LECTURE I: Berkovich Analytic Spaces from the Tropical perspective

Plan: ① Tropical Geometry is a coordinatewise dependent combinatorial shadow of subvarieties of tori / Toric Varieties.

$\rightsquigarrow$ Q1 What happens when we change the embeddings equivariantly?

Q2. Can we find a topological space containing ALL tropicalizations?

A: Berkovich non-Archimedean analytification!

② How to effectively decide which embedding best reflects the geometry of the analytic space?

§1 Setup: non-Archimedean fields

Fix (K, ν) rank-1 valuation $\nu : K^{\times} \longrightarrow \mathbb{R}$ $\left(\begin{array}{l} \nu(ab) = \nu(a) + \nu(b) \\ \nu(a+b) \geq \min\{\nu(a), \nu(b)\} \end{array} \right)$ Extend by $\nu(0) = \infty$

Then, ν induces a Topology on K via a non-Archimedean absolute value (multiplicative norm)

$$|\ |: K \longrightarrow \mathbb{R}_{\geq 0} \qquad |x| = e^{-\nu(x)}$$

Properties:
- $|a| = 0 \iff a = 0$ ($|\ |$ has no kernel)
- $|ab| = |a||b|$ multiplicative
- $|a+b| = e^{-\nu(a+b)} \leq e^{\max\{-\nu(a), -\nu(b)\}} = \max\{|a|, |b|\}$ non-Archimedean Δ-inequality ULTRAMETRIC
  $=$ if $|a| \neq |b|$

Non-Archimedean $=$ $|\underbrace{1 + \cdots + 1}_{n \text{ times}}| \leq 1 \quad \forall n$.

TOPOLOGY: Basis $B_0(x, r) = \{y : |x-y| < r\}$ $\quad (B(x,r) := \{y : |x-y| \leq r\})$

Remark: Open balls are closed!

Pf/ $|x-z| \geq r \Rightarrow$ Pick any $0 < \varepsilon < r$ & see $B_0(z, \varepsilon) \subset K \setminus B_0(x, r)$

Indeed $|x-y| = |\underbrace{(x-z)}_{|\ | \geq r} + \underbrace{(z-y)}_{< \varepsilon < r}| = \max\{|x-z|, |z-y|\} \geq r$ if $y \in B_0(z, \varepsilon)$ □

Equivalently: Every pt y in $B(x, r)$ gives $B(x, r) = B(y, r)$ ($\rightsquigarrow$ Two disks are either disjoint or the same!)

Pf/ Enough to show ($\subseteq$) $|y-z| = |\underbrace{y-x}_{|\ | \leq r} + \underbrace{x-z}_{|\ | \leq r}| \leq r$ for any $z \in B(x, r)$. □

Conclusion: Topology is totally disconnected $\rightsquigarrow$ analysis on these spaces breaks down!

Soln (1) Rigid analytic geometry (Tate) $\rightsquigarrow$ work w/ Grothendieck topology

(2) Berkovich = add pts to K (or to any scheme of finite type $/K$) to fix the nasty Topology

For most applications & properties, we want to assume K is complete w.r.t this abs. value (& alg closed) sometimes

If not, take $\hat{K}$ completion and give it the extended abs. value:

$$\|\lim_\alpha x_\alpha\| := \lim_\alpha |x_\alpha| \qquad \left(\text{show } \big||x_\alpha| - |x_\beta|\big| \leq |x_\alpha - x_\beta| \xrightarrow[\alpha,\beta\to\infty]{} 0 \text{ because } (x_\alpha) \text{ is Cauchy} \right)$$

Show: $(\hat{K}, |\,|)$ is non-Arch. complete abs. value.

$\rightsquigarrow$ $val = -\log|\,|_{\hat{K}}$ is the! valuation on $\hat{K}$ extending v on K.

EXAMPLES: (1) Any K with trivial valuation ($v(x)=0\ \forall x\neq 0$, $|x|=1\ \forall x\neq 0$)

(2) $K = \widehat{\overline{\mathbb{C}((t))}} \subsetneq$ generalized Puiseux series with t-valuation = lowest exp in t of power series.

[finitely many exp in $\mathbb{Q}$ below any bound.] — exponents from a well-ordered subset of $\mathbb{R}$. $|K^\times| = \mathbb{Q}$

(3) $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ w/ p-adic valuation. $v_p(\frac{a}{b}) = v_p(p^r \frac{c}{d}) = r \rightsquigarrow |\frac{a}{b}|_p = p^{-r}$, $p\nmid c$, $p \nmid d$

(2 pts are closed p-adically if their difference is divisible by a large power of p)

Note: $|\mathbb{Q}_p^\times| =$ value gp $= \mathbb{Z} \rightsquigarrow |\overline{\mathbb{Q}_p}^\times| = \mathbb{Q}$. (in general $|\hat{K}^\times| = |K^\times|$)

FACT: $|\mathbb{C}_p^\times| = \mathbb{Q}$. $\rightsquigarrow$ norms $= p^{\mathbb{Q}}$

§2. Berkovich Analytification

$$\begin{cases} K^\circ = \{f\in K : |f|\leq 1\} \equiv \{x\in K \mid v(x)\geq 0\} & \text{valuation ring} \\ K^{\circ\circ} = \{f\in K : |f|<1\} \equiv \{x\in K \mid v(x)>0\} & \text{maximal ideal} \\ \tilde{K} = K^\circ/K^{\circ\circ} \quad \text{residue field} \end{cases}$$

Eg (1) $K^\circ = K \Rightarrow \tilde{K} = \{0\}$. (2) $\tilde{K} = \mathbb{C}$ (3) $\tilde{K} = \overline{\mathbb{F}_p}$.

X = K-scheme of finite type / K $\xrightarrow{\boxed{\text{GAGA FUNCTOR}}}$ X^{an} = Top space + sheaf of analytic functions

focus on this for tropicalization purposes

• Build X^{an} by gluing $(\operatorname{Spec} A)^{an}$ for A f.g. K-algebra, say $A = K[x_1,\dots,x_n]/I$.

Def: $(\operatorname{Spec} A)^{an} = \{\|\ \| : A \to \mathbb{R}_{\geq 0}$ multiplicative seminorms on A extending $|\,|_K\}$

seminorm: $\|a\| = 0 \not\Rightarrow a = 0$.

[In particular: $\|f\| = \|f+I\|$, $\|f\|=0$ for all $f\in I$ (further $\forall f\in\sqrt{I}$)]

TOPOLOGY: Weakest such that $ev_f : \|\ \| \longmapsto \|f\|$ are continuous $\forall f\in A$. (evaluation maps)

$\Rightarrow (\operatorname{Spec} A)^{an} \subset \mathbb{R}_{\geq 0}^A$ with product topology.

A Banach ring w/ norm $\|\ \|$ (Berkovich original setting): • $|1|\leq 1$, $|fg|\leq |f|\,|g|$ • $|f| \leq C\|f\| \rightsquigarrow$ bounded ($|\,|$ continuous), (if multiplicative can take $C=1$) $\rightsquigarrow \mathcal{M}(A) \subseteq \prod_{f\in A}[0,\|f\|]$ compact. HARD: $\mathcal{M}(A) \neq \emptyset$

Note: (1) $X(K) \longrightarrow X^{an}$ ($X = \operatorname{Spec} A$)

$\varphi: A \to K$ K-algebra map $\longmapsto (\|\ \|_\varphi : f \longmapsto |\varphi(f)|)$

(2) Compatible w/ valued field extensions $(L, \mathrm{val}_L) \mid (K, \mathrm{val}_K)$, ie $\mathrm{val}_L|_K = \mathrm{val}_K$

$\rightsquigarrow \exists\ |\ |_L$ extending $|\ |_K$.

$$\begin{tikzcd} X(L) \arrow[r] & X^{an} \\ X(K) \arrow[u] \arrow[ur] \arrow[u, phantom, shift left=4ex, "\circlearrowleft"] & \end{tikzcd}$$

for $X = \operatorname{Spec} A$, and then glue.

EXAMPLE: Skeleton (semi) norms on $(\mathbb{A}^n)^{an}$ induce $\overline{\mathbb{R}}^n \hookrightarrow (\mathbb{A}^n)^{an}$ where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$

given $p \in \overline{\mathbb{R}}^n \rightsquigarrow \delta(p)$ a (semi) norm mult on $K[x_1, \ldots, x_n]$ (w/ values in $\mathbb{R}_{\geq 0}$.)

$$\delta(p) : K[x_1, \ldots, x_n] \longrightarrow \mathbb{R}_{\geq 0} \qquad \text{with} \quad \delta(x_i) = \exp(p_i) \geq 0$$

$$f = \sum_{\text{finite}}{}' c_\alpha x^\alpha \longmapsto \max_\alpha \{ |c_\alpha| \exp(\underbrace{\sum_{i=1}^n \alpha_i p_i}_{= \langle \alpha, p \rangle}) \} = \exp(\underbrace{\operatorname{trop}(f)(p_i)}_{\text{in } \overline{\mathbb{R}}}) \in \mathbb{R}_{\geq 0}$$

KEYS: • Each f has a ! polynomial representative $\Rightarrow \delta(p)$ is well-def & norm.

• $\delta(p)(x_i) = \exp(p_i)\ \forall i$ & it is maximal among all $\|\ \|$ with these prescribed values on the coordinates $x_1, \ldots, x_n$ (in Shilov boundary of $\operatorname{trop}^{-1}(\exp(p_1), \ldots, \exp(p_n))$)

Example: $n = 1$ & $|\ |_K$ trivial $\quad f = \sum_{i=m}^N c_i x^i \quad \rightsquigarrow \|f\| \leq \max_{c_i \neq 0} \{1 \cdot \|x\|^i\}$.(nonArch)

$c_m, c_N \neq 0. \qquad\qquad = \text{ if all } \neq.$

Q: Can we deduce $\|f\|$ from $\|x\|$? ⚠ not always.

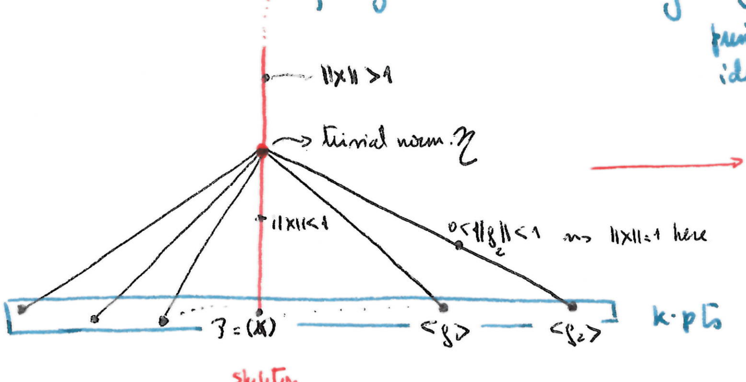
$$\|f\| = \begin{cases} \|x\|^N & \|x\| > 1 \\ \|x\|^m & \|x\| < 1 \\ ? & \|x\| = 1 \end{cases}$$

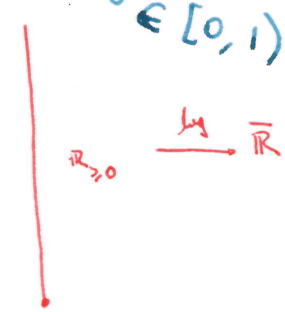
$\|f\| \leq 1\ \forall f \rightsquigarrow \mathfrak{p} = \{ f \mid \|f\| < 1\}$ is max'l ideal $\Rightarrow$ Write $f = g^s h$ for $(g:h) = 1$

$= \langle g \rangle$ b/c $K[x]$ is PID $\qquad\qquad \|h\| = 1$

Then $\|f\| = \|g\|^s$

so $\|f\|$ is uniquely determined by $\underset{\text{prime ideal}}{\mathfrak{p}}$ & $\|g\| \in [0, 1)$.





Topology = on each open segment (branch) $\subseteq \mathbb{R}$.

Nbhd of η = open segments on finitely many branches & others = whole branch

$\Downarrow$

$\boxed{\text{Connected top space}}$

$\exists$! ramification pt (with ∞ branches). Leaves = K-points

• $(\mathbb{A}^1)^{an}$ contracts to η

NEXT TIME: curves, non-trivial valn

- **Description II**: X^{an} is the space of all valuations on X Enough: $X = \operatorname{Spec} A$.

  If $\| \ \| \in X^{an} \Rightarrow \operatorname{Ker}(\| \ \|) := \{a \in A \mid \|a\| = 0\}$ is a prime ideal $\mathfrak{p}$ of A

  (closed under $+$ by non-Arch Δ-ineq, prime by mult property)
  $A\cdot$ " mult. property

  $\rightsquigarrow$ Define $K_{\mathfrak{p}} = \operatorname{Quot}(A/\mathfrak{p})$ is a valued field extension of K with ! abs. value $= \| \ \|$ extending $| \ |_K$.

  Write $\mathcal{V}_{\mathbb{R}}(L) = \{(L, \omega) \mid L|K$ field extension, $\omega =$ valn on L extending $\mathrm{val}_K\}$

  We get $X^{an} \twoheadrightarrow \operatorname{Spec} A$, $\| \ \| \longmapsto \operatorname{Ker}(\| \ \|)$ & fiber over a pt $\mathfrak{p}$ is $\mathcal{V}_{\mathbb{R}}(K_{\mathfrak{p}})$.

Corollary $X^{an} = \bigsqcup_{\mathfrak{p} \in X} \mathcal{V}_{\mathbb{R}}(K_{\mathfrak{p}})$ space of valns.

- $\mathfrak{p}$ closed pt $\Rightarrow \mathfrak{p}$ mxl ideal $\Rightarrow K_{\mathfrak{p}}$ is algebraic over $K \underset{K \text{ complete}}{\Longrightarrow} | \ |_K$ extends uniquely to $K_{\mathfrak{p}}$, so $\mathcal{V}_{\mathbb{R}}(K_{\mathfrak{p}}) = \{| \ |_K\}$
- Replace each non-closed pt w/ $\mathcal{V}_{\mathbb{R}}(K_{\mathfrak{p}})$

- **Description III**: $L|K$ valued field extension

  Consider $(L, \mathrm{val}_L, x \in X(L))$ w/ equivalence relation generated by:

  $(L, \mathrm{val}_L, x) \sim (L', \mathrm{val}_{L'}, x')$ if $\exists\, L \hookrightarrow L'$ st $\mathrm{val}_{L'}|_L = \mathrm{val}_L$ & $X(L) \hookrightarrow X(L')$, $x \longmapsto x'$

  Using this: $\boxed{X^{an} = \{(L, \mathrm{val}_L, x \in X(L))\}/\sim}$ [Ranganathan-Foster papers]

- Use this definition to extend analytification functor to higher rank valuations (HAHN ANALYTIF) ↪ Kiumars Kaveh's TAPAS course

  Higher rank valuation $\equiv |K^\times| \subset \mathbb{R}^k$ with lex order for some $k > 1$.

- $\overline{\mathbb{R}}^k$ has 2 options for topology $\Longrightarrow$ 2 analytifications:

  Option 1: Basis on $\overline{\mathbb{R}} \rightsquigarrow \mathbb{R}^k \cup \{\infty\}^k \subseteq (\overline{\mathbb{R}})^k$ & use subspace topology $\rightsquigarrow X^{\xi}_{\#}$

  Option 2: 

  Basis: $(a, b) = \{x \in \mathbb{R}^k_{lex} : a < x < b\}$, $(a, \underset{\infty^k}{\overset{\shortparallel}{\infty}}] = \{x : a \leq_{lex} x\} \rightsquigarrow X^{\xi}$

Theorem [Berkovich '90]

(1) X^{an} is locally compact, locally path connected

(2) X connected $\Longleftrightarrow X^{an}$ is path connected
$X|K$ sep $\Longleftrightarrow X^{an}$ is Hausdorff
$X|K$ proper $\Longleftrightarrow X^{an}$ is compact

(3) If $| \ |_K$ is non-trivial, then $X(\overline{K})$ is dense in X^{an}

§3: Tropical shadows:

$X \underset{cl}{\xhookrightarrow{i}} Y_\Sigma$ Toric variety where $i(X)$ meets the dense torus $(K^\times)^n = \mathrm{Spec}[y_1^{\pm}, \ldots, y_n^{\pm}]$

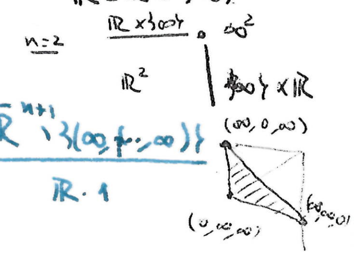
[extended tropicalizations]

(eg $Y_\Sigma = (K^\times)^n$, $\mathbb{A}^n$, $\mathbb{P}^n$ w/ torus $(K^\times)^{n+1}/K^\times$)

Fund Thm of Trop Geom: $\mathrm{Trop}(X, i) = \text{closure}\ \{ \mathrm{val}(X(L)) \}$ $L|K$ nontrivial valn field extension $\} \subseteq \mathrm{Trop}\,Y_\Sigma$

$$\mathrm{Trop}\,Y_\Sigma = \bigsqcup_{\sigma < \Sigma} \mathrm{Trop}(\mathcal{O}_\sigma) \qquad \mathcal{O}_\sigma \cong (K^\times)^{n-\dim\sigma}$$

Eg $\mathbb{A}^n \rightsquigarrow \mathrm{Trop}\,\mathbb{A}^n = \overline{\mathbb{R}}^n$ $\quad \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$



$\mathbb{P}^n \rightsquigarrow \mathrm{Trop}\,\mathbb{P}^n = \mathbb{TP}^n = \dfrac{\overline{\mathbb{R}}^{n+1} \smallsetminus \{(\infty, \ldots, \infty)\}}{\mathbb{R} \cdot 1}$

$$\begin{tikzcd}
X^{an} \arrow[drr, "{(\mathrm{trop}, i)}", "\text{CONT \& SURJ}"'] & \| \ \| \arrow[r, mapsto] & (-\log\|y_1\|, \ldots, -\log\|y_n\|) \\
X(K) \arrow[u, hook] \arrow[rr, "{(\mathrm{trop}, i)}"] & & \mathrm{Trop}(X, i) \subset \mathrm{Trop}(Y_\Sigma) \\
x \arrow[rr, mapsto] & & (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) = (-\log|x_1|, \ldots, -\log|x_n|)
\end{tikzcd}$$

Q: What happens under equivariant re-embeddings into higher toric varieties?

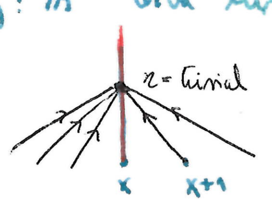
$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[dr, "i'"'] & Y_\Sigma \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] & \supset (K^\times)^n \arrow[d, "\text{monomial map } A \in \mathbb{Z}^{m \times n}"] \\
& Y_{\Sigma'} & \supseteq (K^\times)^m
\end{tikzcd}$$

$\Rightarrow \boxed{\mathrm{Trop}(X, i') = A\ \mathrm{Trop}(X, i)}$

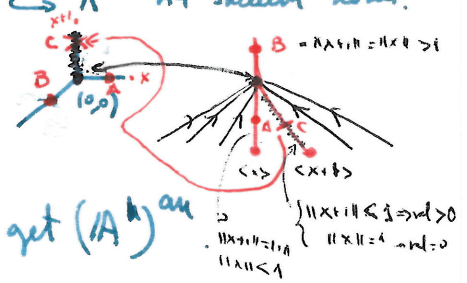
$\rightsquigarrow$ embeddings form an inverse system

Thm [Payne '09] $\quad X^{an} \underset{\text{homeo}}{\simeq} \varprojlim_{X \xhookrightarrow{i} Y_\Sigma} \mathrm{Trop}(X, i)$ $\quad$ (Y_Σ affine sp., quasiproj toric embeddings) [Foster-Gross-Payne] Y_Σ toric var.

Eg: $\mathbb{A}^1$ with trivial valuation $\quad \mathbb{A}^1 \xhookrightarrow{} \mathbb{A}^1 \rightsquigarrow \mathrm{Trop}(X, i) = \overline{\mathbb{R}} \hookrightarrow X^{an}$ via Skeleton norm.



$\rightsquigarrow \quad \mathbb{A}^1 \hookrightarrow \mathbb{A}^2$ $\quad x \longmapsto (x, x+1)$ $\quad y = x+1.$ $\quad \mathrm{Trop}(X, i) =$



$\Rightarrow$ we add one more dimension per branch and in the limit we get $(\mathbb{A}^1)^{an}$

Q Can we see $\mathrm{Trop}(X, i)$ as a closed subset of X^{an} for some i?

Equiv, can we find a continuous section σ to $(\mathrm{trop}, i) : X^{an} \twoheadrightarrow \mathrm{Trop}(X, i)$?

If so, we say the Tropicalization is faithful (alternative versions: required integral affine structure / metric on the image of σ)

<u>Q2</u>: Can we detect faithfulness solely from $\mathrm{Trop}(X,i)$, or local information on initial degenerations of X? but w/o knowing X^{an}!

<u>Q3</u>: If we detect a problem, can we repair the embedding? Effectively?

tropical modifications

- <u>Examples</u>: (Hyperelliptic) curves, Grassmannians.
- <u>Next time</u>:
  - $(\mathbb{A}^1)^{an}$ over non-trivial valuation & classification of points.
  - Metrics & piecewise linear structures (skeletons) on analytic curves.