LECTURE III: Berkovich Analytic Spaces from the Tropical Perspective

Recall: X = K-scheme of f. type / K, $(K, |\ |_K)$ alg. closed, complete non-Arch field ($val(x) = -\log|x|_K \ \ \forall x \in K$)

$i: X \underset{cl}{\hookrightarrow} Y_\Sigma$ - meeting the dense torus $\mathbb{G}_m^n = \operatorname{Spec} K[y_1^{\pm}, \ldots, y_n^{\pm}]$

$X^{an} \ni \|\ \| \longmapsto (\log\|y_1\|, \ldots, \log\|y_n\|)$, $X^{an} \twoheadrightarrow \operatorname{Trop}(X,i) \subseteq \operatorname{Trop} Y_\Sigma$ ($\circlearrowleft$)

$X(K) \hookrightarrow X^{an}$, $X(K) \longrightarrow \operatorname{Trop}(X,i)$

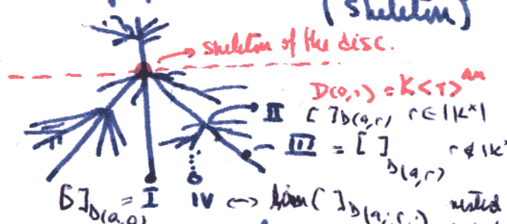
$p \longmapsto (-val(p_1), \ldots, -val(p_n))$ (USE MAX convention)

X^{an} glued from $(\operatorname{Spec} A)^{an} = \{\|\ \|: A \to \mathbb{R}_{\geq 0}$ mult seminorms extending $|\ |_K\}$ where A = f.g. K-algebra. w/ topology of pointwise convergence ($\mathbb{R}_{\geq 0}^A$)

Payne: $\varprojlim_{\text{equiv. cat.}} \operatorname{Trop}(X,i) \underset{\text{homeo}}{\cong} X^{an}$

• Described $(\mathbb{A}^1)^{an}$ & X^{an} for X curve = locally glued from $(\mathbb{A}^1)^{an}$ w/ global topology captured by a (metric) graph = dual graph to semistable regular model of X over K° = valn ring. (Skeleton)


Trivial valn. $K = \mathbb{C}$.



non-trivial valn. $K = \mathbb{C}_p$

$[f]_{D(a,r)} := \sup_{z \in D(a,r)} |f(z)|_K$


E w/ bad reduction

S^1 w/ length $= -val\, j_E$

E el. curve $K^* \twoheadrightarrow E \xrightarrow{\text{an.}} \mathbb{R}/_{-val\, j_E}\mathbb{Z}$ $val(j_E) < 0$

TODAY: focus on K w/ non-trivial valn

Thm (Berkovich '90) X^{an} is ~~Hausdorff~~ locally compact, locally path-connected, $X(K) \subseteq X^{an}$ is dense. (non-triv valn)

Thm (Hrushovski-Loeser '10) $\forall$ X quasi-projective X^{an} is locally contractible & homotopy equivalent to a finite simplicial complex of dim = dim X (= dual complex assoc. to a semistable model of X)

AIM: Use tropical geometry techniques to construct this complex & understand how it maps to a given tropicalization $\operatorname{Trop}(X,i)$.

§1 Curves: [Reference: Baker, Payne, Rabinoff]

simplicial complex = graph = Skeleton

Semistable decomposition [BPR'11] $\exists$ finite set $V \subseteq X^{an} \smallsetminus X(K)$ st

$$X^{an} \smallsetminus V \simeq \bigsqcup_{\text{finite}} \operatorname{Ann}(f, r, R) \sqcup \bigsqcup_{\text{infinite}} D$$

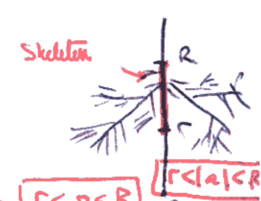
$\hookrightarrow$ length $= \log R - \log r$.

D = open unit disc in $(\mathbb{A}^1)^{an}$

$\operatorname{Ann}(f, r, R) = \{\|\ \| \in (\mathbb{A}^1)^{an} : r < \|f\| < R\}$, $0 \leq r < R$, $f \in K[T]$

and the induced metric on $X^{an} \smallsetminus X(K)$ is independent of all choices

$\boxed{\Sigma = \bigsqcup_{\text{finite}} \text{segments} = \text{skeleton of } X^{an}.}$



Eg: ① $f = T$ $\operatorname{Ann}(T, r, R) = \{r < \|T\| < R\}$

$\|\ \|$ Type I, II or III: $\|\ \| = |\ |_{D(a,\rho)}$ $a \in K$, $\rho \in \mathbb{R}_{\geq 0}$

$$\|T\| = \sup_{z \in D(a,\rho)} |z| = \begin{cases} \rho & \text{if } a = 0 \\ |z-a+a| = |a| > \rho & \text{if } a \neq 0 \quad \rho < |a| \ (0 \notin D(a,\rho)) \\ \rho & \text{if } a \neq 0 \quad \rho > |a| \Rightarrow D(a,\rho) = D(0,\rho) \end{cases}$$

Theorem (BPR'11) For any finite embedded subgraph $\Gamma \subset X^{an} \setminus X(K)$, $\exists$ an embedding $i : X \hookrightarrow Y_\Sigma$ st
1) Trop maps Γ (piecewise linearly) isometrically onto its image
2) Each edge in $\mathrm{trop}^{-1}(\mathrm{Trop}\,\Gamma)$ and disjoint from Γ is contracted to a pt.

The system of all such embeddings is stable & cofinal.

Q: • How to find i?
• Given i, how to certify if its good (homeomorphism or isometry)?

POINCARE-LELONG FORMULA.

• FACT: • For general $i : X \hookrightarrow Y_\Sigma$, the map trop is PL (& surjective if the semistable decomp is adapted to i), but there are integral stretching factors ($= m_{rel}(e')$ for an edge e' in Σ)

$$m_{Trop}(e) = \sum_{e_i' \xrightarrow{Trop} e} m_{rel}(e_i').$$

If a segment e' gets contracted, we write $m_{rel}(e') = 0$.

• (trop, i) is a harmonic map: for any vertex v in X^{an}, $(\mathrm{trop}, i)(\mathrm{Star}_\Sigma v)$ is a balanced fan: for all but finitely many edges emanating from v, $m_{rel}(e) = 0$) ("tangent directions at v.")

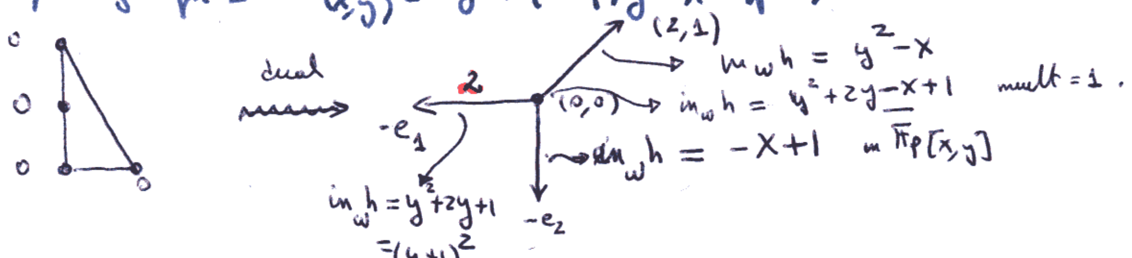
(Type II pts: Star $\simeq \mathbb{P}^1(\tilde{K})$, type III: Star $= \{v\}$, Type IV $= v$ leaf.)

Corollary: If $\Gamma' \subset \mathrm{Trop}(X, i)$ & $m_{Trop}(w) = 1$ for all $w \in \Gamma'$ (including vertices!) Then, there exists a ! subgraph $\Gamma \subset X^{an} \setminus X(K)$ mapping isometrically onto Γ'.

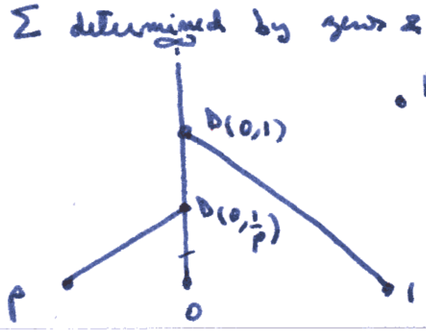
Here: $m_{Trop}(w) := \#$ of components of $in_w X \subseteq (\tilde{K}^*)^n$

EXAMPLE (last time) $K = \mathbb{C}_p$ $\quad X \subset (K^*)^2$ $\quad K[t] \longrightarrow K[x,y]$, $\quad t \longmapsto (x(t), y(t))$ $\quad x(t) = t(t-p)$, $y(t) = t-1$

defining eqn $= h_{(x,y)} = y^2 + (2-p)y - x - (p-1)$



Σ determined by zeros & poles of $x(t), y(t)$ $= 0, 1, p, \infty$



• $\| \; \| = [\;]_{D(0,r)}$, $r < \frac{1}{p}$

$\|x(t)\| = \|t\| \, \|t-p\| = r \cdot \frac{1}{p} = r/p$ ($\|\cdot\|<r$, $\|\cdot\| = \frac{1}{p}$)

$\|y(t)\| = \|t-1\| = 1$

$\}$ $\Rightarrow$ $-\log r$ maps to $(-\log_p r - 1, 0)$ $\longleftarrow$ $-e_1$ $(-2, 0)$

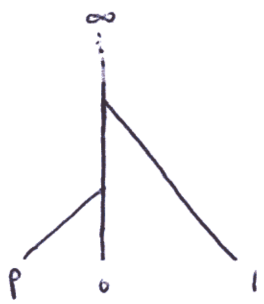
• $\| \; \| = [\;]_{D(p,r)}$, $r < \frac{1}{p}$

$\|x(t)\| = \|t\| \, \|t-p\| = \|t-p+p\| \, r = r/p$ ($<r$, $\| \cdot \| = \frac{1}{p}$)

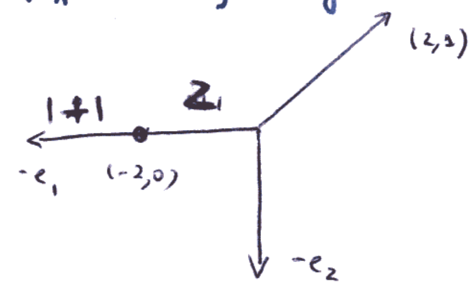
$\|y(t)\| = \|t-1\| = \|t-p+p-1\| = 1$

$\}$ $\Rightarrow$ same image

- $\| \ \| = [\ ]_{D(0,r)}$, $\frac{1}{p} < r < 1$, $D(p,r)$: $\|x(t)\| = \|t\|\ \|t-p\| = r^2$, $\|y(t)\| = \|t-1\| = 1$ $\Big\} \xrightarrow{-\log} (-2\log r, 0) = (2)(\log r, 0)$ ↳ stretching factor
- $\| \ \| = [\ ]_{D(1,r)}$: $\|x(t)\| = \|t\|\ \|t-p\| = 1$, $\|y(t)\| = r$ $\Big\} \xrightarrow{-\log} (0, -\log r)$
- $\| \ \| = [\ ]_{D(0,r)}$, $r > 1$: $\|x(t)\| = r^2$, $\|y(t)\| = \|t-1\| = r$ $\Big\} \xrightarrow{-\log} (-\log r)(2,1)$



Trop $\longrightarrow$



NOT faithful!

Q: How to repair?

- [C Markwig] Use Tropical modifications & lifting of components of $in_\omega h$!

(In search of a prime cone [Kerch-Mann])

§2 Tropical Modifications:

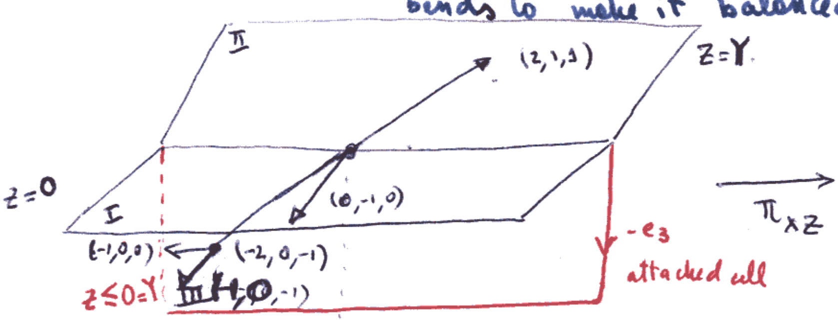
IDEA: Write $f_{(x,y)} :=$ lift of a component of $in_\omega h$ from $\overline{\mathbb{F}}_p[x,y]$ to $\mathbb{C}_p[x,y]$. $m_{Trop}(\omega) > 1$. (cannot be a vertex), ie $m_\omega f_{(x,y)}$ is a component of $in_\omega h$

Embed via $I_{h,f} = \langle h, z - f_{(x,y)} \rangle \subseteq \mathbb{C}_p[x,y,z]$.

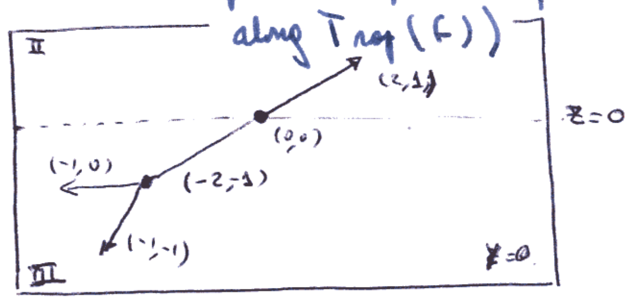
Claim: $Trop\, I_{h,f} \overset{\pi_{xy}}{\twoheadrightarrow} Trop(h)$ & $\exists\, \omega' \mapsto \omega$ with $m_{Trop}(\omega') < m_{Trop}(\omega)$. $Trop\, I_{h,f}$ can be recovered from all 3 projections

Example 1: $z = y + 1$. for ω ⟵ $(0,0)$, $mult_{Trop}(\omega) > 1$. $Trop(X, i) \subseteq$ Tropical surface defined by $z - f_{(x,y)}$

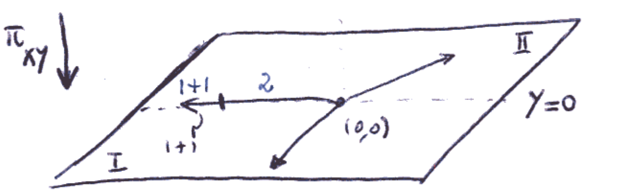
Trop surface = graph of $z = \max\{y, 0\} = Trop(f)$ on $\mathbb{R}^2$ & add cells along the bends to make it balanced (= Tropical modification of $\mathbb{R}^2$ along $Trop(f)$)



π_{xz} $\longrightarrow$

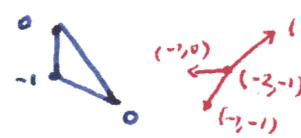


π_{xy} ↓



2 & (1+1) multiplicities are explained by Push-forward formula for multiplicities under monomial maps (Sturmfels Tevelev)

2 = lattice index of $\langle(-2,0)\rangle$ in $\mathbb{Z}^2$



$$\tilde{h}(x,z) = z^2 - 2z + 1 + (2-p)(z-1) - x + 1 - p = z^2 - zp - x$$

If we write $X^{an} \twoheadrightarrow Trop(X, i')$ we get an isometry (check all multiplicities $=1$)

$(x(t), y(t), z(t))$

KEY: For $z = y + a$, for any $a \neq 1 +$ Tail we get

because the initial deg. at $(\omega, 0)$ wasn't improved $\subseteq in_\omega h$.

Witnessing non-faithfulness: $X \xrightarrow{i} \mathbb{G}_m^n$, let $\Sigma \subset X^{an}$ be a skeleton adapted to i.

Lemma [C Markwig] Given a sm. curve. A scheme $\Sigma \xrightarrow{(\mathrm{Trop}, i)} \mathrm{Trop}(X, i)$ is NOT a closed embedding. Then, one of the following conditions hold:

(1) $\mathrm{Trop}(X,i)$ has an edge of higher mult (>1), or a locally reducible vertex v w/ $m_{\mathrm{Trop}}(v) \geq 2$

2) $\mathrm{Trop}(X,i)$ faithfully represents a! subgraph Γ of Σ (an contracts the rest!) ($\mathrm{Star}_{\mathrm{Trop}}\, v$ is the union of 2 $\neq$ balanced fans.)

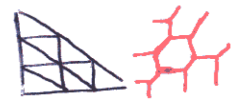
This is enough to develop an algorithm for plane elliptic cubics $E \subseteq \mathbb{G}_m^2$ with $\mathrm{val}(j_E) < 0$.

Say E is defined by a cubic $g(x,y)$

Thm [Katz-Markwig^2 '07] $\mathrm{Trop}(g)$ has no loops (as a graph) or the loop has lattice length $\leq -\mathrm{val}(j_E)$
↳ Eg Weierstrass form $y^2 - x^3 - ax - b$, $a, b \in K$

Equality holds if the vertices on the loop have valence 3 (initial term is not the expected one).
($j_E = A/\Delta$ & val j_E goes up $\Leftrightarrow$ val(Δ) goes up.)

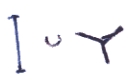
Thm [Chan-Sturmfels] Can be re-embedded in $\mathbb{G}_m^2$ w/ Trop E in honeycomb form



Thm [C-Markwig] Whenever there is a loop or a bounded edge of $m_{\mathrm{Trop}}(e) > 1$, we can linearly-reembed in dim ≤ 4 & see a loop of length $= -\mathrm{val}(j_E)$. ⤳ Algorithmic!
(w/ Tropical modifications)

Pf/. Locally reducible vertices v are dual to trapezoids

I at 1 (up to symmetry, and rotation by 45°)



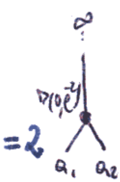
$m_{\mathrm{Trop}}(v) = 2 \Longleftrightarrow \Delta_v(\mathrm{in}_v(g)) = 0$ (FACT: $m_t \Delta = \pm \prod_{v\ \text{vertex}} \Delta_v^{\square}$)
↑ local discriminant

. Trop modification along a horiz, vertical or slope 1 line. will prolong the loop.

. Bounded edges e induce $\mathrm{in}_e g$ with 2 $\neq$ components. A tropical modification associated to one component will unfold e & produce a loop. in $\mathbb{R}^3$. □

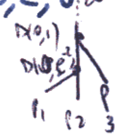
. Examples on slides

. **Exercise**: $K = \widehat{\mathbb{C}\{\{t\}\}}$ $E \subseteq \mathbb{T}^2$ $y^2 = x^3 + x^2 + t^4$ j_E w/ val $= -4$.

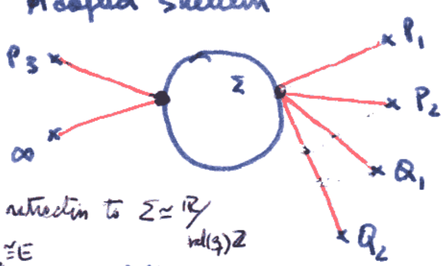


functions $(x) = (Q_1) + (Q_2) - 2(\infty)$ $Q_1 = (0, t^2)$, $Q_2 = (0, -t^2)$ ⤳ val Q_1 = val(Q_2) = 2

$(y) = (P_1) + (P_2) + (P_3) - 3(\infty)$ val(P_i) are zeros of Trop($x^3 + x^2 + t^4$) = 0, 2, 2, ∞

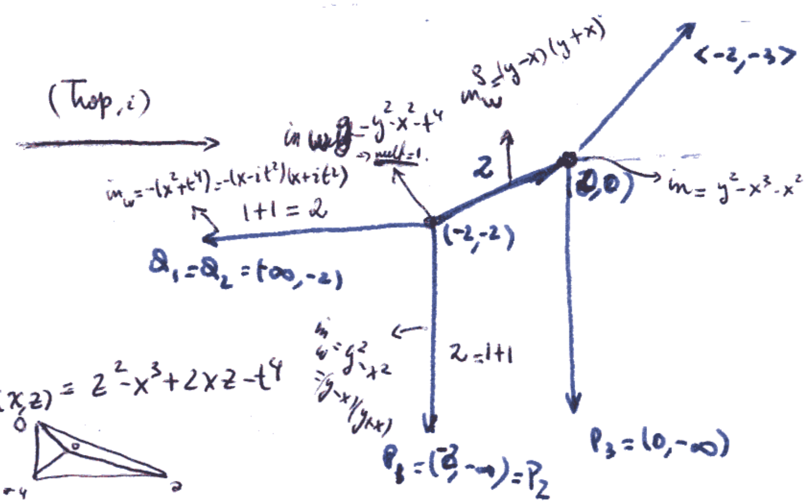


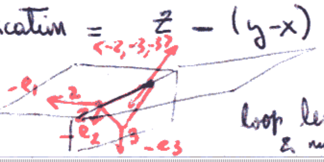
Adapted skeleton



retraction to $\Sigma \simeq \mathbb{R}/\mathrm{val}(g)\mathbb{Z}$
$K^\times / g^{\mathbb{Z}} \cong E$ & follow retraction of each pt to Σ
$\mathrm{val}\, g = -\mathrm{val}\, j_E$

Modification $= z - (y - x)$ gives $g(x,z) = z^2 - x^3 + 2xz - t^4$





loop length = 4, & mult = 1.