}, \mathfrak{m})$: $\underset{\substack{\text{ideal in } \mathcal{O}[x] \\ \text{containing } \mathfrak{p}R_{\mathfrak{p}}}}{\mathfrak{m}\mathcal{O}[x]} = \mathcal{O}[x]$ & $\mathfrak{m}\mathcal{O}[x^{-1}] = \mathcal{O}[x^{-1}]$

so $\exists$ $a_1, \ldots, a_n \in \mathfrak{m}$, $b_0, \ldots, b_m \in \mathfrak{m}$, say $m \leq n$ with n, m minimal

$$1 = \sum_{i=0}^{n} a_i x^i \quad \& \quad 1 = \sum_{j=0}^{m} b_j x^{-j} \qquad (a_n, b_m \neq 0)$$

Write $\displaystyle\sum_{j=1}^{m} b_j x^{-j} = 1 - b_0 \in \mathcal{O} \setminus \mathfrak{m} = \mathcal{O}^\times$ ($\mathcal{O}$ local!)

$$\Rightarrow \sum_{j=1}^{m} \frac{b_j}{1-b_0} x^{-j} = 1 \quad \& \quad c_j := \frac{b_j}{1-b_0} \in \mathfrak{m}.$$

Multiply by x^n: $\displaystyle x^n = \sum_{j=1}^{m} c_j x^{n-j}$

$$\Rightarrow 1 = \sum_{i=0}^{n-1} a_i x^i + \sum_{j=1}^{m} a_n c_j x^{n-j} \qquad \deg \leq n-1 \quad \text{Contr!}$$

Def: $L|K$ field extension, $\mathcal{O}\subset K$ valn ring w/ maxl ideal $\mathfrak{m}$,
$\mathcal{O}'\subset L$ " " " " " $\mathfrak{m}'$

$\mathcal{O}'$ is a <u>prolongation</u> of $\mathcal{O}$ if $\mathcal{O}'\cap K = \mathcal{O}$ ("$\mathcal{O}'$ is an <u>extension</u> of $\mathcal{O}$")

<u>Remark</u>: If so, $\mathfrak{m}'\cap K = \mathfrak{m}'\cap \mathcal{O} = \mathfrak{m}$
$\mathcal{O}'^{\times}\cap K = \mathcal{O}'^{\times}\cap\mathcal{O} = \mathcal{O}^{\times}$

($\subseteq$ ✓ otherwise, unit in $\mathcal{O}\Rightarrow$ unit in $\mathcal{O}'\smallsetminus\mathfrak{m}'$)
($\supseteq$: if $x\in\mathfrak{m}\smallsetminus\mathfrak{m}' \Rightarrow x\in\mathcal{O}'^{\times}$ & $x\in\mathcal{O}$ & $x^{-1}\in\mathcal{O}'\cap K = \mathcal{O}$ $\Rightarrow x\notin\mathfrak{m}$ $\square$)

Also, if $\mathcal{O}'$ is a valuation ring in L $\underset{L|K}{\Rightarrow}$ $\mathcal{O}'\cap K$ is a valuation ring in K w/ maxl ideal $\mathfrak{m}'\cap K$.

<u>Cor 1</u>: $L|K$ field extension, $\mathcal{O}\subseteq K$ valn ring. Then: $\exists$ an extension of $\mathcal{O}$ in L

Pf/ Take $\underset{\mathfrak{m}}{R}=\mathcal{O}\subseteq L$ & use Chevalley's Thm to find $\mathcal{O}'$ valn ring of L, $\mathcal{O}\subset\mathcal{O}'$ & $\mathfrak{m}'\cap\mathcal{O}=\mathfrak{m}$

Now $\mathcal{O}\subseteq\mathcal{O}'\cap K$ valn rings in K & $\mathfrak{m}=\mathfrak{m}'\cap\mathcal{O}=\mathfrak{m}'\cap K$ is the maxl ideal of both rings $\Longrightarrow$ $\boxed{\mathcal{O}=\mathcal{O}'\cap K}$.

$\subseteq$ ✓
$\supseteq$ $x\in\mathfrak{m}'\cap K$ → $x\in\mathcal{O}\Rightarrow x\in\mathfrak{m}'\cap\mathcal{O}=\mathfrak{m}$
→ $\left.\begin{array}{l} x^{-1}\in\mathcal{O}\subseteq\mathcal{O}' \\ x\in\mathcal{O}'\end{array}\right\}\Rightarrow x\notin\mathfrak{m}'$ <u>Contr!</u>

<u>Cor 2</u>: K field, $R\subseteq K$ subring. $\mathfrak{m}$ = maxl ideal of $\mathcal{O}$

$\mathbb{V}_R := \{\mathcal{O}\subseteq K$ valuation ring with $R\subset\mathcal{O}$ & $\mathfrak{m}\cap R$ is a maxl ideal of $R\}$
($\mathbb{V}_R\neq\emptyset$ by Chevalley's Thm)

Then: $\overline{R}$ = int closure of R in $K = \displaystyle\bigcap_{\mathcal{O}\in\mathbb{V}_R}\mathcal{O}$.

Pf/ Recall: valn rings are normal so $\overline{R}\subseteq\mathcal{O}$ $\forall\,\mathcal{O}\in\mathbb{V}_R$.

- We show: if $x\in K\smallsetminus\overline{R}\Rightarrow x\notin$ (RHS) in several steps.

<u>STEP 1</u>: $x\notin\overline{R}[x^{-1}]$.

Otherwise: $x = b_0+b_1x^{-1}+\dots+b_mx^{-m}$ $b_i\in\overline{R}$ $\underset{\cdot x^m}{\Longrightarrow}$ $x^{m+1}-\sum_{i=0}^{m} b_i x^{m-i}=0$

$\Rightarrow x$ is integral over $\overline{R}\Rightarrow$ integral over R $\Rightarrow x\in\overline{R}$ <u>Contr!</u>

<u>STEP 2</u>: x^{-1} not unit in $\overline{R}[x^{-1}]$ $\Longrightarrow$ $x^{-1}\in\mathfrak{M}'$ for <u>some</u> maximal ideal $\mathfrak{M}'$ of $\overline{R}[x^{-1}]$

By Chevalley's Thm $\exists$ $\overline{R}[x^{-1}]\subseteq\mathcal{O}\subseteq K$ valn ring. with $\mathfrak{m}\cap\overline{R}[x^{-1}]=\mathfrak{M}'$ (*)

But $x^{-1}\in\mathfrak{m}\Rightarrow\boxed{x\notin\mathcal{O}}$

<u>STEP 3</u>: Show $\mathcal{O}\in\mathbb{V}_R$, ie $R\cap\mathfrak{m}$ is maxl ideal in R. $\Rightarrow x\notin$ (RHS) ✓

- $\pi: \overline{R}\hookrightarrow\overline{R}[x^{-1}]\twoheadrightarrow\boxed{\overline{R}[x^{-1}]/\mathfrak{M}'}$ field is a surjective homomorphism because $x^{-1}\in\mathfrak{M}'$

$\Rightarrow\mathfrak{m}\cap\overline{R} \underset{\text{by }(*)}{=} \mathfrak{M}'\cap\overline{R}$ is a maximal ideal of $\overline{R}$ (b/c $\mathfrak{M}'$ is maxl in $\overline{R}[x^{-1}]$ & $\overline{\pi}$ surj.

- $\mathfrak{m}\cap R$ is a prime ideal in R ($R\hookrightarrow\mathcal{O}$). So $\dfrac{R}{\mathfrak{m}\cap R}\subset\boxed{\dfrac{\overline{R}}{\mathfrak{m}\cap\overline{R}}}$ field

<u>Note</u> $R\subset\overline{R}$ integral so $\dfrac{R}{\mathfrak{m}\cap R}\subset\dfrac{\overline{R}}{\mathfrak{m}\cap\overline{R}}$ is integral $\Longrightarrow$ $\dfrac{R}{\mathfrak{m}\cap R}$ is a field

$\Rightarrow\mathfrak{m}\cap R$ is maxl ideal in R $\square$

$x\in R/_{\mathfrak{m}\cap R}\Rightarrow u=x^{-1}$ $u^n+\sum a_i u^i=0$ $\cdot x^n$

Cor 3: $L|K$ & $\mathcal{O}\subset K$ valuation ring. Then $\mathcal{O}^L = \bigcap_{\substack{\mathcal{O}' \text{ extn} \\ \text{of } \mathcal{O} \text{ in } L}} \mathcal{O}'$

Pf/ $W_{\mathcal{O}}(L)$ = extensions of $\mathcal{O}$ in L

Ranks of valuations & convex sets of Γ (Γ = ordered ab grp.)

Def: A subgroup $\Delta\subset\Gamma$ is convex if for each $\gamma\in\Gamma$ with $0\leq\gamma\leq\delta\in\Delta \Rightarrow \gamma\in\Delta$

- $\{\Delta : \Delta \text{ convex subgp of } \Gamma\}$ is ordered by inclusion (totally) (Exercise: $\exists x\in\Delta_1\setminus\Delta_2 \Rightarrow \Delta_2\subseteq\Delta_1$)
  $\ni \{0\}, \Gamma$
- $\operatorname{rank}(\Gamma) :=$ maximal size of a chain of convex subgroups of Γ.

Eg: $\Gamma = \mathbb{Z}^2_{lex}$ $0 \subsetneq \Delta_1 = \{0\}\times\mathbb{Z} \subsetneq \Delta_2 = \Gamma$ rank $=2$ Thm (Hahn): $\exists\, \rho: \Gamma \hookrightarrow \mathbb{R}^{\operatorname{rank}\Gamma}$ order preserving hom

- $\Delta\subseteq\Gamma$ convex subgroup $\Rightarrow \Gamma/\Delta$ is an ordered ab gp ($\gamma+\Delta\leq\gamma'+\Delta \iff \gamma\leq\gamma'$)

Pf: • Well-defined: $\gamma\leq\gamma'$ but $\exists\, \delta,\delta'\in\Delta$ with $\delta+\gamma > \delta'+\gamma' \Rightarrow \underbrace{\delta-\delta'}_{\in\Delta} > \gamma'-\gamma\geq 0$
$\underset{\Delta \text{ convex}}{\Rightarrow} \gamma'-\gamma\in\Delta$ (conclusion: independent of representatives of cosets)
  • Addition is monotone ✓

KEY property to show: $v: K^\times \twoheadrightarrow \Gamma$ valn with $\mathcal{O}=\mathcal{O}_v$, $\Gamma\neq\{0\}$ ($\iff \mathcal{O}\neq K$)

$$\left\{\begin{matrix}\text{convex subgroups}\\ \text{of } \Gamma\end{matrix}\right\} \xleftrightarrow[(**)]{1\text{-to-}1} \operatorname{Spec}(\mathcal{O}) = \left\{\begin{matrix}\mathfrak{P}\subset R \text{ prime}\\ \text{ideals}\end{matrix}\right\} \xleftrightarrow[(*)]{1\text{-to-}1} \begin{matrix}\mathcal{O}\subseteq\mathcal{O}'\subseteq K\\ \text{(valn) rings.}\end{matrix}$$

(*) $\mathcal{O}\subsetneq\mathcal{O}'\subset K$ ring. Then $\mathcal{O}'$ is a valn ring ($x\notin\mathcal{O}' \Rightarrow x\notin\mathcal{O} \Rightarrow x^{-1}\in\mathcal{O}\subsetneq\mathcal{O}'$)
$(\mathcal{O},\mathfrak{m})$, $(\mathcal{O}',\mathfrak{m}')$. Then: $\mathfrak{m}'\subsetneq\mathfrak{m}$ ($x\in\mathfrak{m}' \Rightarrow x^{-1}\notin\mathcal{O}' \Rightarrow x^{-1}\notin\mathcal{O} \Rightarrow x\in\mathfrak{m}$)
So $\mathfrak{m}' = i^{-1}(\mathfrak{m}') \subseteq \mathcal{O} \overset{i}{\hookrightarrow} \mathcal{O}'$ is a prime ideal in $\mathcal{O}$ $\Rightarrow$ Localize $\mathcal{O}$ at $\mathfrak{m}'\in\operatorname{Spec}(\mathcal{O})$

Lemma: $\mathcal{O}_{\mathfrak{m}'} = \mathcal{O}'$

Pf/ Pick $x\in\mathcal{O}'$. If $x\in\mathcal{O}$, write $x=\frac{x}{1}\in\mathcal{O}_{\mathfrak{m}'}$. If $x\notin\mathcal{O}$, then $x^{-1}\in\mathfrak{M}\setminus\mathfrak{m}'$, write $x=\frac{1}{x^{-1}}$ ($x\in(\mathcal{O}')^\times$) $\in\mathcal{O}_{\mathfrak{m}'}$. □

• Conversely, given $\mathfrak{P}\in\operatorname{Spec}(\mathcal{O})$, get $\mathcal{O}\subseteq\underset{\text{ring}}{\mathcal{O}_{\mathfrak{P}} := \mathcal{O}'\subset K}$ with maximal ideal $\mathfrak{P}\mathcal{O}_{\mathfrak{P}}\subset\mathfrak{M}$

(**) **Lemma**: (1) $\Delta \longmapsto \mathfrak{P}_\Delta = \{x\in K \mid v(x) > \delta \quad \forall \delta\in\Delta\}$

(2) $\Delta_{\mathfrak{P}} = \{\gamma\in\Gamma : \gamma, -\gamma < v(x) \text{ for all } x\in\mathfrak{P}\} \longleftarrow \mathfrak{P}$

Gives the inclusion-reversing correspondence (**).

In particular, if $\operatorname{rank}(\Gamma)<\infty$ then $\boxed{\operatorname{rank}(\Gamma) = \dim\mathcal{O}}$

Krull dim

L2 (9)

Pf/ (1). By valn condition: $\mathfrak{P}_\Delta$ is an ideal of $\mathcal{O}$ ($0 \in \Delta$ subgp.)

• $\mathfrak{P}_\Delta$ is prime: Pick $x, y \in \mathcal{O}$: $xy \in \mathfrak{P}_\Delta$ so $v(xy) > \delta \ \forall \delta \in \Delta$.

Assume $x \notin \mathfrak{P}_\Delta \Rightarrow \exists \delta > 0 \quad 0 < v(x) \leq \delta$
$y \notin \mathfrak{P}_\Delta \Rightarrow \exists \delta' > 0 \quad 0 < v(y) \leq \delta'$ $\Big\} \Rightarrow 0 < v(xy) \leq \delta + \delta' \in \Delta$ Contr!

Clear: $\Delta \subseteq \Delta' \Rightarrow \mathfrak{P}_\Delta \supseteq \mathfrak{P}_{\Delta'}$.

• $\Delta_\mathfrak{P}$ is convex ✓ ($0 \leq \gamma \leq \delta$ $\quad \delta \in \Delta \Rightarrow \delta, -\delta < v(x) \ \forall x \in \mathfrak{P} \Rightarrow \gamma < v(x) \ \forall x \in \mathfrak{P}$, $-\gamma < 0 \leq v(x)$ ✓ $\} \Rightarrow \gamma \in \Delta_\mathfrak{P}$ ✓)

(2). $\Delta_\mathfrak{P}$ is a subgroup of Γ: $0 \in \Delta_\mathfrak{P}$ b/c $\mathfrak{P} \subseteq \mathfrak{m}$ ✓; $-\Delta_\mathfrak{P} = \Delta_\mathfrak{P}$ by def ✓

$\Delta_\mathfrak{P} + \Delta_\mathfrak{P} \subseteq \Delta_\mathfrak{P}$: Since $\leq$ is a total order & $\Delta_\mathfrak{P}$ is convex, enough to show $0 \leq \delta \in \Delta_\mathfrak{P} \Rightarrow \delta + \delta \in \Delta_\mathfrak{P}$

Indeed: $\gamma, \gamma' \in \Delta_\mathfrak{P}$ Then $0 \leq \gamma + \gamma' \leq 2\max\{\gamma, \gamma'\} \in \Delta_\mathfrak{P} \Rightarrow \gamma + \gamma' \in \Delta_\mathfrak{P}$
or $0 \leq -(\gamma + \gamma') \leq 2\max\{-\gamma, -\gamma'\} \Rightarrow -(\gamma + \gamma') \in \Delta_\mathfrak{P} \Rightarrow \gamma + \gamma' \in \Delta_\mathfrak{P}$.

• v surjective, so $\delta = v(x)$ for $x \in \mathcal{O}$. Assume $v(x^2) = \delta + \delta \notin \Delta_\mathfrak{P} \Rightarrow$ By def

$\exists y \in \mathfrak{P}$ with $v(y) \leq v(x^2)$ so $v(x^2 y^{-1}) \geq 0$ so $x^2 y^{-1} \in \mathcal{O}$

Hence $x^2 = y(x^2 y^{-1}) \in \mathfrak{P} \underset{\mathfrak{P} \text{ prime}}{\Longrightarrow} x \in \mathfrak{P} \Rightarrow \delta = v(x) \notin \Delta_\mathfrak{P}$ Contr!

Clear: $\mathfrak{P} \subset \mathfrak{P}' \Rightarrow \Delta_\mathfrak{P} \supseteq \Delta_{\mathfrak{P}'}$

• (1) & (2) are inverse to each other, i.e. $\underset{\subseteq \text{(clear)}}{\Delta = \Delta_{\mathfrak{P}_\Delta}}$ & $\underset{\subseteq \text{(clear)}}{\mathfrak{P} = \mathfrak{P}_{\Delta_\mathfrak{P}}}$.

• Pick $\gamma \in (\Delta_{\mathfrak{P}_\Delta})_{\geq 0} \overset{?}{\Longrightarrow} \gamma \in \Delta_{\geq 0}$.

If not, write $\gamma = v(y)$ for $y \in \mathcal{O}$ Then $0 \leq \gamma' < \gamma \ \forall \gamma' \in \Delta_{\geq 0}$ by convexity,
so $y \in \mathfrak{P}_\Delta$ & $\gamma \notin \Delta_{\mathfrak{P}_\Delta}$. Contr!

- Both are subgroups so then $\Delta_{\mathfrak{P}_\Delta} \subseteq \Delta$ ✓

• Pick $x \in \mathfrak{P}_{\Delta_\mathfrak{P}} \Rightarrow$ If $\exists y \in \mathfrak{P}$ with $v(y) \leq v(x)$, then $x = \underset{\in \mathfrak{P}}{y} \underset{\in \mathcal{O}}{\frac{x}{y}} \in \mathfrak{P}$ ✓

Otherwise $\forall y \in \mathfrak{P}$: $v(x) > v(x) \Rightarrow v(x) \in \Delta_\mathfrak{P}$ but then $x \notin \mathfrak{P}_{\Delta_\mathfrak{P}}$ Contr! □

Observation 1:

• $v: K^\times \twoheadrightarrow \Gamma$ & $\Delta \leq \Gamma$ convex subgroup $\rightsquigarrow \mathfrak{P} = \mathfrak{P}_\Delta \in \mathrm{Spec}(\mathcal{O}_v)$ & Γ/Δ ordered ab gp.

• $\exists v_\mathfrak{P}: K^\times \twoheadrightarrow K^\times/\mathcal{O}_\mathfrak{P}^\times$ val. assoc to $\mathcal{O}_\mathfrak{P} \supset \mathcal{O}$; $\Gamma \simeq K^\times/\mathcal{O}^\times$.

• $\pi: K^\times/\mathcal{O}^\times \twoheadrightarrow K^\times/\mathcal{O}_\mathfrak{P}^\times \quad x\mathcal{O}^\times \longmapsto x\mathcal{O}_\mathfrak{P}^\times$ order preserving ($x\mathcal{O}^\times \leq y\mathcal{O}^\times$ iff $yx^{-1} \in \mathcal{O} \subset \mathcal{O}_\mathfrak{P}$ $\Downarrow$ $x\mathcal{O}_\mathfrak{P}^\times \leq y\mathcal{O}_\mathfrak{P}^\times$ ✓)

$\mathrm{Ker}(\pi) = \mathcal{O}_\mathfrak{P}^\times/\mathcal{O}^\times = \{\frac{a}{b}\mathcal{O}^\times,\ a, b \in \mathcal{O} \setminus \mathfrak{P}\} \Rightarrow 0 \leq v(a), v(b) < \delta$ for some $\delta \in \Delta \Rightarrow v(a), v(b) \in \Delta$

So $v(\frac{a}{b}) = v(a) - v(b) \in \Delta$ (Δ gp) $\overset{\text{def } \Delta = \Delta_\mathfrak{P}}{}$ Conclusion: $\mathrm{Ker}\,\pi = \Delta$ & $v_\mathfrak{P}(K^\times) = \Gamma/\Delta$

$\Rightarrow v_\mathfrak{P} = \pi \circ v: K^\times \twoheadrightarrow \Gamma \twoheadrightarrow \Gamma/\Delta$ $\quad \boxed{v_\mathfrak{P}(x) = v(x) + \Delta}$

In addition: $\pi_\mathfrak{z} : \mathcal{O}_\mathfrak{z} \twoheadrightarrow \mathcal{O}_\mathfrak{z}/\mathfrak{z}\mathcal{O}_\mathfrak{z} =: K_\mathfrak{z}$ is a field & $\pi_\mathfrak{z}(\mathcal{O})$ is a valuation ring in $K_\mathfrak{z}$.

(Reason: $x = \pi_\mathfrak{z}(w)$, $w \in \mathcal{O}_\mathfrak{z}$ $\begin{cases} w \in \mathcal{O} \Rightarrow x \in \pi_\mathfrak{z}(\mathcal{O}) \\ w \notin \mathcal{O} \Rightarrow w^{-1} \in \mathcal{O} \Rightarrow x^{-1} \in \pi_\mathfrak{z}(\mathcal{O}). \end{cases}$)

$\rightsquigarrow \bar{v} : \mathcal{O}_\mathfrak{z}/\mathfrak{z}\mathcal{O}_\mathfrak{z} \twoheadrightarrow \Delta \cup \{\infty\}$ $\quad \bar{v}(\bar{x}) = v(x)$ is a well-defined valuation

with $\mathcal{O}_{\bar{v}} = \pi_\mathfrak{z}(\mathcal{O}) = \mathcal{O}/\mathfrak{z}$ & $\mathfrak{m}_{\bar{v}} = \pi_\mathfrak{z}(\mathfrak{m}) = \mathfrak{m}/\mathfrak{z}$ so $\mathcal{O}_{\bar{v}}/\mathfrak{m}_{\bar{v}} \simeq \mathcal{O}/\mathfrak{m}$.

Conversely: Given a valuation $v' : K^\times \twoheadrightarrow \Gamma'$ & $\bar{v} : (\underbrace{\mathcal{O}_{v'}/\mathfrak{m}_{v'}}_{\text{residue field of } v'})^\times \twoheadrightarrow \Delta'$ valuation,

define a "composition" of v' with $\bar{v}$ (= new valuation ~~on~~ on K)

- $\mathcal{O} = \pi^{-1}(\mathcal{O}_{\bar{v}})$ where $\pi : \mathcal{O}_{v'} \twoheadrightarrow \mathcal{O}_{v'}/\mathfrak{m}_{v'} \supseteq \mathcal{O}_{\bar{v}}$ subring

  $\mathcal{O}$ is a valuation ring of K (because both $\mathcal{O}_{v'}$ & $\mathcal{O}_{\bar{v}}$ are)

(Pf/ If $x \in \mathcal{O}_{v'} \setminus \mathcal{O} \Rightarrow \pi(x) \in \mathcal{O}_{v'}/\mathfrak{m}_{v'} \setminus \mathcal{O}_{\bar{v}} \xrightarrow{\text{v. ring}} \Rightarrow \pi(x)^{-1} = y + \mathfrak{m}_{v'} \in \mathcal{O}_{\bar{v}}$ for some $y \in \mathcal{O}$

where $(y + \mathfrak{m}_{v'})(x + \mathfrak{m}_{v'}) = xy + \mathfrak{m}_{v'} = 1 + \mathfrak{m}_{v'} \Rightarrow xy \in 1 + \mathfrak{m}_{v'} \subseteq \mathcal{O}_{v'}^\times \Rightarrow xy = u \in \mathcal{O}_{v'}^\times$

So $x, y \in \mathcal{O}_{v'}^\times$ & $x^{-1} = yu^{-1} \in \mathcal{O}_{v'} \Rightarrow \pi(x^{-1}) = \pi(x)^{-1} \in \mathcal{O}_{\bar{v}}$ so $x^{-1} \in \mathcal{O}$ □)

Then, pick $v : K^\times \twoheadrightarrow \Gamma$ valn. with $\mathcal{O}_v = \mathcal{O} \subset \mathcal{O}_{v'}$ so $\mathcal{O}_{v'} \longleftrightarrow \Delta \subseteq \Gamma$ convex subgroup

& $\Gamma' = \Gamma/\Delta$ ($v' = v_\mathfrak{z}$ for $\mathfrak{z} = \mathfrak{z}_\Delta$), $\mathcal{O}_{v'}/\mathfrak{m}_{v'} = \mathcal{O}/\mathfrak{m}$.

Q: How many valuation ring extensions are there?

- Key result 1: $L|K$ algebraic extension, $[L:K]_{sep} < \infty$. Then, for $\mathcal{O} \subset K$ valn. ring, there are finitely many extensions of $\mathcal{O}$ to L & this number is $\leq [L:K]_{sep}$. The rank of the valuation is preserved.

- Special case: $K|k$ algebraic extension, v valn that is trivial on k, $k \subseteq \mathcal{O}_v \subseteq K$ then $\mathcal{O}_v = K$ ($k^\times \subset \mathcal{O}_v^\times$) but $K = \mathrm{Quot}(\mathcal{O}_v)$ & $\mathcal{O}_v$ is integrally closed, so $\mathcal{O}_v = K$ & so v is trivial!

Observation: $K|k$ has non trivial valuations $\Longleftrightarrow$ $\mathrm{tr\,deg}(K, k) > 0$
(ie $v|_k$ trivial but $v|_K$ non-triv)

- Key result 2: v valuation on $K|k$ (:= trivial on k). Then:
(Dimension Ineq.)

$$\boxed{\mathrm{tr\,deg}(K, k) \geq \mathrm{rank}(v(K^\times)) + \mathrm{tr\,deg}(\mathcal{O}_v/\mathfrak{m}_v, k)}$$

- Sanity check: $K = k(x)$ w/ x-adic valn. $v(K^\times) = \mathbb{Z}$ so rank 1
Example 1 $\quad \mathrm{tr\,deg}(K, k) = 1$, $\mathrm{tr\,deg}(\mathcal{O}_v/\mathfrak{m}_v, k) = 0$ | $\mathcal{O}_v = k[x]_{(x)}$, $\mathfrak{m}_v = (x)k[x]_{(x)}$, $\mathcal{O}_v/\mathfrak{m}_v \simeq k$

Application: Valuations over function fields

$V \subseteq \mathbb{A}^n_k$ irreducible variety of dim r $\Rightarrow$ $k[V] = k[y_1, \ldots, y_n]/q$ (coord ring of V), q prime ideal

$K = k(V) := \mathrm{Quot}(k[V])$ <u>function field associated to V</u>

Def: Let v be a valuation on $k(V)/k$ (ie $v|_k \equiv 0$) st

(1) $\mathcal{O}_v = \{ x \in k(V) \mid v(x) \geq 0 \} \supset k[V]$ ("valn is non-neg on V")

(2) $\mathrm{tr\,deg}(\mathcal{O}_v/\mathfrak{m}_v, k) = r-1$ $(= \dim V - 1)$ $\boxed{r = \dim V = \mathrm{tr\,deg}(k(V), k)}$

We call v a <u>prime divisor</u> of $K|k$ (aka prime valuations).

Note: $\mathfrak{P} := k[V] \cap \mathfrak{m}_v$ is a prime ideal in $k[V]$ (because $\iota: k[V] \hookrightarrow \mathcal{O}_v \supset \mathfrak{m}_v$, $\mathfrak{P} = \iota^{-1}(\mathfrak{m}_v)$)

Then $\dim \mathfrak{P} \leq r-1$

Pf/ $k[V]/\mathfrak{P} \hookrightarrow \mathcal{O}_v/\mathfrak{m}_v$ field (Int domain) $\Rightarrow$

$$\begin{tikzcd}
\mathrm{Quot}(k[V]/\mathfrak{P}) \arrow[r, hook, "j"] & \mathcal{O}_v/\mathfrak{m}_v \\
k \arrow[u, "\mathrm{tr\,deg} \leq r-1"] \arrow[ur, "\mathrm{tr\,deg} = r-1"'] &
\end{tikzcd}$$

$\dim \mathfrak{P} = \mathrm{tr\,deg}(\mathrm{Quot}(k[V]/\mathfrak{P}), k)$ b/c $k[V]/\mathfrak{P}$ is a f.g k-algebra. $\square$

- If $\dim \mathfrak{P} = r-1$, then the top arrow j defines an algebraic extension, so V is a <u>divisorial valuation</u>. Otherwise, we say v is a <u>prime divisor</u> of <u>the 2nd kind</u>.

- If $\pi : k[y_1, \ldots, y_n] \twoheadrightarrow k[V]$, then $\tilde{\mathfrak{P}} := \pi^{-1}(\mathfrak{P}) \supset q$ is a prime ideal defining an <u>irreducible</u> subvariety W of V of dim $= \dim \mathfrak{P}$. We say that the valn v is <u>centered at W on V</u>.

THM: Given $W \subset V \subset \mathbb{A}^n$ irred $\dim W = \dim V - 1$

Let $S_{W,V} = \{$ prime divisors on $k(V)/k$ st $\mathcal{O}_v \supset k[V]$ & v centered at W on $V \}$

Then: (1) $0 < |S_{W,V}| < \infty$ (divisorial valns on $k(V)$ non neg on V exist & are finite!)

(2) $\forall v \in S_{W,V}$, $\mathcal{O}_v/\mathfrak{m}_v$ is algebraic over $k(W)$ of finite degree

<u>Proof</u>: Lecture III.

Frankfurt School (07/04-08/16)

Lecture III (Effective?) Divisorial Valuations

Recall: $V \subseteq \mathbb{A}^n_k$ irred. r-dim'l variety & $W \subseteq V$ irred., dim $= r-1$.

(Eg: V surface in 3-space. W curve, but not necessarily describable by 1 extra equation)

A valuation v on $k(V)$ is divisorial with center W if (1) $\mathcal{O}_v \supset k[V]$

$k(V)$ = function field of V = Quot$(k[V])$

(2) tr deg$(\mathcal{O}_v/m_v, k) = r-1$

(3) $\mathfrak{P} = m_v \cap k[V]$ defines W.

(in general (1),(2) ⇒ $m_v \cap k[V]$ is prime of dim $\leq r-1$)

NOTE: Away from $\mathfrak{P}$ in $k[V]$ val $= 0$, so $\mathcal{O}_v \supset k[V]_{\mathfrak{P}}$ (not in m_v!)

THM 1: Given $W \subset V \subseteq \mathbb{A}^n_k$ as above.

(a) There are finitely many divisorial valns on $k(V)$ with center W.

(b) For all valns v in (1), $\mathcal{O}_v/m_v$ is a finite algebraic extension over $k(W)$

($\Rightarrow \mathcal{O}_v/m_v$ is a "function field" of trans degree $r-1$ $= k(a_1, \dots, a_n)$, $k(a_1 \dots$)

Note: Recall Dimension Ineq from Lecture II:

$$\underbrace{\text{tr deg}(\mathcal{O}_v/m_v, k)}_{= r-1} + \underbrace{\text{rank}(v(k(V)^\times))}_{\geq 1 \ (v \text{ is non trivial})} \leq \underbrace{\text{tr deg}(k(V), k)}_{= r}$$

$\Rightarrow \boxed{\text{rank} = 1}$ Also $k(W) = \text{Quot}(k[V]/\mathfrak{P}) \hookrightarrow \mathcal{O}_v/m_v$ ⇒ algebraic (tr deg $= 0$)

(Why finite?)

tr deg $= r-1$ (dim W) over k; tr deg $= r-1$

Pf/ (a) Existence: follows from Chevalley's Thm $k[V] \subset \exists\, \mathcal{O} = \mathcal{O}_v \subset k(V)$

$m \cap k[V] = \mathfrak{P}$ prime — m

(1), (3) ✓

(2): Use the dim ineq $k(W) \hookrightarrow \mathcal{O}_v/m_v$ & $= r-1$ by Dim. ineq.

tr deg $= r-1$ over k ⇒ tr deg $\geq r-1$

[Effective Algorithm?]

Finiteness: Use Noether Normalization. ⇝ want to find a local eqn for W in $k[V]$.

Why? Motivating situation: V normal on W = $k[V]_{\mathfrak{P}}$ is normal ie. int. closed

Then: $k[V]_{\mathfrak{P}}$ int closed, Noeth, local & dim $\mathfrak{P}$ ⟹ $k[V]_{\mathfrak{P}}$ is

⟹ $k[V]_{\mathfrak{P}} \subset \mathcal{O}_v \subseteq k(V)$ & dim $\mathfrak{P} = 1$ so $\boxed{\mathcal{O} = k[V]_{\mathfrak{P}}}$ (codim $W = 1$ Lecture II a DVR

In $k[V]_{\mathfrak{p}}$, $\mathfrak{p}k[V]_{\mathfrak{p}} = (\pi)$ for some π & val was the π-adic valn.
↑ one eqn. to locally define W!

The valuation is ! and we can find it if we can compute π!. (a "generic" comb of the gens of $\mathfrak{P}$?)

In the general case, use Noether Norm to make it into this setting!.

Noether Normaliz on $W \subset V \subset \mathbb{A}^n$ ⇝ chain of ideals $\tilde{\mathfrak{p}} \supseteq \tilde{q} \supseteq 0$ in $k[y_1,\dots,y_n]$ (lifts of $\mathfrak{p}$ to $k[y_1,\dots,y_n]$).

Then $\exists\ x_1,\dots,x_n \in k[y_1,\dots,y_n] = R$ alg indep $/k$ st. $R \supset S = k[x_1,\dots,x_n]$
and (1) R is a f.gen. S-module.

(2) $\tilde{\mathfrak{p}} \cap S = (x_r,\dots,x_n)$, $r = \dim\tilde{\mathfrak{p}}+1$, $\tilde{q}\cap S = (x_{r+1},\dots,x_n)$, $r = \dim q$ ($\tilde{0}\cap S = 0$)

$$\begin{tikzcd}
k[V]/\mathfrak{p} = k[W] & k[V] \arrow[l, two heads] & k[y_1,\dots,y_n] = k[\mathbb{A}^n] \arrow[l, two heads] \\
k[X]/(x_r,\dots,x_n) = k[x_1,\dots,x_{r-1}] \arrow[u, hook, "\text{finite.}"'] & k[x_1,\dots,x_r] \arrow[l, two heads] \arrow[u, hook, "\text{finite}"'] & k[x_1,\dots,x_n] \arrow[l, two heads] \arrow[u, hook, "\text{finite.}"']
\end{tikzcd}$$

$\Rightarrow$ Each top ring is f.g module over bottom ring.

$k[x_1,\dots,x_r] = k[X]/(x_{r+1},\dots,x_n)$, $\quad 0 \leftarrow x_{r+1},\dots,x_n$

$0 \leftarrow x_r$

The finiteness means they are integral! $\Rightarrow$ they induce finite algebraic field extensions!. $\Rightarrow$ finiteness will follow from finiteness of valns on bottom.

Write $\mathfrak{p}_0 := \mathfrak{p} \cap k[x_1,\dots,x_r]$

[Claim: $\mathfrak{p}_0 = \langle x_r \rangle$ from the diagram.]

$k[V] \supset \mathfrak{p} \supset \boxed{\cancel{\text{then}}\ \mathfrak{p}'}$ — GOING DOWN

finite.

intd. $k[x_1,\dots,x_r] \supset \mathfrak{p}_0 \supset$ if $\exists\ \mathfrak{p}_0'$

$\mathfrak{p}$ minimal $\Rightarrow$ going-Down Lemma says $\mathfrak{p}_0$ is minimal as well!

Now: $k[x_1,\dots,x_r]$ is UFD & $\mathfrak{p}_0$ is minimal prime $\Rightarrow$ principal $\mathfrak{p}_0 = (f)$

$\Rightarrow \exists!$ valuation on $k(x_1,\dots,x_r)$ with center $\mathfrak{p}_0$ in $k[x_1,\dots,x_r]$

($\boxed{\mathcal{O}_v = k[x_1,\dots,x_r]_{\mathfrak{p}_0}} \supset k[x_1,\dots,x_r]$ & $\mathfrak{m}_v \cap k[x_1,\dots,x_r] = \mathfrak{p}_0$.

$k(V) \supset \boxed{\mathcal{O}_v'}$ finitely many!

finite

$k(x_1,\dots,x_n) \supset \mathcal{O}_v$

effective way?

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<u>Missing point</u>: If v is a prime valn, then $\mathcal{O}_v/\mathfrak{m}_v$ is a function field over k of transc degree $r-1$

($\Rightarrow$ we can write it as $k(z_1,\dots,z_s) = \underbrace{k(z_1,\dots,z_p)}_{\text{finite algebraic}/k}(z_{s-r},\dots,z_s)$ $\Rightarrow$ finite algebraic extension of $k(w)$.

Pf/ Pick $x_1,\dots,x_{r-1}\in\mathcal{O}_v$ whose cosets in $\mathcal{O}_v/\mathfrak{m}_v$ are alg indep$/k$.

$\Rightarrow x_1,\dots,x_{r-1}$ are alg indep $/k$.

Extend this to a transc basis of $k(V)/k$ by adding 1 more element $x_r\in\mathcal{O}_v$.

$$\begin{array}{l} K = k(V) \\ \quad | \quad \Rightarrow \text{finite} \\ k(\underline{x}) = k(x_1,\dots,x_r) \quad \text{tr deg}=r \\ \quad | \quad \rightarrow \text{tr deg } r \\ \quad k \end{array}$$

Take $v' = v|_{k(x_1,\dots,x_r)}$ valn on $k(\underline{x})$ $\Rightarrow \mathcal{O}_v \supset \mathcal{O}_{v'} \supset k[\underline{x}]$ ($x_i\in\mathcal{O}_v\setminus\mathfrak{m}_v$)

$\mathfrak{m}_{v'} = \mathcal{O}_{v'}\cap\mathfrak{m}_v$

$K|_{k(\underline{x})}$ is finite $\underset{\text{THM.}}{\Longrightarrow}$ $\mathcal{O}_{v'}/\mathfrak{m}_{v'} \hookrightarrow \mathcal{O}_v/\mathfrak{m}_v$ is finite

($\& \; v'(k(x)^\times) \subset v(K)$ has finite index)
$\quad$ gp $\quad$ gp.

$$\begin{tikzcd} \mathcal{O}_{v'}/\mathfrak{m}_{v'} \arrow[r, hook, "\text{alg}"] & \mathcal{O}_v/\mathfrak{m}_v \\ k \arrow[u, "\text{tr deg}=r-1"] \arrow[ur, "\text{tr deg}=r-1"'] & \end{tikzcd}$$

$\Rightarrow v'$ is a prime valn on $k(\underline{x})|k$.

<u>STEP 1</u>: Prove the result for v' & $k(\underline{x})$.

Write $\mathfrak{P}' = k[\underline{x}]\cap\mathfrak{m}_{v'}$ is prime. & $k[x]/\mathfrak{P}' \hookrightarrow \mathcal{O}_{v'}/\mathfrak{m}_{v'} \hookrightarrow \mathcal{O}_v/\mathfrak{m}_v$.

$\boxed{\not\ni \; x_1,\dots,x_{r-1}}$ (b/c $\notin\mathfrak{m}_v$) $\xrightarrow[x_1,\dots,x_{r-1}\text{ alg indep}/k]{}$ tr deg $\geq r-1$ so $= r-1$

$k \to k[x]/\mathfrak{P}'$, $\;k\xrightarrow{\text{tr deg}=r-1}\mathcal{O}_{v'}/\mathfrak{m}_{v'}$, $\;k\to\mathcal{O}_v/\mathfrak{m}_v$

<u>Claim</u>: $\mathfrak{P}'$ is a <u>minimal</u> prime in $k[x_1,\dots,x_r]$.

Indeed, pick $\mathfrak{P}\subsetneq\mathfrak{P}'$, <u>WTS</u>: $\mathfrak{P}=0$. But $\dim\mathfrak{P} > \dim\mathfrak{P}' = r-1$
$\qquad r = \dim k[x] \geq$ $\qquad\Rightarrow \dim\mathfrak{P}=0$
so $\mathfrak{P}=0$ ✓

Now: $k[x]$ is Noeth & int closed. $\underset{\mathfrak{P}\text{ minimal}}{\Longrightarrow}$ $k[x]_\mathfrak{P}$ is DVR of rank 1.

$\underbrace{k[x]_\mathfrak{P}\subset\mathcal{O}_{v'}}_{\text{max'l. subring } (\mathfrak{P}\text{ minimal})}\subsetneq k(x)$ $\Rightarrow \mathcal{O}_{v'} = k[x]_\mathfrak{P}$ & v' is discrete rank 1.

$$\mathcal{O}_{v'}/\mathfrak{m}_{v'} = \frac{k[x]_\mathfrak{P}}{\mathfrak{P}k[x]_\mathfrak{P}} = \mathrm{Quot}\left(\frac{k[x]}{\mathfrak{P}}\right)$$
is a f. field $/k$.

<u>STEP 2</u>: $v'(k(x)^\times) \subset v(K^\times)$ finite indep subgroup
$\qquad\quad\;\; \| \qquad\qquad\quad \downarrow$
$\qquad\quad\;\; \mathbb{Z} \qquad\qquad$ rank 1 $\qquad \Rightarrow v$ is also a DVR.

f. field $\mathcal{O}_{v'}/\mathfrak{m}_{v'} \xhookrightarrow{\text{finite}} \mathcal{O}_v/\mathfrak{m}_v$ $\Rightarrow \mathcal{O}_K/\mathfrak{m}$ is also a function field over k. $\square$

Q: Algorithm for divisorial valuations on V with center W $(\operatorname{codim}_V W = 1)$?

Need 2 things: • effective Noether. Nzm → OK! (Eisenbud's Comm Alg book)

• effective extension of a valn on L to a (finite) alg. extension $K|L$.

$L = k(x_1, \ldots, x_r)$, v_{x_r}; $K = k(V)$.

Things we can use: ① If $K|L$ algebraic has finite separability deg $\Rightarrow$ # extensions $\leq [K:L]_S$.

CONJUGACY THM:

② If $K|L$ is a (finite) normal extension & $G = \operatorname{Aut}(K|L)$ then if $\mathcal{O}$ is a valn ring in L, any two valn. rings $\mathcal{O}', \mathcal{O}''$ in K extending $\mathcal{O}$ are conjugated

⇝ If we know 1 & G, we know them all!

• If $K|L$ is purely insep & normal, any valn $\mathcal{O}$ extends UNIQUELY to K.

$$\min(\alpha, L) = X^n + \cdots + a_0 = \prod \text{all roots} = N(\alpha) \in L.$$

$$\Rightarrow \operatorname{val}(\alpha) = \frac{\operatorname{val}(\overset{\in L}{N(\alpha)})}{n}$$

($\cap K$ (normal)) (all roots are conj $\Rightarrow$ they must have the same value!)

③ $K|L$ algebraic & $\mathcal{O}' \subset K$, $\mathcal{O} \subset L$ valn. extensions. then:

$$v': K \twoheadrightarrow \Gamma_2 = K^\times/\mathcal{O}'^\times$$
$$v: L \twoheadrightarrow \Gamma_1 = L^\times/\mathcal{O}^\times$$

(1) We get $\Gamma_1 \hookrightarrow \Gamma_2$ as ordered subgroups, & furthermore Γ_2/Γ_1 is a finite torsion gp. index = ramification index.

⇒ In particular, if $\mathcal{O}$ is DVR ⇒ $\mathcal{O}'$ is DVR
• rank Γ_2 = rank Γ_1

(2) $\mathcal{O}/m \hookrightarrow \mathcal{O}'/m'$ is an algebraic extension. degree =: residue degree

$$\boxed{\text{res deg} \cdot \text{ram index} \leq [K:L]}$$

Fact: $\sum_{i=1} \text{res deg}(\mathcal{O}'_i, \mathcal{O}_i) \cdot \text{ram index}(\mathcal{O}'_i, \mathcal{O}) \leq [K:L]$

Eg. 2-adic val. from $\mathbb{Q}$ to $\mathbb{Q}(\sqrt{17})$. $\begin{cases} \operatorname{val}(\sqrt{17}) = \frac{1}{2}\operatorname{val}(17) = \frac{1}{2} \\ \operatorname{val}(a + b\sqrt{17}) = ? \end{cases}$

④ If $K|L$ algebraic and $R = \overline{\mathcal{O}}^K$ integral closure of $\mathcal{O}$ in K & $\mathcal{O}'$ extends $\mathcal{O}$ to K; $\mathcal{O}'$–$\mathcal{O}$ valrings. Then $\mathfrak{P} = m' \cap R$ prime gives $R_{\mathfrak{P}} = \mathcal{O}'$.

(Know: $R \subsetneq \mathcal{O}'$, $m \subset \mathfrak{P}$)

$$\mathcal{O}' \supseteq m'$$
$$R \;—\; \boxed{\mathfrak{P} = m' \cap R} \text{ prime in } R \text{ containing } m.$$
INTEGRAL
$$\mathcal{O} \supseteq m = m' \cap \mathcal{O}$$

∃ algorithms for computing integral closure!

Need an algorithm to find $\mathfrak{P}$ ↔ Minimal Primes in the primary decomposition of m inside R

Projective varieties? X irred <u>normal</u> proj variety$/k$, $K = k(X) \supset k$. it's function field

[Eg: $X = \mathbb{P}^2$ $K = k\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right)$.]

$X \hookrightarrow \mathbb{P}^n$ homog ideal I.

1) Assume we have an interesting valn v on K, write $\mathcal{O} = \mathcal{O}_v$, $\dim \mathcal{O} = r < \infty$. $\Gamma = \Delta_0 \supsetneq \Delta_1 \supsetneq \cdots \supsetneq \Delta_r = \{0\}$ convex subgroups

By our Corresp. Thm from Lecture 1: $0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{m}_v$ maxl chain of prime ideals

induces chain of valuation rings $\mathcal{O}_{\mathfrak{p}_i}$ (with value gps of rank i $= \Gamma/\Delta_i$)

(*) $$K = \mathcal{O}_{(0)} \supsetneq \mathcal{O}_{\mathfrak{p}_1} \supsetneq \mathcal{O}_{\mathfrak{p}_2} \supsetneq \cdots \supsetneq \mathcal{O}_{\mathfrak{m}_v} = \mathcal{O}$$

Projective varieties are <u>proper</u>, so they satisfy the <u>valuative criterion</u> for $(\mathcal{O}_v \subset K)$

$$\begin{tikzcd}
\{0\} = \operatorname{Spec} K \arrow[r, "\underline{x}"] \arrow[d] & X \arrow[d] \\
\{0, \mathfrak{m}_v\} = \operatorname{Spec} \mathcal{O}_v \arrow[r] \arrow[ur, dashed, "\exists! \sigma_{x,v}"] & \operatorname{Spec} k
\end{tikzcd}$$

$(\underline{x} \in X(K))$ $\rightsquigarrow$ $\sigma_{x,v}(\mathfrak{m}_v) \in X$ is a pt $=$ closed pt

$\Rightarrow$ its (Zariski) closure in X is the <u>center</u> of v on X.

We compose these maps w/ the chain (*) & get

$$\sigma_{x, v_{i+1}}(\mathfrak{m}_{v_{i+1}}) \in \overline{\sigma_{x,v_i}(\mathfrak{m}_{v_i})} =: \text{center}(v_i)$$ so $\text{center}(v_i) \supseteq \text{center}(v_{i+1})$

In particular: $\text{center}(v_0) = \overline{\text{gen pt}} = X$

$\cup^{\neq}$

$\text{center}(v_1) = \overline{\sigma_{x v_1}(\mathfrak{p}_1 \mathcal{O}_{\mathfrak{p}_1})} \subsetneq X$ but inclusions for $i \geq 1$ might be $=$

② **Valuations from admissible flags:** $(\dim X = r)$

A full flag $Y_\bullet$ of irred. subvarieties $X = Y_0 \supsetneq Y_1 \supsetneq \cdots \supsetneq Y_{r-1} \supsetneq Y_r$ is <u>admissible</u> if
- $\operatorname{codim}_X(Y_i) = i$ for all $1 \leq i \leq r$, → find i many local eqns for Y_i $= 1$ for Y_i in Y_{i-1} around Y_r (around Y_r)
- Y_i is normal and smooth at Y_r for all $0 \leq i \leq r-1$.

The flag is <u>good</u> if Y_i is smooth for all $i = 0, \ldots, r$.

We define r many rank 1 valuations on $K = K(X) \ni \Phi$

$\Rightarrow$ $v_1(\phi) := \operatorname{ord}_{Y_1}(\Phi) = \text{dim. valn}_{Y_1}(\Phi)$ & set $\Phi_1 = \left.\dfrac{\phi}{g_1^{v_1(\phi)}}\right|_{Y_1}$ → kill things that don't vanish along Y_1

↳ local eqn $g_1 = 0$ for Y_1 in Y_0 (around Y_r)

$\Rightarrow$ $v_2(\Phi) = \operatorname{ord}_{Y_2}(\Phi_1)$, $\Phi_2 = \left.\dfrac{\Phi_1}{g_2^{v_2(\phi)}}\right|_{Y_2}$ for g_2 local eqn for Y_2 in Y_1

We continue to find v_i $i=3,\dots,r$ $\rightsquigarrow v=(v_1,\dots,v_r)$ is the flag valn.

By construction: each v_i has center Y_i & is $\geq$ on Y_{i-1} (killed $g_i^{\square}$ at each step!)

Props (Ciliberto, Farnik, Küronya, Lozovanu, Roé & Shramov)

A valuation of max'l rank $=\dim(X)$ whose flag of centers is admissible is equivalent to the flag valuation.

Frankfurt School (07/09 – 08/16)

LECTURE III (Appendix): Prime Valuations of the 2^{nd} kind

Recall $K = k(V)$ where $V \subseteq \mathbb{A}^n_k$ is an irred. variety of dim r

A valuation on $k(V)/k$ is a prime divisor of $K|k$ if $\mathcal{O}_v \supseteq k[V]$, $\operatorname{tr\,deg}(\mathcal{O}_v/m_v, k) = r-1$

$\mathfrak{P} := k[V] \cap m_v$ is a prime ideal of dim $\leq r-1$ & defines $W \subset V$ irred subv.
The valn v is centered at W on V. If dim $= r-1$ the valuation is divisorial & essentially $v(f)$ is the order of vanishing of $f \in k(V)$ at W. (proof of THM 1)

Q: What if $\dim \mathfrak{P} < r-1$? Can any value be achieved? A: YES!

THM 2: $W \subset V$ $\dim V = r$ irred. $k[W] = k[V]/\mathfrak{P} \leadsto$ prime.

$S_{W,V} = \{$ valns v on $k(V)/k$ prime divisors & $m_v \cap k[V] = \mathfrak{P}\}$

Then (1) $S_{W,V} \neq \emptyset$ & (2) $\displaystyle\bigcap_{v \in S_{W,V}} \mathcal{O}_v$ = integral closure of $k[V]_{\mathfrak{P}}$ in $k(V)$

Pf/. If $\dim W = r-1$, we have $S_{W,V} \neq \emptyset$ (and finite!) by THM 1 (Lecture II)

Recall: (Lecture II) $R_{\mathfrak{P}} \subset K$ ring $\Rightarrow \overline{R_{\mathfrak{P}}} = \displaystyle\bigcap_{v:\, \mathcal{O}_v \supset R_{\mathfrak{P}}} \mathcal{O}_v = \bigcap_{\substack{v:\, \mathcal{O}_v \supset R \\ m_v \cap R = \mathfrak{P}}} \mathcal{O}_v$. but $\operatorname{tr\,deg}(\mathcal{O}_v/m_v) \leq r-1$! (by DIM INEQ!)

r an index, Then $\dim W = r-1 \leq \operatorname{tr\,deg}(\mathcal{O}_v/m_v, k) \underset{\boxed{\text{Dim} \leq}}{\leq} \operatorname{tr\,deg}(k(V), k) - \underset{\geq 1}{\text{rank } v} \Rightarrow v \in S_{W,V}$ because $=$ holds

r

$\subseteq \{$all valns on $k[V]$ centered at $\mathfrak{P}\}$ so (RHS) $\subseteq$ (LHS) ✓

(1). If $\dim W < r-1$, $S_{W,V} \subsetneq$ {all valns on $k[V]$ centered at $\mathfrak{P}$}

We first show $S_{W,V} \neq \emptyset$: (Construct a valn & localize so that W's a prime divisor) (Noetherian)

STEP 1: Write $\mathfrak{P} = \langle w_1, \ldots, w_h \rangle \subset R = k[V]$

$R_i := R\left[\frac{w_1}{w_i}, \ldots, \frac{w_h}{w_i}\right] \subset k(V) \leadsto R_i \mathfrak{P} = R_i w_i$ $\left(w_j = \frac{w_j}{w_i} w_i \;\; \forall j\right)$ & $R w_i \cap R \supseteq \mathfrak{P}$ $\forall i$

Claim: $\exists\, i$ st. $R_i w_i \cap R = \mathfrak{P}$. ($\supseteq$ is clear)

Pf/ By Chevalley's Extension Thm, find v on $k(V)/k$ st $(k \subset) R \subset \mathcal{O}_v$ & $m_v \cap R = \mathfrak{P}$

Pick i satisfying $0 < v(w_i) = \min_{1 \leq j \leq h} \{v(w_j)\}$ Then $R_i \subset \mathcal{O}_v$ since $v\left(\frac{w_j}{w_i}\right) \geq 0$ $\forall j$.

$\hookrightarrow \{R \subset \mathcal{O}_v,\ w_i \in \mathfrak{P} < m_v$

And $R_i w_i \subset m_v$ Write $\mathfrak{P}' := R_i \cap m_v$ Then $\mathfrak{P}' \cap R = \mathfrak{P}$ ($m_v \cap R = \mathfrak{P}$) & $\mathfrak{P}' \supset R_i w_i$

So $R_i w_i \cap R \subset \mathfrak{P}$ as desired!

WLOG. Assume $i = 1$ (if not, relabel)

- w_1 is not a unit in R_1 $(\%w \; 1 \in \mathfrak{P})$; R_1 Noetherian

$\Rightarrow$ Write $R_1 w_1 = \bigcap$ primary ideals in R_1 & restriction to R preserves primary decomp. (*)

$$\mathfrak{P} = R_1 w_1 = \bigcap \text{ primary ideals in } R_1 \text{ assoc to } \mathfrak{P} \quad (! \text{ Ass prime to } \mathfrak{P})$$

By uniqueness of minimal primary components, $\exists$ a minimal $\mathfrak{P}_1$ in R_1 st

$R \cap \mathfrak{P}_1 = \mathfrak{P}$ and by $R_1 w_1 = R_1 \mathfrak{P}$, $\mathfrak{P}_1 \supset R_1 w_1$

Note: We cannot use Going-Up lemma because $R \subset R_1$ is NOT integral.

- **STEP 2**: . $R \subset R_1 \subset \operatorname{Quot}(R) = K$, so $\dim R_1 = \dim R = r$ $(\operatorname{tr\,deg}(K_{(1)}, k))$ (f.g. k alg)

$\mathfrak{P}_1$ is minimal over $(w_1) \neq 0 \overset{\uparrow}{\Rightarrow} \dim R_1/\mathfrak{P}_1 = r-1$.
Principal Ideal Thm

- **STEP 3**: By Chevalley's Thm, pick v_1 valn on $k(V)/k$ s.t. $R_1 \subset \mathcal{O}_{v_1}$

then: $R_1/\mathfrak{P}_1 \subset \mathcal{O}_{v_1}/m_{v_1} \subset \operatorname{Quot}(R_1/\mathfrak{P}_1)$ $\Rightarrow$ $\operatorname{tr\,deg}(\mathcal{O}_{v_1}/m_{v_1}, k) = r-1$. $\mathfrak{P}_1 = m_{v_1} \cap R_1$.

(f.g. k-alg.) [b/c $\operatorname{Quot}(R) = K$]

Also $m_{v_1} \cap R = m_{v_1} \cap R_1 \cap R = \mathfrak{P}_1 \cap R = \mathfrak{P}$ $\longrightarrow$ $\boxed{v_1 \in S_{W,V}}$ ✓

2) Next, we show (RHS) $\subseteq$ (RHS), equivalently, if $z \in k(V) \setminus$ (RHS) $\Rightarrow z \notin$ (LHS)

Since $z \notin \overline{R_{\mathfrak{P}}} \Rightarrow z \notin R$. Write $y = \frac{1}{z}$ & $R' = R[y] \supseteq R$.

By ~~Chevalley's Thm~~ characteriz. of $\overline{R_{\mathfrak{P}}}$ using valns $\exists$ v valn on $k(V)/k$ with $\mathcal{O}_v \supset k[V] = R$, $m_v \cap R = \mathfrak{P}$,
& $z \notin \mathcal{O}_v \Rightarrow v(z) < 0$

$\Rightarrow v(y) > 0$ so $y \in m_v$.

Then $\mathcal{O}_v \supset R[y]$ & $\mathfrak{P} \subset m_v \cap R[y] =: \mathfrak{P}'$ prime & $\mathfrak{P}' \cap R = \mathfrak{P}$.

$\dim R[y] = \operatorname{tr\,deg}(K, k) = r = \dim R$
(f.g. k-alg.)

Replace R & $\mathfrak{P}$ w/ R' & $\mathfrak{P}'$ in the proof of (1) $\Rightarrow \exists$ $v' \in S_{\mathfrak{P}', R'}$, ie

valn on $k(V)/k$ with $\mathcal{O}_{v'} \supset R'$, $m_{v'} \cap R' = \mathfrak{P}'$ & $\operatorname{tr\,deg}(\mathcal{O}_{v'}/m_{v'}, k) = \dim R' = r-1$

$\mathcal{O}_{v'} \supset R' \supseteq R$; $m_{v'} \cap R = \mathfrak{P}$ $\Rightarrow v' \in S_{W,V}$

Since $y \in \mathfrak{P}' \Rightarrow v'(y) > 0 \Rightarrow v'(z) < 0 \Rightarrow z \notin$ (LHS). $\square$

Lecture IV: Introduction to Berkovich Analytification

Plan: ① Trop geometry is a coordinate-wise dependent combinatorial shadow of subvarieties of tori / T.V. $\rightsquigarrow$ • What happens when we change the embeddings?
• Can we find a topological space containing ALL tropicalizations?
A: Berkovich analytification.

② How to decide which embedding is better?

§1 Setup.

Fix (K, v) valuation of rank 1 non-Archimedean $v : K^* \longrightarrow \mathbb{R}$.

• v induces a topology in K via an absolute value $|\ | : K \longrightarrow \mathbb{R}_{\geq 0}$ $\quad |x| = e^{-v(x)}$

Properties: • $|a| = 0 \iff a = 0$
• $|ab| = |a|\,|b|$ (multiplicative)

Non-arch Δ-ineq $\rightarrow$ • $|a+b| = e^{-v(a+b)} \leq e^{\max\{-v(a), -v(b)\}} = \max\{|a|, |b|\}$
ULTRAMETRIC $\quad = \text{ if } |a| \neq |b|$

($|\underbrace{1+1 \cdots +1}_{n}| \leq 1$ for any n = non-Archimedean absolute value.

Topology: Basis $B(x,r) = \{y : |x-y| < r\}$.

Claim: Open balls are closed! ($|x-z| \geq r \Rightarrow$ Pick $0 < \varepsilon < r$, so $B(z,\varepsilon) \subseteq K \smallsetminus B(x,r)$)
$|x-y| = |(x-z) + (z-y)| = \max = |x-z| \geq$, with $|x-z| \geq r$, $|z-y| < \varepsilon$

Every pt in $B(x,r)$ is it's center $B(x,r) = B(y,r)$ iff $|x-y| < r$.

$\rightsquigarrow$ $(K, |\ |)$ totally disconnected $\rightarrow$ Analysis in this spaces breaks down!

[We can assume K is complete wrt this abs. value, if not take $\hat{K}$ & $\operatorname{val} = -\log|\ |_{\hat{K}}$ gives the ! valuation on $\hat{K}$ extending val$_K$.

Eg: • K w/ trivial valn
• $K = \widehat{\mathbb{C}((t))} = \mathbb{C}\{\{t^{\mathbb{R}}\}\}$ (exp well-ordered) w/ t-val.
• $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$ p-adic valn.

Berkovich's Theory adds enough pts to the space $(K, X$ variety / scheme$)$ to fix the nasty topology.

§2: Analytification à la Berkovich

K field w/ rank 1 valuation $\operatorname{val} = -\log|\ |_K$, $k = \mathcal{O}_v/\mathfrak{m}_v$ residue field.

Idea: X K-scheme of f. type /K. $\underset{\text{[GAGA functor]}}{\rightsquigarrow}$ X^{an} = Top space +
(w Gubler's Talk) (gluing of Spec(A) A f.g k-alg)

Do it on affine schemes & glue! Fix A f.g K-algebra $A = \dfrac{K[x_1, \ldots, x_n]}{I}$
$K[x]$ coord ring of $X \subseteq \mathbb{A}^n$.

Defn: $(\operatorname{Spec} A)^{an} = \{ \|\ \| : A \longrightarrow \mathbb{R}_{\geq 0}$ multiplicative seminorms on A extending $|\ |_k \}$

$\hookrightarrow \|a\|=0 \not\Rightarrow a=0.$

$\|f\| = \|f+I\|$ & $\|f\|=0$ for all $f \in I$., further $\|f\|=0\ \forall f \in \sqrt{I}$.

Topology: Weakest such that $ev_f : \|\ \| \longmapsto \|f\|$ are cont. $\forall f \in A$. ($X^{an} \subset \mathbb{R}_{\geq 0}^{K[X]}$ w/ prod top)

Note: ① $X(K) \longhookrightarrow X^{an}$ & compatible with ~~extends to~~ field extensions $L|K$.

$\varphi : A \to K \longmapsto \|\ \|_\varphi : f \mapsto |\varphi(f)|_k$
k-alg map

$$\begin{array}{ccc} X(L) & \longrightarrow & X^{an} \\ \uparrow & \circlearrowleft & \nearrow \\ X(K) & & \end{array}$$

(val on L extending val_k ⇝ $|\ |_L$ extending $|\ |_k$)

Description II:

② X^{an} as a space of all valuations on $X = \operatorname{Spec} A$:

If $\|\ \| \in X^{an} \Rightarrow \ker(\|\ \|) := \{a \in A \mid \|a\| = 0\}$ is a prime ideal $\mathfrak{p}$ of A.
(seminorm!) (closed under $+$ in the ideal, $A\cdot$, prime is clear)

We set $K_{\mathfrak{p}} = \operatorname{Quot}(A/\mathfrak{p})$ is a field extension of K with ! absolute value $= \|\ \|$ extending $|\ |_k$.

Write $\nu_{\mathbb{R}}(L) = \{ (L, \omega) \mid$ $L|K$ field extension, ω = valn on L extending $val_K \}$

This gives $X^{an} \twoheadrightarrow \operatorname{Spec} A$ & fiber over a pt $\mathfrak{p}$ is $\nu_{\mathbb{R}}(K_{\mathfrak{p}})$.
$\|\ \| \longmapsto \operatorname{Ker}(\|\ \|)$ (prime ideal)

Corollary: $\boxed{X^{an} = \bigsqcup_{\mathfrak{p} \in X} \nu_{\mathbb{R}}(K_{\mathfrak{p}})}$ (space of valuations) ⇝

- $\mathfrak{p}$ closed pt $\Rightarrow \mathfrak{p}$ = mxl ideal $\Rightarrow K_{\mathfrak{p}}$ is algebraic over $K \underset{K \text{ complete}}{\Rightarrow} |\ |_k$ extends uniquely to $K_{\mathfrak{p}}$ so $\nu_{\mathbb{R}}(K_{\mathfrak{p}}) = \{ |\ |_k \}$
- Replace each non-closed pt w/ $\nu_{\mathbb{R}}(K_{\mathfrak{p}})$.

Description III: $L|K$ valued field extension. ($val_L|_K = val_K$)

Consider $(L, val_L, x \in X(L))$ w/ equivalence relation generated by $(L, val_L, x) \sim (L', val_{L'}, x')$ if $\exists\ L \hookrightarrow L'$ s.t. $val_{L'}|_L = val_L$ & $X(L) \hookrightarrow X(L')$, $x \longmapsto x'$.

Using this $\boxed{X^{an} = \{ (L, val_L, x \in X(L)) \} /\sim}$

[Ranganathan-Foster series of papers]

- We can use this definition, to extend to higher rank valuations, (HAHN ANALYTIFICATION) Only issue: $\Gamma = val(K^\times)$ of rank $k < \infty$

view $\Gamma \subset \mathbb{R}^k$. $\mathbb{R}^k \cup \{\infty^k\}$ has 2 options for Topology $\Rightarrow$ 2 analytifications

OPT 1: Basis on $\overline{\mathbb{R}}$ ⇝ $\mathbb{R}^k \cup \{\infty\}^k \subseteq (\overline{\mathbb{R}})^k$. & use subspace top ⇝ $X^{\S}$

OPT 2: Basis: $(a,b) = \{ x \in \mathbb{R}^k_{lex} : a < x < b \}$, $(a, \infty] = \{ x : a \leq x \}$ ⇝ $X^{\#}$

Theorem [Berkovich '90]

(1) X^{an} is locally compact, locally path connected

(2) X connected $\iff$ X^{an} is path connected

X/K sep. $\iff$ X^{an} is Hausdorff

X/K proper (compact) $\iff$ X^{an} is compact

→ (3) If $|\,|_K$ is non-trivial, then $X(\overline{K})$ is dense in X^{an}.

Example: Skeleton (semi) norms on $(\mathbb{A}^n)^{an}$ induce $\overline{\mathbb{R}}^n \hookrightarrow (\mathbb{A}^n)^{an}$

Given: $\rho \in \overline{\mathbb{R}}^n \rightsquigarrow \delta(\rho) : K[x_1,\dots,x_n] \longrightarrow \mathbb{R}_{\geq 0}$

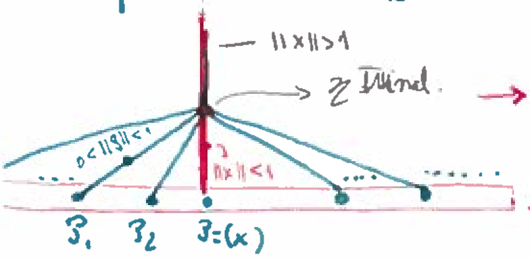
$$\underset{\text{finite}}{f = \sum c_\alpha x^\alpha} \longmapsto \max_\alpha \Big\{ |c_\alpha| \exp\Big(\sum_{i=1}^{n} \alpha_i \rho_i\Big)\Big\}$$

$$= \exp\big(\underbrace{\operatorname{Trop}(f)(\rho_i)}_{\text{in } \mathbb{R}}\big) \in \mathbb{R}_{\geq 0}$$

KEY: • Each f has a ! polynomial rep-n $\Rightarrow \delta(\rho)$ is a well-defined norm

• $\delta(\rho)(x_i) = \exp(\rho_i)$ $\forall i$ & it's maximal among all $\|\,\|$ w/ these values at the coordinates $x_1,\dots,x_n$

Examples ① $n=1$ & $|\,|_K$ trivial. $\|f\| = \Big|\sum_{i=m}^{N} c_i x^i\Big\| \leq \max_{c_i \neq 0} \{\|x\|^i\}$. and $=$ if all $\neq$.



skeleton → $[0,\infty) \sim \overline{\mathbb{R}}$ log.

leaves = K-points

$c_m, c_N \neq 0$

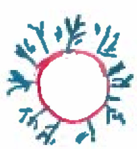
$$\|f\| = \begin{cases} \|x\|^N & \text{if } \|x\|>1 \\ \|x\|^m & \text{if } \|x\|<1 \\ ? & \text{if } \|x\|=1 \quad (*) \Rightarrow \|f\| \leq 1. \end{cases}$$

(*) $\underset{\text{prime}}{\mathfrak{P}} = \{f : \|f\|<1\} = \langle g \rangle$ ($K[x]$ PID) $\Rightarrow f = g^s h$ $(h:g)=1$ $\rightarrow \|f\| = \|g\|^s$ $\quad \|\,\| \longleftrightarrow (\mathfrak{P}, \|g\|)$ $0<\ <1$

② $|\,|_K$ non-trivial $\Rightarrow$ dense set of branch points (as ζ)

• In general, for X/K curve: X^{an} is locally homeo to $(\mathbb{A}^1)^{an}$ but may have non-trivial global topology, captured by a finite graph = skeleton (associated to models* of X over $\mathcal{O}_v$ (W Gubler's Talk).) (X^{an} deformation retracts onto skeleton)

• Eg: E Elliptic curve. (j_E = j-invariant) $\in K$



skeleton $= S^1$ BAD red.

skeleton = pt

GOOD red.

• $X^{an} \setminus$ leaves has a natural metric structure. skeleton has length $= \{-\operatorname{val}(j_E)$.

§3 Relation To Tropicalization:

$X \overset{i}{\underset{cl}{\hookrightarrow}} Y_\Sigma$ T.V. where $i(X)$ meets the dense torus $(K^\times)^n = \mathrm{Spec}(K[y_1^{\pm},\dots,y_n^{\pm}])$

(eg $Y_\Sigma = (K^\times)^n$, $\mathbb{A}^n$, $\mathbb{P}^n$ w/ torus $(K^\times)^{n+1}/K^\times$)

$$\begin{array}{ccc} X^{an} & \xrightarrow[\text{CONT}]{(\mathrm{trop}, i)} & (-\log\|y_1\|,\dots,-\log\|y_n\|) \\ \updownarrow & \circlearrowleft & \\ X(K) & \longrightarrow & \mathrm{Trop}(X,i) \subseteq \mathrm{Trop}\,Y_\Sigma = \bigsqcup_\sigma \mathrm{Trop}(\mathcal{O}_\sigma) \\ \underline{x} & \longmapsto & (\mathrm{val}(x_1),\dots,\mathrm{val}(x_n)) \\ & & \| \\ & & (-\log|x_1|,\dots,-\log|x_n|) \end{array}$$

$\mathcal{O}_\sigma \simeq (K^\times)^{n-\dim\sigma}$

Diane's Lecture I: $\mathrm{Trop}(X,i) = cl\{\mathrm{val}(X(L))\}$

What happens under reembeddings into higher-dim toric var. (& equivariant maps)

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "i'"'] & Y_\Sigma \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] & \supset (K^\times)^n \arrow[d] \\ & Y_{\Sigma'} & \supseteq (K^\times)^m \end{tikzcd}$$

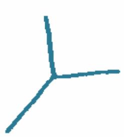
monomial map $A \in \mathbb{Z}^{m\times n}$ $\qquad \mathrm{Trop}(X,i') = A\,\mathrm{Trop}(X,i)$

⇝ form an inverse system

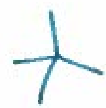
Payne '09: X^{an} is homeomorphic to the inverse limit of all tropicalizations $\mathrm{trop}(X \hookrightarrow Y_\Sigma)$

Works also for quasi-projective varieties

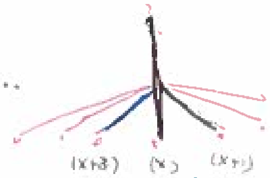
Eg:



$(x+y+1) \subseteq \mathbb{A}^1$ $\quad y=-(x+1)$ $\longleftarrow$



$x+y+1=0$, $x+z+2=0$ $\subseteq \mathbb{A}^2$ $\quad y=-(x+1)$, $z=-(x+2)$ $\longleftarrow \cdots$



Q: Can we see $\mathrm{Trop}(X,i)$ as a closed subset of X^{an} for some i? What about metric str? (FAITHFUL TROPICALIZATION)

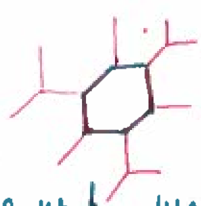
Thm [Baker-Payne-Rabinoff] Γ finite subgraph of $X^{an} - X(K)$. Then, there exists a closed embedding $X \hookrightarrow Y(\Delta)$ toric quasiproj var. such that Γ maps isometrically onto its image in $\mathrm{Trop}(X)$.

[Baker-Werner-Rabinoff] Analogous results for higher dimensional varieties.

. Tropical multiplicity (= # components of $\mathrm{in}_w X$) $=1$ in a torus (+ boundary conditions for T.V)

ensure we can lift $\mathrm{trop}(X,i)$ to X^{an} continuously.

Eg: Elliptic cubic curves w/ bad reduction



[KMM], [BPR]
trivalent loop $\Rightarrow$ length(loop) $= -\mathrm{val}(j_E)$

Q: What if not trivalent? $-\mathrm{val}(j_E)$ could go up! We can do a local discriminant calculation at a vertex w to detect & repair problems [-, Markwig] ⇝ Trop Modifications