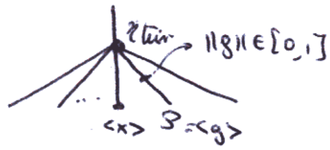
LECTURE II: Berkovich Analytic Spaces from the Tropical Perspective

<u>Last time</u>: $(K, |\,|_K)$ non-Archimedean field ($|\,|_K$ multiplicative norm) K complete & alg closed.

X = K-scheme of f. type, say $X = \mathrm{Spec}(A)$

$X^{an} = \{\|\ \|: A \longrightarrow \mathbb{R}_{\geq 0}$ multiplicative seminorms extending $|\,|_K\}$.

$(\mathbb{A}^1)^{an}$ w/ $|\,|_K$ trivial



<u>TODAY</u>: Analytic curves, metrics & skeletons. (with examples!)

§1 $(\mathbb{A}^1)^{an}$ with K non trivially valued:

<u>Example</u>: $K = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ ($\mathbb{Q}_p$ = completion of $\mathbb{Q}$ wrt <u>p-adic norm</u>: $|\frac{a}{b}|_p = |p^r \frac{a'}{b'}| = p^{-r}$ $p \nmid a', p \nmid b'$

• <u>Construct (~~semi~~)norms on $K[T]$</u>

$\mathbb{D}(a,r) = \{z \in K \mid |z-a| \leq r\} \subseteq K$ for $a \in K$. (<u>disc</u>)

<u>Def</u> $|\,|_{\mathbb{D}(a,r)}$ on $K[T]$ as $\boxed{|f|_{\mathbb{D}(a,r)} = \sup_{z \in \mathbb{D}(a,r)} |f(z)|}$ (sup-norm)

<u>Claim</u>: $|\,|_{\mathbb{B}(a,r)}$ is multiplicative $\Rightarrow |\,|_{\mathbb{B}(a,r)}$ defines a pt in $(\mathbb{A}^1)^{an}$.

Pf/ Use Gauss' Lemma to factor $f = c \prod_{i=1}^{N} (T - \alpha_i)$ <u>easy</u>: can assume $c=1$ ($|f| = |c|\,|\frac{f}{c}|$)

. If $\alpha_i \notin \mathbb{D}(a,r)$ $|z - \alpha_i| = |\underbrace{z-a}_{\leq r} + \underbrace{a-\alpha_i}_{> r}| = |a - \alpha_i|$ $\forall z \in \mathbb{D}(a,r)$

. If $\alpha_i \in \mathbb{D}(a,r) = \mathbb{D}(\alpha_i, r) \Rightarrow |z - \alpha_i|$ is maximal over any pt in the "boundary of the ball" & takes value $= r$.

$\Rightarrow |f|_{\mathbb{D}(a,r)} = r^{\#\{\alpha_i \in \mathbb{D}(a,r)\}} \cdot \prod_{\alpha_i \notin \mathbb{D}(a,r)} |a - \alpha_i|$ & $|\,|_{\mathbb{D}(a,r)}$ is multiplicative □.

<u>Note</u>: (1) If $r \notin |K^\times|$, then the sup will <u>not</u> be achieved if f has a root inside $\mathbb{B}(a,r)$.

(2) ~~View~~ $K \hookrightarrow (\mathbb{A}^1)^{an}$ as $a \mapsto |\,|_{\mathbb{B}(a,0)}$ evaluation at a.

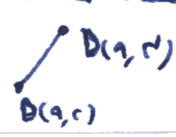
(3) If $r > 0$, $|\,|_{\mathbb{D}(a,r)}$ is a <u>norm</u> $|T - b|_{\mathbb{B}(a,r)} = \begin{cases} r \neq 0 & \text{if } b \in \mathbb{D}(a,r) \smallsetminus \{a\}. \\ |b-a| \neq 0 & \text{if } b \notin \mathbb{D}(a,r). \end{cases}$

(4) $|\,|_{\mathbb{D}(a,r)}$ are <u>all distinct</u>: $\left(\begin{array}{l} \cdot \text{ Balls are disjoint } |T-a'|_{\mathbb{B}(a,r)} = |a-a'| \text{ but } |T-a'|_{\mathbb{B}(a',r')} = r' \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad > r' \\ \cdot\ \mathbb{B}(a,r) \subsetneq \mathbb{B}(a,r') \text{ pick } b \in \mathbb{B}(a,r') \smallsetminus \mathbb{B}(a,r) \text{ \& } f = |T-a|. \end{array} \right)$

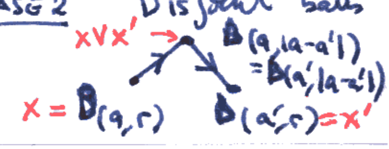
<u>Q</u>: Path connecting $|\,|_{\mathbb{B}(a,r)}$ to $|\,|_{\mathbb{D}(a',r')}$? <u>A</u>: Order discs by containment.
(Includes r or $r' = 0$)

<u>CASE 1</u> $a = a'$ & $r \leq r'$: $[0,1] \longrightarrow (\mathbb{A}^1)^{an}$

$t \longmapsto |\,|_{\mathbb{D}(a, tr' + (1-t)r)}$



<u>CASE 2</u> Disjoint balls



- Missing pts if K is not spherically complete ($\exists$ decreasing sequences of nested discs w/ empty intersection)

Say $B(a_i, r_i)$ is such a sequence $\rightsquigarrow$ $f \longmapsto \lim\limits_{i\to\infty} |f|_{B(a_i,r_i)}$ is a multiplicative seminorm on $K[T]$ extending $|\ |_K$.

(In fact, it defines a norm)

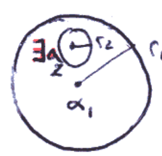
Example: $\mathbb{C}_p$ is not spherically complete

Pf/ $\mathbb{C}_p \cap D(0,1)$ is not sph. complete. Write $\overline{\mathbb{Q}}^{\,alg} \cap D(0,1) = \{\alpha_j\}_{j\geq 1}$. (countable & dense in $D(0,1)$)

Use $|\mathbb{C}_p^\times| = \mathbb{Q}$ to construct a sequence $r_1 > r_2 > \dots \geq 0$ in $\mathbb{Q}$ st. $D(a_i,r_i) \supseteq D(a_{i+1}, r_{i+1})$

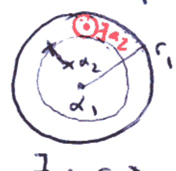
$\alpha_1 = a_1,\ a_2, \dots$ in $\mathbb{C}_p \cap D(0,1)$ $\quad \alpha_1, \dots, \alpha_s \notin D(a_{s+1}, r_{s+1})$

CASE 1



$r_2 = \min\{r_1, \frac{1}{2}|a-\alpha_1|\}$

CASE 2



$\exists\, a_2 \in D(\alpha_1, r_1) \setminus D(\alpha_1, |\alpha_1 - \alpha_2|)$

$r_2 = \min\{r_1, \frac{1}{2}|a-\alpha_2|, \frac{1}{2}|a-\alpha_1|\}$

Berkovich's Classification Thm: These are all the points of $(\mathbb{A}^1)^{an}$:$\longleftrightarrow$ Nested sequences of ~~balls~~ discs in K $D(a_i, r_i)$

(1) Type I: $\lim\limits_{i\to\infty} r_i = 0 \underset{K\text{ complete}}{\Rightarrow} \bigcap_i D(a_i,r_i) = \{a\} \subseteq K$

(2) Type II $\lim\limits_{i\to\infty} r_i = r \in |K^\times|$ & $\exists\, a \in \bigcap_i D(a_i, r_i) \Rightarrow pt = |\ |_{D(a,r)}$ (rat'l)

(3) Type III $r \notin |K^\times|$ & $\exists\, a \in \bigcap_i D(a_i, r_i)$ (irrational)

(4) Type IV: $\bigcap_i D(a_i, r_i) = \emptyset$ (in particular, $r > 0$ b/c K is complete)

- ~~Metric outside type I~~ ~~...~~

- Two nested sequences give the same seminorm iff $\begin{cases} \text{cofinal (they interlace) \& both give empty intersection} \\ \text{they have the same non-empty intersection} \end{cases}$

Partial order on $(\mathbb{A}^1)^{an}$: $[\ ]_x \leq [\ ]_y \Leftrightarrow [f]_x \leq [f]_y \ \forall f \in K[T]$

Equivalently: $[\ ]_x = |\ |_{D(a,r)}$ & $[\ ]_y = |\ |_{D(a',r')}$ then $[\ ]_x \leq [\ ]_y \Leftrightarrow D(a,r) \subseteq D(a',r')$

(Can extend to Type IV)

$\rightsquigarrow$ Can find $x \vee y$

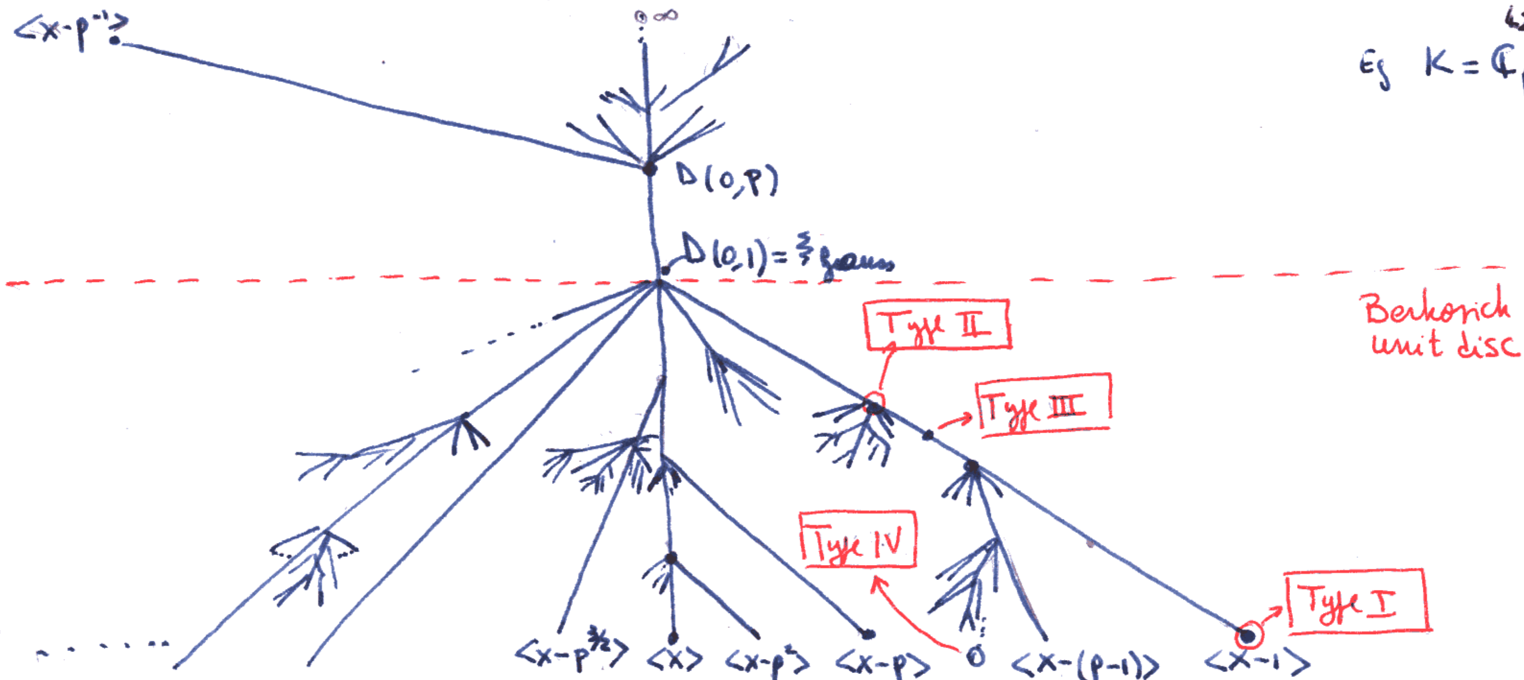
- Metric outside Type I: If $x \leq y$: $d([\ ]_x, [\ ]_y) = \log\left(\dfrac{\text{diam}[\ ]_y}{\text{diam}[\ ]_x}\right)$

where $\text{diam}[\ ]_x = \lim r_i$ if $x \longleftrightarrow \{D(a_i, r_i)\}$

In general $d([\ ]_x, [\ ]_y) = d([\ ]_x, [\ ]_{x\vee y}) + d([\ ]_{x\vee y}, [\ ]_x)$.

$\Rightarrow$ Type I: points are infinitely far away

Eg $K = \mathbb{C}_p$



- It's an $\mathbb{R}$-Tree but Topology induced by the metric outside Type I is ~~not~~ FINER THAT the subspace topology! (neighborhood of ξ_{gauss}: all but finitely many branches (which are replaced by proper segments). with the metric topology: no Type I points in the nbhd of ξ_{gauss})
- Branching at each point $x \longleftrightarrow$ tangent vectors at x = equivalence classes of paths $[x,y]$ from x with $y \neq x$.

  $[x,y_1] \sim [x,y_2]$ if they share a common initial segment.

- Type III only 2 branches.
- Type II = infinitely many branches $\longleftrightarrow \mathbb{P}^1(\tilde{K})$ (Eg ξ_{gauss})

§2 Berkovich discs:

- Berkovich Unit disc = $\{[\ ]_x \mid [\ ]_x \leq \xi_{gauss}\}$

Why? Analytification of a Banach algebra/K = $K\langle T\rangle := \{\sum_{i=0}^{\infty} a_i T^i \in K[[T]] \mid \lim_{i\to\infty} |a_i| = 0\}$

$\boxed{\text{norm on } K\langle T\rangle = \|f\| = \max_i (|a_i|)}$ = Gauss norm. (formal power series converging on $D(0,1)$)

$\boxed{K\langle T\rangle^{an} = \{\|\ \|_x : K\langle T\rangle \to \mathbb{R}_{\geq 0} \text{ mult. seminorms bounded by } \|\ \|\} \ \&\ [0]_x = 0, [1]_x = 1, [f+g]_x \leq [f]_x + [g]_x \quad \hookrightarrow \exists C_x : \|f\|_x \leq C_x \|f\| \ \forall f \in K\langle T\rangle}$
pointwise convergent topology.

Easy: Can always take $C_x = 1$ ($\forall n$: $[f]_x^n = [f^n]_x \leq C_x \|f^n\| = C_x \|f\|^n \to [f]_x \leq (C_x)^{1/n} \|f\|$ Take $n \to \infty$.)

- $[c]_x = |c| \ \forall c \in K$. (If $c=0$ ✓ . $[c^{-1}]_x \leq \|c^{-1}\| = \|c\|^{-1} = |c|^{-1}$, $[c]_x \leq \|c\| = |c|$. Product gives $1 \leq 1$ so = holds.)

Lemma: $[f+g]_x \leq \max([f]_x, [g]_x)$ with = if $[f]_x \neq [g]_x$ ⇝ non-Archimedean

Pf/ $([f+g]_x)^n = [(f+g)^n]_x = [\sum_{k=0}^{n} \binom{n}{k} f^k g^{n-k}]_x \leq \sum_{k=0}^{n} \underbrace{|\binom{n}{k}|_K}_{\leq 1} [f]_x^k [g]_x^{n-k} \leq (n+1)(\max\{[f]_x,[g]_x\})^n$

$\Rightarrow [f+g]_x \leq (n+1)^{1/n} \max\{[f]_x, [g]_x\}$ take $\lim_{n\to\infty}$ ✓

If $[f]_x < [g]_x$, then $[g]_x \leq \max\{[g+f]_x, [-f]_x\} \Rightarrow$ = holds! $\quad f = \sum a_i (T-a)^i$

Note: for any $a \in D(0,1)$, $r \leq 1$: $|\ |_{D(a,r)} \in K\langle T\rangle^{an}$ because $|f|_{D(a,r)} = \sup_{z \in D(a,r)} |f(z)| = \sup |a_i| r^i$ by max Modulus Prple.

Topology with subbasis: $U(f,\alpha) = \{ [\ ]_x \mid [f]_x < \alpha \}$
$V(f,\alpha) = \{ [\ ]_x \mid [f]_x > \alpha \}$ $\forall f \in K\langle T\rangle, \alpha \in \mathbb{R}$.

Weierstrass Preparation Thm: Can factor any $f \in K\langle T\rangle$ uniquely as $f = c\prod_{j=1}^{m}(T-a_j)\, u(T)$
where $a_j \in D(0,1)$ $\forall j$, $u(T)$ is a unit, ie $u(T) = 1 + \sum_{i=1}^{\infty} a_i T^i$ $|a_i|<1$ $\forall i \geq 1$ & $\lim_{i\to\infty} |a_i| = 0$.

$\Longrightarrow$ $[\ ]_x$ is determined by values on linear polynomials $(T-a)$ for $a \in D(0,1)$

Proof of Berkovich's classification on $(K\langle T\rangle)^{an}$:

$\mathcal{F} = \{ D(a, [T-a]_x) : a \in D(0,1) \}$

Claim: $\mathcal{F}$ is totally ordered by containment.

If $[T-b]_x \geq [T-a]_x \Rightarrow |b-a| = [b-a]_x \leq \max\{[T-b]_x, [T-a]_x\} = [T-b]_x$ (*)

and $=$ if $[T-b]_x > [T-a]_x$. so $a \in D(b, [T-b]_x)$ & $D(a, [T-a]_x) \subseteq D(b, [T-b]_x)$

Set $\boxed{r := \inf_{a\in D(0,1)} [T-a]_x}$ and choose $a_i \in D(0,1)$ st $r_i := [T-a_i]_x$ form a decreasing sequence converging to r.
by (*) $D(a_1,r_1) \supset D(a_2,r_2) \supset \ldots$

Claim 2: $[T-a]_x = \lim_{i\to\infty} [T-a]_{D(a_i,r_i)}$ $\forall a \in D(0,1)$

Pf/ By definition of r: $[T-a]_x \geq r$.

- If $[T-a]_x = r \leq r_i \Rightarrow$ $\forall i$: $r_i = [T-a_i]_x \geq |a_i - a|$ (use (*) $b = a_i$) so $a \in D(a_i, r_i)$ $\forall i$

and $[T-a]_{D(a_i,r_i)} = \sup_{z\in D(a_i,r_i) = D(a,r_i)} |z-a| = r_i \xrightarrow[i\to\infty]{} r = [T-a]_x$ ✓

- If $[T-a]_x > r$, for $i \gg 1$ $[T-a]_x > [T-a_i]_x$ & by (*) $[T-a]_x = |a-a_i|$
so $|a-a_i| > r_i$ and $[T-a]_{D(a_i,r_i)} = \sup_{z\in D(a_i,r_i)} |z-a| = |a-a_i| = [T-a]_x \xrightarrow[i\to\infty]{} [T-a]_x$
stable seq.

By continuity $[f]_x = \lim_{i\to\infty} [f]_{D(a_i,r_i)}$ $\forall f \in K\langle T\rangle$. & $D(a_i,r_i)$ is a nested sequence!

- If the nested sequence has non-empty intersection (say $a \in \bigcap D(a_i,r_i)$), then
$r \leq [T-a]_x = \lim_{i\to\infty} [T-a]_{D(a_i,r_i)} \leq \lim_{i\to\infty} r_i = r$ $\Rightarrow$ $[T-a]_x = r$ & $D(a,r) \in \mathcal{F}$ is minimal

We can take $a_i = a$ & $r_i = r$ (inf is a min in the def of r) $\Rightarrow$ $[f]_x = [f]_{D(a,r)}$
($\Rightarrow$ Type II or III if $r>0$.)

If $r=0$ $[\ ]_x$ is Type I. □

Prop: Berkovich unit disc $= \varprojlim \Gamma$ for $\Gamma = \bigcup_{i=1}^{m} [\zeta_{a_i,r_i}, \zeta_{gauss}]$

$\Gamma \leq \Gamma'$ gives a retraction map $r_{\Gamma',\Gamma} : \Gamma' \longrightarrow \Gamma$ $\ni y$ ↪ lines of semi-norms in Berk unit disc
$x \longmapsto$ First point where the path $[x,y]$ meets Γ.

- $r_{\Gamma',\Gamma}(x) = x \iff x \in \Gamma$
- maps are compatible $\Gamma \leq \Gamma' \leq \Gamma'' \Rightarrow r_{\Gamma'',\Gamma} = r_{\Gamma',\Gamma} \circ r_{\Gamma'',\Gamma'}$.

Berkovich disc $D(0,R) = K\langle R^{-1}T\rangle^{an}$ where $K\langle R^{-1}T\rangle = \{\sum c_k T^k : \lim_{k\to\infty} R^k |c_k| = 0\}$

$\|f\|_R = \max_k R^k |c_k|$ is Banach norm

$(\mathbb{A}^1)^{an} = \bigcup_{R>0} D(0,R)$. (natural inclusion via $K\langle R^{-1}T\rangle \to K\langle r^{-1}T\rangle$ for $r<R$.

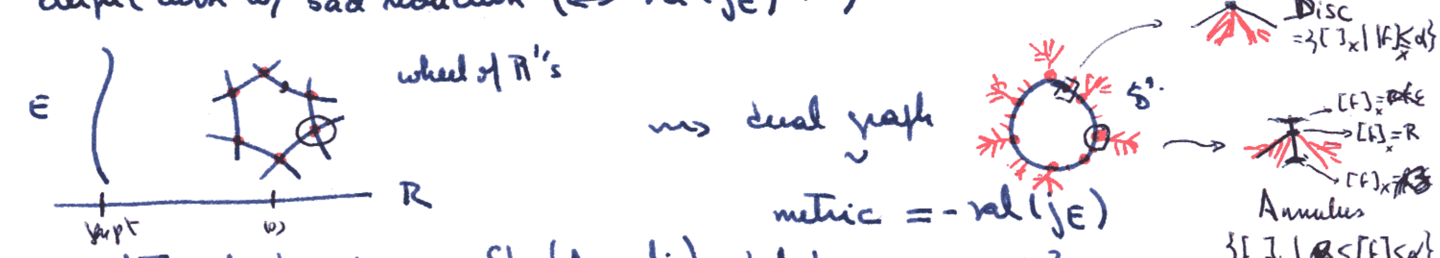
§3. Curves:

X smooth complete curve $/K$

General picture: X^{an} is locally homeomorphic to $(\mathbb{A}^1)^{an}$ with global topology captured by a finite PL object = skeleton

Skeletons are obtained as dual graphs of semistable regular models over K^o

Eg: E elliptic curve w/ bad reduction ($\iff val(j_E)<0$)



. Skeleton = obtained by gluing$_{\text{f. many}}$ Sk.(Annuli) $\subseteq \mathbb{A}^1$ $= \{1_{D(0,\rho)} \; r<\rho<R\}$ = length = $\log R - \log r$.

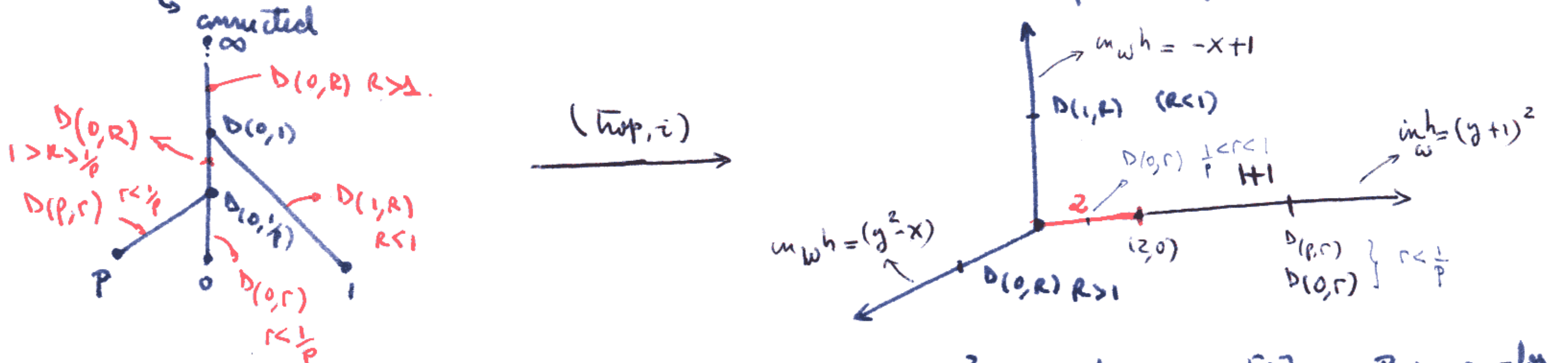
. Connected components of $X^{an} \setminus$ Skeleton $\cong_{\text{open}}$ Berkovich discs $:= \{[\;]_z \in (\mathbb{A}^1)^{an}_{K[t]} : [t]_z < 1\}$ (pt $\rightsquigarrow$ unrooted)

Retraction maps each component to the ! $[\;]_z$ in it's closure (eg

Example 1 [BPR] $K = \mathbb{C}_p$ $\quad X \subset (K^\times)^2$ defined by $\begin{cases} x(t) = t(t-p) = f \\ y(t) = t-1 = g. \end{cases}$ $\quad$ Eqn: $h = y^2 + (2-p)y - x + (p-1)$

$K[t] \xleftarrow{(f,g)} K[x,y]$ $\qquad$ $\operatorname{Trop}(X) = \{(-\log|f(t)|_p, -\log|g(t)|_p) \;\; t \in \mathbb{C}_p^\times\}$

(extended) Skeleton $\subseteq (\mathbb{P}^1)^{an}$ determined by the zeros and poles of f & g. $= 0, 1, p, \infty$.
↳ connected



Check value of each $[\;]_x$ on f & g. $\quad R>1$ $[f]_{D(0,R)=D(p,R)} = R^2 \mapsto -2\log_p R$, $\quad [g]_{D(0,R)=D(1,R)} = R \mapsto -\log_p R$

$$\begin{cases} [f]_{D(p,r)} = [t(t-p)]_x = [t]_x \cdot r = [\underbrace{t-p}_{<r<1/p} + \underbrace{p}_{|\;|=1/p}]_x \cdot r = \frac{r}{p} \xmapsto{-\log_p} 1 - \log_p r \geq 2 \\ [g]_{D(p,r)} = [t-1]_x = [\underbrace{t-p}_{r<1/p<1} + \underbrace{p-1}_{|\;|=1}] = 1 \mapsto 0 \end{cases}$$

$[f]_{D(0,r)} = r[t-p]_x = \frac{r}{p} \mapsto 1 - \log_p r$, $\quad [g]_{D(0,r)} = [t-1]_x = 1 \mapsto 0$

$\frac{1}{p}<R<1$ $\quad [f]_{D(0,R)} = R[t-p]_x = R^2 \mapsto \cancel{-2}\; 2\log_p R \in [0,2]$ $\quad ; [g]_{D(0,R)} = 1 \mapsto 0$

$[f]_{D(1,R)} = 1\,[t-p] = [\underbrace{t-1}_{|\;|<R<1} + \underbrace{p-1}_{|\;|=1}] = 1 \mapsto 0$, $\quad [g]_{D(1,R)} = R \mapsto -\log R \geq 0$