LECTURE IV: Berkovich Analytic Spaces from the Tropical perspective

GOALS of Today: ① Discuss faithful tropicalizations in dim > 1 via a concrete example $= Gr(2,n)$.

② Use combinatorial methods (eg cluster algebras) to study initial degenerations of $Gr(k,n)$ & its matroid stratification

§1. Setup

$(K, |\ |_K)$ non-Arch. valued field, complete, $K = \overline{K}$. ; $[n] = \{1, \ldots, n\}$

Def: $Gr(d,n) = \{ L \subset \mathbb{A}^n \text{ subspace of dim } d\} \simeq M_n(K)_{rk\, d} \big/ GL_d(K)$ → choice of basis.

We use the Plücker embedding $\varphi : Gr(d,n) \hookrightarrow \mathbb{P}_K^{\binom{n}{d}-1}$

$M_n(K) \ni X \longmapsto \Lambda^d X = \left(\det(X^{(i_1 \ldots i_d)})\right)_{i_1 < \ldots < i_d}$

Write $I_{d,n}$ for its defining (Plücker) ideal in $K[p_B, B \in \binom{[n]}{d}]$

- $Gr(d,n)$ can be decompose into its torus orbits (vanishing of certain Plücker coords)

$$\boxed{Gr_J(d,n) = \varphi^{-1}(E_J) \qquad E_J := \{ p \in \mathbb{P}_K^{\binom{n}{d}-1} : p_B = 0 \Leftrightarrow B \in J\} \quad J \subseteq \binom{[n]}{d}}$$

Lemma: $Gr_J(d,n) \neq \emptyset \iff J^c$ are the basis of a realizable matroid over K of rk $= d$ on $[n]$.

We call this the matroid stratification [Gelfand-Goresky-MacPherson-Serganova '87].

⚠ • Not a stratification in general: $\overline{Gr_J(d,n)} \neq \bigcup Gr_{J'}(d,n)$ (*)

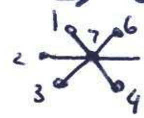
$J \subseteq J' \rightsquigarrow J'^c = \mathcal{B}(F) \subseteq \mathcal{B}(G) = J^c$

$F \leq G$ weak order of matroids

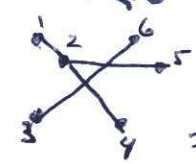
Eg $d=3$, $n=7$, $K = \mathbb{C}$ [Sturmfels '89]

G with non-basis $\{1,4,7\}$, $\{2,5,7\}$, $\{3,6,7\}$

F " " $\{1,2,4\}$, $\{i,j,7\}$ $1 \leq i < j \leq 6$



$= G$



$\{7\}$ loop.

$= F$

Shows $\overline{Gr_J(d,n)} \cap Gr_{J'}(d,n) = \emptyset$ $J = \mathcal{B}(G)^c$, $J' = \mathcal{B}(F)^c$

- Each stratum need not be irreducible, nor reduced, with singularities (⟺ Mnëv's Universality Theorem)

Example of rk 3 matroid on 9 elements where $K[Gr_{\mathcal{B}(G)^c}(3,9)]$ is not CM.

- **GOOD news!** Things are nice for $\boxed{d=2, \ \& \ (d,n)=(3,6)}$ ⇝ FIX: Positroid Stratification [Knutson-Lam-Speyer]

Proposition ([C, Haebich, Werner]$_{d=2}$, [Corey, C.]) Assume $d=2$ or $(d,n)=(3,6)$. Then

① The matroid stratification is a stratification of $Gr(d,n)$, ie $=$ holds in (*)

② Each $Gr_J(d,n)$ & $\overline{Gr_J(d,n)}$ are irreducible & reduced with defining prime ideals: $I(\overline{Gr_J(d,n)}) = I_{d,n} + \langle p_B : B \in J \rangle \subseteq K[p_B]$ (ie, no need to saturate or take radicals)

NOTE: Well known fact: $I(Gr_J(d,n)) \subseteq K[p_B : B \in J][p_B^{\pm} \ B \notin J] \Leftarrow \left\langle \sum \operatorname{sgn}(j,\sigma,\tau) p_{\sigma \cup j}\, p_{\tau \setminus j} \text{ where } \left\{ \begin{array}{l} |\sigma| = d-1 \\ |\tau| = d+1 \\ rk(\sigma) = d-1 \\ rk(\tau) = d, \ \sigma \not\subset \tau \end{array} \right\} \right\rangle$ is generated by quadrics.

⇝ Restrict our attention to these 2 families!

→ [CC] $in_w\, Gr_J(2,n)$ are Laurent polynomial rings on subsets of cluster variables.

§ 2 Tropical Grassmannian

Def $\operatorname{Trop} Gr(d,n) \subseteq \mathbb{TP}^{\binom{n}{d}-1} = \dfrac{\overline{\mathbb{R}}^{\binom{n}{d}-1} \smallsetminus \{(-\infty,\dots,-\infty)\}}{\mathbb{R}\cdot\mathbb{1}}$ $\quad \overline{\mathbb{R}} = \mathbb{R}\cup\{-\infty\}$

given by the Plücker embedding (with MAX convention)

- $\mathbb{G}_m^n \circlearrowright Gr(d,n)$, $Gr_J(d,n)$ by rescaling each column by t_i $\left[\mathbb{G}_m^{n-\#loop(G)}\text{ acts if } J^c = \mathcal{B}(G)\right]$

$\underline{t} * p_B = \left(\prod_{i\in B} t_i\right) p_B$. b/ i loop $\Leftrightarrow$ $\text{colm } i = \underline{0}$

- Can tropicalize each strata & glue $\operatorname{Trop} Gr_J(d,n) \subseteq \mathbb{T}\left(\mathbb{G}_m^{J^c}/\mathbb{G}_m\right) \times \{-\infty\}^J$.

Q1: Combinatorial (polyhedral) structure? eg f-vector, homology, ...

Structure is present if we mod out by the lineality space $\cong \mathbb{R}^{n-\#\text{loops}(G)}$ fr $J^c = \mathcal{B}(G)$
$\langle \sum_{i \in B \in \mathcal{B}(G)} e_B : i \text{ not a loop} \rangle$

Lemma: $\operatorname{Trop} Gr_J(d,n) / \mathbb{R}^{n-\#\text{loops}(G)}$ is a pointed fan (unless it's a linear space)

- Alternatively: $Gr_\emptyset(d,n)$ is dense in $Gr(d,n)$, so $\operatorname{Trop} Gr(d,n) = \overline{\operatorname{Trop} Gr_\emptyset(d,n)}$

Pf/ $Gr_\emptyset(d,n)^{an}$ is dense in $Gr(d,n)^{an}$ & $\operatorname{trop}: Gr(d,n)^{an} \twoheadrightarrow \operatorname{Trop} Gr(d,n)$ is cont. $\square$

Q2:

$$\begin{tikzcd} Gr(d,n)^{an} \arrow[drr, two heads, "\substack{\text{trop}\\ \text{cont \& surj}}"] & & \\ Gr(d,n)(K) \arrow[u, hook] \arrow[rr] & \circlearrowleft & \operatorname{Trop} Gr(d,n) \end{tikzcd}$$

$\exists\, \sigma: \operatorname{Trop} Gr(d,n) \longrightarrow Gr(d,n)^{an}$

continuous section to trop?

$\omega \longmapsto [\sigma(\omega)(p_B)]_B \equiv \omega \mod \mathbb{R}\cdot\mathbb{1}$

Thm [C-Häbich-Werner'13, Draisma-Postinghel'14] Answer is YES for $Gr(2,n)$!

Furthermore: (1) All tropical multiplicities on $\operatorname{Trop} Gr(2,n)$ equal 1. Equivalently,

if $w \in \operatorname{Trop} Gr_J(2,n)$, this says $\operatorname{in}_{\pi_J(w)} Gr_J(2,n) \subseteq k[p_B^{\pm} \mid B \notin J]$ is a prime ideal
$\quad\quad \rotatebox{90}{$\subseteq$}\; \mathbb{G}_m^{J^c}/\mathbb{G}_m$ $\qquad \pi_J: \mathbb{TP}^{\binom{n}{d}-1} \dashrightarrow \mathbb{TP}^{|J|-1}$

(in this case, it's an affine space!)

(2) $\forall\, w \in \operatorname{Trop} Gr(2,n)$: $\operatorname{trop}^{-1}(\omega) \subseteq Gr(2,n)^{an}$ has a ! distinguished pt ρ satisfying

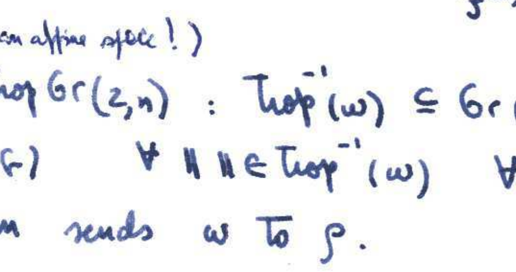
$\|f\| \leq \rho(f)$ $\quad \forall\, \|\;\| \in \operatorname{trop}^{-1}(\omega)$ $\quad \forall\, f \in K[p_B]/I_{2,n}$ $\quad$ (Shilov boundary pt of $\operatorname{trop}^{-1}(\omega)$)

$\Rightarrow$ The section sends ω to ρ.

Open Problem: Is the image of the section an extended skeleton of $Gr(2,n)^{an}$?

($\Leftrightarrow$ Comes from a model / K° (strictly semistable)

Enhancement $(\mathfrak{X}, H)$



special fiber: SNC divisor

Skeleton $= S(\mathfrak{X})$; PL space, closed in X^{an} of dim $\leq \dim(X)$ [Berkovich '99]

dual complex to $\mathfrak{X}_s$
$\updownarrow$
simplicial structure on $S(\mathfrak{X})$

- X^{an} strongly deformation retracts onto $S(\mathfrak{X})$
- Extended Skeleton $= S(\mathfrak{X}, H)$ $\quad$ H Cartier divisor on $\mathfrak{X}$ ~~ons~~ boundary divisor $H_{\phi_1} \cup \dots \cup H_{\phi_r}$

Extended skeleton: strata $S = \{H_{i_1}, \ldots H_{i_p}\}$ meeting transversally of $\mathcal{X}_s \leftrightarrow T$

$[\, D = H + \mathcal{X}_s \,]$ $\qquad \Delta_S \simeq \Delta_T \times \mathbb{R}^p_{\geq 0}$

$\exists\ X^{an} \setminus \mathfrak{H}^{an} \longrightarrow S(\mathcal{X}, H)$ canonical retraction map.

<u>Thm</u> (Gubler-Rabinoff-Werner '14) If $X \xhookrightarrow{i} \mathbb{G}_m^n$ (X irreducible) and <u>ALL</u> tropical multiplicities are 1, then each $\mathrm{trop}^{-1}(w)$ has a ! Shilov boundary pt & this defines a continuous section σ to (trop, i).

<u>Hard</u>: extending this result to toric varieties is very delicate (and often fails).

<u>Sufficient conditions</u> [GRW'15] $\forall \tau$: $X \cap \mathcal{O}(\tau)$ is equidimensional of dim d_τ or $\emptyset$ ($d_0 = \dim X$)

$X \underset{\text{close}}{\xhookrightarrow{i}} Y_\Sigma \supseteq \mathbb{G}_m^n$ (dense torus) • $\exists$ covering $\mathrm{Trop}(X \cap \mathbb{G}_m^n) = \bigcup_{\dim P = d} P$ with $P \subseteq N_{\mathbb{R}}$ polyhedron st

$\Sigma = \{\tau\}$ pointed fan. $\forall \tau$: if recession cone (P) (= tail cone) $\cap$ relint$(\tau) \neq \emptyset \Rightarrow \pi_\tau(P)$ in $N_{\mathbb{R}}/\langle\tau\rangle$ has dim $= d_\tau$.

<u>Still open</u>: Is the image of σ an extended skeleton of X^{an}?

§2 Proof of Thm [CHW]:

$$I_{2,n} = \langle p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \rangle \quad \text{Plücker relations.}$$

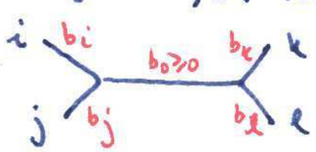
<u>Theorem</u> (Speyer-Sturmfels) $\mathrm{Trop}\, Gr_0(2,n) \subseteq \mathrm{Trop}\, \mathbb{G}_m^{\binom{n}{2}}/\mathbb{G}_m = \mathbb{R}^{\binom{n}{2}}/\mathbb{R}\cdot 1$ is the space of phylogenetic trees $\mathcal{T}_n$ on n leaves, labelled 1 through n of [Billera-Holmes-Vogtmann].

$\mathcal{T}_n \ni (T, b) = \{T$ graph w/ no cycles & no deg 2 vertices, n labelled leaves
$b: E(T) \longrightarrow \mathbb{R}$ & $b(e) \geq 0$ if $e \in E(T)$ not adjacent to any leaf$\}$

$\rightsquigarrow$ fan structure $\quad \mathcal{C}$ cone $\longleftrightarrow T$ tree

$\mathcal{C}_T \subseteq \mathcal{C}_{T'} \longleftrightarrow T$ is a coarsening of T' (contract non leaf edges)

<u>Why?</u> $(T, b) \rightsquigarrow \underline{w} \in \mathbb{R}^{\binom{n}{2}}$ as $w_{pq} = \sum'_{e: p \to q} b(e) \quad (p < q)$

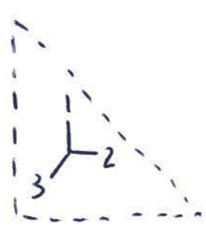


$w_{ij} = b_i + b_j$
$w_{ik} = b_i + \underset{\geq 0}{b_0} + b_k$, etc $\qquad$ [for all $i<j<k<l$.]

4 pt condition = tropical Plücker relns = $\max\{w_{ij} + w_{kl}, \underline{w_{ik} + w_{jl}}, \underline{w_{il} + w_{jk}}\}$ attained twice

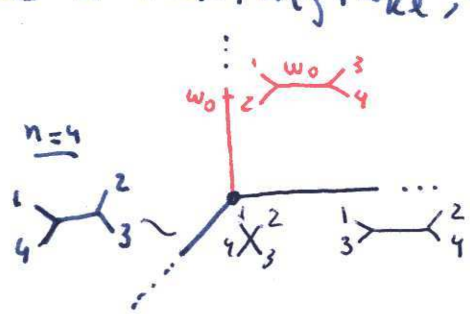
<u>Examples</u>:

<u>$n=3$</u>



$= \mathbb{R}^3/\mathbb{R}\cdot 1$

<u>$n=4$</u>



$\times\ \mathbb{R}^4/\mathbb{R}\cdot 1$

IDEA: Construct an open affine cover $Gr(2,n) = \bigcup U_{ij}$ $U_{ij} = \varphi^{-1}(\ldots)$

- $U_{ij} \cong \operatorname{Spec} K\left[\frac{p_{ik}}{p_{ij}}, \frac{p_{jk}}{p_{ij}} : k \neq i,j\right] \simeq \mathbb{A}^{2(n-2)}$
- $(\mathbb{A}^N)^{an} \supseteq \overline{\mathbb{R}}^N$ via the <u>Skeleton norms</u>: $p \in \mathbb{R}^N \rightsquigarrow \delta(p): K[x_1, \ldots, x_N] \longrightarrow \mathbb{R}_{\geq 0}$

$$\sum_\alpha c_\alpha x^\alpha \longmapsto \max_\alpha \{|c_\alpha| \exp(\langle p, \alpha\rangle)\}$$

$$\Rightarrow \begin{tikzcd}
Gr(2,n)^{an} \arrow[r, "trop"] & Trop\, Gr(2,n) \arrow[l, dashed, bend left, "\sigma"] \\
U_{ij}^{an} \arrow[u, phantom, "\cup"] \arrow[r, "Trop"] \arrow[dr, "Trop"'] & Trop\, U_{ij} = \{ w \in Trop\, Gr(2,n) : w_{ij} \neq -\infty \} \arrow[u, phantom, "\cup"] \arrow[d, "\pi_{\{ik, jk\}}"] \\
& \overline{\mathbb{R}}^{2(n-2)} \arrow[ul, dashed, bend left]
\end{tikzcd}$$

$2(n-2)$ words $= \frac{p_{ik}}{p_{ij}}, \frac{p_{jk}}{p_{ij}}$

$\exists\, \sigma = \delta_{(p)}$ skeleton norm $p = (w_{k\ell} - w_{ij})_{k\ell \in \{i-, j-\}}$

NEED = $\boxed{\sigma_{(w)}\left(\frac{p_{k\ell}}{p_{ij}}\right) = \exp(w_{k\ell} - w_{ij}) \quad \forall\, k\ell \neq ij}$

$\binom{n}{2} - 1$ words of dense tori G_m $= \left\{ \frac{p_{k\ell}}{p_{ij}} \quad k\ell \neq ij \right\}$

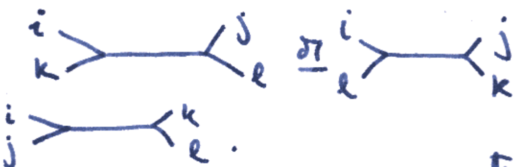
BUT $\quad \frac{p_{k\ell}}{p_{ij}} = \frac{p_{ik}}{p_{ij}} \frac{p_{j\ell}}{p_{ij}} - \frac{p_{i\ell}}{p_{ij}} \frac{p_{jk}}{p_{ij}}$ from the Plücker relns.

$$\sigma_{(p)}\left(\frac{p_{k\ell}}{p_{ij}}\right) = \max\{(w_{ik} - w_{ij}) + w_{j\ell} - w_{ij},\ w_{i\ell} - w_{ij} + w_{jk} - w_{ij}\}$$

$$\overset{?}{/\!/} \qquad = \max\{w_{ik} + w_{j\ell},\ w_{i\ell} + w_{jk}\} - 2w_{ij} \qquad (*)$$

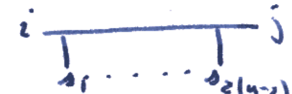
$w_{k\ell} - w_{ij}$

$\Longleftrightarrow$ (1) $\underline{w} \in \overline{\mathcal{C}_T}$ where T contains the quartet



or

(2) one of $w_{ik}, w_{jk}, w_{i\ell}$ or $w_{j\ell} = -\infty$.

- The condition $(*)$ should hold $\underline{\forall}\, k\ell$: This happens $\Leftrightarrow T = $  caterpillar tree wrt i,j

to $w \in \mathbb{R}^{\binom{n}{2}} / \mathbb{R}\cdot\mathbb{1}$

If T is not the caterpillar (wrt i,j) then we must trade $\frac{p_{ik}}{p_{ij}}, \frac{p_{j\ell}}{p_{ij}}, \frac{p_{i\ell}}{p_{ij}}$ or $\frac{p_{jk}}{p_{ij}}$ by $\frac{p_{k\ell}}{p_{ij}}$.

This choice is forced by non-infinity coordinates on $\underline{w}$ (J) & the tree T with $w \in \overline{\mathcal{C}_T}$

Eg $w_{ik} \neq -\infty \Longrightarrow$ work with $\frac{p_{j\ell}}{p_{ij}} = \left(\frac{p_{ik}}{p_{ij}}\right)^{-1} \left(\frac{p_{k\ell}}{p_{ij}} + \frac{p_{i\ell}}{p_{ij}} \frac{p_{jk}}{p_{ij}}\right)$ & $\sigma_{(w)}\left(\left(\frac{p_{ik}}{p_{ij}}\right)^{-1}\right) = w_{ij} - w_{ik}$

$w \in \overline{\mathcal{C}_T}$ $\quad J = \{k\ell : w_{k\ell} = -\infty\}$

- Combinatorial rule:  $= T$

  - Define a total order $<$ on the leaves of each T_a st $k < \ell < v \Rightarrow \{k,\ell\}$ or $\{\ell, v\}$ are cherries of the quartet $\{i,j,k,v\}$

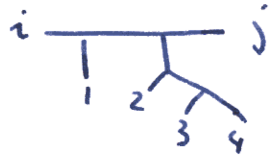
  - For each T_a we pick $2(\#\text{leaves } T_a)$ coordinates I_a among $\{ik, j\ell, k\ell,\ k,\ell \in T_a\}$ ie NOT 

satisfying exactly one of the following:

(1) ik & $jk \in I_a$ & $\forall\, \ell < k$ either $i\ell$ or $j\ell \in J$.

(2) $ik \notin I_a$, $j\ell \in I_a$ $\forall\, \ell \in T_a$ & $\exists\, t < k$ $t \in T_a$ with $it, jt \notin J$. If t is maximal with this property, then $kt \in I_a$

(3) Same as (2) but swap the role of i & j.

Eg 

$I_1 = \{i1, j1\}$

$J=\emptyset$ $I_2 = \{i2, j2, i3, 23, i4, 34\}$

$J\{23\}$ $I_2 = \{i2, j2, i3, j3, i4, 24\}$

Take $I := \bigcup_a I_a$ & should:

(1) $\{\frac{p_{kl}}{p_{ij}} \;\; kl \in I\}$ is algebraically indep in $K(U_{ij})$

(2) Each Plücker coordinate lies in $A = K[\frac{p_{kl}}{p_{ij}} \;\; kl\in I][(\frac{p_{kl}}{p_{ij}})^{-1} : kl \in I \cap J^c \cap \{is, js : s \neq i,j\}]$

Using these coordinates we obtain a section to trop $\sigma^{(ij)}_{J,T} : \overline{\mathcal{C}}_T^T \cap \mathrm{Trop}\, U_{ij} \cap \{w_{ik} = -\infty : ik \in J\} \cap \{w_{jk} = -\infty : jk \in J\}$ with image in $(\mathrm{Spec} A)^{an} \subset (Gr_J(2,n) \cap U_{ij})^{an}$, via the skeleton norms

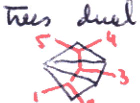
By construction • map is independent of all choices of $<$ & I. (& T)

• maps glue to give $\sigma^{(ij)} : \mathrm{Trop}\, U_{ij} \longrightarrow U_{ij}^{an}$.

• " " " " $\sigma : \mathrm{Trop}\, Gr(2,n) \longrightarrow \mathrm{Trop}\, Gr(2,n)$

Hard: Continuity when moving from one J to another $J' \supset J$ (need to show the term achieving the max contains no negative exponents for $\frac{p_{kl}}{p_{ij}}$ if $kl \in J'$).

ALTERNATIVE PROOF: Use • cluster algebras (joint w/ Corey) • Structure on $\mathrm{Trop}\, Gr_J(2,n)$. ⇝ for continuity

Trees dual to triangulations of n-gon 

rotate labels: 

Write $J = B_1 \sqcup \cdots \sqcup B_m \sqcup B_0$ $\quad$ B_i = parallel elements w/o loops

$\hookrightarrow$ loops of the matroid

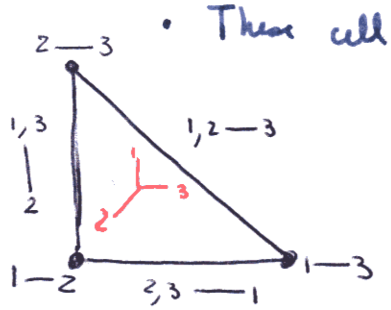
• Up to reordering the columns of each matrix in $Gr_J(2,n)$ & write:

$$X = \begin{array}{cccccc} B_1 & B_2 & & B_m & B_0 \\ \big(\;\; & \big| \;\; & \big| \cdots\cdots \big| & \;\; \big| & \;\; 0 \;\; \big) \end{array} \qquad \text{Each block} = (v_1 \mid \lambda_1 v_1 \mid \cdots \mid \lambda_r v_1) \quad \lambda_i \in K^*$$

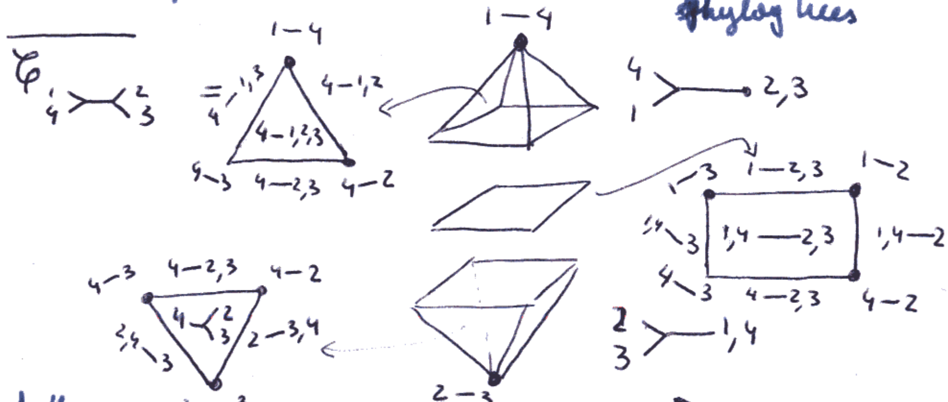
Project to the first entry of each block to get a pt in $Gr_\emptyset(2,m) \times \mathbb{G}_m^{\binom{n-B_0}{2}}$

Thm [CC]. • $\mathrm{Trop}\, Gr_J(2,n)$ = space of phylogenetic trees on m leaves labelled by $B_1, \ldots, B_m$

• These cell-structures are compatible & glue to $\mathrm{Trop}\, Gr(2,n)$ =: compact space of phylog trees

Eg: 

f-vector $= (3, 3, 1)$



$\mathrm{Trop}\, Gr(2,4)$ = gluing of 3 subdivisions of the octahedron along 8 boundary Δ's.

f-vector $= (6, 12, 11, 7, 3)$ ⇝ $\chi = 1$ ✓