Local Systems in Algebraic Geometry 2024

Abstract

These are the lectures notes of the instructional conference "Local Systems in Algebraic Geometry" held May 7-10, 2024 at Ohio State University (organized by Stefan Patrikis, Dave Anderson, Angelica Cueto, and Jennifer Park). The conference featured two mini-courses of four lectures each given by Daniel Litt ("Nonabelian cohomology and applications") and Alexander Petrov ("The p-adic Riemann-Hilbert correspondence"), and supplemented by background lectures given by PhD student and postdoc participants. For $n = 1 \ldots 8$, talk (i.e., section in this document) $2n-1$ is the background lecture for talk $2n$, which may also rely on some earlier talks.

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For more information about the conference and retreat, see https://people.math.osu.edu/cueto.5/RTG/rtg24/RTGConference24.html. For lecture videos from the conference, see https://www.youtube.com/@OSU_RTG_AGNT.

Contents

1 The classical Riemann-Hilbert correspondence. Speaker: Christian Klevdal. Notes by Yifei Zhang and Jake Huryn

Let X be a connected manifold, and let $\underline{\mathbb{C}}$ be the constant sheaf on X attached to $\mathbb{C}$. More concretely we have

$$\underline{\mathbb{C}}(U) := \{f : U \to \mathbb{C} | f \text{ locally constant}\}.$$

Definition 1.1. A local system on X is a sheaf of $\mathbb{C}$ vector spaces $\mathcal{F}$ such that $\mathcal{F}|_U \simeq \underline{\mathbb{C}}^r$ for U in some open cover of X. The category of local systems on X is denoted by $\mathrm{Loc}_{\mathbb{C}}(X)$.

Example 1.2.
- Let $f : Y \to X$ be a smooth proper submersion. By Ehresmann's theorem, f is a locally trivial fibration, so $R^i f_* \underline{\mathbb{C}}$ is a local system on X.
- Let $\Delta^* \subset \mathbb{C}$ be a punctured open disk. Let $f : E \to \Delta^*$ be the family of elliptic curves where the fibre over $q \in \Delta^*$ is $E_q := \mathbb{C}^\times / q^{\mathbb{Z}}$ (this is isomorphic to $\mathbb{C}/(\mathbb{Z} + t\mathbb{Z})$ via the exponential map). Let $\mathbb{L} := (R^1 f_* \underline{\mathbb{C}})^\vee$. We have $\mathbb{L}_q \simeq H_1(E_q, \mathbb{C})$. This is generated by e_1, e_2 where e_1 is a loop that lifts to the loop around 0 in $\mathbb{C}^\times$ and e_2 is a loop that lifts to a loop from 1 to q.

Let $\mathcal{F}$ be a local system on X, and let $\gamma : [0,1] \to X$ be a path, then we have a canonical isomorphism $\mathcal{F}_{\gamma(0)} \simeq \mathcal{F}_{\gamma(1)}$ as both of them are naturally isomorphic to $H^0([0,1], \gamma^* \mathcal{F})$. Using that a locally constant sheaf on a simply connected space is constant, this induces a monodromy representation

$$\pi_1(X, x) \to \mathrm{GL}(\mathcal{F}_x).$$

Example 1.3. For $\mathbb{L}$ as above, we get $\pi_1(\Delta^*) \simeq \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$. The generator γ acts on $H_1(E_q, \mathbb{C})$ by sending e_1 to e_1 and e_2 to $e_1 + e_2$

Theorem 1.4. *The following is an equivalence of categories*

$$\begin{aligned} \mathrm{Loc}_{\mathbb{C}}(X) &\xrightarrow{\sim} \mathrm{Rep}(\pi_1(X, x)) \\ \mathcal{F} &\longmapsto \mathcal{F}_x \end{aligned}$$

Remark 1.5. $\mathrm{Loc}_{\mathbb{C}}(X)$ is a purely topological invariant of X. The Riemann Hilbert correspondence will relate this to a category that is defined by the analytic structure of X.

Now let X be a complex manifold.

Definition 1.6. A module with integrable connection (MIC) on X is a pair $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ is a coherent sheaf with respect to $\mathcal{O}_X$, the sheaf of holomorphic functions, and

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$$

satisfies the Leibniz rule: $\nabla(fs) = s \otimes df + f\nabla(s)$ for $f \in \mathcal{O}_X(U), s \in \mathcal{E}(U)$, and the flatness condition: $\nabla^2 : \mathcal{E} \to \mathcal{E} \otimes \Omega^2_X$ (given by ∇ composed with the induced map from $\mathcal{E} \otimes \Omega_X$ to $\mathcal{E} \otimes \Omega^2_X$ sending $s \otimes w$ to $\nabla(s) \wedge w + s \otimes dw$) is 0.

Example 1.7.
- Take $X = \mathbb{C}^\times$, and let $\alpha \in \mathbb{C}$. Consider $(\mathcal{O}_X, \nabla_\alpha)$ with $\nabla_\alpha f = df - \alpha \frac{dz}{z}$. This gives a module with integrable connection.
- Let $\mathcal{V} \in \mathrm{Loc}_{\mathbb{C}}(X)$. Then $(\mathcal{V} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X, \nabla)$ with $\nabla = 1 \otimes d$ is an MIC because $d^2 = 0$.

Let $X \subset \mathbb{P}^1(\mathbb{C})$ be open, and let

$$\frac{d}{dz} \vec{f} = A(z) \vec{f}$$

be a rank-n homogeneous linear system of differential equations. Then there is an associated connection $(\mathcal{O}^n_X, \nabla)$ with $\nabla(\vec{f}) = d\vec{f} - A(z)\vec{f}dz$. The solutions to this system of equations correspond to global sections of $(\mathcal{O}^n_X)^{\nabla = 0} := \ker(\nabla)$.

Remark 1.8. An equation like $f^{(n)} + a_{n-1}(z)f^{(n-1)} + \cdots + a_0(z) = 0$ is encoded by the companion matrix associated to the polynomial $x^n + a_{n-1}(z)x^{n-1} + \cdots + a_0(z)$.

Theorem 1.9. *We have an equivalence of categories*

$$\begin{aligned} \mathrm{MIC}_{\mathbb{C}}(X) &\xrightarrow{\sim} \mathrm{Loc}_{\mathbb{C}}(X) \\ (\mathcal{E}, \nabla) &\longmapsto \mathcal{E}^{\nabla=0}, \end{aligned}$$

and the quasi-inverse is given by $\mathcal{V} \mapsto (\mathcal{V} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X, \nabla)$.

This is the analytic RH correspondence. Main point of the proof: Locally $(\mathcal{E}, \nabla)$ looks like a solution to a differential equation, so $\mathcal{E}^{\nabla=0}$ is a local solution by existence and uniqueness for ODEs (by [Kat70, Proposition (8.8)], MICs are vector bundles).

Now let X be a smooth variety over $\mathbb{C}$. Then the category $\mathrm{MIC}(X)$ is defined in the same way, doing everything with the algebraically defined sheaves. We have

$$\begin{tikzcd} \mathrm{MIC}(X) \arrow[rr, "\mathrm{sol}"] \arrow[dr] & & \mathrm{Loc}_{\mathbb{C}}(X) \\ & \mathrm{MIC}(X^{an}) \arrow[ur] & \end{tikzcd}$$

Question 1.10 (what should have been Hilbert's 21st problem)**.** Is sol essentially surjective?
Case 1: X proper. In this case Serré's GAGA says $\mathrm{Coh}(X) \to \mathrm{Coh}(X^{an})$ is an equivalence of categories, then so is $\mathrm{MIC}(X) \to \mathrm{MIC}(X^{an})$. (One has to be careful here: a connection is not $\mathcal{O}_{X^{an}}$-linear but only $\mathbb{C}$-linear. One has to slightly reinterpret connections; see Daniel's mathoverflow post [hl])
Case 2: X not proper. Fix $j : X \hookrightarrow \overline{X}$ an open immersion with $\overline{X}$ smooth proper and $\overline{X} \setminus X = D$ a strict normal crossing, which is guaranteed to exist by Hironaka's resolution of singularity. This means for any $x \in \overline{X}$ there is a neighborhood $U \subset \overline{X}$ of x and $U \to \mathbb{A}^n$ étale with $D|_U \simeq V(t_1 \cdots t_k)|_U = V(t_1 \cdots t_k) \times_{\mathbb{A}^n} U$ (this says $D|_U$ is the pullback of union of some coordinate hyperplane in $\mathbb{A}^n$). We use the sheaf of logarithmic differentials $\Omega^1_{\overline{X}}(\log D) \subseteq j_*\Omega^1_X$ given by $\Omega^1_{\overline{X}}(\log D)$ restricted to U as above being free on $\{\frac{dt_1}{t_1}, \cdots, \frac{dt_k}{t_k}, dt_{k+1}, \cdots, dt_n\}$

Definition 1.11.

1. An MIC on $\overline{X}$ with log poles along D is $(\overline{\mathcal{E}}, \overline{\nabla})$ with $\overline{\mathcal{E}} \in \mathrm{Coh}(\overline{X})$ and with $\overline{\nabla} \colon \overline{\mathcal{E}} \to \overline{\mathcal{E}} \otimes \Omega^1_{\overline{X}}(\log(D))$ again satisfying the Leibniz condition and flatness condition.
2. The essential image of the restriction $j^* : \mathrm{MIC}(\overline{X}, D) \to \mathrm{MIC}(X)$ is $\mathrm{MIC}^{\mathrm{reg}}(X)$, connection with regular singularity at ∞.

Remark 1.12. In the definition of MIC, the existence of such ∇ forces $\mathcal{E}$ to be locally free by Katz, but in this definition, such $\overline{\nabla}$ won't guarantee $\overline{\mathcal{E}}$ to be locally free anymore.

Fact 1.13. $\mathrm{MIC}^{\mathrm{reg}}(X)$ does not depend on the the choice of $\overline{X}$.

Example 1.14.

- $\nabla_\alpha(f) = df - \alpha\frac{dz}{z}$ on $\mathbb{C}^\times$ has a regular singularity at ∞: if $t = z^{-1}$, $dt/t = -dz/z$.
- Consider $\mathbb{L}$ on Δ^* in the Example 1.2, $(\mathcal{V}, \nabla)$ the associated MIC. V is a free module that is globally generated by $e_1 \otimes 1, e_1 \otimes \log(z) - e_2 \otimes 1$. Then we let $\overline{\mathcal{V}}$ be free of rank 2 on Δ with $\overline{\nabla}[f_1, f_2] = d[f_1, f_2] + [f_2 dz/z, 0]$ which has log pole at 0. We have that $(\mathcal{V}, \nabla)$ is the restriction of $(\overline{\mathcal{V}}, \overline{\nabla})$ just defined.

Theorem 1.15. *Analytification is an equivalence of categories*

$$\mathrm{MIC}^{\mathrm{reg}}(X) \to \mathrm{MIC}(X^{an}).$$

Key points of the proof: Given $(\mathcal{E}_{an}, \nabla_{an}) \in \mathrm{MIC}(X^{an})$, we want to extend it to $\mathrm{MIC}(\overline{X}^{an}, D^{an})$ (which is defined similarly as in the algebraic setting and can be shown to be equivalent to $\mathrm{MIC}(\overline{X}, D)$ by GAGA) such that

$$H^0(\overline{X}^{an}, \overline{\mathcal{E}}_{an})^{\overline{\nabla}=0} \twoheadrightarrow H^0(X^{an}, \mathcal{E}_{an})^{\nabla=0}.$$

This is called Deligne's canonical extension. Since the category MIC has internal Hom, and the actual Hom set is the global flat section of the internal Hom, this gives the fullness. It is faithful since the analytification functor is faithful.

For essential surjectivity: Let's assume for simplicity $\dim(X) = 1$. Then we only need to extend along the punctured disk to the whole disk and glue. On Δ^*, we have the equivalence between MIC on Δ^* and representation of $\pi_1(\Delta^*)$. Let's say $(\mathcal{E}_{an}, \nabla_{an})$ corresponds to $\pi_1(\Delta^*) \to \mathrm{GL}(E)$ which sends the generator γ to A. Choose $B \in \mathrm{End}(E)$ such that $A = \exp(2\pi i B)$. Take $\overline{\mathcal{E}}_{an}$ to be a free module on Δ with $\overline{\nabla}$ being $v \otimes df - f(B(v) \otimes dz/z)$. Then the flat sections are $q^B \cdot v$ where $q^B := \exp(B \log z)$. Check that the monodromy representation of this on Δ^* is the same as $(\mathcal{E}_{an}, \nabla_{an}) : \gamma \cdot q^B = Aq^B$. Now we can deanalytify.

2 Nonabelian cohomology and applications, lecture 1. Speaker: Daniel Litt. Notes by Luke Wiljanen.

2.1 Pre–History: Non-abelian cohomology and examples

Question 2.1. Let X/K be a variety. How does the topology of X reflect its geometry? ...its arithmetic?

- Abelian: Let X/K be smooth and proper.
  - $K = \mathbb{C}$: For $H^*_{Hodge}(X)$, there is a Hodge structure and the Hodge conjecture.
  - K finitely generated: For $H^*_{\ell\text{-adic}}(X)$, there is a Galois action and the Tate conjecture.
  - K finitely generated: For $H^*_{dR}(X)$, there is a conjugate filtration, Hodge filtration, and the Ogus conjecture.
- Relative abelian: Let $X \to S$ be smooth and proper. One can associate an abelian invariant $R^i\pi_*\Lambda$. There is an action of $\pi_1(S, s)$ on $(R^i\pi_*\Lambda)_s$. One can ask questions about it, and there are various conjectures.
- Non-abelian: For X, we have $\pi_1(X)$. Since this is a complicated object, we slightly abelianize and look at its representations $Rep(\pi_1(X))$. The idea is that questions we can ask in the abelian setting, we can ask in this non-abelian setting, and vice versa.
- Relative non-abelian: For $X \to S$ smooth and proper and $s \in S$. There is an exact sequence

  $$\pi_1(X_s) \to \pi_1(X) \to \pi_1(S, s) \to 1,$$

  and this induces an outer action $\pi_1(S, s) \to Out(\pi_1(X_s))$. We get an action of $\pi_1(S, s)$ on $Rep(\pi_1(X_s))$. We'll study this non-abelian monodromy representation.

In this talk, we will consider $X = \mathbb{P}^1_{\mathbb{C}} \setminus D$ and $\mathbb{P}^1_S \setminus D_{univ} \to S = \mathrm{Conf}^n(\mathbb{CP}^1)$ where D_{univ} is a divisor whose fiber over $x_1 + \cdots + x_n$ is $\{x_1, \ldots, x_n\}$.

2.2 Projective Line Removing Some Points

Consider $\mathbb{CP}^1 \setminus \{x_1, \ldots, x_n\}$. Its fundamental group has a presentation

$$\pi_1(\mathbb{CP}^1 \setminus \{x_1, \ldots, x_n\}) \cong \langle \gamma_1, \ldots, \gamma_n \mid \gamma_1 \cdots \gamma_n = \mathrm{id} \rangle$$

where γ_i comes from a loop around x_i. From this presentation we have an identification

$$\frac{\mathrm{Hom}(\pi_1(\mathbb{CP}^1 \setminus D), GL_r(\mathbb{C}))}{\sim} \cong \frac{\{(A_1, \ldots, A_n) \in GL_r(\mathbb{C})^n \mid \prod_{i=1}^n A_i = \mathrm{id}\}}{\text{simultaneous conjugation}}.$$

Finding such matrices $A_1, \ldots A_n$ which product to the identity is straight-forward since you can solve for A_n in terms of $A_1, \ldots, A_{n-1}$. The problem becomes more interesting when we impose some constraints.

Question 2.2. Fix conjugacy classes $C_1, \ldots C_n \subseteq GL_r(\mathbb{C})$.

1. (Existence) Does there exist $A_1, \ldots, A_n \in GL_r(\mathbb{C})$ such that $\prod_{i=1}^n A_i = \mathrm{id}$ and $A_i \in C_i$ for all i? (Deligne–Simpson Problem)

2. (Uniqueness) When is a solution unique up to simultaneous conjugation? (If so, $(A_1, \ldots, A_n)$ is called a "rigid tuple", "rigid representation", or "rigid local system".) From [Kat96], a tuple $(A_1, \ldots, A_n) \in GL_r(\mathbb{C})^n$ yields by middle convolution a tuple $(A'_1, \ldots, A'_n) \in GL_{r'}(\mathbb{C})^n$. Under a suitable specification of parameters and a middle tensor product by rank 1 local systems, middle convolution maps rigid tuples to rigid tuples and is such that $r' < r$ if $r \geq 2$. Middle convolution is invertible. By reduction to the case of rank 1, Katz is thus able to classify (irreducible) rigid local systems on the punctured line.

3. (Monodromy) The group $\pi_1(\mathrm{Conf}^n(\mathbb{CP}^1)$ acts on $Rep(\pi_1(\mathbb{CP}^1 \setminus \{x_1, \dots x_n\}))$ by

$$\sigma_i : (A_1, \dots, A_n) \mapsto (A_1, \dots, A_{i-1}, A_i A_{i+1} A_i^{-1}, A_i, A_{i+2}, \dots, A_n)$$

where $\pi_1(\mathrm{Conf}^n(\mathbb{P}^1)) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle$. We then get an induced action where we mod out by simultaneous conjugation on the right. What are the dynamics for this action? What are the finite orbits?[1]

2.3 ODE (de Rham side)

There is a category of modules with integral connection

$$\mathrm{MIC}(\mathbb{P}^1, D) = \left\{ (\mathcal{E}, \nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{\mathbb{P}^1}(\log D)) \text{ flat bundles on } \mathbb{P}^1 \text{ with regular singularities along } D \right\}.$$

Example 2.3. Suppose that $\infty \notin D$, and that $B_1, \dots B_n \in \mathfrak{gl}_r(\mathbb{C})$ are such that $\sum B_i = 0$. Let $\mathcal{E} = \mathcal{O}^r_{\mathbb{P}^1}$, and $\nabla = d + \sum \frac{B_i}{z - x_i} dz$. This is a Fuchsian ODE.

Remark 2.4. The matrix A_j conjugate to the matrix $\exp(2\pi i B_i)$.

Question 2.5. As you vary x_i, how does one change B_i so that the monodromy representation stays the same?

Answer 2.6 (Schlesinger 1912)**.** The B_i have to satisfy a differential equation:

$$\begin{cases} \frac{\partial B_i}{\partial x_j} = \frac{[B_i, B_j]}{x_i - x_j} & i \neq j \\ \sum_i \frac{\partial B_i}{\partial x_j} = 0 & \text{for all } j \end{cases}$$

Remark 2.7. Schlesinger wasn't the first person to try to write this down. In 1905, Fuchs did the case $n = 4$, $B_i \in \mathfrak{sl}_2$. In this case, the equation is called the Panlevé VI equation.

There is a correspondence

$$\left\{ \text{finite } \pi_1(\mathrm{Conf}^n)\text{-orbits on } Rep(\pi_1(\mathbb{CP}^1 \setminus D)) \right\} \longleftrightarrow \{\text{algebraic solutions to Schlesinger equation}\}$$

Classification of algebraic solutions when $n = |D| = 4$, $r = 2$ (Hitchen, Dubrovin, Mazzocco, Boalch, Kitaev, Lisovyy, Tykhyy):

- 4 continuous families,
- 1 countable infinite family, and
- 45 exceptional.

The computer aided proof relies on effective version of Manin–Mumford for tori.

Question 2.8. Can we classify finite $\pi_1(\mathrm{Conf}^n(\mathbb{CP}^1))$-orbits on 2-dimensional representations of

$$\pi_1(\mathbb{CP}^1 \setminus \{x_1, \dots, x_n\})?$$

Answer 2.9. Almost.

Definition 2.10. We say $(A_1, \dots, A_n) \in SL_2(\mathbb{C})^n$ is interesting if

1. It has finite $\pi_1(\mathrm{Conf}^n)$-orbit.
2. The subgroup $\langle A_1, \dots, A_n \rangle \subseteq SL_2(\mathbb{C})$ is Zariski–dense.

[1] A complete classification of finite orbits appears in recent work of Bronstein and Maret: see https://arxiv.org/abs/2409.04379.

3. None of the A_i are $\pm$id.

4. It doesn't move in a continuous family of finite orbits (Corlette–Simpson).

Theorem 2.11 (Lam–Landesmann–Litt). *Suppose that $(A_1, \ldots, A_n)$ is interesting and that some A_i has infinite order. Then, there exists $\alpha_1, \ldots, \alpha_n, \lambda \in \mathbb{C}^\times$ such that $(\alpha_1 A_1, \ldots, \alpha_n A_n) = MC_\lambda(B_1, \ldots, B_n)$ where MC_λ is Katz's middle convolution operator and where $\langle B_1, \ldots, B_n \rangle \subseteq GL_{n-2}(\mathbb{C})$ is a finite complex reflection group.*

Definition 2.12. $B \in GL_r(\mathbb{C})$ is a pseudo-reflection if B has finite order and the rank of $B - \mathrm{id}$ is 1. A finite complex reflection group (FCRG) is a finite subgroup of $GL_r(\mathbb{C})$ generated by pseudo-reflections.

Finite complex reflection groups were completely classified by Shephard and Todd. There is 1 infinite family and 34 exceptional ones.

Corollary 2.13. *Let $(A_1, \ldots, A_n)$ be interesting with some A_i of infinite order. Then, $n \leq 6$.*

The upshot is that $MC_\lambda(B_1, \ldots, B_n) \subseteq R^1\pi_*\underline{\mathbb{C}}$ where $\pi : Y \to \mathbb{P}^1 \setminus \{x_1, \ldots, x_n\}$ is an explicit family of curves.

3 Étale fundamental groups and local systems. Speaker: Gleb Terentiuk. Notes by Luke Wiljanen.

3.1 Étale fundamental groups

Goal 3.1. For a connected scheme X with a geometric point $\overline{x} \in X(\overline{K})$, construct a profinite group $\pi_1^{\text{ét}}(X, \overline{x})$.

Motivation 3.2. For X a reasonable topological space, there is a correspondence

$$\{\text{covering spaces over } X\} \leftrightarrow \{\pi_1(X,x)\text{–sets}\}$$

Fact 3.3. Let $F : G - \text{Sets} \to \text{Sets}$ be the forgetful functor. There is a natural map $G \to \operatorname{Aut}(F)$ which is an isomorphism.

Fact 3.4. Let $F : \text{Finite} - G - \text{Sets} \to \text{FiniteSets}$ be the forgetful functor. Then,

$$\operatorname{Aut}(F) \simeq \varprojlim_{\substack{N \le G \\ \text{finite index} \\ \text{normal subgroup}}} G/N = \widehat{G}$$

where $\widehat{G}$ is the profinite completion of G.

Let FÉt_X be the category of finite étale X–schemes, and let $F_{\overline{x}} : \text{FÉt}_X \to \text{FiniteSets}$ be the functor sending a finite étale X scheme $Y \to X$ to the finite set $|Y_{\overline{x}}|$, the underlying topological space of the fiber product $Y_{\overline{x}} = Y \times_X \operatorname{Spec} \overline{K}$.

Definition 3.5. The étale fundamental group of X relative to a geometric point $\overline{x}$ is

$$\pi_1^{\text{ét}}(X, \overline{x}) = \operatorname{Aut}(F_{\overline{x}}).$$

Remark 3.6. (1) If $\overline{x}, \overline{y} \in X(\overline{K})$, then $\pi_1^{\text{ét}}(X, \overline{x}) \simeq \pi_1^{\text{ét}}(X, \overline{y})$.

(2) We have an equivalence of categories

$$\text{FÉt}_X \xrightarrow{\sim} \left\{\text{finite } \pi_1^{\text{ét}}(X, \overline{x})\text{–sets}\right\}.$$

(3) Given $f : X \to Y$ and a geometric point $\overline{x} \in X(\overline{K})$, let $\overline{y} = f \circ \overline{x} \in Y(\overline{K})$. There is a natural map $\pi_1^{\text{ét}}(X, \overline{x}) \to \pi_1^{\text{ét}}(Y, \overline{y})$.

Given a finite type scheme X over $\mathbb{C}$, let $X^{an} = X(\mathbb{C})$. Then, $(Y \to X) \mapsto (Y^{an} \to X^{an})$ gives an equivalence between FÉt_X and $\{\text{finite covering spaces of } X^{an}\}$ by the Riemann existence theorem.

Corollary 3.7. *With X as above, the natural map $\pi_1^{top}(X^{an}, x)^\wedge \xrightarrow{\sim} \pi_1^{ét}(X, \overline{x})$ is an isomorphism.*

Example 3.8. Let $X = \operatorname{Spec}(K)$ with a geometric point $\overline{x}$ given by $K \hookrightarrow \overline{K}$. Then, we have an isomorphism

$$\pi_1^{\text{ét}}(X, \overline{x}) \simeq \operatorname{Gal}(\overline{K}/K)$$

between the étale fundamental group of X and the absolute Galois group of K.

Example 3.9. Let F be a finitely generated field over $\mathbb{Q}$. Consider $\mathbb{G}_{m,\overline{F}}$. Let $Y \to \mathbb{G}_{m,\overline{F}}$ be a finite étale map. By composing with the inclusion $\mathbb{G}_{m,\overline{F}} \to \mathbb{P}^1_{\overline{F}}$, we get a map $Y \to \mathbb{P}^1_{\overline{F}}$. This map extends to a smooth compactifaction $Y \to \overline{Y}$. We now look at the resulting map $\overline{\varphi} : \overline{Y} \to \mathbb{P}^1_{\overline{F}}$. An argument using Riemann–Hurwitz shows that $g_{\overline{Y}} = 0$ and that there are exactly two points of ramification. Namely, with ramification points $y_1, \dots, y_s$ above 0 and ramfication points $z_1, \dots, z_r$ above ∞ of ramification degrees $e_1, \dots, e_s$ and $d_1, \dots, d_r$, respectively, we have

$$2g_{\bar{Y}} - 2 = -2n + \sum(e_i - 1) + \sum(d_j - 1) = -(r+s)$$

where n is the degree. Consequently, $g_{\bar{Y}} = 0$. Hence, $\overline{Y} \cong \mathbb{P}^1_{\overline{F}}$. It follows that Y can be identified with $\mathbb{G}_{m,\overline{F}}$, so that the map $Y \to \mathbb{G}_{m,\overline{F}}$ is the map $x \mapsto x^n$. The group of automorphisms of the nth power map is identified with the group of nth roots of unity. So, $\mathrm{Aut}(Y|\mathbb{G}_{m,\overline{F}}) \simeq \mu_n(\overline{F})$. So,

$$\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{F}}) \simeq \varprojlim \mu_n(\overline{F}).$$

This comes with an action of $G_F = \mathrm{Gal}(\overline{F}/F)$. We write $\widehat{\mathbb{Z}}(1)$ for $\varprojlim \mu_n(\overline{F})$ with this Galois action.

3.2 Local systems

Definition 3.10. A $\mathbb{Q}_\ell$-local system on X is a continuous homomorphism $\pi_1^{\text{ét}}(X, \overline{x}) \to \mathrm{GL}_n(\mathbb{Q}_\ell)$.

Example 3.11. A main source of local systems comes from the following setting: Let $X \to S$ be smooth and proper, and assume that ℓ is invertible on the base, i.e., $\ell \in \mathcal{O}(S)^\times$. Then, we have $\mathbb{Q}_\ell$-local systems $\pi_1^{\text{ét}}(S, \overline{s}) \to \mathrm{GL}(H^i_{\text{ét}}(X_{\overline{s}}, \mathbb{Q}_\ell))$.

Theorem 3.12. *Let F be a finitely generated field over $\mathbb{Q}$, and let $\rho : G_{F((t))} \to \mathrm{GL}_n(\mathbb{Q}_\ell)$. Then, $\rho|_{G_{\overline{F}((t))}}$ is quasi-unipotent. That is, if σ topologically generates $G_{\overline{F}((t))} \cong \hat{\mathbb{Z}}$, then $\rho(\sigma)^N - 1$ is nilpotent for some N.*

Proof. Since $1 + \ell^2 M_n(\mathbb{Z}_\ell) \subseteq \mathrm{GL}_n(\mathbb{Q}_\ell)$ is open, there exists a finite extension $K/F((t))$ such that

$$G_K \subseteq \rho^{-1}(1 + \ell^2 M_n(\mathbb{Z}_\ell)).$$

Then, $\overline{F}((t)) \subseteq K_{nr}$ where K_{nr} is the maximal unramified extension of K. Since $K_{nr}/\overline{F}((t))$ is a finite extension, some power of the topological generator $\sigma \in G_{\overline{F}((t))}$ topologically generates $G_{K_{nr}}$, i.e., there is some $n \in \mathbb{N}$ such that σ^n topologically generates $G_{K_{nr}}$.

Let $K_\ell \subseteq \overline{K}$ be the field obtained by adjoining all ℓ-power roots of a uniformizer to K_{nr}. Then, G_{K_ℓ} is a prime to ℓ profinite group, so $\rho|_{G_{K_\ell}}$ is trivial. Thus, ρ factors through $\mathrm{Gal}(K_\ell/K)$. We have the short exact sequence

$$1 \to \mathrm{Gal}(K_\ell/K_{nr}) \to \mathrm{Gal}(K_\ell/K) \to \mathrm{Gal}(K_{nr}/K) \to 1.$$

Let $\chi : \mathrm{Gal}(K_{nr}/K) \to \mathbb{Z}_\ell^\times$ be the ℓ-adic cyclotomic character.

For $s \in \mathrm{Gal}(K_\ell/K_{nr})$, we see that s and $s^{\chi(t)}$ are conjugate for all $t \in \mathrm{Gal}(K_{nr}/K)$. Then, write $X = \log(\rho(s))$. We have that X and $\chi(t)X = \log(\rho(s)^{\chi(t)})$ are conjugate. Since X and $\chi(t)X$ are conjugate, they have the same characteristic polynomials. But, we describe a relationship between the characteristic polynomials. Namely, if $\sum_{i=0}^n a_i(M)y^{n-i}$ is the characteristic polynomial of a matrix $M \in \mathrm{M}_n(\mathbb{Q}_\ell)$, then $a_i(X) = a_i(\chi(t)X) = \chi(t)^i a_i(X)$. Since F is finitely generated over $\mathbb{Q}$, if $i > 0$, then there exists t such that $\chi(t)^i \neq 1$. Hence, $a_i(X) = 0$ for $i > 0$. Thus, the characteristic polynomial of X is y^n. Therefore, X is nilpotent, and $\exp(X) = \rho(s)$ is unipotent. □

Corollary 3.13. *Let $X \to S = \overline{S} \setminus \{s'\}$ be over $\mathbb{C}$ be smooth projective, where $\overline{S}$ is a smooth projective curve. Let $\operatorname{Spec} \mathbb{C}((t)) \to S$ be around s', i.e., look at $\mathcal{O}_{\overline{S},s'} \to \widehat{\mathcal{O}}_{\overline{S},s'} \cong \mathbb{C}[[t]]$, which gives $\operatorname{Spec} \mathbb{C}[[t]] \to \operatorname{Spec}(\mathcal{O}_{\overline{S},s'}) \to \overline{S}$, and localize to get $\operatorname{Spec} \mathbb{C}((t)) \to S$. As in Example 3.11, we get a homomorphism $\pi_1^{\acute{e}t}(S, s) \to \mathrm{GL}(H^i_{\acute{e}t}(X_s, \mathbb{Q}_\ell))$. Then, $G_{\mathbb{C}((t))} \to \pi_1^{\acute{e}t}(S, s) \to \mathrm{GL}(H^i_{\acute{e}t}(X_s, \mathbb{Q}_\ell))$ is quasi-unipotent.*

Proof. Find F finitely generated over $\mathbb{Q}$ and a smooth projective spreading out $\mathfrak{X} \to \mathfrak{S}$ over F such that the fiber product with $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(F)$ recovers $X \to S$. Then, $G_{\mathbb{C}((t))} \to \mathrm{GL}(H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell))$ extends to

$$G_{F((t))} \to \mathrm{GL}(H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell)).$$

We have $G_{\mathbb{C}((t))} \simeq G_{\overline{F}((t))}$, and so the theorem implies $G_{\mathbb{C}((t))} \to \pi_1^{\text{ét}}(S, s) \to \mathrm{GL}(H^i_{\text{ét}}(X_s, \mathbb{Q}_\ell))$ is quasi-unipotent. □

4 The p-adic Riemann-Hilbert correspondence, lecture 1. Speaker: Alexander Petrov. Notes by Mehmet Basaran.

Fix a prime p. Let $S/\mathbb{C}$ be a connected smooth variety, and let $f\colon X \to S$ be smooth and proper. Then $R^i f_*\mathbb{Z}$ form a local system on $S(\mathbb{C})$, where i is an arbitrary nonnegative integer.

Let A be a commutative ring (most of the times one of $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\overline{\mathbb{Q}_p}$). We define

$$\left\{\begin{matrix}\text{local systems } \mathbb{L}\\ \text{of free } A\text{−modules}\\ \text{of rank } n \text{ on } S/\mathbb{C}\end{matrix}\right\}_{/\cong} = \left\{\begin{matrix}\pi_1(S(\mathbb{C}),s)\xrightarrow{\rho_{\mathbb{L}}}\mathrm{GL}_n(A)\\ \text{up to conjugation}\end{matrix}\right\}.$$

Definition 4.1. A local system of A-modules $\mathbb{L}$ is of geometric origin, if there is a Zariski open $U \hookrightarrow S$ and a smooth proper family $f\colon X \to U$ such that $\mathbb{L}|_{U(\mathbb{C})}$ is a direct summand of $R^i f_* A$.

Conjecture 4.2 (Litt, during the lecture)**.** We may replace U by S in the above definition.

Question 4.3. How can we classify local systems of geometric origin?

Remark 4.4. For any $\mathbb{Z}$-local system $\mathbb{L}$ on $S(\mathbb{C})$ there is a proper fibration of complex manifolds $f\colon Y \to S(\mathbb{C})$ such that $\mathbb{L}$ is a direct summand of $R^1 f_*\mathbb{Z}$.

Definition 4.5. A $\mathbb{Z}_p$-local system $\mathbb{L}$ on $S(\mathbb{C})$ is called arithmetic, if there exists a finitely generated over $\mathbb{Q}$ field $F \subset \mathbb{C}$, and a variety S_0/F with $S_0 \times_F \mathbb{C} \simeq S$, such that $\mathbb{L}$ extends to an étale local system $\widetilde{\mathbb{L}}$ on S_0. In a diagram: Here, the homomorphism $\rho_{\widetilde{\mathbb{L}}}$ needs to be continuous.

$$\begin{tikzcd}
 & \pi_1\left(S(\mathbb{C}),s\right) \arrow[rrr, "\rho_{\mathbb{L}}"] \arrow[d] & & & \mathrm{GL}_n\left(\mathbb{Z}_p\right) & \\
 & \pi_1^{\acute{e}t}\left(S,s\right) \arrow[d, "\simeq"] & & & & \\
1 \arrow[r] & \pi_1^{\acute{e}t}\left(S_{0,\overline{F}}\right) \arrow[r] & \pi_1^{\acute{e}t}\left(S_0,s\right) \arrow[r] \arrow[urr, dashed, "\rho_{\widetilde{\mathbb{L}}}"] & G_F \coloneqq \mathrm{Gal}\left(\overline{F}/F\right) \arrow[r] & 1 &
\end{tikzcd}$$

This definition can be formulated verbatim for $\mathbb{Q}_p$ or $\overline{\mathbb{Q}_p}$ in place of $\mathbb{Z}_p$.

Remark 4.6. If a $\overline{\mathbb{Q}_p}$-local system $\mathbb{L}$ is of geometric origin, it is arithmetic.

Example 4.7. Take $S = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$. Then a $\overline{\mathbb{Q}_p}$-local system $\mathbb{L}$ is arithmetic if and only if for the corresponding representation $\rho_{\mathbb{L}}\colon \pi_1\left(S(\mathbb{C}),s\right) \to \mathrm{GL}_n\left(\overline{\mathbb{Q}_p}\right)$, the matrix $\rho_{\mathbb{L}}$ is quasi-unipotent, where γ is a generator of $\pi_1\left(S(\mathbb{C}),s\right) = \mathbb{Z}$ (cf. Theorem 3.12).

Conjecture 4.8 (relative Fontaine-Mazur)**.** For every semi-simple $\overline{\mathbb{Q}_p}$-local system $\mathbb{L}$ on $S(\mathbb{C})$, $\mathbb{L}$ is arithmetic if and only if $\mathbb{L}$ is of geometric origin.

Example 4.9. For $S = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}$ the conjecture is true: In this case, $\mathbb{L}$ being arithmetic implies that it has finite monodromy (i.e. it is trivialized by a finite étale cover). If $f\colon X \to S$ is such a finite étale cover, then $\mathbb{L}$ is a direct summand of $f_*\overline{\mathbb{Q}_p}$, and thus of geometric origin.

Now we illustrate how Conjecture 4.8 can be viewed as a non-abelian analogue of the Tate conjecture. We work with $\overline{\mathbb{Q}_p}$-local systems on $S_{0,\overline{F}}$ as in Definition 4.5. Then

$$\left\{\overline{\mathbb{Q}_p}\text{-local systems of rank } n \text{ on } S_{0,\overline{F}}\right\} = H^1_{\acute{e}t}\left(S_{0,\overline{F}}, \mathrm{GL}_n\left(\overline{\mathbb{Q}_p}\right)\right).$$

There is an action of G_F on the right-hand side. In this setting it holds that $\mathbb{L}$ is arithmetic if and only if the class $[\mathbb{L}]$ in the right-hand side has finite orbit under G_F. Now assume that S is smooth and proper. Then there is a map

$$\mathrm{cl}\colon Z^i(S) \otimes \mathbb{Q}_p \to H^{2i}_{\acute{e}t}\left(S_{0,\overline{F}}, \mathbb{Q}_p(i)\right)$$

where $Z^i(S)$ consists of algebraic cycles of codimension i.

Conjecture 4.10 (Tate conjecture)**.** The image of the above map is

$$\mathrm{im}\,(\mathrm{cl}) = \left\{ x \in H^{2i}_{\acute{e}t}\left(S_{0,\overline{F}}, \mathbb{Q}_p(i)\right) \middle|\ x \text{ has finite orbit under } G_F \right\}.$$

4.1 p-adic Hodge theory

Remark 4.11. If Conjecture 4.8 holds, then all semi-simple arithmetic $\mathbb{L}$ should underlie a VHS. In the following we investigate where this VHS would come from.

Let X be a smooth proper variety over $\mathbb{Q}_p$. Then $G_{\mathbb{Q}_p}$ acts on $H^n_{\acute{e}t}\left(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p\right)$. There is a functor

$$D_{dR}\colon \left\{ \substack{\text{finite dimensional continuous} \\ \text{representations of } G_{\mathbb{Q}_p} \text{ on a } \mathbb{Q}_p\text{-vector space}} \right\} \to \left\{ \substack{\text{vector spaces } V/\mathbb{Q}_p \text{ with a filtration} \\ 0=F^bV\subset\cdots\subset F^iV\subset F^{i-1}V\subset\cdots\subset F^aV=V} \right\}$$

such that $D_{dR}\left(H^n_{\acute{e}t}\left(X_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p\right)\right) = H^n_{dR}\left(X/\mathbb{Q}_p\right)$ with filtration $F^\bullet_{\text{Hodge}}$, $F^i_{\text{Hodge}}/F^{i+1}_{\text{Hodge}} = H^{n-i}\left(X, \Omega^i_X\right)$.

4.2 p-adic Riemann-Hilbert correspondence

First we summarize the complex Riemann-Hilbert correspondence. Let $Y/\mathbb{C}$ be a smooth proper variety, and let $f\colon Z \to Y$ be a smooth and proper family of varieties. Then there is a bijection

$$\{\mathbb{C}\text{-local systems on } Y(\mathbb{C})\} \xrightarrow{\sim} \{\text{vector bundles with flat connection on } Y\}$$

such that

$$R^i f_* \mathbb{C} \mapsto \left(\mathcal{H}^i_{dR}\left(Z/Y\right), \nabla_{GM}\right).$$

Now to introduce a p-adic Riemann-Hilbert correspondence, let $S/\mathbb{Q}_p$ be a smooth variety. There is a functor

$$D_{dR}\colon \{\text{étale } \mathbb{Q}_p\text{-local systems on } S\} \xrightarrow{\sim} \left\{ \substack{\text{vector bundles } E/S \text{ with} \\ \text{a flat connection } \nabla\colon E\to E\otimes\Omega^1_S \text{ on } Y \\ \text{and with a filtration} \\ F^b=0\subset\cdots\subset F^i\subset F^{i+1}\subset\cdots\subset F^a=E \\ \text{such that } F^i/F^{i+1} \text{ is a vector bundle} \\ \text{and } \nabla\left(F^i\right)\subset F^{i-1}\otimes\Omega^1_S} \right\}$$

such that for $f\colon X \to S$ smooth proper, we get $D_{dR}\left(R^n f_* \mathbb{Q}_p\right) \simeq \left(\mathcal{H}^i_{dR}\left(X/S\right), \nabla_{GM}, F^\bullet_{\text{Hodge}}\right)$.

Remark 4.12. This functor D_{dR} cannot preserve ranks and be monoidal (meaning that $D_{dR}\left(\mathbb{L}_1 \otimes \mathbb{L}_2\right) = D_{dR}\left(\mathbb{L}_1\right) \otimes D_{dR}\left(\mathbb{L}_2\right)$). To see this, take $S = \mathrm{Spec}\,(K)$ with $K = \mathbb{Q}_p\left(\mu_p\right)$ for $p > 2$, and $V = \mathbb{Q}_p(-1) = \chi^{-1}_{\text{cycl}} = H^2_{\acute{e}t}\left(\mathbb{P}^1_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_p\right)$. Then

$$D_{dR}(V) = H^2_{dR}\left(\mathbb{P}^1_K\right) = K$$

with filtration $0 = F^2 \subset F^1 = K$. If D_{dR} preserved ranks and was monoidal, then

$$D_{dR}\left(\chi^{-1/2}_{\text{cycl}}\right)^{\otimes 2} = D_{dR}(V) = \left(K, 0 = F^2 \subset F^1 = K\right).$$

This is impossible, since $D_{dR}\left(\chi^{-1/2}_{\text{cycl}}\right)$ also needs to be a one dimensional vector space where the filtration jumps at some point, and thus the filtration of $D_{dR}\left(\chi^{-1/2}_{\text{cycl}}\right)^{\otimes 2}$ would have to jump at an even index, but the jump is at index 1.

5 Variations of Hodge structure and Higgs bundles. Speaker: Yilong Zhang. Notes by Min Shi.

5.1 Content

1. Example: one parameter family of elliptic curves;
2. Hodge structures;
3. From variations of Hodge structures to Higgs bundle.

5.2 Motivation

Treat the compact, oriented surface with $g = 1$ as a complex torus $\mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}\langle e_1, e_2\rangle$. Choose a basis $\{\delta, \gamma\}$ for $H_1(T, \mathbb{Z})$, so that $H_1(T, \mathbb{Z}) = \mathbb{Z}\delta + \mathbb{Z}\gamma$. On the complex torus, there is a canonical holomorphic 1-form dz, with $dz \in H^1_{\mathrm{dR}}(T) \bigotimes_{\mathbb{R}} \mathbb{C} \cong (H_1(T, \mathbb{Z})^\vee) \bigotimes_{\mathbb{Z}} \mathbb{C}$.

One can integrate dz against δ and γ to get two complex numbers (periods) $\int_\delta dz$, $\int_\gamma dz$, which depend on e_1, e_2. However, what matters is not these two numbers, but rather their ratio, so we can regard $\left[\int_\delta dz : \int_\gamma dz\right] \in \mathbb{P}^1$. Up to some choices, this gives a map from the isomorphism classes of elliptic curves over $\mathbb{C}$ to $\mathbb{A}^1(\mathbb{C})$, which is part of the reason that the moduli space of elliptic curves over $\mathbb{C}$ has $\mathbb{P}^1(\mathbb{C})$ as the set of complex points.

5.3 Hodge structures

Let X be a compact Kähler manifold (for instance, the complex points of a smooth projective variety). Then $H^n(X, \mathbb{Q})$ has a Hodge structure, i.e., $H^n(X, \mathbb{C}) = H^n(X, \mathbb{Q}) \bigotimes_{\mathbb{Q}} \mathbb{C}$ admits a Hodge decomposition of complex vector spaces

$$\begin{gathered} H^n(X, \mathbb{C}) \cong H^{n,0}(X) \oplus H^{n-1,1}(X) \oplus \cdots \oplus H^{0,n}(X) \\ \text{satisfying } \overline{H^{p,n-p}(X)} = H^{n-p,p}(X). \end{gathered}$$

Equivalently, $H^n(X, \mathbb{C})$ admits a filtration called the Hodge Filtration:

$$0 = F^{n+1}H^n \subseteq F^n H^n \subseteq F^{n-1}H^n \subseteq ... \subseteq F^1 H^n \subseteq F^0 H^n = H^n(X, \mathbb{C})$$

satisfying

1. $F^p H^n \cap \overline{F^{n-p+1}H^n} = \{0\}$;
2. $F^p H^n \bigoplus \overline{F^{n-p+1}H^n} = H^n(X, \mathbb{C})$.

To recover the Hodge decomposition, take $H^{p,n-p}(X) = F^p H^n \cap \overline{F^{n-p}H^n}$.

Example 5.1. For $T =$ complex torus, $\mathrm{span}(dz) = F^1 H^1(T, \mathbb{C}) = H^{1,0}(T) \cong \mathbb{C}$.

5.3.1 Where does the Hodge Filtration come from?

We have a resolution of $\underline{\mathbb{C}_X}$ by the holomorphic de Rham complex $\Omega^\bullet_X$

$$\underline{\mathbb{C}_X} \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \to \Omega^2_X \to \cdots \to 0,$$

where Ω^i_X is placed in the i-th degree.

By the abstract de Rham theorem,

$$H^n_{\mathrm{sing}}(X, \mathbb{C}) \cong H^n(X, \underline{\mathbb{C}_X}) \cong \mathbb{H}^n(X, \Omega^\bullet_X).$$

Define $F^p\Omega^j_X := \Omega^j_X$ if $p \leq j \leq n$, and $F^p\Omega^j_X := 0$ otherwise. Then define the Hodge Filtration

$$F^pH^n(X,\mathbb{C}) := \mathrm{Im}(\mathbb{H}(X, F^p\Omega^\bullet_X) \xrightarrow{\phi} \mathbb{H}(X, \Omega^\bullet_X))$$

.

By the $\partial\overline{\partial}$-lemma, the inclusion of the holomorphic de Rham complex into the complex of (real) smooth differentials $(\Omega^\bullet_X, d) \to (\mathcal{A}^\bullet, d)$ is a quasi-isomorphism. Then the Hodge decomposition combined with Dolbeault's theorem implies the degeneration of the Hodge-to-de Rham (Fröhlicher) spectral sequence at the E_1-page. In particular, ϕ is injective, and

$$F^pH^n(X,\mathbb{C}) = \mathbb{H}^n(X, F^p\Omega^\bullet_X).$$

Also as a consequence of the degeneration at the E_1-page,

$$F^pH^n(X,\mathbb{C})/F^{p+1}H^n(X,\mathbb{C}) \cong H^{n-p}(X, \Omega^p_X)$$

5.4 Real Variation of Hodge structures

Let B be a complex manifold and $\mathbb{V}^n$ a local system on B. A real variation of Hodge structures is a filtration by holomorphic subbundles on $\mathbb{V}^n \bigotimes_{\underline{\mathbb{C}_B}} \mathcal{O}_B$

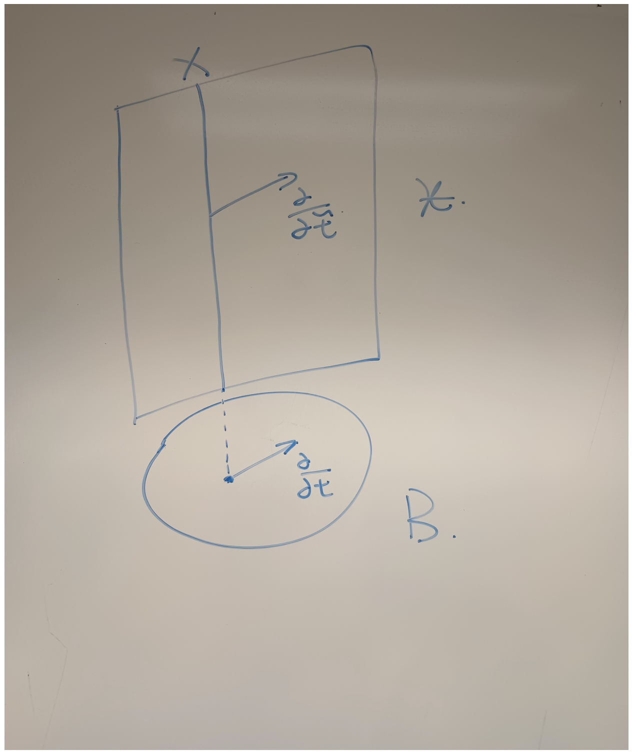
$$\mathbb{F}^k\left(\mathbb{V}^n \bigotimes_{\underline{\mathbb{C}_B}} \mathcal{O}_B\right) \subseteq \mathbb{F}^{k-1}\left(\mathbb{V}^n \bigotimes_{\underline{\mathbb{C}_B}} \mathcal{O}_B\right) \subseteq \ldots \subseteq \mathbb{F}^1\left(\mathbb{V}^n \bigotimes_{\underline{\mathbb{C}_B}} \mathcal{O}_B\right) \subseteq \mathbb{F}^0\left(\mathbb{V}^n \bigotimes_{\underline{\mathbb{C}_B}} \mathcal{O}_B\right) = \mathbb{V}^n \bigotimes_{\underline{\mathbb{C}_B}} \mathcal{O}_B.$$

satisfying

1. $\mathbb{F}^p \cap \overline{\mathbb{F}^{n-p+1}} = \{0\}$;
2. $\mathbb{F}^p \bigoplus \overline{\mathbb{F}^{n-p+1}} \cong \mathbb{V}^n \bigotimes_{\underline{\mathbb{C}_B}} \mathcal{O}_B$
3. Griffiths transversality: there is a flat connection ∇ on $\mathbb{V}^n \bigotimes_{\underline{\mathbb{C}_B}} \mathcal{O}_B$ such that

$$\nabla : \mathbb{F}^p \to \mathbb{F}^{p-1} \bigotimes \Omega^1_B.$$

To be more specific, suppose $\mathbb{V}^n = R^nf_*\mathbb{C}$, where $f : \mathcal{X} \to B$ is a smooth family of compact Kähler manifolds. By Ehresmann's theorem, locally, $\mathcal{X}$ is diffeomorphic to $X_0 \times B$. A picture is drawn below, where X is the fiber over a point 0 in B, $\frac{\partial}{\partial t}$ is a tangent vector at 0 in B, and $\frac{\partial}{\partial \tilde{t}}$ is a lift of the tangent vector $\frac{\partial}{\partial t}$.



By Kodaira's theory on deformation of complex structure, the 1st order deformation is governed by a $(0,1)$ form in $T^{1,0}(X_0)$:

$$\kappa = \Sigma_{\alpha,\beta} f_{\alpha\beta} \frac{\partial}{\partial z_\alpha} \bigotimes d\bar{z}_\beta.$$

Now Griffiths transversality means that for $\omega \in \mathbb{F}^p \mathcal{A}^n_{\mathcal{X}/B}$ on fiber direction with at least p dz_i's, the image of ω under the Gauss-Manin connection is in $\mathbb{F}^{p-1}H^n(X_0, \mathbb{C})$. In fact, locally, the map $H^{n-p}(X, \Omega^p_X) \to H^{n-p+1}(X, \Omega^{p-1}_X)$ induced by ∇ can be written as $\nabla(\alpha) = \alpha \cup \kappa$.

5.4.1 What is a variation of Hodge structure?

Fix a complex vector space $H^n(X, \mathbb{C})$. Then a variation of Hodge structures is a family of Hodge structures on it that varies in a certain way and satisfies certain axioms.

Example 5.2. The Legendre family of elliptic curves is $\{y^2 = x(x-1)(x-t), t \in \mathbb{P}^1 - \{0, 1, \infty\}\}$, whose singularities are roughly described in Figure 1. Pick a basepoint $t_0 \in \mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$. The holomorphic 1-forms are $\omega_t = \left[\frac{dx_t}{y_t}\right] \in H^1(X_t, \mathbb{C})$, which spans $F^1H^1(X_t, \mathbb{C}) \subseteq H^1(X_t, \mathbb{C}) \cong H^1(X_{t_0}, \mathbb{C})$. After desingularization, the minimal resolution can be depicted in Figure 2, with two rational curves meeting transversely above each 0 and 1, and an I_2^* fiber (7 components, with three multiplicity-two fibers colored in red) above ∞.

The corresponding Picard-Fuchs equation is (for details, see [Lit13, section 1.4.2])

$$\omega_t'' = -\frac{1}{4t(t-1)}\omega_t + \frac{2t-1}{t(t-1)}\omega_t'$$

By ODE theory, locally around 0, solutions take the form: $f(t)\log(t) + g(t)$, where f and g are holomorphic.

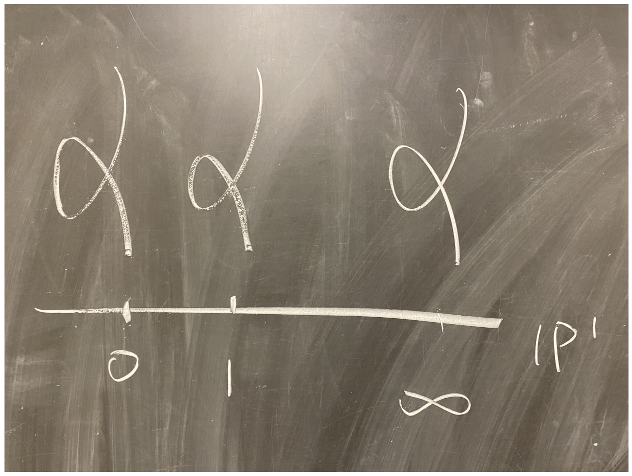


Figure 1: Legendre family

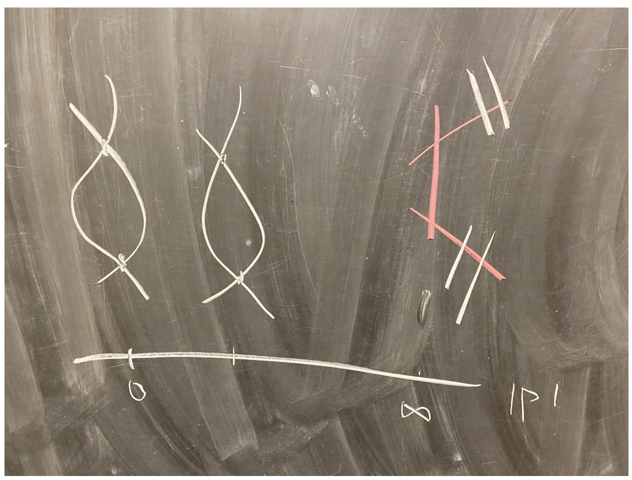


Figure 2: Kodaira-Néron model of Legendre family

The local monodromy operators on $H_1(E_{t_0}, \mathbb{Z})$ are $T_0 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, T_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ At infinity, the monodromy is $= \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$, which is the inverse of the product $T_0 T_1$.

6 Nonabelian cohomology and applications, lecture 2. Speaker: Daniel Litt. Notes by Jake Huryn.

6.1 Introduction

Let X be a smooth projective variety over $\mathbb{C}$.

Conjecture 6.1 (Hodge)**.** The image of the cycle-class map $Z^i(X) \otimes_{\mathbb{Z}} \mathbb{Q} \to H^{2i}(X, \mathbb{Q}(i))$ is $H^{2i}(X, \mathbb{Q}(i)) \cap H^{0,0}(X, \mathbb{C}(i))$.[2]

The "non-Abelian Hodge conjecture" is:

Conjecture 6.2 (Simpson)**.** Any $\mathbb{Q}$-local system on X underlying a polarizable $\mathbb{Z}$-VHS is of geometric origin in the sense of Definition 4.1.

One goal of the rest of the talk will be to explain the parallel between these two conjectures, which will require us to fill in the missing entry in the following table of analogies.

Abelian	non-Abelian
$H^n(X, \mathbb{C})$	$\mathrm{Hom}(\pi_1(X), \mathrm{GL}_r(\mathbb{C}))/\cong$
$H^n_{\mathrm{dR}}(X)$	$\mathrm{MIC}(X)/\cong$
$\bigoplus_{p+q=n} H^{p,q}(X)$	???

Remark 6.3. Recall that the isomorphisms between objects in different rows of this table are highly transcendental!

6.2 Higgs bundles

Continue to assume X is a smooth projective variety over $\mathbb{C}$. (This permits us, by GAGA, to ignore the difference between algebraic and holomorphic coherent modules on X.)

Definition 6.4. A *Higgs bundle* on X is a pair $(\mathcal{E}, \theta)$, where $\mathcal{E}$ is a vector bundle on X and $\theta\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X$ (the *Higgs field*) is an $\mathcal{O}_X$-linear map such that $\theta^2 = 0$, i.e. the following composition vanishes:[3]

$$\mathcal{E} \xrightarrow{\theta} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\theta \otimes \mathrm{id}} \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X \twoheadrightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^2_X$$

Remark 6.5.
1. It is necessary to note the difference between a Higgs field and an integrable connection; the former is $\mathcal{O}_X$-linear, rather than $\mathbb{C}$-linear, hence does *not* satisfy the Leibniz rule.
2. The $\mathcal{O}_X$-linear nature of the Higgs field allows $\mathbb{C}^\times$ to act on the category of Higgs bundles via $t \cdot (\mathcal{E}, \theta) := (\mathcal{E}, t\theta)$.

Example 6.6.
1. If $\mathcal{E}$ is any vector bundle on X, then $(\mathcal{E}, 0)$ is a Higgs bundle on X (but essentially *never* a MIC!).
2. Suppose $(\mathcal{E}, F^\bullet, \nabla)$ is a $\mathbb{C}$-VHS. Since $\nabla(F^i(\mathcal{E})) \subseteq F^{i-1}(\mathcal{E})$, we get a map $\mathrm{gr}(\nabla)\colon \mathrm{gr}(\mathcal{E}) \to \mathrm{gr}(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^1_X$. The integrability of ∇ implies that $(\mathrm{gr}(\mathcal{E}), \mathrm{gr}(\nabla))$ is a Higgs bundle.

Definition 6.7. Let $\mathcal{E}$ be a vector bundle bundle. The *slope* of $\mathcal{E}$ is $\mu(\mathcal{E}) := \deg_H(\mathcal{E})/\mathrm{rank}_{\mathcal{O}_X}(\mathcal{E})$. (We fix an ample divisor to define the degree function.) We say that a Higgs bundle $(\mathcal{E}, \theta)$ is *stable* if for any proper sub-Higgs bundle[4] $(\mathcal{E}', \theta)$, we have $\mu(\mathcal{E}') < \mu(\mathcal{E})$, and *semistable* if $\mu(\mathcal{E}') \leq \mu(\mathcal{E})$.

[2] The (i) here is the Tate twist in Hodge cohomology; it subtracts i from both superscripts in the Hodge decomposition, so that $H^{2i}(X, \mathbb{Q}(i)) \cap H^{0,0}(X, \mathbb{C}(i)) = H^{2i}(X, \mathbb{Q}) \cap H^{i,i}(X, \mathbb{C})$. This is done simply to ensure that the notation parallels the usual formulation of the Tate conjecture.

[3] If Ω^1_X is free on $\omega_1, \ldots, \omega_d$, then $\theta = \theta_1\omega_1 + \cdots + \theta_d\omega_d$ for some $\theta_i \in \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$, and the condition $\theta^2 = 0$ means that the θ_i commute.

[4] Meaning $\mathcal{E}'$ is a vector bundle which is a proper subsheaf of $\mathcal{E}$ preserved by θ, i.e. $\theta(\mathcal{E}') \subseteq \mathcal{E}' \otimes_{\mathcal{O}_X} \Omega^1_X$.

Example 6.8. If $\mathcal{E}$ is a direct sum of line bundles, then $\mu(\mathcal{E})$ is the average degree of the line bundles, and $(\mathcal{E}, 0)$ is semistable if and only if all of the line bundles have the same degree.

Higgs bundles are related to local systems via the following theorem, which was contributed to by Narasimhan–Seshadri, Hitchin, Corlette, Donaldson, Simpson, ... See [Sim91, §1] for citations and more discussion of the following theorem and remark.

Theorem 6.9. *There is a bijection between isomorphism classes of irreducible $\mathbb{C}$-local systems on X and isomorphism classes of stable Higgs bundles with vanishing Chern classes.*[5] *This bijection satisfies (viewing local systems as MICs on X via Riemann–Hilbert):*

1. *If $(\mathcal{E}, \nabla)$ is a MIC whose monodromy representation is unitary (i.e. the closure of its image is compact), the corresponding Higgs bundle is $(\mathcal{E}, 0)$. Conversely, any Higgs bundle with vanishing Higgs field arises in this way.*

2. *If $(\mathcal{E}, F^\bullet, \nabla)$ is a $\mathbb{C}$-VHS, the Higgs bundle corresponding to $(\mathcal{E}, \nabla)$ is $(\mathrm{gr}(\mathcal{E}), \mathrm{gr}(\nabla))$.*

3. *It is compatible with pullback and smooth proper pushforward.*

This is the non-Abelian Hodge decomposition (the "???" in the "table of analogies" above).

Remark 6.10.
1. The proof of Theorem 6.9 passes through a category of "harmonic bundles" (somehow analogous to the proof of the Hodge decomposition via harmonic forms).

2. A Higgs bundle $(\mathcal{E}, \theta)$ arises (via Theorem 6.9) from a $\mathbb{C}$-VHS if and only if $(\mathcal{E}, \theta) \cong (\mathcal{E}, t\theta)$ for each $t \in \mathbb{C}^\times$.

To summarize: let $M_{\mathrm{B}}(X, r)$ be the moduli space of semisimple rank-r local systems on X, let $M_{\mathrm{dR}}(X, r)$ be the moduli space of semisimple rank-r MICs on X, and let $M_{\mathrm{Dol}}(X, r)$ be the moduli space of rank-r *polystable* Higgs bundles on X with vanishing Chern classes ("polystable" meaning "direct sum of stable"). Then we have the following isomorphisms:

$$M_{\mathrm{B}}(X, r)(\mathbb{C}) \cong M_{\mathrm{dR}}(X, r)(\mathbb{C}) \cong M_{\mathrm{Dol}}(X, r)(\mathbb{C}), \tag{1}$$

and the final object has an action of $\mathbb{C}^\times$ whose fixed points correspond to $\mathbb{C}$-VHSs. The first isomorphism is in general only holomorphic, while the second is in general only real-analytic.

Example 6.11. We explain the isomorphisms (1) in the case $\dim(X) = r = 1$. Let g be the genus of X.

1. (Representations). We have $M_{\mathrm{B}}(X, 1) \cong H^1(\pi_1(X), \mathbb{C}^\times) \cong H^1(X, \mathbb{C}^\times) \cong \mathbb{C}^{\times, 2g}$. Using the exponential exact sequence, this moduli space is (holomorphically) equivalent to $H^1(X, \mathbb{C})/H^1(X, \mathbb{Z})$.

2. (MICs). By interpreting MICs as objects coming from Čech cohomology, we get an isomorphism $M_{\mathrm{dR}}(X, 1) \cong \mathbb{H}^1(X, \mathcal{O}_X^\times \xrightarrow{\mathrm{dlog}} \Omega_X^1) \cong H^1_{\mathrm{dR}}(X)/H^1(X, \mathbb{Z})$. If $g \geq 1$, this space admits a nontrivial algebraic map to the Abelian variety $\mathrm{Pic}^0(X)$ via $(\mathcal{L}, \nabla) \mapsto \mathcal{L}$; in fact, it is an $H^0(X, \Omega_X^1)$-torsor over $\mathrm{Pic}^0(X)$. In particular, $M_{\mathrm{dR}}(X, 1)$ cannot be algebraically isomorphic to $M_{\mathrm{B}}(X, 1)$ if $g \geq 1$, since $\mathbb{C}^{\times, 2g}$ admits no nontrivial algebraic map to $\mathrm{Pic}^0(X)$.

3. (Higgs bundles). Since $\dim(X) = 1$, the Higgs-field condition $\theta^2 = 0$ is meaningless, so a Higgs field on a line bundle $\mathcal{L}$ over X is just an $\mathcal{O}_X$-linear map $\mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \Omega_X^1$, which in turn is the same as a global section of Ω_X^1. Thus $M_{\mathrm{Dol}}(X, 1) \cong \mathrm{Pic}^0(X) \times H^0(X, \Omega_X^1)$. The isomorphism between this and $M_{\mathrm{dR}}(X, 1)$ is just the Hodge decomposition of $H^1_{\mathrm{dR}}(X)$. However, the $H^0(X, \Omega_X^1)$-torsor $M_{\mathrm{dR}}(X, 1) \to \mathrm{Pic}^0(X)$ is in general nontrivial, in which case M_{dR} is not algebraically isomorphic to $M_{\mathrm{Dol}}(X, 1)$.

[5] If X is a curve, the condition of vanishing Chern classes just means the vector bundle has degree 0.

Finally, we return to the Hodge conjecture. Recall that an $\mathbb{R}$-Hodge structure H is endowed with a $\mathbb{C}^\times$-action in which z acts on $H^{p,q}$ via $z^{-p}\overline{z}^{-q}$, so that $H^{0,0} = H^{\mathbb{C}^\times}$. In view of Remark 6.10(2), we can rephrase Simpson's non-Abelian Hodge conjecture as:

Conjecture 6.12. Every element of $M_{\mathrm{B}}(X, r)(\mathbb{Q}) \cap M_{\mathrm{Dol}}(X, r)(\mathbb{C})^{\mathbb{C}^\times}$ which admits a $\mathbb{Z}$-structure is of geometric origin.

6.3 Geometry of the Dolbeault moduli space

A key observation for non-Abelian Hodge theory is that the Higgs field, being $\mathcal{O}_X$-linear, has a notion of "characteristic polynomial".[6] (A reference for this section is [Sim91, §§1–2].)

Let $(\mathcal{E}, \theta)$ be a Higgs bundle. From θ we get an action of the tangent sheaf T_X, hence $\mathrm{Sym}^*(T_X)$, on $\mathcal{E}$ (the action of the tensor algebra factors through the symmetric algebra since $\theta^2 = 0$). Thus we may view $\mathcal{E}$ as a sheaf on the relative spectrum $\mathrm{Tot}(\Omega^1_X) \coloneqq \mathrm{Spec}_X(\mathrm{Sym}^*(T_X))$.

Question: What is its support? (These will be the eigenvalues of θ, and the fibers the eigenspaces of θ.)

Let $\mathcal{A}_r \coloneqq \bigoplus_{i=1}^r H^0(X, \mathrm{Sym}^i(\Omega^1_X))$. The *Hitchin map* $h\colon M_{\mathrm{Dol}}(X, r) \to \mathcal{A}_r$ sends $(\mathcal{E}, \theta)$ to the "characteristic polynomial of θ", the element whose ith component is $\mathrm{tr}(\bigwedge^i \theta\colon \bigwedge^i \mathcal{E} \to \bigwedge^i \mathcal{E} \otimes_{\mathcal{O}_X} \mathrm{Sym}^i(\Omega_X))$. This map has some useful and remarkable properties:

Theorem 6.13. *1. Each fiber of h corresponds to $(\mathcal{E}, \theta)$ supported on a fixed subscheme of $\mathrm{Tot}(\Omega^1_X)$ finite flat over X, called the spectral variety.*

2. Because (1) identifies each fiber of h with a moduli space of certain semistable sheaves on the spectral variety, h is a proper map.

3. By (2), the limit $\lim_{t\to 0}(\mathcal{E}, t\theta)$ always exists in $M_{\mathrm{Dol}}(X, r)$ and is $\mathbb{C}^\times$-stable (but need not equal $(\mathcal{E}, 0)$, as the latter might not be semistable).

Corollary 6.14. *As a consequence of Theorem 6.13(3) and Remark 6.10(2), any semisimple $\mathbb{C}$-local system can be deformed to a $\mathbb{C}$-VHS.*

6.4 Rank-2 local systems

As an application of the theory, we have the following theorem of Simpson [Sim91, p. 340, Theorem 10] (generalized to the quasiprojective case by Corlette–Simpson [CS08, Theorem 1]).

Theorem 6.15. *Let X be a smooth projective variety over $\mathbb{C}$, and let $\mathbb{V}$ be a $\mathbb{C}$-local system on X whose monodromy group is Zariski-dense in $\mathrm{SL}_2(\mathbb{C})$. Then one of the following holds*

1. There exists a map $f\colon X \to Y$ with Y a smooth Deligne–Mumford curve[7] *and a $\mathbb{C}$-local system $\mathbb{W}$ on Y such that $\mathbb{V} = f^*\mathbb{W}$.*

2. $\mathbb{V}$ is rigid and of geometric origin.

Proof sketch. First of all, if $\mathbb{V}$ is rigid, Corollary 6.14 implies $\mathbb{V}$ is a direct summand of a $\mathbb{C}$-VHS, and as Daniel Litt's Lecture 3 (§10) will discuss (from a very different perspective!), $\mathbb{V}$ is in fact a $\mathbb{C}$-direct summand of a $\mathbb{Z}$-VHS. Now since $\mathbb{V}$ is of rank 2, there are at most two interesting pieces of the Hodge filtration, hence $\mathbb{V}$ comes from a family of Abelian varieties [Sim92, Corollary 4.9] (by the equivalence of categories involving Abelian varieties over $\mathbb{C}$ and certain Hodge structures).

[6] For understanding this section, it is extremely instructive to first try to answer the following questions: Consider a map $f\colon V \to V \otimes_{\mathbb{C}} W$ of $\mathbb{C}$-vector spaces satisfying $f^2 = 0$ (as in the definition of a Higgs field). What is an eigenvalue, eigenvector, or eigenspace of f? What is the trace, determinant, or characteristic polynomial of f? Viewing f as a linear map $W^\vee \to \mathrm{End}_{\mathbb{C}}(V)$, we get a $\mathrm{Sym}^*(W^\vee)$-module structure on V; what is the support, and what are the fibers, of the corresponding coherent sheaf $\widetilde{V}$ on $\mathrm{Spec}(\mathrm{Sym}^*(W^\vee)) \cong \mathbb{A}^{\dim(W)}_{\mathbb{C}}$?

[7] See [CS08, §§2–3] and [Sim91, p. 340], although the reader should ignore this technicality.

Now suppose $\mathbb{V}$ is not rigid. We want to describe where the curve Y comes from (and how non-Abelian Hodge theory gets involved). Let $\mathrm{Spec}(R)$ be the $\mathbb{C}$-scheme whose set of A-points is $\mathrm{Hom}(\pi_1(X), \mathrm{SL}_2(A))$ (cf. Definition 9.1 below). Then the monodromy representation of $\mathbb{V}$ takes the form

$$\pi_1(X) \xrightarrow{\rho} \mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathbb{C}).$$

for some map $R \to \mathbb{C}$. It suffices to show that ρ factors through the fundamental group of a DM curve. Note that if $t \in \mathrm{Spec}(R)(\mathbb{C})$ is a closed point not lying in a fixed countable union of proper closed subvarieties of $\mathrm{Spec}(R)$, then the composition ρ_t of ρ with the induced map $\mathrm{SL}_2(R) \to \mathrm{SL}_2(\mathbb{C})$ will have kernel equal to that of ρ and have Zariski-dense image. Thus it suffices to pick one such t and show that ρ_t factors through a surjective map $\pi_1(X) \to \pi_1(Y)$ with Y a DM curve.

To describe what t to pick, observe that the image of $\mathrm{Spec}(R)$ in $M_{\mathrm{B}}(X, r)$ is affine (being the quotient of $\mathrm{Spec}(R)$ by the GL_2-action; see Remark 9.3(4) below) and of positive dimension (since $\mathbb{V}$ is not rigid; see loc. cit.). Moreover, we have a homeomorphism $M_{\mathrm{B}}(X, r)(\mathbb{C}) \cong M_{\mathrm{Dol}}(X, r)(\mathbb{C})$, and since h is proper, the image of $\mathrm{Spec}(R)(\mathbb{C})$ in $M_{\mathrm{Dol}}(X, r)(\mathbb{C})$ cannot live in the fiber above 0. Thus we may pick t such that the Higgs bundle $(\mathcal{E}, \theta)$ corresponding to ρ_t via Theorem 6.9 satisfies $h(\mathcal{E}, \theta) \neq 0$. Then $\det(\theta) \neq 0$ in $H^0(X, \mathrm{Sym}^2(\Omega^1_X))$, and also $\mathrm{tr}(\theta) = 0$ since ρ_t has image contained in $\mathrm{SL}_2(\mathbb{C})$. (The latter implication is not obvious but follows from the proof of Theorem 6.9.)

Now let $Z \subseteq \mathrm{Tot}(\Omega^1_X)$ be the spectral variety of $(\mathcal{E}, \theta)$. Since $\mathrm{tr}(\theta) = 0$, it is the space of square roots of $\det(\theta)$. Since $\det(\theta)$ is locally a square, Z is finite and flat over X, and its desingularization $\widetilde{Z}$ carries a tautological 1-form ω, the "global square root" of $\det(\theta)$. It induces a map[8] $\widetilde{Z} \to \mathrm{Alb}(\widetilde{Z})$ and a tautological 1-form on $\mathrm{Alb}(\widetilde{Z})$ which pulls back to ω. Let $\mathrm{Alb}(\widetilde{Z}) \twoheadrightarrow A$ be the smallest quotient by an Abelian subvariety such that ω is pulled back from A.

Claim: The image of $\widetilde{Z}$ in A is a curve.

Granted this, one takes the quotient of $\widetilde{Z} \to A$ by a $\mathbb{Z}/2$-action and then the Stein factorization to obtain a map $\widetilde{X} \to Y$ with $\widetilde{X}$ birational to X and Y a curve. Next, one shows that $\widetilde{X} \to Y$ factors through $X \to Y$. Finally, giving Y an appropriate structure of a DM curve yields the desired map. See [Sim91, p. 345–347] and [CS08, Lemma 3.1] for details.

Finally, we outline the proof of the claim. Assume the image of $\widetilde{Z}$ in A is not a curve. By a simple argument, the zero-locus of ω maps to a finite set of points in A [Sim91, p. 343, Lemma 13], so the assumption implies that, after a suitable birational modification, the zero-locus of ω has codimension ≥ 2 in X. By a Lefschetz-type theorem, we may then find a projective curve C in X such that $\omega|_C$ is nowhere vanishing and the map $\pi_1(C) \to \pi_1(X)$ is surjective. Now let $\widetilde{C} \to C$ be the connected (double or single) cover determined by $\widetilde{Z} \to X$. Since ω has a globally defined square root on $\widetilde{C}$, the eigenspaces of $(\mathcal{E}, \theta)|_{\widetilde{C}}$ are globally defined, i.e. $(\mathcal{E}, \theta)|_{\widetilde{C}}$ is a direct sum of two Higgs line bundles. But $\widetilde{C}$ is a projective variety, so we may use Theorem 6.9(3) to conclude that $\rho_t|_{\widetilde{C}}$ is a direct sum of two local systems. So the image of $\rho_t|_{\widetilde{C}}$ is contained in a torus in $\mathrm{SL}_2(\mathbb{C})$ and is an index-two normal subgroup of the image of ρ_t (normal since $\widetilde{C} \to C$ is a Galois cover). This contradicts Zariski density. See [Sim91, p. 344–345] for details. ∎

Remark 6.16.
1. Jost–Zuo have proven a similar result for local systems of higher rank [JZ97, p. 497, Theorem 3.1].
2. This whole story, at least in rank 2, has an analogue in the non-Archimedean world. In particular, one can give a proof, which is different to what will be discussed in Daniel Litt's Lecture 3 (§10), of the integrality of rigid local systems from this "non-Archimedean Corlette–Simpson" point of view.

Lastly we state a result due to Landesman–Litt which connects back to Daniel Litt's Lecture 1 (§2) and whose proof uses techniques of non-Abelian Hodge Theory (in particular, Corollary 6.14 above). Recall that given a smooth proper map $X \to S$ and $s \in S$, there is an action of $\pi_1(S, s)$ on the moduli space $\mathrm{Rep}(\pi_1(X_s))$ of representations of $\pi_1(X_s)$. In the following, we take X to be the moduli space $\mathcal{M}_{g,n}$ of genus-g curves with n punctures and $X \to S$ to be the universal such curve.

[8] Recall that $\mathrm{Alb}(\widetilde{Z})$ denotes the *Albanese variety*, the Abelian variety given by the complex torus $H^0(\widetilde{Z}, \Omega^1_{\widetilde{Z}})^\vee / H_1(\widetilde{Z}, \mathbb{Z})$.

Theorem 6.17 ([LL24]). *If $\rho\colon \pi_1(\Sigma_{g,n}) \to \mathrm{GL}_r(\mathbb{C})$ is such that*

1. *$r^2 \leq g$ and*
2. *the conjugacy class of ρ has finite $\pi_1(\mathcal{M}_{g,n})$-orbit,*

then ρ has finite image.

7 p-adic Hodge theory. Speaker: Alice Lin. Notes by Stefan Nikoloski.

7.1 The field C and Tate twists

Let K be a complete discretely valued field of characteristic 0 with a perfect characteristic p residue field.

Definition 7.1. $C = \mathbb{C}_p \coloneqq \widehat{\overline{K}}$, the p-adic completion of $\overline{K}$.

Fact 7.2.

- C is algebraically closed.
- By continuity $G_K \coloneqq \mathrm{Gal}(\overline{K}/K)$ acts on C, preserving the absolute value $|\cdot|$.

Definition 7.3. We set $\mathbb{Z}_p(1) \coloneqq \varprojlim \mu_{p^n}$. By a choice of a system of compatible p-th power roots of unity we get an isomorphism

$$\begin{tikzcd}
\mathbb{Z}_p(1) = \varprojlim \mu_{p^n} = (\cdots \arrow[r] & \mu_{p^n} \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mu_{p^2} \arrow[r, "x \to x^p"] \arrow[d] & \mu_p) \arrow[d] \\
\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n = (\cdots \arrow[r] & \mathbb{Z}/p^n \arrow[r] & \cdots \arrow[r] & \mathbb{Z}/p^2 \arrow[r, "\bmod p"] & \mathbb{Z}/p)
\end{tikzcd}$$

The p-adic cyclotomic character $\chi : \Gamma_K \to \mathbb{Z}_p^\times$ is defined so that for all n and for all $\sigma \in G_K$ we have $\sigma(\zeta_{p^n}) = \zeta_{p^n}^{\chi(\sigma)}$. Hence, we can view $\mathbb{Z}_p(1)$ as $\mathbb{Z}_p$ with a G_K-action by χ.

Definition 7.4. The r-th Tate twist for $r \in \mathbb{Z}$ is given by

$$\mathbb{Z}_p(r) \coloneqq \begin{cases} \mathbb{Z}_p(1)^{\otimes r}, & r \geq 0 \\ \mathbb{Z}_p(-r)^\vee, & r < 0 \end{cases}$$

In general, for a $\mathbb{Z}_p[G_K]$-module M we define the r-th Tate twist of M, to be $M(r) \coloneqq M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$.

Theorem 7.5 ([Tat67, Theorem 1 and Theorem 2])**.**

(1) $H^0_{\mathrm{cts}}(G_K, C) = C^{G_K} = K$ *and* $H^1_{\mathrm{cts}}(G_K, C)$ *is a* 1*-dimensional vector space over* K*.*

(2) For $r \neq 0$*,* $H^0_{\mathrm{cts}}(G_K, C(r)) = C(r)^{G_K} = 0$ *and* $H^1_{\mathrm{cts}}(G_K, C(r)) = 0$*.*

Remark 7.6. The statements about H^1_{cts} are incorrect if we replace C with $\overline{K}$.

7.2 Hodge–Tate decomposition

The p-adic Hodge–Tate theory is motivated by studying $H^1_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ for smooth projective varieties X. In particular, we can ask what these Galois representations can recover about the geometry of X?

We recall that if we have a smooth proper scheme Y over $\mathbb{C}$ then we have the classical Hodge decomposition

$$H^n_{\mathrm{sing}}(Y(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_j H^{n-j}(Y, \Omega^j_Y)$$

In [Tat67], Tate showed that an analogous decomposition exists for $H^1_{\text{ét}}$ of an abelian variety over a p-adic field with a good reduction. He also conjectured that such a decomposition exists beyond abelian varieties.

Theorem 7.7 (Hodge–Tate decomposition, [Fal88, Chapter III, Theorem 4.1])**.** *Let X be a smooth proper K-scheme, $n \geq 0$*

$$\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \bigoplus_{j \in \mathbb{Z}} C(j)\right)^{G_K} \simeq \bigoplus_j H^{n-j}(X, \Omega^j_{X/K})$$

as graded K-vector spaces.

Remark 7.8. Alternatively, using the Serre–Tate Lemma ([BC, Lemma 2.3.1]) the statement of the theorem can be rewritten as

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \simeq \bigoplus_q H^{n-q}(X, \Omega^q_{X/K}) \otimes_K C(-q)$$

Remark 7.9. The theorem tells us that $H^n_{\text{ét}}$ recovers the Hodge numbers.

We point out that as in the classical case where we only get the decomposition after tensoring with $\mathbb{C}$, the analogous thing happens in this case after tensoring with $\bigoplus_j C(j)$. This is the first example of a period ring.

Definition 7.10. The ring $B_{\text{HT}} \coloneqq \bigoplus_j C(j)$ is called the Hodge–Tate period ring.

7.2.1 Abelian variety example

Let A be an abelian variety and $n = 1$, then the Hodge–Tate decomposition becomes:

$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C \simeq H^1(A, \mathcal{O}_A) \otimes_K C \oplus H^0(A, \Omega^1_{A/K}) \otimes_K C(-1)$$

We get a Galois equivariant map $\alpha_A : H^1(A, \mathcal{O}_A) \otimes_K C \to H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$, which we can explicitly construct.

7.3 Constructing α_A

Suppose A/K has good reduction, i.e. it extends $\mathcal{A}/\mathcal{O}_K$. We let $\mathcal{O}_C = \widehat{\mathcal{O}_{\overline{K}}}$.

Definition 7.11. We define the $\mathcal{O}_C$-scheme $\mathcal{A}_\infty = \varprojlim_n (\cdots \xrightarrow{[p]} \mathcal{A}_1 = \mathcal{A}_{\mathcal{O}_C} \xrightarrow{[p]} \mathcal{A}_0 = \mathcal{A}_{\mathcal{O}_C})$ as a limit of affine morphisms.

We let $\pi : \mathcal{A}_\infty \to \mathcal{A}_0$ be the projection map coming from the limit. Then π will induces a map $\pi^* : R\Gamma(\mathcal{A}_{\mathcal{O}_C}, \mathcal{O}_{\mathcal{A}_{\mathcal{O}_C}}) \to R\Gamma(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})$. We now observe that $\mathcal{A}[p^n](C)$ act on $\mathcal{A}_n$ by translation and this action is compatible with the transition maps. Hence, we get a $T_p(A) = \varprojlim \mathcal{A}[p^n]$-action on $\mathcal{A}_\infty$. Then, the map $\pi : \mathcal{A}_\infty \to \mathcal{A}_0$ is $T_p(A)$-equivariant, where the action on $\mathcal{A}_0$ is trivial, as all the p-torsion points will be mapped to 0 in $\mathcal{A}_0$.

Remark 7.12. We need to work over $\mathcal{O}_C$, or at least over $\mathcal{O}_{\overline{K}}$ in order to make sure that the p^n-torsion points of A are defined over the underlying ring.

$T_p(A)$ acts trivially on $\mathcal{A}_{\mathcal{O}_C}$ and the sheaf cohomology, so by functoriality we get a composition of maps

$$R\Gamma(\mathcal{A}_{\mathcal{O}_C}, \mathcal{O}_{\mathcal{A}_{\mathcal{O}_C}}) \to R\Gamma_{\text{cts}}(T_p(A), R\Gamma(\mathcal{A}_{\mathcal{O}_C}, \mathcal{O}_{\mathcal{A}_{\mathcal{O}_C}})) \to R\Gamma_{\text{cts}}(T_p(A), R\Gamma(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty}))$$

Fact 7.13 ([Bha, Proposition 2.2.1])**.** The natural map $\mathcal{O}_C \to R\Gamma(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})$ given by sending the constant sections $\mathcal{O}_C$ to $H^0(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})$ is an isomorphism after p-adic completion.

Using this fact, after p-adic completion we get a map

$$R\Gamma(\mathcal{A}_{\mathcal{O}_C}, \mathcal{O}_{\mathcal{A}_{\mathcal{O}_C}}) \to R\Gamma_{\mathrm{cts}}(T_p(A), \mathcal{O}_C) = R\Gamma_{\mathrm{cts}}(\pi_1^{\text{ét}}(A), \mathcal{O}_C) \to R\Gamma_{\text{ét}}(A_C, \mathcal{O}_C)$$

where the last map is induced by the map from the étale site to the finite étale site. Finally, taking H^1 and tensoring with C we get

$$\alpha_A : H^1(A, \mathcal{O}_A) \otimes_K C \to H^1_{\text{ét}}(A_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} C$$

7.4 Étale–de Rham comparison

We recall that for a smooth proper $\mathbb{C}$-scheme Y of dimension d we have:

$$H^n_{\mathrm{sing}}(Y(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H^n_{\mathrm{dR}}(Y/\mathbb{C})$$

given by the Poincaré duality and the "period" pairing

$$H^n_{\mathrm{dR}}(Y(\mathbb{C})/\mathbb{C}) \times H_{2d-n}(Y(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow \mathbb{C}, \qquad (\omega, Z) \to \int_Z \omega$$

In general, these "periods" $\int_Z \omega$ are highly transcendental, which explains why we have to tensor with $\mathbb{C}$ to get the identification.

Theorem 7.14 (p-adic de Rham comparison, [Fal89, Theorem 8.1])**.** *For X a proper smooth K-scheme, $n \geq 0$ there is a canonical isomorphism*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq H^n_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}}$$

compatible with the G_K-action and filtration.

Remark 7.15. In general, de Rham cohomology isn't equipped with a G_K-action, while on the other side the étale cohomology doesn't come with a filtration. The role of B_{dR} is to fill in the gaps of what is missing on both sides of this isomorphism.

The ring B_{dR} was constructed by Fontaine and he conjectured that such an isomorphism exists

Definition 7.16 ([Fon82, Chapter 2])**.** The de Rham period ring B_{dR} has

(1) filtration such that the associated graded ring is B_{HT}.

(2) G_K-action such that $B_{\mathrm{dR}}^{G_K} = K$.

Definition 7.17.

$$D_{\mathrm{dR}} : \left\{ \begin{matrix} \text{finite dimensional} \\ \mathbb{Q}_p - \text{representations of } G_K \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{filtered} \\ K - \text{vector spaces} \end{matrix} \right\}$$

$$V \longrightarrow D_{\mathrm{dR}}(V) \coloneqq (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$$

With this definition in hand, we can rephrase the p-adic de Rham comparison as

$$D_{\mathrm{dR}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \simeq H^n_{\mathrm{dR}}(X/K)$$

as filtered K-vector spaces.

7.5 Construction of B_{dR}^+, B_{dR}

Recall that $\mathcal{O}_C = \widehat{\mathcal{O}_{\overline{K}}}$.

Definition 7.18. The tilt $\mathcal{O}_C^\flat$ of $\mathcal{O}_C$ is defined as

$$\mathcal{O}_C^\flat \coloneqq \varprojlim_{x \to x^p} \mathcal{O}_C = \left\{ (x^{(0)}, x^{(1)}, \dots) \in \prod \mathcal{O}_C \;\middle|\; \left(x^{(n+1)}\right)^p = x^{(n)} \right\}$$

It is isomorphic to $\varprojlim_{x \to x^p} \mathcal{O}_C/(p)$.

Fact 7.19. $\mathcal{O}_C^\flat$ is perfect of characteristic p. It is a domain with a continuous G_K-action.

Let $A_{\mathrm{inf}} \coloneqq W(\mathcal{O}_C^\flat)$, the ring of Witt vectors of $\mathcal{O}_C^\flat$. We then define a map θ that makes the diagram below commute.

$$\begin{tikzcd}
\sum [\underline{x}_i] p^i \arrow[r] \arrow[ddd, bend right] & \sum x_i^{(0)} p^i \arrow[ddd, bend left] \\
A_{\mathrm{inf}} = W(\mathcal{O}_C^\flat) \arrow[r, two heads, "\theta"] \arrow[d, "\bmod p"'] & \mathcal{O}_C \arrow[d, "\bmod p"] \\
\mathcal{O}_C^\flat \arrow[r] & \mathcal{O}_C/(p) \\
\underline{x}_0 = (x_0^{(0)}, x_1^{(1)}, \dots) \arrow[r] & x_0^0 \pmod p
\end{tikzcd}$$

where $[\underline{x}_i]$ are multiplicative lifts of $\underline{x}_i \in \mathcal{O}_C^\flat$.

We can now extend θ to $\theta_{\mathbb{Q}} : A_{\mathrm{inf}}[1/p] \twoheadrightarrow \mathcal{O}_C[1/p] = C$. The kernel of $\theta_{\mathbb{Q}}$ is principal and G_K-stable.

Fact 7.20. $B_{\mathrm{dR}}^+ \coloneqq (A_{\mathrm{inf}}[1/p])\widehat{\ }_{(\ker \theta_{\mathbb{Q}})}$ is a complete DVR with maximal ideal $\ker \theta_{\mathbb{Q}}$ and has a Γ_K-action.

Definition 7.21. $B_{\mathrm{dR}} \coloneqq \mathrm{Frac}(B_{\mathrm{dR}}^+)$. It has a $\mathbb{Z}$-grading by powers of the maximal ideal of B_{dR}^+ and a G_K-action.

8 The p-adic Riemann-Hilbert correspondence, lecture 2. Speaker: Alexander Petrov. Notes by Yifei Zhang.

Recall the complex Riemann–Hilbert correspondence. Given a complex manifold X and a $\mathbb{C}$-local system $\mathbb{L}$, we get $\mathrm{RH}(\mathbb{L}) := (\mathbb{L} \otimes_{\underline{\mathbb{C}}} \mathcal{O}_X, 1 \otimes d) \in \mathrm{MIC}(X)$. Now let $U \subseteq X$ be open; what is $\mathrm{RH}(\mathbb{L})(U)$? Let $f : \tilde{U} \to U$ be a universal cover; then

$$\mathrm{RH}(\mathbb{L})(U) = \mathrm{RH}(\mathbb{L})(\tilde{U})^{\pi_1(U)} = (\Gamma(\tilde{U}, f^*\mathbb{L}|_U) \otimes_{\mathbb{C}} \mathcal{O}(\tilde{U}))^{\pi_1(U)}$$

where the π_1-action on $\Gamma(\tilde{U}, f^*\mathbb{L}|_U) \otimes_{\mathbb{C}} \mathcal{O}(\tilde{U})$ is diagonal.

Motivation 8.1. Notice that $\Gamma(\tilde{U}, f^*\mathbb{L}|_U)$ is the representation of $\pi_1(U)$ associated to $\mathbb{L}|_U$ and $\mathcal{O}(U)$ is an "interesting" ring with a $\pi_1(U)$-action (as B_{dR} is an "interesting" ring with G_K-action). Moreover, $\mathrm{RH}(\mathbb{L})(U)$ is an $\mathcal{O}(\tilde{U})^{\pi_1(U)} = \mathcal{O}(U)$-module.

Now let K be a discretely valued p-adic field. We want to study the category $\mathcal{C}$ of étale $\mathbb{Q}_p$-local systems on $\mathrm{Spec}(K)$ which is equivalent to the category of finite-dimensional continuous $\mathbb{Q}_p$-representations of G_K. As a first approximation to the Riemann-Hilbert functor in this setting following the motivation above, we have

$$\begin{aligned} \mathcal{C} &\xrightarrow{\quad D_{\mathrm{dR}} \quad} \mathrm{FilVec}_{\mathrm{K}}^{\mathrm{f.d.}} \\ V &\longmapsto (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}. \end{aligned}$$

Think of B_{dR} as a humongous representation and $D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K} \simeq \mathrm{Hom}_{G_K}(V^\vee, B_{\mathrm{dR}})$ as detecting how often $V^\vee$ appears in it. Its filtration is $F^i_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} F^i B_{\mathrm{dR}})^{G_K}$. However this D_{dR} is unsatisfactory because it kills many representations.

Example 8.2. Let $\chi : G_K \to \mathbb{Z}_p^\times$ be the cyclotomic character, $a \in \mathbb{Z}_p$ such that χ^a makes sense (for example a can be any element of $\mathbb{Z}_p$ whenever K has p-th (or 4-th if $p = 2$) roots of unity). Then

$$D_{\mathrm{dR}}(\chi^a) = \begin{cases} K, F^{-a} = K, F^{-a+1} = 0 & a \in \mathbb{Z} \\ 0 & a \notin \mathbb{Z} \end{cases}$$

Fact 8.3. $\dim_K D_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p}(V)$.

Definition 8.4. V is de Rham if $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p}(V)$ (for example, étale cohomology is de Rham).

To "improve" D_{dR}, consider $K_\infty = K(\zeta_{p^\infty})$. We want to define a functor

$$\mathrm{RH}^+ : \mathcal{C} \longrightarrow \left\{ \begin{matrix} \text{finite free } K_\infty[[t]]\text{-modules } M \text{ equipped with} \\ \nabla : M \to M \otimes K_\infty[[t]] \frac{dt}{t} \text{ s.t.} \\ \nabla(am) = a\nabla(m) + m \otimes da \\ \text{for any } a \text{ in } K_\infty[[t]] \text{ and any } m \text{ in } M. \end{matrix} \right\}$$

We can further compose RH^+ with the inverting t functor

$$\left\{ \begin{matrix} \text{finite free } K_\infty[[t]]\text{-modules } M \text{ equipped with} \\ \nabla : M \to M \otimes K_\infty[[t]] \frac{dt}{t} \text{ s.t.} \\ \nabla(am) = a\nabla(m) + m \otimes da \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{finite dimensional } K_\infty((t))\text{-vector spaces } M \text{ equipped with} \\ \text{connection with regular singularity} \end{matrix} \right\}$$

to get $\mathrm{RH}^+[1/t]$. RH^+ preserves rank and is a tensor functor that satisfies[9]

$$D_{\mathrm{dR}}(V) \otimes_K K_\infty = \mathrm{RH}^+(V)[1/t]^{\nabla=0}.$$

In this language, V is de Rham if and only if $\mathrm{RH}^+(V)[1/t]$ is isomorphic to $(K_\infty((t))^{\dim V}, d)$, i.e. is trivial as a bundle with connection on the "punctured disc".[10]

[9]This immediately explain the dimensional inequality, as in the MIC case, dimension of flat sections is less than the rank of the bundle.

[10]This means the module is generated by flat sections

Recall that B_{dR} has a separated exhaustive decreasing $\mathbb{Z}$-indexed filtration whose i-th graded piece is $C(i)$. What if we try other things? In view of the "universal cover" analogy, taking $\overline{K}$ and C as $\mathcal{O}(\tilde{U})$ respectively, we have the following

Example 8.5.

- $(V \otimes_{\mathbb{Q}_p} \overline{K})^{G_K} = \bigcup_{L/K \text{ finite Galois}} (V \otimes_{\mathbb{Q}_p} L)^{G_K} = \bigcup_L V^{G_L} \otimes_{\mathbb{Q}_p} K$ =vectors in $V \otimes K$ acted-on through a finite quotient of G_K. For the second equality, we $L \simeq \oplus_{\sigma \in G(L/K)} K\sigma$ as Gal_K-module (follows from normal basis theorem), so $(V \otimes_{\mathbb{Q}_p} L) \simeq \oplus_{\sigma \in G(L/K)} V \otimes_{\mathbb{Q}_p} K\sigma$. Hence $(V \otimes_{\mathbb{Q}_p} L)^{G_K} = ((V \otimes_{\mathbb{Q}_p} L)^{G_L})^{G(L/K)} = (V^{G_L} \otimes_{\mathbb{Q}_p} L)^{G(L/K)} = (\oplus_{\sigma \in G(L/K)} V^{G(L)} \otimes_{\mathbb{Q}_p} K\sigma)^{G(L/K)}$. It's readily verifiable that it consists of elements in the form of $\sum_\sigma \sigma(v)\sigma$ for $v \in V^{G_L} \otimes_{\mathbb{Q}_p} K$.
- What about $(V \otimes_{\mathbb{Q}_p} C)^{G_K}$? Consider $H_K = G(\overline{K}/K_\infty) = ker(\chi : G_K \to \mathbb{Z}_p^\times)$ and $\Gamma_K := G_K/H_K \hookrightarrow \mathbb{Z}_p^\times$.

Theorem 8.6 (Tate-Sen). *Let W be a f.d. $C = C$-vector space with an H_K-semilinear continous action (e.g. $W = V \otimes_{\mathbb{Q}_p} C$). Then $W \simeq C^{\dim(W)}$ as a (semilinear) H_K-representation.*

Remark 8.7. On the contrary, for V a $\mathbb{Q}_p$-representation of G_K, $V \otimes_{\mathbb{Q}_p} \overline{K} \simeq \overline{K}^{\dim_{\mathbb{Q}_p} V}$ as semilinear G_K-modules only if the G_K-action on V factors through some finite quotients.[11] Indeed, one can verify that if the G_K-action on V does not factor through some finite quotients, then $V \otimes_{\mathbb{Q}_p} \overline{K}$ does not have a G_K-invariant basis.

To see the theorem, we use the following two results. In fact, these two results powers the p-adic hodge theory.

Lemma 8.8 (special case of étale descent). *Let $R \to S$ be a finite Galois étale map of rings. M_S a projective S-module equipped with a semilinear action of $G := G(S/R)$. Then $M_S \simeq M \otimes_R S$ for some R-module M as modules with S-semilinear G-action.*

Theorem 8.9 (Almost Purity,Tate-Faltings-Scholze). *Let L/K_∞ be a finite extension. Then $\mathcal{O}_{K_\infty} \to \mathcal{O}_L$ is almost étale: $\Omega_{\mathcal{O}_L/\mathcal{O}_{K_\infty}}$ is annihilated by $\mathfrak{m}_L$.*[12]

Remark 8.10. The almost purity is true for $K'_\infty = K(p^{1/p^\infty})$ as well. Moreover, upto p-adic completion, perfectoid fields are exactly those for which almost purity holds.

Example 8.11. For an odd p, let $K = \mathbb{Q}_p, L = K'_\infty(\sqrt{p})$. Note $\mathbb{Z}_p \subset \mathbb{Z}_p[\sqrt{p}]$ is not étale: $\Omega_{\mathbb{Z}_p[\sqrt{p}]/\mathbb{Z}_p}$ is

$$\mathbb{Z}_p[\sqrt{p}]dp/(\sqrt{p}d\sqrt{p} = 0).$$

Note that $\mathcal{O}_L \ni p^{1/(2p^n)} = p^{1/2}(p^{1/p^n})^{-(p^n-1)/2}$. Thus we get

$$dp^{1/2} = d(p^{1/2p^n}(p^{1/p^n})^{(p^n-1)/2}) = (p^{1/p^n})^{(p^n-1)/2} dp^{1/(2p^n)}.$$

$dp^{1/(2p^n)}$ is annihilated by $p^{1/2p^n}$, hence so is $dp^{1/2}$. This shows $dp^{1/2}$ is annihilated by element with arbitrarily small valuation, hence by $\mathfrak{m}_L$. Similar argument shows that $dp^{1/2p^n}$ is annihilated by $\mathfrak{m}_L$ for any n, which means $\Omega_{\mathcal{O}_L/\mathcal{O}_{K'_\infty}}$ is annihilated by $\mathfrak{m}_L$.

A formal argument takes care of Tate-Sen using almost purity and étale descent: specifically, one gets Tate-Sen for modules over $\mathcal{O}_C/p^n$, then one takes the limit.

Now $V \otimes_{\mathbb{Q}_p} C \simeq C^{\oplus \dim V}$ as an H_K-module, so $(V \otimes_{\mathbb{Q}_p} C)^{H_K} \simeq \widehat{K_\infty}^{\dim V}$ as Γ_K-modules. To turn this into a $K_\infty[[t]]$-module, the first step is

Decompletion: there is a Γ_K-module $H(V)$ over K_∞ such that $H(V) \otimes_{K_\infty} \widehat{K_\infty} \simeq (V \otimes_{\mathbb{Q}_p} C)^{H_K}$ as a Γ_K-module, and we make the following

[11] The converse of this statement is true as well, which is a special case of Lemma 8.8. Alternatively, it follows from Hilbert 90: If the G_K-action factors through $G(L/K)$ for finite Galois L/K, then the Hilbert 90 ($H^1(G(L/K), \mathrm{GL}_n(L)) = 1$) is saying $V \otimes_{\mathbb{Q}_p} L \simeq L^{\dim_{\mathbb{Q}_p} V}$ as modules with semilinear $G(L/K)$-action.

[12] The almost purity theorem and the étale descent lemma are used to show the vanishing of the $H^1(H_K, \mathrm{GL}_d(C))$ which proves the theorem.

Definition 8.12.

$$\phi : H(V) \to H(V)$$

is defined by differentiating the Γ_K action:

$$\phi(v) = \lim_{\gamma \to 1} \frac{\gamma(v) - v}{\chi(\gamma) - 1}.$$

ϕ is a K_∞-linear operator because for any $x \in K_\infty$, every γ close enough to 1 will fix x.

Goal 8.13. We want to define a functor RH^+ from $\mathcal{C}$ to the category of $K_\infty[[t]]$-modules with $\nabla : \mathrm{RH}^+(V) \to \mathrm{RH}^+(V) \otimes K_\infty[[t]]\frac{dt}{t}$ s.t. $\nabla(am) = a\nabla(m) + m \otimes da$. What we defined so far is $\mathrm{RH}^+(V)/t = H(V)$ and $t\nabla$ mod t is ϕ.

9 Rigid local systems. Speaker: Jake Huryn. Notes by Jake Huryn.

This talk takes place in the following setting, the notation of which we use throughout.

Setting: Let X be a smooth connected quasiprojective scheme over $\mathbb{C}$, fix a point $x \in X(\mathbb{C})$, and set $\Gamma \coloneqq \pi_1(X, x)$. Let $X \hookrightarrow \overline{X}$ be an embedding into a smooth connected projective scheme over $\mathbb{C}$ such that $D \coloneqq \overline{X} \setminus X$ is a normal crossings divisor, and let $D_1, \ldots, D_n$ be the irreducible components of D. Finally, for each i, let $T_i \in \Gamma$ be a "counterclockwise loop around D_i".

To describe T_i explicitly, let D^{sing} be the singular locus of D, and put $U \coloneqq \overline{X} \setminus D^{\mathrm{sing}}$. Fix an open ball Δ_i in $U(\mathbb{C})$ meeting D_i, but no other D_j, and a point x_i in the "punctured ball" $\Delta_i^* \coloneqq \Delta_i \setminus D(\mathbb{C})$. Then $\pi_1(\Delta_i^*, x_i) \cong \mathbb{Z}$, and there is a "counterclockwise" generator determined by the orientation on Δ_i^*; let T_i be the image of this generator under the map $\pi_1(\Delta_i^*, x_i) \to \Gamma$ induced by choosing a path in $X(\mathbb{C})$ from x to x_i. It is important that the *conjugacy class* of T_i does not depend on the choices.

9.1 Rigidity

The goal of this section is to formalize the following "definition": a local system ρ is <u>rigid</u> if it cannot be deformed to a non-isomorphic local system while its determinant and "local monodromy" (the conjugacy classes of the $\rho(T_i)$) remain fixed. To do this, we define "representation varieties" which parameterize local systems on X.

Definition 9.1. Fix the data of

- a positive integer r and a field K.
- a finite set Θ of homomorphisms $\Gamma \to K^\times$.
- a tuple $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_n)$ of locally closed subschemes of $\mathrm{GL}_{r,K}$ which are finite unions of conjugacy classes.

Define a functor $\mathrm{Rep}(\Gamma, r)\colon \mathrm{Ring} \to \mathrm{Set}$ by sending R to the set of homomorphisms $\Gamma \to \mathrm{GL}_r(R)$. This is an affine scheme of finite type over $\mathbb{Z}$ because Γ is finitely generated:[13] if $\{\gamma_1, \ldots, \gamma_m\}$ is a generating set of Γ closed under inverses, then $\mathrm{Rep}(\Gamma, r)$ is isomorphic to the closed subscheme of GL_r^n cut out by the equations $\prod_j g_{i_j} = 1$ whenever $\prod_j \gamma_{i_j} = 1$ in Γ. (Since GL_r^n is a Noetherian scheme, finitely many such equations suffice to define $\mathrm{Rep}(\Gamma, r)$.)

Define $\mathrm{Rep}(\Gamma, r; \Theta, \mathcal{C})\colon \mathrm{Ring}_K \to \mathrm{Set}$ to be the functor which sends R to the set of $\rho \in \mathrm{Rep}(\Gamma, r)(R)$ such that $\det(\rho) \in \Theta$ and, for each $i \in \{1, \ldots, N\}$ and K-algebra morphism $\varphi\colon R \to F$ with F a field, one has $\varphi(\rho(T_i)) \in \mathcal{C}_i(F)$. This is a locally closed subscheme of[14] $\mathrm{Rep}(\Gamma, r)_K$, which we call a <u>representation variety</u>.

Observe that $\mathrm{Rep}(\Gamma, r; \Theta, \mathcal{C})$ has an action of $\mathrm{GL}_{r,K}$ by conjugation. Also, it is independent of the choices made in defining the T_i.

Definition 9.2. Assume that $\#\Theta = 1$ and each $\mathcal{C}_i$ is a conjugacy class. A local system $\rho \in \mathrm{Rep}(\Gamma, r; \Theta, \mathcal{C})(K)$ is *rigid* if its orbit under the action of $\mathrm{GL}_{r,K}$ (i.e. the set-theoretic image of the morphism $\mathrm{GL}_{r,K} \to \mathrm{Rep}(\Gamma, r; \Theta, \mathcal{C})$ given by $g \mapsto g\rho g^{-1}$) is a connected component of $\mathrm{Rep}(\Gamma, r; \Theta, \mathcal{C})$ for some compactification $\overline{X}$ as above.

Remark 9.3. (For an exposition of items (4–6) below, see [KP22, §4].)

1. In the context of Definition 9.2, if ρ is rigid, then its $\mathrm{GL}_{r,K}$-orbit is a connected component in any representation variety obtained by finitely enlarging the set Θ.

[13] Indeed, it is well known that the complex manifold $X(\mathbb{C})$ admits a finite triangulation. Alternatively, one can show using Morse theory that $X(\mathbb{C})$ is homotopy-equivalent to a finite CW complex [Mor78, p. 137].

[14] Since $\mathcal{C}_i$ is open in $\overline{\mathcal{C}_i}$, if we let Z and Z' be, respectively, the closed subschemes of $\mathrm{Rep}(\Gamma, r; \Theta)$ (where we impose a determinant condition but no local monodromy condition) defined by the identities $\rho(T_i) \in \overline{\mathcal{C}_i}(R)$ and $\rho(T_i) \in (\overline{\mathcal{C}_i} \setminus \mathcal{C}_i)(R)$, then $\mathrm{Rep}(\Gamma, r; \Theta, \mathcal{C}) = Z \setminus Z'$; indeed, its R-points are the morphisms $\mathrm{Spec}(R) \to Z$ whose image does not meet Z'.

2. In the context of Definition 9.2, if ρ is rigid and semisimple, then its $\mathrm{GL}_{r,K}$-orbit is a connected component in the representation variety $\mathrm{Rep}(\Gamma, r; \Theta, \overline{\mathcal{C}})$ obtained by replacing each conjugacy class $\mathcal{C}_i$ with its closure $\overline{\mathcal{C}_i}$. This follows from the fact that $\mathrm{Rep}(\Gamma, r; \Theta, \mathcal{C})$ is open in $\mathrm{Rep}(\Gamma, r; \Theta, \overline{\mathcal{C}})$ and that the orbit of ρ in the entire parameter space $\mathrm{Rep}(\Gamma, r)_K$ is closed by [Ric88, Theorem 3.6] (which requires semisimplicity).[15]

3. The "connected component" condition in Definition 9.2 (or in items (1–2) of the present remark) may be checked on geometric points endowed with the Zariski topology.

4. An actual *moduli space* (i.e. parameterizing isomorphism types) of local systems on X is obtained by taking a quotient of a representation variety by its $\mathrm{GL}_{r,K}$-action. This can be done using a stack quotient or a GIT quotient. In the latter case, the $\mathcal{C}_i$ must be closed, so the representation variety is affine, say, $\mathrm{Spec}(\mathcal{O})$; the GIT quotient is then $\mathrm{Spec}(\mathcal{O}^{\mathrm{GL}_{r,K}})$, which is a coarse moduli scheme for semisimple local systems. From these perspectives, the orbit of a local system is a connected component of a representation variety if and only if it is an isolated point of the corresponding moduli space.

5. The reformulation of rigidity in terms of isolated points of a moduli space leads to the following notion: a local system is <u>cohomologically rigid</u> if it is a *smooth* (i.e. reduced) isolated point of the appropriate moduli space. In other words, it has no infinitesimal deformations. The terminology is due to the fact that if $\underline{M}$ is the moduli scheme of $\mathrm{Rep}(\Gamma, r; \Theta, \mathcal{C})$ where each $\mathcal{C}_i$ is a conjugacy class, and $\rho \in \mathrm{Rep}(\Gamma, r; \Theta, \mathcal{C})(K)$, the tangent space of $\underline{M}$ at $[\rho]$ is

$$T_{[\rho]}\underline{M} = \mathrm{Ker}\left(\mathrm{res}\colon H^1(\Gamma, \mathfrak{sl}_r(K)) \to \bigoplus_{i=1}^{n} H^1(T_i^{\mathbb{Z}}, \mathfrak{sl}_r(K))\right)$$

where $\mathfrak{sl}_r(K)$ is a representation of Γ via $\gamma \cdot \Xi \coloneqq \rho(\gamma)\Xi\rho(\gamma)^{-1}$.

6. One can (and it is interesting and useful to) generalize the above discussion by replacing GL_r with other connected reductive groups. Much of what is said below extends beyond GL_r.

9.2 Simpson's conjecture

Example 9.4. For the purposes of the rest of the talk, it suffices to consider representation varieties of the following form. Set $K = \overline{\mathbb{Q}}$, fix an integer $d \geq 1$, and assume each $\mathcal{C}_i$ is a conjugacy class of quasi-unipotent matrices (meaning some power is unipotent). Let Θ be the set of all homomorphisms $\Gamma \to K^\times$ of order dividing d (i.e. satisfying $\theta^d = 0$), and let $\overline{\mathcal{C}} \coloneqq (\overline{\mathcal{C}_1}, \ldots, \overline{\mathcal{C}_n})$. Then we write $\mathrm{Rep}(\Gamma, r; d, \overline{\mathcal{C}})$ for $\mathrm{Rep}(\Gamma, r; \Theta, \overline{\mathcal{C}})$, which is closed in $\mathrm{Rep}(\Gamma, r)_{\overline{\mathbb{Q}}}$. By Remark 9.3(1–2), the rigidity of a semisimple local system $\rho \in \mathrm{Rep}(\Gamma, r; d, \overline{\mathcal{C}})(K)$ can be checked in this larger representation variety.

The following is an elaboration of [Sim92, p. 9, Conjecture].

Conjecture 9.5. Let $\rho\colon \Gamma \to \mathrm{GL}_r(\mathbb{C})$ be an irreducible rigid $\mathbb{C}$-local system on X. Assume that

1. ρ has quasi-unipotent local monodromy.

2. ρ has finite-order determinant.

Then ρ is of geometric origin in the sense of Definition 4.1.

Remark 9.6. A local system of geometric origin satisfies the properties (1) and (2) above. Indeed, (1) is the quasi-unipotent monodromy theorem (see e.g. [sga72, Theorem 1.2]; the case when X is a curve was explained in Corollary 3.13 above) and (2) is [Del71, Corollaire 4.2.8.iii(b)].

[15] To see this, use the fact that the orbit (being the continuous image of the irreducible scheme $\mathrm{GL}_{r,K}$) is irreducible. It is an easy exercise to give a counterexample if ρ is not semisimple, or if one replaces $\overline{\mathcal{C}_i}$ by a more general finite union of conjugacy classes including $\mathcal{C}_i$ (even when ρ is taken to be *irreducible*).

Remark 9.7. We outline some evidence for Conjecture 9.5. Let ρ be as therein; then ρ "looks geometric" in the following ways:

1. ρ is a $\mathbb{C}$-direct factor of a $\mathbb{Q}$-VHS. This was proven by Simpson for X projective [Sim92, p. 56, Theorem 5] and T. Mochizuki in general [Moc06, Lemma 10.13].

2. ρ is arithmetic in the sense of Definition 4.5. This was essentially proven by Simpson for X projective [Sim92, p. 55, Theorem 4] and Esnault–Groechenig in general [EG18, Proposition 3.1]. See §9.3 below.

3. if ρ is *cohomologically* rigid, then it is integral, i.e. conjugate to an $\mathcal{O}_K$-local system for some number field K. This was proven by Esnault–Groechenig [EG18, Theorem 1.1]. See Daniel Litt's Lecture 3 (§10) below.

These facts should be compared with the statements of Simpson's non-Abelian Hodge conjecture (Conjecture 6.12) and the relative Fontaine–Mazur conjecture (Conjecture 4.8).

Also, there are some X for which Simpson's conjecture is known to hold:

4. ρ comes from geometry if X is open in $\mathbb{P}^1$. This was proven by Katz using his method of "middle convolution" [Kat96, Theorem 8.4.1].

5. ρ comes from geometry if $X = \mathcal{A}_{g,n}$ is the moduli space of principally polarized Abelian varieties of dimension $g \geq 2$. In this case, *every* local system on X is rigid and of geometric origin. See §9.4 below.

The rest of this talk will be devoted to explaining some details of (2) and (5) above.

9.3 Arithmeticity

Fix ρ as in Conjecture 9.5. Let $d \geq 1$ be an integer satisfying $\det(\rho)^d = 1$, and for each i, let $\mathcal{C}_i$ be the conjugacy class of $\rho(T_i)$. Let R be a finitely generated subring of $\mathbb{C}$ such that $X \hookrightarrow \overline{X}$ spreads out to an open immersion $X_R \hookrightarrow \overline{X}_R$ over R with $\overline{X}_R$ again smooth projective and $\overline{X}_R \setminus X_R$ a normal crossings divisor. Let k be the fraction field of R.

Since $\mathrm{Rep}(\Gamma, r; d, \overline{\mathcal{C}})$ is defined over $\overline{\mathbb{Q}}$ (see Example 9.4), each of its connected components contains a $\overline{\mathbb{Q}}$-point, so by rigidity, we may conjugate ρ to have image contained in $\mathrm{GL}_r(\overline{\mathbb{Q}})$. Since Γ is finitely generated, ρ further factors through $\mathrm{GL}_r(\mathcal{O}_K[1/N])$ for some number field K and positive integer N. Let λ be any finite place of K not dividing N. Since $\mathrm{GL}_r(\mathcal{O}_{K_\lambda})$ is profinite, the composition

$$\Gamma \xrightarrow{\rho} \mathrm{GL}_r(K) \hookrightarrow \mathrm{GL}_r(\mathcal{O}_{K_\lambda})$$

factors through the profinite completion of Γ, which we identify via Riemann's existence theorem (Corollary 3.7) with the étale fundamental group $\pi_1^{\text{ét}}(X, x)$; this we in turn identify with $\pi_1^{\text{ét}}(X_{\overline{k}}, x)$, using that $\pi_1^{\text{ét}}$ is invariant under base-change between algebraically closed fields. Thus we obtain a representation

$$\rho_\lambda \colon \pi_1^{\text{ét}}(X_{\overline{k}}, x) \to \mathrm{GL}_r(K_\lambda)$$

for any such λ, i.e. an *étale* local system on $X_{\overline{k}}$.

Theorem 9.8. *There exists a finite extension $k(\lambda)$ of k such that ρ_λ extends to a étale local system*[16] $\pi_1^{\text{ét}}(X_{k(\lambda)}, x) \to \mathrm{GL}_r(\overline{K_\lambda})$ *on $X_{k(\lambda)}$.*

Proof. We give a proof in the case when X is projective, i.e. $D = \varnothing$, so any condition on local monodromy is vacuous (but see Remark 9.9 below). We may assume $X(k) \neq \varnothing$ and moreover (by choosing a suitable étale path between geometric points) that x lies above an element of $X(k)$. This splits the exact sequence

$$1 \to \pi_1^{\text{ét}}(X_{\overline{k}}, x) \to \pi_1^{\text{ét}}(X, x) \to \mathrm{Gal}_k \to 1.$$

[16] By a standard argument using the Baire category theorem, any continuous representation $\pi_1^{\text{ét}}(X_{k(\lambda)}, x) \to \mathrm{GL}_r(\overline{K_\lambda})$ takes values in $\mathrm{GL}_r(L)$ for a finite extension L/K_λ.

Given $\sigma \in \mathrm{Gal}_k$, define $\rho_\lambda^\sigma \colon \pi_1^{\text{ét}}(X_{\overline{k}}, x) \to \mathrm{GL}_r(K_\lambda)$ by $\gamma \mapsto \rho_\lambda(\sigma\gamma\sigma^{-1})$, where we identify σ with its image in $\pi_1^{\text{ét}}(X, x)$ via a fixed section of the short exact sequence. This induces a map $\mathrm{Gal}_k \to \mathrm{Rep}_r(\Gamma, d)(K_\lambda)$ given by $\sigma \mapsto \rho_\lambda^\sigma|_\Gamma$. This is continuous for the λ-adic topology on the target since Gal_k acts continuously on $\pi_1^{\text{ét}}(X_{\overline{k}}, k)$, and it sends 1 to ρ, so there is an open neighborhood of 1 in Gal_k which maps into the Zariski-connected component of ρ. Replacing k by a finite extension, we may assume all of Gal_k is sent to the connected component of ρ; then for any $\sigma \in \mathrm{Gal}_k$, rigidity implies that $\rho_\lambda^\sigma|_\Gamma$ is $\mathrm{GL}_r(\overline{K_\lambda})$-conjugate to ρ, hence (by the density of Γ in $\pi_1^{\text{ét}}(X_{\overline{k}}, x)$) that ρ_λ^σ is $\mathrm{GL}_r(\overline{K_\lambda})$-conjugate to ρ_λ.

For each $\sigma \in \mathrm{Gal}_k$, let $P_\sigma \in \mathrm{GL}_r(\overline{K_\lambda})$ be a matrix such that $\rho_\lambda^\sigma = P_\sigma \rho_\lambda P_\sigma^{-1}$. Let $\overline{P_\sigma}$ denote the image of P_σ in $\mathrm{PGL}_r(\overline{K_\lambda})$. Then, since ρ_λ is irreducible, Schur's lemma implies that $\overline{P_\sigma}$ does not depend on the choice of P_σ. The function $\sigma \mapsto \overline{P_\sigma}$ is a therefore a homomorphism and is, by a simple argument, continuous. Now let Λ be the image of $1 + \mathfrak{a}^{r\times r}$ in $\mathrm{PGL}_r(\overline{K_\lambda})$, where $\mathfrak{a}$ is an ideal of $\mathcal{O}_{\overline{K_\lambda}}$ small enough that $1+\mathfrak{a}$ contains no roots of unity of order $\leq r$. Then the canonical map $(1+\mathfrak{a}^{r\times r}) \cap \mathrm{SL}_r(\overline{K_\lambda}) \twoheadrightarrow \Lambda$ is an isomorphism of topological groups, so after enlarging k such that $\overline{P_\sigma} \in \Lambda$ for all σ, we may lift each $\overline{P_\sigma}$ uniquely to $(1+\mathfrak{a}^{r\times r}) \cap \mathrm{SL}_r(\overline{K_\lambda})$, and thereby assume that $\sigma \mapsto P_\sigma$ is a continuous homomorphism $\mathrm{Gal}_k \to \mathrm{SL}_r(\overline{K_\lambda})$.

We claim that, after making these enlargements, $k(\lambda) = k$ works. Indeed, define

$$\pi_1^{\text{ét}}(X_k, x) \to \mathrm{GL}_r(\overline{K_\lambda}), \qquad \gamma\sigma \mapsto \rho_\lambda(\gamma)P_\sigma \tag{2}$$

for all $\gamma \in \pi_1^{\text{ét}}(X_{\overline{k}}, x)$ and $\sigma \in \mathrm{Gal}_k$. One checks, using the isomorphism $\pi_1^{\text{ét}}(X_k, x) \cong \pi_1^{\text{ét}}(X_{\overline{k}}, x) \rtimes \mathrm{Gal}_k$ and the definition of P_σ, that (2) is in fact a continuous homomorphism. $\square$

Remark 9.9. (Not included in the talk.) Let $D_{1,R}, \dots, D_{n,R}$ be the irreducible components of $\overline{X}_R \setminus X_R$. The difficulty in generalizing the proof to the quasiprojective case lies in showing that $\rho_\lambda^\sigma|_\Gamma$ remains in the same (appropriately chosen) representation variety as ρ. In other words, one has to study the local monodromy of $\rho_\lambda^\sigma|_\Gamma$. To do this, one has to identify the local monodromy of $\rho_\lambda^\sigma|_\Gamma$ (defined topologically as above) with the image of

$$\pi_1^{\text{ét}}\left(\mathrm{Spec}\left(\mathrm{Frac}\left(\mathcal{O}^\wedge_{\overline{X}_{\overline{k}}, D_{i,\overline{k}}}\right)\right)\right) \to \pi_1^{\text{ét}}(X_{\overline{k}}, x) \xrightarrow{\rho_\lambda^\sigma} \mathrm{GL}_r(K_\lambda)$$

(cf. the statement of Corollary 3.13). See e.g. the proof of [EG18, Lemma 3.2], which begins by reducing to the curve case.

9.4 Superrigidity

Let $\mathcal{H}_g$ be Siegel's upper half space, the set of $g \times g$ complex symmetric matrices with positive-definite imaginary part. Let $\Gamma_g(n)$ be the kernel of the reduction map $\mathrm{Sp}_{2g}(\mathbb{Z}) \to \mathrm{Sp}_{2g}(\mathbb{Z}/n)$. Then $\Gamma_g(n)$ acts on the complex manifold $\mathcal{H}_g$; let $\mathcal{A}_{g,n}$ be the (topological) quotient $\Gamma_g(n)\backslash\mathcal{H}_g$. If $n \geq 3$, then $\mathcal{A}_{g,n}$ has the structure of a smooth quasiprojective variety, is the moduli space for principally polarized Abelian varieties of dimension g with level-n structure (i.e. a symplectic basis of n-torsion), and $\Gamma_g(n)$ acts on $\mathcal{H}_g$ in a sufficiently nice way that $\pi_1(\mathcal{A}_{g,n}) \cong \Gamma_g(n)$. (The last two statements hold in general if one instead takes a stack quotient.)

Example 9.10. Here are some local systems on $\mathcal{A}_{g,n}$. There is a "tautological" local system coming from the inclusion $\Gamma_g(n) \hookrightarrow \mathrm{GL}_{2g}(\mathbb{C})$. If f denotes the universal family of Abelian varieties over $\mathcal{A}_{g,n}$, then $R^1 f_* \underline{\mathbb{C}}$ is a local system on $\mathcal{A}_{g,n}$. (Actually, it can be shown that these are the same local system.)

The following is a version of "superrigidity" for $\Gamma_g(n)$, which gives us a source of rigid local systems.

Theorem 9.11. *Suppose $g \geq 2$. For any field K of characteristic 0 and representation $\rho\colon \Gamma_g(n) \to \mathrm{GL}_r(K)$, we have $H^1(\Gamma_g(n), \rho) = 0$. Consequently, every local system on $\mathcal{A}_{g,n}$ is cohomologically rigid.*

Remark 9.12. A stronger version of superrigidity allows one to prove that, if $g \geq 2$, any $\rho\colon \Gamma_g(n) \to \mathrm{GL}_r(\mathbb{C})$ as above is isomorphic to $\rho_1 \otimes \rho_2$, where ρ_1 is the restriction to $\Gamma_g(n)$ of an algebraic representation $\mathrm{Sp}_{2g}(\mathbb{C}) \to \mathrm{GL}_r(\mathbb{C})$ and ρ_2 factors through a finite group. Since the category of algebraic representations of $\mathrm{Sp}_{2g}(\mathbb{C})$ is generated by a single faithful representation, and finite-order local systems are of geometric origin, it follows that *every* local system on $\mathcal{A}_{g,n}$ is of geometric origin. For an exposition of this, see [Put], which treats the analogous question for SL_n, $n \geq 3$ (though the same argument works for Sp_{2g}, $g \geq 2$).

The statements above can be derived from the solution to the congruence subgroup problem for Sp_{2g} ($g \geq 2$) by Bass–Milnor–Serre [BMS67]. However, they are special cases of very deep "superrigidity" theorems of Margulis which apply to general irreducible lattices in semisimple Lie groups of rank ≥ 2. For example, Theorem 9.11 is a special case of [Mar91, Chapter IX, Theorem 6.15(ii)].

10 Nonabelian cohomology and applications, lecture 3. Speaker: Daniel Litt. Notes by Kyle Binder

Let X be a smooth, projective variety over a finitely-generated field k and $\ell \neq \operatorname{char} k$.

Sasha discussed the similarity between the Tate Conjecture and the Relative Fontaine–Mazur Conjecture.

Conjecture 10.1 (Tate Conjecture)**.** An element $\xi \in H^{2i}\left(X_{\bar{k}}, \mathbb{Q}_\ell(i)\right)$ is in the image of

$$cl\colon Z^i\left(X_{\bar{k}}\right)_{\mathbb{Q}_\ell} \to H^{2i}\left(X_{\bar{k}}, \mathbb{Q}(i)\right)$$

if and only if the orbit $\mathrm{Gal}_k \cdot \xi$ is finite.

Analogously in the non-abelian case we have the following conjecture.

Conjecture 10.2 (Non-abelian Tate Conjecture, Relative Fontaine–Mazur/Petrov Conjecture)**.** A continuous, semi-simple representation

$$\rho\colon \pi_1^{\text{ét}}\left(X_{\bar{k}}\right) \to \mathrm{GL}_n\left(\overline{\mathbb{Q}_\ell}\right)$$

is of geometric origin if and only if $\mathrm{Gal}_k \cdot [\rho]$ is finite (equivalently, by Jake's talk, ρ is arithmetic).

What we now have is two conjectural characterizations of representations of geometric origin: being a direct factor of a $\mathbb{Z}$-VHS and having finite orbit $\mathrm{Gal}_k \cdot [\rho]$. Conjecturally, these two characterizations should be equivalent, and this is also an open problem.

One piece of weak evidence for the equivalence is the following.

Proposition 10.3. *Suppose we have the Cartesian diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d, "\widetilde{\pi}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"', "\textit{smooth, proper}"] \\ \widetilde{S} \arrow[r, "f", "univ.cover"'] & S \end{tikzcd}$$

of varieties over $\mathbb{C}$ and $\mathbb{V}$ a $\mathbb{Q}$-local system on $\widetilde{X}$ such that $\mathbb{V}_{|\text{very general fibre of } \widetilde{\pi}}$ underlies a $\mathbb{Z}$-VHS. (Very general here means on the complement of a countable number of analytic closed subvarieties of $\widetilde{S}$.)

Then $\mathbb{V} \otimes \mathbb{Q}_\ell$ (viewed as a local system on $X_{\overline{\mathbb{C}(S)}}$) has finite orbit over $\mathrm{Gal}_{\mathbb{C}(S)}$.

Proof. We want to show that $\pi_1(S,s) \cdot \left[\mathbb{V}_{|\text{fibre}}\right]$ is finite. Consider $f^{-1}(s)$; for all $s' \in f^{-1}(s)$ we get a local system $\mathbb{V}_{s'}$ on X_s. This is because we get local system on $\widetilde{X}_{s'}$ which is isomorphic to X_s by base-change. By assumption, $\mathbb{V}_{s'}$ underlies a $\mathbb{Z}$-VHS. By unwinding definitions, the family $\{\mathbb{V}_{s'}\}$ is precisely $\pi_1(S,s) \cdot \left[\mathbb{V}_{|\text{fibre}}\right]$. Using the following theorem of Deligne, this implies the orbit is finite.

Theorem 10.4 ([Del87, Théorème 0.5])**.** *The set of isomorphism classes of rank r $\mathbb{Q}$-local systems on a smooth, quasi-projective variety over $\mathbb{C}$ which underlie a $\mathbb{Z}$-VHS is finite.*

$\square$

10.1 Some Predictions

1. **Conjecture 10.5** (Simpson's Conjecture)**.** Rigid local systems are of geometric origin.

   As Jake explained, this conjecture makes many predictions. If we take any property satisfied by a local system of geometric origin, we can try to verify that rigid local systems satisfy the property. For example, take the property of being a direct factor of a $\mathbb{Z}$-VHS. Jake explained that Mochizuki showed in the quasi-projective case that rigid local systems are direct factors of a $\mathbb{Q}$-VHS. In order to conclude the rigid local system is a direct factor of a $\mathbb{Z}$-VHS, we are left to show that the rigid local system is integral, i.e., that the monodromy is defined over $\mathcal{O}_K$ for a number field K.

The main goal of this talk is to prove this statement for cohomologically rigid local systems.

Theorem 10.6 ([EG18, Theorem 1.1])**.** *Irreducible, cohomologically rigid local systems with finite-order determinant are integral.*

Remark 10.7. Implicit in the statement of cohomological rigidity is the group GL, but it is natural to ask similar things for a different connected, reductive group G. By [KP22, Theorem 1.2], the analogue of Esnault–Groechenig's integrality result holds for G-cohomologically rigid local systems.

Remark 10.8. One can ask if Esnault–Groechenig's theorem implies Simpson's Conjecture, *viz.* are all rigid local systems also cohomologically rigid? In this simplest phrasing, de Jong, Esnault, and Groechenig construct a rigid but non-cohomologically rigid local system ([dJEG22]). However, one may ask the sharper question of whether all rigid local systems are G-cohomologically rigid (here G is the Zariski-closure of monodromy); the answer to this question is unknown.

2. Another property of representations of geometric origin in our context is the relation between choices of the prime number ℓ.

   Conjecture 10.9 (Deligne)**.** Let $X/\mathbb{F}_q$ be a normal variety and $\ell \neq \operatorname{char} \mathbb{F}_q$ a prime. If $\mathcal{E}$ is an irreducible $\overline{\mathbb{Q}}_\ell$-sheaf on X with finite determinant, then there is a number field $E \subseteq \overline{\mathbb{Q}}_\ell$ such that:

   (a) For all closed points $x \in |X|$ we have $\det(1 - F_x t \mid \mathcal{E}_x) \in E[t]$, where F_x is the geometric Frobenius of the point x.

   Why is this predicted? The fact that the local system is defined on $X/\mathbb{F}_q$ means the corresponding representation is extended from $\pi_1\left(X_{\overline{\mathbb{F}}_q}\right)$ to $\pi_1\left(X_{\mathbb{F}_q}\right)$. In particular, the representation is arithmetic, so we expect the representation to come from geometry and the cohomology of a family of varieties. But in this case, the characteristic polynomials of Frobenius count points on the various fibres. So if we take the total cohomology, the characteristic polynomials should live in $\mathbb{Z}[t]$. However, our conjectures only assume this is a direct summand of the total cohomology, so the coefficients of these characteristic polynomials live in some number field.

   (b) For all places λ' of E not above $\operatorname{char} \mathbb{F}_q$, there is an $E_{\lambda'}$-sheaf $\mathcal{E}'$ with

   $$\det(1 - F_x t \mid \mathcal{E}') = \det(1 - F_x t \mid \mathcal{E}).$$

   As the local system conjecturally has geometric origin, this predicts the choice of prime ℓ (the place λ' corresponding to a different choice) when taking cohomology doesn't have much effect, as it is still just counting points in some fixed family of varieties.

   (c) Roots of $\det(1 - F_x t \mid \mathcal{E})$ should be integral over $\mathbb{Z}\left[\frac{1}{\operatorname{char} \mathbb{F}_q}\right]$.

   This is some property that is satisfied when $\mathcal{E}$ comes from geometry. It is saying that the Frobenius eigenvalues of the sheaf should behave like the Frobenius eigenvalues acting on cohomology.

   (d) $\mathcal{E}$ is pure of weight 0.

This conjecture is actually mostly known, as we will see in the following two results.

The first is a result coming from Lafforgue's proof of the Langland's Program for Function Fields for GL_n [Laf02]. Via this proof, Lafforgue proves the Relative Fontaine-Mazur Conjecture (Conjecture 10.2) for curves over finite fields, as he is able to realize that arithmetic local systems on some curve come from the cohomology of a stack of shtukas.

Theorem 10.10 ([Laf02, Théorème VII.6])**.** *If* $\dim X = 1$*, Conjecture 10.9 holds.*

The second result proves Deligne's conjecture when the normality assumption is replaced with the stronger condition of the variety being smooth.

Theorem 10.11 (Lafforgue, Deligne, Drinfeld, see [Dri12])**.** *If* X *is smooth (and* $\dim X \geq 1$*), Conjecture 10.9 holds.*

The proof works by a reduction to the theorem in dimension 1. For each curve on X, we can find some local system $\mathcal{E}'$ as in part (b) of the conjecture. Then the job is to show that these local systems glue together, and this is non-obvious.

Remark 10.12. One way of thinking about the characteristic polynomials of Frobenius is in terms of the Weil Conjectures. Another way of thinking about these is that the characteristic polynomials $\det(1 - F_x t \mid \mathcal{E})$ determine $\mathcal{E}$ at least up to semi-simplification by the Čebotarev Density Theorem, so the $\mathcal{E}'$ in part (b) of the conjecture is unique. However, this relation between $\overline{\mathbb{Q}}_\ell$-local systems and $\overline{\mathbb{Q}}_{\ell'}$-local systems is not functorial. In fact, this bijection should not be expected to be a functor, as in one category, the morphisms are $\overline{\mathbb{Q}}_\ell$-vector spaces, while in the other they are $\overline{\mathbb{Q}}_{\ell'}$-vector spaces.

Remark 10.13. In Deligne's Conjecture 10.9, we restricted to the case of X over a finite field. However, the Relative Fontaine–Mazur Conjecture 10.2 is stated for any finitely-generated field, so we should expect Deligne's Conjecture to also be true in this case.

3.

Conjecture 10.14 (Non-abelian Variational Conjecture)**.** Suppose $X \longrightarrow S/\mathbb{C}$ is a smooth and proper map and that $\mathbb{V}$ is a local system of geometric origin on a fibre X_s. Then the Relative Fontaine–Mazur Conjecture gives a prediction when the local system $\mathbb{V}_{s'}$ on a nearby fibre is of geometric origin.

Specifically, suppose the $\pi_1(S, s) \cdot \mathbb{V}$ orbit is finite.

Then:

(a) There is a dominant étale map $S' \longrightarrow S$ such that $\mathbb{V}$ extends to $X_{S'}$.
(This part is proven and is just a way of rephrasing that the orbit $\pi_1(S, s) \cdot \mathbb{V}$ is finite. It makes no mention of $\mathbb{V}$ having geometric origin on X_s.)

(b) The local system $\mathbb{V}_{|\text{general fibre of } X_{S'} \to S}$ is of geometric origin.

Let us briefly say why the Relative Fontaine–Mazur Conjecture predicts this. Assuming the conjecture, what one should check in verifying the local system $\mathbb{V}_{|\text{general fibre of } X_{S'} \to S}$ is of geometric origin is that this local system has finite orbit under the Galois action of the generic point of S over some finitely-generated field. Roughly, this Galois group is composed of the Galois group of S and the Galois group of the finitely-generated field. The Galois group of S by assumption induces a finite orbit, while the Galois group of the field should be independent of fibre; hence if it holds for one fibre it should hold for nearby fibres.

Some predictions coming from this conjecture are known. For example, if the local system on one fibre in this family underlies a $\mathbb{Z}$-VHS then the same is true on all fibres. This is a theorem of Katzarkov, Pantev ([KP02]) and Jost, Zuo ([JZ01]). One may also use Mochizuki's work in [Moc06] to derive the result using parabolic Higgs bundles.

10.2 Integrality

For the rest of this talk we will explain how the Theorem of Lafforgue, Deligne, and Drinfeld (Theorem 10.11) implies the result of Esnault–Groechenig (Theorem 10.6).

Lemma 10.15 (Simpson)**.** *Let X be a smooth, projective variety over $\mathbb{C}$, and suppose $\mathbb{V}$ is rigid. Then $\mathbb{V}$ is defined over $\mathcal{O}_K\left[\frac{1}{N}\right]$ where K is a number field and $N \in \mathbb{Z}_{>0}$.*

Proof. By assumption, $[\mathbb{V}] \in M_B(X,r)(\mathbb{C})$ is an isolated point of the character variety (which is a finite-type $\mathbb{Q}$-scheme), so the residue field of this point is finite over $\mathbb{Q}$. Then by some algebra the representation can be defined over a slightly larger number field. To show we only have to invert finitely many primes, we use the deep result that $\pi_1(X)$ is finitely generated. Therefore the representation can be defined by finitely many matrices and hence only finitely many numbers. □

In order to prove the result of Esnault–Groechenig, all we have to do is remove the $\frac{1}{N}$ appearing in the lemma. To do this we utilize Lafforgue, Deligne, and Drinfeld's result about switching between primes.

Proof of Esnault–Groechenig. (For more details, see [EG18, Theorem 1.1]). Let $L(r,d)$ be a set of irreducible, cohomologically rigid, complex local systems on X, of rank r, with determinant of order dividing d, one for each isomorphism class. This is a finite set, so Simpson's Lemma implies the existence of an integer N such that every local system in $L(r,d)$ is defined over $\mathcal{O}_K\left[\frac{1}{N}\right]$. Choose primes $\ell \nmid N$ and $\ell' \mid N$. For an element $\mathcal{V} \in L(r,d)$ take the associated representation

$$\rho_{\mathcal{V}} \colon \pi_1(X) \longrightarrow \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]\right).$$

For a place of K, $\lambda \mid \ell$, we can then take the λ-adic completion

$$\rho_{\mathcal{V},\lambda} \colon \pi_1(X) \longrightarrow \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]\right) \longrightarrow \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}\right).$$

As $\mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}\right)$ is a profinite group, this map factors through the profinite completion of $\pi_1(X)$, which is $\pi_1^{\text{ét}}(X)$. So we have the commutative diagram

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[dr] & \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]\right) \arrow[r] & \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}\right) \\ & \pi_1^{\text{ét}}(X) \arrow[ur] & \end{tikzcd}$$

By rigidity, Jake explained how this representation can be extended to a model of X over a finitely-generated $\mathbb{Q}$-algebra. A similar argument shows the representation can be extended to a smooth model over R which is finitely-generated over $\mathbb{Z}$.

$$\begin{tikzcd} \pi_1(X) \arrow[r] \arrow[dr] & \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]\right) \arrow[r] & \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}\right) \\ & \pi_1^{\text{ét}}(X) \arrow[ur] \arrow[r] & \pi_1^{\text{ét}}(X_R) \arrow[u, "\text{arithmeticity}"'] \end{tikzcd}$$

Now suitably pick a closed point $\mathfrak{p}$ in $\mathrm{Spec}(R)$ which is not above ℓ, then reduce mod $\mathfrak{p}$. [17]

This gives the commutative diagram

[17] Added by the note-taker: The closed point $\mathfrak{p} \mid p$ must satisfy the following conditions, with p uniform across all representations from $L(r,d)$ (see [EG18, Bottom of p. 4285]).

1. p is prime to the order of the image of the residual representation
$$\pi_1^{\text{ét}}(X) \to \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}/\mathfrak{m}\right).$$
2. $p \nmid \ell$.
3. $p \nmid N$.

$$
\begin{tikzcd}
\pi_1(X) \arrow[r] \arrow[dr] & \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]\right) \arrow[r] & \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge_\lambda}\right) \\
& \pi_1^{\text{ét}}(X) \arrow[ur] \arrow[r] & \pi_1^{\text{ét}}\left(X_R\right) \arrow[u] \\
& & \pi_1^{\text{ét}}\left(X_{\mathfrak{p}}\right) \arrow[u] \arrow[uu, bend right, "\rho_{\mathcal{V},\mathfrak{p},\lambda}"']
\end{tikzcd}
$$

Note that $X_{\mathfrak{p}} := X_R \times_R R/\mathfrak{p}$ is now a variety over a finite field.

By Part (b) of Theorem 10.11 (see 2b of Section 10.1), there is a representation

$$\rho_{\mathcal{V},\mathfrak{p},\ell'} \colon \pi_1^{\text{ét}}\left(X_{\mathfrak{p}}\right) \longrightarrow \mathrm{GL}_n\left(\overline{\mathbb{Q}}_{\ell'}\right)$$

which is a companion to $\rho_{\mathcal{V},\lambda}$. This yields, by "unspecializing" the representation, a $\overline{\mathbb{Q}}_{\ell'}$-representation of $\pi_1^{\text{ét}}(X)$ and therefore of $\pi_1(X)$ which we call $\rho_{\mathcal{V},\ell'}$. [18]

If we have chosen the ℓ and $\mathfrak{p}$ sufficiently nicely, we claim we have constructed a bijective map from the isomorphism classes of irreducible, cohomologically rigid, rank r, with determinant of order dividing d, $\overline{\mathbb{Q}}_\ell$-representations to the isomorphism classes of those $\overline{\mathbb{Q}}_{\ell'}$-representations. In fact, it is enough to show this mapping is an injection: $\overline{\mathbb{Q}}_\ell$ and $\overline{\mathbb{Q}}_{\ell'}$ are isomorphic as fields, so there are the same number of isomorphism classes of irreducible, cohomologically rigid, rank r, with determinant of order dividing d representations of $\pi_1(X)$ over either field.

Assuming the claim, we will have shown that for any $\ell' \mid N$ there is an integral model of the representation $\rho_{\mathcal{V}}$. This is because $\pi_1^{\text{ét}}(X)$ is a profinite, compact group, so any representation factors through a maximal compact subgroup of $\mathrm{GL}_n\left(\overline{\mathbb{Q}}_{\ell'}\right)$. Using the integrality criterion of [Bas80, Corollaries 2.3 and 2.5], this is enough to show the result.

In order to verify the claim, we need to check two things:

1. We need the representation $\rho_{\mathcal{V},\ell'}$ (and $\rho_{\mathcal{V},\mathfrak{p},\ell'}$) to be irreducible, cohomologically rigid, rank r, with determinant of order dividing d, in order to have a map between the appropriate class of $\overline{\mathbb{Q}}_\ell$-representations and the appropriate class of $\overline{\mathbb{Q}}_{\ell'}$-representations.

2. We need this map to be injective.

The obstruction to cohomological rigidity is

$$H^1\left(X_{\overline{\mathfrak{p}}}, \mathrm{End}^0\left(\rho_{\mathcal{V},\mathfrak{p},\ell'}\right)\right).$$ [19]

4. p is prime to the order of the determinant of the representation

$$\rho_{\mathcal{V}} \colon \pi_1(X) \longrightarrow \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]\right).$$

[18] Added by the note-taker: Let us explain how this yields a $\overline{\mathbb{Q}}_{\ell'}$-representation of $\pi_1(X)$. Let $\overline{\mathfrak{p}}$ be a geometric point of $\mathrm{Spec}(R)$ over $\mathfrak{p}$.

We consider the companion to $\rho_{\mathcal{V},\mathfrak{p},\lambda}$,

$$\rho_{\mathcal{V},\mathfrak{p},\ell'} \colon \pi_1^{\text{ét}}\left(X_{\mathfrak{p}}\right) \to \mathrm{GL}_n\left(\overline{\mathbb{Q}}_{\ell'}\right).$$

It is easy to check that $\rho_{\mathcal{V},\mathfrak{p},\ell'}$ has finite determinant. Then we pull back $\rho_{\mathcal{V},\mathfrak{p},\lambda}$ to a representation of $\pi_1^{\text{ét}}\left(X_{\overline{\mathfrak{p}}}\right)$, then to a $\overline{\mathbb{Q}}_\ell$-representation of $\pi_1^{\text{ét}}(X)$ via the surjective specialization map

$$sp \colon \pi_1^{\text{ét}}(X) \twoheadrightarrow \pi_1^{\text{ét}}\left(X_{\overline{\mathfrak{p}}}\right).$$

Finally, we pull back again to get a representation of $\pi_1(X)$.

[19] Added by the note-taker: The verification of the other properties of $\rho_{\mathcal{V},\ell'}$ follows from first verifying the representation $\rho_{\mathcal{V},\lambda}$ factors through $\rho_{\mathcal{V},\mathfrak{p},\lambda}$, and this is where the assumptions on $\mathfrak{p}$ are necessary. Let $\overline{\mathfrak{p}}$ be a geometric point of $\mathrm{Spec}(R)$ over $\mathfrak{p}$. Recall the specialization isomorphism

$$sp \colon \pi_1^{\text{ét},p'}(X) \xrightarrow{\cong} \pi_1^{\text{ét},p'}\left(X_{\overline{\mathfrak{p}}}\right),$$

The second part on injectivity has obstructions

$$H^0\left(X_{\mathfrak{p}}, \mathrm{Hom}\left(\rho_1, \rho_2\right)\right)$$

for each pair of distinct representations $\rho_1, \rho_2 \colon \pi_1\left(X_{\mathfrak{p}}\right) \to \mathrm{GL}_n\left(\overline{\mathbb{Q}}_{\ell'}\right)$.

By Part (d) of Theorem 10.11 (Part 2d of Section 10.1) both of these obstructions are cohomology groups of pure local systems; hence, the obstructions are pure. Therefore the dimensions of the obstructions can be read off from the respective L-functions. In particular, the dimension of the obstructions are independent of the prime λ and therefore vanish.

□

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where $\pi_1^{\text{ét},p'}$ denotes the prime-to-p quotient.

We first want to show the representation

$$\rho_{\mathcal{V},\lambda} \colon \pi_1^{\text{ét}}\left(X\right) \to \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}\right)$$

factors through $\pi_1^{\text{ét},p'}\left(X_{\overline{\mathfrak{p}}}\right)$. By the prime-to-$p$ specialization isomorphism, it is enough to show we have the factorization

$$\pi_1^{\text{ét}}\left(X\right) \to \pi_1^{\text{ét},p'}\left(X\right) \to \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}\right).$$

This is done in two steps. First, the condition that p is prime to the order of the image of the residual representation

$$\pi_1^{\text{ét}}\left(X\right) \to \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}/\mathfrak{m}\right).$$

ensures the factorization

$$\pi_1^{\text{ét}}\left(X\right) \to \pi_1^{\text{ét},p'}\left(X\right) \to \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}/\mathfrak{m}\right).$$

Second, the kernel of the residue map

$$\mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}\right) \to \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}/\mathfrak{m}\right)$$

is pro-ℓ, so the assumption that $p \nmid \ell$ yields the factorization

$$\pi_1^{\text{ét}}\left(X\right) \to \pi_1^{\text{ét},p'}\left(X\right) \to \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}\right) \to \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}/\mathfrak{m}\right).$$

Using the assumption that p is prime to the order of the determinant of the representation

$$\rho_{\mathcal{V}} \colon \pi_1(X) \longrightarrow \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]\right),$$

one similarly shows the determinant of

$$\pi_1^{\text{ét},p'}\left(X_{\overline{\mathfrak{p}}}\right) \to \mathrm{GL}_n\left(\mathcal{O}_K\left[\frac{1}{N}\right]^{\wedge\lambda}\right)$$

is finite.

By replacing $\mathfrak{p}$ with a point with larger residue field, one can show this representation of $\pi_1^{\text{ét},p'}\left(X_{\overline{\mathfrak{p}}}\right)$ factors through $\pi_1^{\text{ét},p'}\left(X_{\mathfrak{p}}\right)$ and the resulting representation, which is $\rho_{\mathcal{V},\mathfrak{p},\lambda}$, has finite determinant. (This is [EG18, Proposition 3.1].)

11 Rigid-Analytic Geometry. Speaker: Zeyu Liu. Notes by Mehmet Basaran

11.1 Rigid analytic space

Definition 11.1. A nonarchimedean field is a field K that is complete with respect to a non-archimedean absolute value $|\cdot|$, i.e. $|\cdot|$ satisfies

(i) $|x| = 0$ iff $x = 0$

(ii) $|xy| = |x||y|$

(iii) $|x + y| \leq \max(|x|, |y|)$

Example 11.2. $K = \mathbb{Q}_p, \mathbb{F}_p((T)), \mathbb{C}((T))$

From now on we fix a nonarchimedean field K with ring of integers $\mathcal{O} = \{x \in K : |x| \leq 1\}$, maximal ideal $\mathfrak{m}_K = \{x \in K : |x| < 1\}$, and residue field $k = \mathcal{O}/\mathfrak{m}_K$.

Definition 11.3 (naïve definition)**.** X/K is called an n-dimensional compact manifold, if

(i) X is an analytic variety over K,

(ii) X is compact, and

(iii) $\dim X = n$.

Theorem 11.4 ([Ser65, Théorème (1)])**.** *Assume that K is discretely valued and $q := |k| < \infty$. Then there are only $q - 1$ many isomorphism classes of compact n-dimensional manifolds over K.*

Because of this theorem, the naïve definition of a compact p-adic manifold is not very interesting. Therefore we need new ideas to work in rigid analytic geometry. Mimicking the setup in algebraic geometry outlined below, where polynomial algebras are the building blocks, we start working in rigid analytic geometry by first defining Tate algebras, which will serve as building blocks in this case.

Classical algebraic geometry	Rigid analytic geometry
schemes/$\mathbb{C}$	rigid spaces/K
$\cup$	$\cup$
affine schemes	affinoid rigid spaces
$\uparrow$	$\uparrow$
finite type algebras	affinoid algebras
$\cup$	$\cup$
polynomial algebras	Tate algebras

11.1.1 Tate algebras

Tate algebras can be thought of functions on the unit disk.

In the complex case $\mathbb{C}$, holomorphic functions on the open unit ball are given by power series $f(z) = \sum a_n z^n$ with restrictions on the coefficients a_n.

In the nonarchimedean case K, we consider power series $f(T) = \sum a_n T^n$ with coefficients $a_n \in K$. This power series $f(T)$ converges on the closed unit ball B if and only if $|a_n| \to 0$. This is true, since restricting to B gives

$$\left| \sum_{n \geq k}^{\infty} a_n T^n \right| \leq \max_{n \geq k} |a_n T^n| \leq \max_{n \geq k} |a_n| \,.$$

Definition 11.5. For $n \geq 1$, the n-th variable Tate-algebra of K is

$$T_n \coloneqq T_n(K) \coloneqq \left\{ \sum_J a_J X^J : \ |a_J| \to 0, \text{ for } \|J\| \to \infty \right\},$$

where $J = (j_1, \ldots, j_n) \in \mathbb{N}^n$ is a multi-index, $\|J\| = \sum_{i=1}^n j_i$, and $X^J = x_1^{j_1} \cdot \cdots \cdot x_n^{j_n}$. It will also be denoted by $K \langle X_1, \ldots, X_n \rangle$. So $T_n(K)$ is the subring of formal power series that converge on B^n.

Example 11.6.

- $\mathbb{Q}_p \langle X_1, \ldots, X_n \rangle$
- $C \langle X_1, \ldots, X_n \rangle$, where $C = \mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$

In both of these examples, elements of the Tate algebras can be viewed as functions on the unit ball $B^n(C) = \{(x_1, \ldots, x_n) : \ x_i \in C, |x_i| \leq 1\}$.

- $\mathbb{Q}_p \left\langle X, X^{-1} \right\rangle = \left\{ \sum_{i=-\infty}^{\infty} a_i X^i : \ \lim |a_i| \to 0 \text{ as } |i| \to \infty \right\}$.

Elements in this Tate algebra can be viewed as functions on the unit circle $\{a \in C : \ |a| = 1\}$.

Definition 11.7. Given a function $f = \sum a_J X^J \in T_n(K)$, we define its Gauß norm $\|f\|$ to be $\|f\| \coloneqq \max_J |a_J|$.

Lemma 11.8. *$T_n(K)$ with the Gauß norm is a Banach algebra, i.e. for all $f, g \in T_n(K)$ and $c \in K$ we have*

(i) $\|f\| = 0$ *if and only if* $f = 0$,

(ii) $\|f + g\| \leq \max\{\|f\|, \|g\|\}$,

(iii) $\|cf\| = |c| \|f\|$,

(iv) $\|fg\| \leq \|f\| \|g\|$, *and*

$T_n(K)$ is complete with respect to this norm.

For any point $\underline{x} \in B^n\left(\overline{K}\right)$ and any function $f \in T_n(K)$, we get a point $f(\underline{x}) \in \overline{K}$. So for fixed $\underline{x} \in B^n\left(\overline{K}\right)$ we can define a norm

$$\begin{aligned} \| \cdot \|_{\underline{x}} \colon T_n(K) &\longrightarrow \overline{K} \xrightarrow{|\cdot|} \mathbb{R} \\ f &\longmapsto |f(\underline{x})| \,. \end{aligned}$$

Then (see [Bos14, Section 2.2 Proposition 5])

$$\|f\| = \sup_{\underline{x} \in B^n\left(\overline{K}\right)} \|f\|_{\underline{x}} = \sup_{\underline{x} \in B^n\left(\overline{K}\right)} |f(\underline{x})| \overset{\text{max. mod. principle}}{=} \max_{\underline{x} \in B^n\left(\overline{K}\right)} |f(\underline{x})| \,.$$

Proposition 11.9. *The Tate algebra $T_n(K)$ has the following properties:*

(i) It is Noetherian, regular, and a UFD.

(ii) Every ideal $I \subset T_n(K)$ is closed with respect to the Gauß norm.

(iii) For all maximal ideals $\mathfrak{m} \in T_n(K)$, the residue field $T_n(K)/\mathfrak{m}$ is finite over K.

(iv) The map $B^n\left(\overline{K}\right) \to M(T_n(K)) \coloneqq \{$maximal ideals of $T_n(K)\}$ is surjective.

Proof. See [Con, Theorem 1.1.5]. ∎

11.1.2 Affinoid algebras

Definition 11.10. A K-algebra A is an affinoid K-algebra, if there is a surjective maps α of K-algebras $\alpha\colon T_n(K) \to A$.

Remark 11.11. (i) Any affinoid K-algebra A is Noetherian.

(ii) Given a surjection $\alpha\colon T_n(K) \to A$ we can define a norm on A as follows: For $x \in A$ we set

$$\|x\|_\alpha = \inf_{\substack{y \in T_n(K) \\ \alpha(y)=x}} \|y\|.$$

This norm depends on the choice of α, but the topology defined by it does not.

Definition 11.12. Let A be an affinoid K-algebra and write $X = M(A)$. Let $f_1, \ldots, f_n, g \in A$ such that $(f_1, \ldots, f_n, g) = A$. Define rational subdomains as sets of the form

$$X\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \right\rangle \coloneqq \{x \in X\colon \ \|f_i(x)\| \le \|g(x)\| \text{ for all } i\} \subset X,$$

where $f(x)$ denotes the image of f in the residue field A/x of the maximal ideal x, and the norm $\|\cdot\|$ on A/x is the unique extension of the norm on K. Moreover, if $(f_1, \ldots, f_n) = A$, we define

$$X_i \coloneqq X\left\langle \frac{f_1}{f_i}, \ldots, \frac{f_{i-1}}{f_i}, \frac{f_{i+1}}{f_i}, \ldots, \frac{f_n}{f_i} \right\rangle.$$

Then $X = \cup X_i$, and this is called a rational cover. We define a Grothendieck topology on $M(A)$ to be the one generated by these X_i.

One can define a structure sheaf $\mathcal{O}$ on $X = M(A)$ that sends X to A and any rational subdomain $X\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \right\rangle$ to an affinoid algebra $A\left\langle \frac{f_1}{g}, \ldots, \frac{f_n}{g} \right\rangle \coloneqq A\langle T_1, \ldots, T_n\rangle / (gT_i - f_i)$ (see [Bos14, Chapters 3-5] for details).

To any affinoid K-algebra A we can associate the space $\mathrm{Sp}(A) = (X = M(A), \mathcal{O}_X)$ consisting of the topological space $M(A)$ with its structure sheaf. Such a space $\mathrm{Sp}(A)$ is called an affinoid rigid space. More generally, a rigid space over K is one that locally looks like an affinoid rigid space

The goal now is to associate to an algebraic variety over K a rigid space over K. Pick $c \in K$ with $|c| > 1$ (e.g. $c = 1/p$ for $K = C$), and define the affinoid K-algebra

$$T_n^{(i)} \coloneqq T_n\left(|c|^i\right) \coloneqq K\left\langle \frac{X_1}{c^i}, \ldots, \frac{X_n}{c^i} \right\rangle.$$

Elements in $T_n^{(i)}$ can be viewed as functions on the n-dimensional closed ball of radius $|c|^i$. Then

$$T_n^{(0)} \hookleftarrow T_n^{(1)} \hookleftarrow \cdots \hookleftarrow T_n^{(k)} \hookleftarrow \cdots \hookleftarrow K[X_1, \ldots, X_n]$$

and

$$B^n \coloneqq \mathrm{Sp}\left(T_n^{(0)}\right) \hookrightarrow \mathrm{Sp}\left(T_n^{(1)}\right) \hookrightarrow \ldots$$

Each of these are affinoid rigid spaces and we define

$$(\mathbb{A}_K^n)^{\mathrm{rig}} \coloneqq \bigcup_{i=0}^{\infty} \mathrm{Sp}\left(T_n^{(i)}\right)$$

to be the rigidification of the n-dimensional affine space. If instead the algebraic variety is of the form $\mathrm{Spec}\left(K[X_1, \ldots, X_n]/\alpha\right)$, we can repeat the above process with $T_n^{(i)}/\alpha$ in place of $T_n^{(i)}$, and we get the rigidification

$$\left(\mathrm{Spec}\left(K[X_1, \ldots, X_n]/\alpha\right)\right)^{\mathrm{rig}} = \bigcup_{i=0}^{\infty} \mathrm{Sp}\left(T_n^{(i)}\alpha\right).$$

This rigidification process gives a well-defined functor from the category of algebraic varieties over K to the category of rigid spaces over K independent of the choice of c and the choice of α (see [Bos14, Section 5.4 Corollary 5] for more details).

11.2 Raynaud generic fiber functor

Let $\mathcal{O}$ be a discrete valuation ring with uniformizer $\pi \in \mathcal{O}$. The goal is to define a functor

$$\{\text{admissible formal schemes}/\mathcal{O}\} \to \{\text{Rigid spaces over}/K\}$$

Definition 11.13. A topological $\mathcal{O}$-algebra $\mathcal{A}$ is called admissible, if

(i) $\mathcal{A}$ is topologically finite type over $\mathcal{O}$, i.e.

$$\mathcal{A} \simeq \mathcal{O}\left\langle X_1, \ldots, X_n \right\rangle / \alpha,$$

where $\mathcal{O}\left\langle X_1, \ldots, X_n \right\rangle$ is defined as Tate algebras were defined over K before; and

(ii) $\mathcal{A}$ has no π-torsion, so $\mathcal{A}$ is flat over $\mathcal{O}$.

We realize the desired functor above by sending an admissible formal scheme $\mathrm{Spf}\left(\mathcal{O}\left\langle X_1, \ldots, X_n \right\rangle / \alpha\right)$ to $\mathrm{Sp}\left(T_n/\alpha\right)$, where $T_n = \left(\mathcal{O}\left\langle X_1, \ldots, X_n \right\rangle\right)\left[\frac{1}{\pi}\right]$ (see [Con, Section 3.3] for details).

Example 11.14. Let $X = \mathrm{Sp}\left(C\left\langle T \right\rangle\right)$. To show that X has many nonsplit finite étale extensions, we show that $H^1_{\acute{e}t}\left(X, \mu_p\right)$ is very large. To see this, consider the Kummer sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_p & \longrightarrow & \mathbb{G}_m & \longrightarrow & \mathbb{G}_m & \longrightarrow & 0, \\ & & & & x & \longmapsto & x^p & & \end{array}$$

which is exact in the étale topology ([Sta24, Tag 03PL]). Therefore we get a sequence

$$0 \longrightarrow C\langle T\rangle^\times / \left(C\langle T\rangle^\times\right)^p \longrightarrow H^1_{\acute{e}t}\left(X, \mu_p\right) \longrightarrow H^1_{\acute{e}t}\left(X, \mathbb{G}_m\right).$$

Since X has no nontrivial étale line bundles, the latter part $H^1_{\acute{e}t}\left(X, \mathbb{G}_m\right)$ vanishes, and thus $H^1_{\acute{e}t}\left(X, \mu_p\right) \simeq C\langle T\rangle^\times / \left(C\langle T\rangle^\times\right)^p$, which is nontrivial. For example it has the nonzero element $1 + p^{1/1000000000}T$. This can be seen by looking at the coefficients a_m of any p-th root. They satisfy the recursive relation

$$a_m = -\frac{(m-1)p - 1}{pm} p^{1/1000000000} a_{m-1},$$

and since

$$\left| -\frac{(m-1)p - 1}{pm} p^{1/1000000000} \right| \geq 1,$$

the coefficients a_m do not tend to 0. In contrast, if we did the same computation with $\mathrm{Spec}\left(C[X]\right)$ in place of $\mathrm{Sp}\left(C\left\langle T \right\rangle\right)$, then we get

$$0 = C[T]^\times / \left(C[T]^\times\right)^p \simeq H^1_{\acute{e}t}\left(\mathrm{Spec}\left(C[T], \mu_p\right)\right).$$

12 The p-adic Riemann-Hilbert correspondence, lecture 3. Speaker: Alexander Petrov. Notes by Stefan Nikoloski.

We continue with our notation from earlier where $K_\infty = \bigcup_n K(\mu_{p^n})$, $H_K = \mathrm{Gal}(\overline{K}/K_\infty)$ and $\Gamma_K = G_K/H_K$, which is realized as an open subgroup of $\mathbb{Z}_p^\times$ via the cyclotomic character χ_{cyc}. We recall that we defined a functor

$$H : \left\{ \begin{smallmatrix} \mathbb{Q}_p-\text{representations} \\ \text{of } G_K \end{smallmatrix} \right\} \longrightarrow \left\{ \begin{smallmatrix} K_\infty-\text{vector spaces } M \\ \phi : M \longrightarrow M \end{smallmatrix} \right\} \tag{$*$}$$

as follows: Starting with a $\mathbb{Q}_p$-representation V of G_K, we consider $(V \otimes_{\mathbb{Q}_p} C)^{H_K}$, which by the almost purity theorem is a vector space over $\widehat{K_\infty}$ with the same dimension as V. We also had a decompletion $H(V)$, a vector space over K_∞ such that $H(V) \otimes_{K_\infty} \widehat{K_\infty} \simeq (V \otimes_{\mathbb{Q}_p} C)^{H_K}$. This was a functorial construction and $H(V)$ comes with a K_∞-semilinear Γ_K-action. We then define a K_∞-linear operator[20] on $H(V)$ by:

$$\phi(v) = \lim_{\substack{\gamma \to 1 \\ \gamma \in \Gamma_K}} \frac{\gamma(v) - v}{\chi_{\mathrm{cyc}}(\gamma) - 1}$$

The purpose of ϕ is to capture the Γ_K-action, although we remark that ϕ could lose a bit of information. For example, if Γ_K acts through a finite quotient on $H(V)$, then any γ close enough to 1 will act trivially on $H(V)$ and hence $\phi \equiv 0$.

Example 12.1. Let $V = \chi_{\mathrm{cyc}}^a = \langle e \rangle$, where $a \in \mathbb{Z}_p$ is close enough to 1 that we can make sense of the power of the cyclotomic character. Since H_K acts trivially on V we have that $(V \otimes_{\mathbb{Q}_p} C)^{H_K} = (e \otimes 1) \cdot \widehat{K_\infty}$ and from this $H(V) = (e \otimes 1) \cdot K_\infty$. Now

$$\phi(e \otimes 1) = \lim_{\gamma \to 1} \frac{\chi_{\mathrm{cyc}}^a(\gamma) - 1}{\chi_{\mathrm{cyc}}(\gamma) - 1} (e \otimes 1) = a(e \otimes 1)$$

From this we deduce that ϕ acts on $H(V)$ as multiplication by a.

Definition 12.2. The eigenvalues of ϕ acting on $H(V)$ are called the Hodge-Tate weights of V.

Motivation 12.3. The motivation for this definition comes from the Hodge-Tate decomposition. We recall that for X/K smooth and proper and $V = H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ we have the decomposition

$$V \otimes_{\mathbb{Q}_p} C \simeq \bigoplus_i H^{n-i}(X, \Omega_X^i) \otimes_K C(-i)$$

Therefore, $V \otimes_{\mathbb{Q}_p} C$ looks like a direct sum of cyclotomic characters. By the same computations as in Example 12.1 we get that

$$H(V) = \bigoplus_i H^{n-i}(X, \Omega_X^i) \otimes_K K_\infty$$

with ϕ acting on the graded piece indexed by i as multiplication by $-i$. In other words, ϕ remembers the Hodge numbers on the n-th cohomology.

As we want to study the functor D_{dR} we instead consider $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{H_K}$. We recall that B_{dR}^+ has a filtration whose associated graded ring is a direct sum of Tate twists of C. Therefore, on the associated graded pieces the action of H_K is same as the action on C. Moreover, after a choice of a system of compatible roots of unity there is a canonical element $t \in B_{\mathrm{dR}}^+$ such that $g \in G_K$ acts on t by $g(t) = \chi_{\mathrm{cyc}}(g)t$. We note that the filtration on B_{dR}^+ is given in terms of power of t, i.e. $F^i B_{\mathrm{dR}}^+ = t^i B_{\mathrm{dR}}^+$.

[20]On an element of K_∞, Γ_K acts via a finite quotient. Hence every γ close enough to 1 will act trivially on this element, giving us the K_∞-linearity of ϕ.

Given the de Rham comparison theorem we can easily see the existence of such t. Consider the projective line $\mathbb{P}^1_K$. We have $H^2_{\text{ét}}(\mathbb{P}^1_{\overline{K}}) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \simeq H^2_{\mathrm{dR}}(\mathbb{P}^1_K) \otimes_K B_{\mathrm{dR}}$. By Serre's duality $H^2_{\mathrm{dR}}(\mathbb{P}^1_K)$ is isomorphic to K, so the right-hand side is isomorphic to B_{dR}. The image of 1 under this isomorphism has to land in the G_K-invariants of $H^2_{\text{ét}}(\mathbb{P}^1_{\overline{K}}) \otimes_K B_{\mathrm{dR}}$, which is isomorphic to $(\mathbb{Q}_p(-1) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$. After a choice of a generator of $\mathbb{Q}_p(-1)$, the image of 1 will be of the form $1 \otimes t$, which gives us the desired element t of B^+_{dR}.

As before there is a decompletion $RH^+(V)$ such that $RH^+(V) \otimes_{K_\infty[\![t]\!]} (B^+_{\mathrm{dR}})^{H_K} \simeq (V \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}})^{H_K}$, where $RH^+(V)$ is a free $K_\infty[\![t]\!]$-module of rank $\dim_{\mathbb{Q}_p} V$, which is also equipped with a Γ_K-action. We remark that $K_\infty[\![t]\!]$ maps into $(B^+_{\mathrm{dR}})^{H_K}$, as H_K acts trivially on both K_∞ and t. Moreover, the projection onto the zeroth graded piece gives rise to the commutative diagram

$$\begin{array}{ccc} RH^+(V) \otimes_{K_\infty[\![t]\!]} (B^+_{\mathrm{dR}})^{H_K} & \xrightarrow{\simeq} & (V \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}})^{H_K} \\ \downarrow & & \downarrow \\ H(V) \otimes_{K_\infty} \widehat{K_\infty} & \xrightarrow{\simeq} & (V \otimes_{\mathbb{Q}_p} C)^{H_K} \end{array}$$

We want to study the Γ_K-action on $RH^+(V)$. To do that we can define an operator ϕ on $RH^+(V)$ which will be compatible with the action on ϕ on the base ring $K_\infty[\![t]\!]$, which in turn is given by the earlier derivative formula. To see how ϕ will interact with the $K_\infty[\![t]\!]$-module structure on $RH^+(V)$, we first compute its action on $K_\infty[\![t]\!]$. As G_K acts by the cyclotomic character on t we get that $\phi(t) = t$. Therefore, as ϕ is a K_∞-linear[20] derivation on $K_\infty[\![t]\!]$ we get that $\phi = t\partial_t$ on $K_\infty[\![t]\!]$. Now generalizing the construction of ϕ on $H(V)$ we can construct an operator ϕ on $RH^+(V)$ such that

$$\phi(am) = a\phi(m) + t\partial_t(a)m$$

for all $a \in K_\infty[\![t]\!]$ and $m \in RH^+(V)$. We can then equip $RH^+(V)$ with a connection $\nabla : RH^+(V) \to RH^+(V) \otimes_{K_\infty[\![t]\!]} K_\infty[\![t]\!] \frac{\mathrm{d}t}{t}$ defined by $\nabla = \phi \otimes \frac{\mathrm{d}t}{t}$. We now have that

$$D_{\mathrm{dR}}(V) \otimes_K K_\infty \simeq RH^+(V)[1/t]^{\nabla=0}$$

Proposition 12.4. *V is de Rham if and only if $RH^+(V)[1/t]$ is spanned by flat sections as a $K_\infty(\!(t)\!)$-vector space.*

Example 12.5. We revisit our earlier example. As in Example 12.1 let $V = \chi^a_{\mathrm{cyc}}$. Since H_K acts trivially on V, as above we can immediately see that $RH^+(V)$ is a free $K_\infty[\![t]\!]$-module of rank 1. Moreover, as in Example 12.1 we get that ϕ will act by multiplication by a on a basis element e of $RH^+(V)$. We remark that ϕ is a K_∞-linear operator, but not a $K_\infty[\![t]\!]$-linear operator. It will satisfy the derivation rule mentioned above and therefore for $f \in K_\infty[\![t]\!]$ we have:

$$\phi(fe) = f\phi(e) + t\partial_t(f)e = afe + t\partial_t(f)e = (af + t\partial_t(f)) \cdot e$$

Hence, after identifying $RH^+(V)$ with $K_\infty[\![t]\!]$ we get that ϕ acts on $K_\infty[\![t]\!]$ as $a + t\partial_t$. From here $\nabla = \mathrm{d} + a\frac{\mathrm{d}t}{t}$. Now, since $RH^+(V)[1/t] \simeq K_\infty(\!(t)\!)$ is a 1-dimensional vector space, by Proposition 12.4 χ^a_{cyc} is de Rham if and only if $K_\infty(\!(t)\!)$, equipped with the connection $\nabla = \mathrm{d} + a\frac{\mathrm{d}t}{t}$ has a non-zero flat section. As this differential equation has a solution in $K_\infty(\!(t)\!)$ if and only if $a \in \mathbb{Z}$ [21], we get that χ^a_{cyc} is de Rham if and only if $a \in \mathbb{Z}$.

12.1 The Relative Setting

Let X be a smooth variety over K. Our goal is to define an analogue of the de Rham functor for local systems

[21] If $a \in \mathbb{Z}$, obviously t^{-a} is a non-zero solution. Conversely, let $f = \sum_{i=N}^{\infty} a_i t^i$ be a non-zero solution to $\nabla = 0$. Then, $0 = \mathrm{d}f + a\frac{\mathrm{d}t}{t} = \sum_{i=N}^{\infty} a_i(i+a)t^{i-1}dt$. As $a_N \neq 0$ we must have $a = -N \in \mathbb{Z}$.

$$D_{\mathrm{dR}} : \left\{ \substack{\mathbb{Q}_p\text{−local} \\ \text{systems on } X} \right\} \longrightarrow \left\{ \substack{\text{vector bundles } E \text{ on } X \\ \text{with flat connection} \nabla : E \to E \otimes \Omega^1_X \\ \text{and filtration } \cdots \subset F^i E \subset F^{i-1} E \subset \cdots \\ \text{such that } \nabla(F^i E) \subset F^{i-1} E \otimes \Omega^1_X} \right\}$$

satisfying the following:

Properties 12.6.

1. *If* $X = \operatorname{Spec} K$*, then* $D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K}$

2. *If* $f : Y \to X$ *is a morphism of smooth varieties and* $\mathbb{L}$ *is a local system on* X*, then* $f^* D_{\mathrm{dR}}(\mathbb{L}) = D_{\mathrm{dR}}(f^* \mathbb{L})$

Definition 12.7. A local system $\mathbb{L}$ is called de Rham if $\operatorname{rank}_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{L}) = \operatorname{rank}_{\mathbb{Q}_p} \mathbb{L}$.

Theorem 12.8 ([LZ17, Theorem 1.1])**.** *Let* X *be a smooth connected variety over* K*. Let* $\mathbb{L}$ *be a* $\mathbb{Q}_p$*-local system on* X*. Suppose there exists a point* $x \in X(K)$ *such that the* G_K*-representation* $\mathbb{L}_x$ *is de Rham. Then for any finite field extension* L/K *and every* $y \in X(L)$*,* $\mathbb{L}_y$ *is de Rham.*

Proof. The proof follows rather formally from Properties 12.6. $\mathbb{L}_x$ being de Rham is equivalent to the equality $\dim_K D_{\mathrm{dR}}(\mathbb{L}_x) = \dim_{\mathbb{Q}_p}(\mathbb{L}_x)$ being satisfied. By the properties mentioned above this equality becomes $\operatorname{rank}_{\mathbb{Q}_p} D_{\mathrm{dR}}(\mathbb{L}) = \operatorname{rank}_{\mathbb{Q}_p}(\mathbb{L})$. Applying this equivalence in the reverse direction for any $y \in X(L)$ we get the result. $\square$

12.2 p-adic Simpson and p-adic Riemann-Hilbert

Let X be a smooth variety over K. To construct the relative dR functor, the first step is to build relative versions of the functors H and RH^+. There is a functor

$$H : \left\{ \substack{\mathbb{Q}_p\text{−local} \\ \text{systems on} X} \right\} \longrightarrow \left\{ \substack{\text{Higgs bundles } M/X_{K_\infty} \\ \theta : M \to M \otimes \Omega^1_{X_{K_\infty}} \\ \mathcal{O}_{X_{K_\infty}}\text{−linear endomoprhism } \phi : M \to M \\ \text{such that } \theta \circ \phi = (\phi - 1) \circ \theta} \right\}$$

satisfying the following:

Properties 12.9.

1. *For every* $x \in X(K)$ *and local system* $\mathbb{L}$ *on* X *we have* $H(\mathbb{L})_x = H(\mathbb{L}_x)$*, where the functor* H *on the right is the one defined in* ($*$).

2. *The functor* H *preserves ranks, i.e* $\operatorname{rank}_{\mathbb{Q}_p}(\mathbb{L}) = \operatorname{rank}_{\mathcal{O}_{X_{K_\infty}}} H(\mathbb{L})$.

3. *The characteristic polynomial of* ϕ *has constant coefficients.*

Example 12.10. Let $f : Y \to X$ be a smooth proper morphism of varieties and $\mathbb{L} = R^n f_* \mathbb{Q}_p$. Then

$$H(\mathbb{L}) = \bigoplus_i R^{n-i} f_* \Omega^i_{Y/X} \otimes_K K_\infty$$

Moreover, the Higgs field is $\theta : R^{n-i} f_* \Omega^i_{Y/K} \to R^{n-i+1} f_* \Omega^{i-1}_{Y/K} \otimes \Omega^1_X$. On the i-th graded piece ϕ acts as multiplication by $-i$. This means that θ maps the $(-i)$-eigenspace of ϕ to the $(-i+1)$-eigenspace of ϕ, which gives us the relation $\theta \circ \phi = (\phi - 1) \circ \theta$.

Remark 12.11. The relation $\theta \circ \phi = (\phi - 1) \circ \theta$ implies that θ is nilpotent in the sense that this bundle has a filtration such that on the associated graded module θ acts as 0 [22]. Additionally, the relation implies that $(M, \theta) \simeq (M, \lambda\theta)$ for any $\lambda \in K_\infty^\times$, as in the setting of variations of Hodge structures.

[22] One such filtration is given by the eigenspaces of ϕ. The relation $\theta \circ \phi = (\phi - 1) \circ \theta$ means that θ will map the λ-eigenspace of ϕ to the $(\lambda - 1)$-eigenspace of ϕ, which we already saw to be the case in Example 12.10.

Remark 12.12. If X is over C and not over K, then we get neither ϕ nor the nilpotence of θ. However, there is a more general p-adic Simpson correspondence over C.

Let $X_{K_\infty}[\![t]\!]$ be the locally ringed spaces whose underlying topological space is X_{K_∞} and whose structure sheaf is $\mathcal{O}_{X_{K_\infty}}[\![t]\!]$. We also have a functor

$$RH^+ : \left\{ \begin{smallmatrix} \mathbb{Q}_p\text{-local} \\ \text{systems on } X \end{smallmatrix} \right\} \longrightarrow \left\{ \begin{smallmatrix} \text{vector bundles } E \text{ on } X_{K_\infty}[\![t]\!] \\ \text{with a flat connection } \nabla: E \to E \otimes \frac{1}{t}\left(\Omega^1_{X_{K_\infty}}[\![t]\!] \otimes \mathcal{O}_{X_{K_\infty}}[\![t]\!]\, \mathrm{d}t\right) \\ \text{satisfying the Leibnitz rule} \end{smallmatrix} \right\}$$

Example 12.13. We consider the same example as above with the local system $\mathbb{L} = R^n f_* \mathbb{Q}_p$. We then have

$$RH^+(\mathbb{L}) = \sum F^i_{\mathrm{Hodge}} \mathcal{H}^n_{\mathrm{dR}}(Y/X) \otimes t^{-i} \mathcal{O}_{X_{K_\infty}}[\![t]\!] \subset \mathcal{H}^n_{\mathrm{dR}}(Y/K) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_{K_\infty}}(\!(t)\!)$$

and the connection is $\nabla = \nabla_{GM} \otimes \mathrm{d}$, where ∇_{GM} is the Gauss-Manin connection and d is the usual derivation.

We can now decompose the connection ∇ appearing in the definition of the RH^+ functor as $\nabla = \nabla^{\mathrm{geom}} + \nabla^{\mathrm{arithm}}$, where ∇^{geom} is the projection to the summand $\Omega^1_{X_{K_\infty}}[\![t]\!]$ and ∇^{arithm} is the projection to the summand $\mathcal{O}_{X_{K_\infty}}[\![t]\!]\, \mathrm{d}t$. Moreover, since ∇ is flat, these two will have to commute with each other. With this notation we remark that the target category of RH^+ is equivalent to

$$\left\{ \begin{smallmatrix} \text{vector bundles } E \text{ on } X_{K_\infty}[\![t]\!] \\ \text{with a connection } \nabla^{\mathrm{geom}}: E \to E \otimes \frac{1}{t} \Omega^1_{X_{K_\infty}}[\![t]\!] \\ \text{and } \mathcal{O}_{X_{K_\infty}}\text{-linear endomorphism } \phi: E \to E \\ \text{such that } \phi(tm) = t\phi(m) + m \text{ and } [\nabla^{\mathrm{geom}}, \phi] = 0 \end{smallmatrix} \right\}$$

where $\phi = t \cdot \nabla^{\mathrm{arithm}}$. From this equivalence we can connect the two functors and in particular we get

$$H(\mathbb{L}) = RH^+(\mathbb{L})/t \qquad \theta = \nabla^{\mathrm{geom}} \pmod t \qquad \phi = t \cdot \nabla^{\mathrm{arithm}} \pmod t$$

Remark 12.14. The functors H and RH^+ are far from equivalence of categories. Indeed, for any finite image representation as explained at the beginning the endomorphism ϕ will be trivial. Hence, any such representation will be sent to the trivial object.

13 Algebraic differential equations in characteristic $p > 0$. Speaker: Ziquan Yang. Notes by William C. Newman.

13.1 Algebraic de Rham Cohomology

Definition 13.1. Suppose $\pi\colon X \to S$ is a smooth morphism of schemes. For $(\mathcal{E}, \nabla) \in \mathrm{MIC}(X/S)$, define the complex[23]

$$\Omega^{\bullet}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \coloneqq (\cdots \to \Omega^i_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \xrightarrow{\nabla_i} \Omega^{i+1}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E} \to \dots)$$

where

$$\nabla_i(\omega \otimes e) = d\omega \otimes e + (-1)^i \omega \wedge \nabla(e).$$

Define the (relative) <u>algebraic de Rham cohomology</u>

$$H^q_{\mathrm{dR}}(X/S, (\mathcal{E}, \nabla)) \coloneqq R^q \pi_*(\Omega^{\bullet}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}).$$

These are quasi-coherent sheaves on the base S.

Remark 13.2. Alternatively, one can define for $H^q_{\mathrm{dR}}(X/S, (\mathcal{E}, \nabla))$ as the qth derived functor of

$$\mathrm{MIC}(X/S) \to \mathrm{QCoh}(S)$$

$$(\mathcal{E}, \nabla) \mapsto \pi_*(\mathcal{E}^{\nabla = 0})$$

These definitions are shown to be equivalent in [Gro68]. Note that an element of $\mathrm{MIC}(S/S)$ is a pair $(\mathcal{E}, \nabla)$ with $\mathcal{E} \in \mathrm{QCoh}(S)$ and $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_S} \Omega_{S/S} = 0$. Hence, the natural map $\mathrm{MIC}(S/S) \to \mathrm{QCoh}(S)$ is an equivalence of categories. We sometimes will write $\mathrm{MIC}(S/S)$ instead of $\mathrm{QCoh}(S)$.

We focus mainly on

$$H^n_{\mathrm{dR}}(X/S) \coloneqq H^n_{\mathrm{dR}}(X/S, (\mathcal{O}_X, d)).$$

Remark 13.3. For $S = \mathrm{Spec}(k)$, $H^n_{\mathrm{dR}}(X/k)$ satisfies some usual cohomology properties, including Poincare duality, Kunneth property, and a cycle class map $Z^i(X) \to H^{2n}(X/k)$. When k has characteristic 0, $X \mapsto H^*_{\mathrm{dR}}(X/k)$ gives a Weil cohomology theory ([Sta24, Tag 0FWC]).

Lemma 13.4. *Let $\mathcal{A}$ be an abelian category with countable direct sums and enough injectives. Given a left exact functor $T\colon \mathcal{A} \to \mathcal{B}$ and a complex $K^{\bullet}$ of objects in $\mathcal{A}$ with a filtration*

$$\cdots \supseteq F^i(K^{\bullet}) \supseteq F^{i+1}(K^{\bullet}) \supseteq \cdots,$$

whose graded pieces $\mathrm{gr}^i K^n = 0$ for $|i|$ sufficiently large, we have a spectral sequence $E_1^{p,q}$ converging to $R^{p+q}T(K^{\bullet})$, where

$$E_1^{p,q} = R^{p+q}T(\mathrm{gr}^p K^{\bullet})$$

and differential $d\colon E_1^{p,q} \to E_1^{p+1,q}$ coming from the boundary map in the long exact sequence coming from applying RT to

$$0 \to \mathrm{gr}^{p+1}(K^{\bullet}) \to F^p K^{\bullet}/F^{p+2}K^{\bullet} \to \mathrm{gr}^p(K^{\bullet}) \to 0$$

We get an induced filtration on $R^n T(K^{\bullet})$ by setting $F^i R^n T(K^{\bullet})$ to be the image of

$$R^q \pi_*(F^i K^{\bullet}) \to R^q \pi_*(K^{\bullet}).$$

[23] Note this complex is not simply the tensor product of the de Rham complex $\Omega^{\bullet}_{X/S}$ with $\mathcal{E}$, as the differentials depend on the connection ∇.

Definition 13.5. For any complex $K^\bullet$, one can define the so-called "stupid filtration" by

$$F^i(K^j) = \begin{cases} 0 & j < i \\ K^j & j \geq i \end{cases}$$

The Hodge-to-de Rham spectral sequence is the spectral sequence $E_r^{p,q}$ computing $R^i\pi_*(\Omega^\bullet_{X/S})$ via Lemma 13.4 using the stupid filtration. The induced filtration F on $H^i_{\mathrm{dR}}(X/S) = R^i\pi_*(\Omega^\bullet_{X/S})$ is called the Hodge filtration.

We can compute the E_1-page of the Hodge-to-de Rham spectral sequence explicitly:

$$E_1^{p,q} = R^{p+q}(\mathrm{gr}^p(\Omega^\bullet_{X/S})) = R^{p+q}\pi_*(\Omega^p_{X/S}[-p]) = R^q\pi_*(\Omega^p_{X/S}).$$

In our setting, we cannot take the complex conjugate of the Hodge filtration F to obtain another filtration of $R^q\pi_*(\Omega^\bullet_{X/S})$. However, we do have the following notion:

Definition 13.6. The conjugate spectral sequence is the "second spectral sequence of hypercohomology"

$${}_{\mathrm{con}}E_2^{p,q} := R^p\pi_*(\mathcal{H}^q(\Omega^\bullet_{X/S})) \implies R^{p+q}\pi_*(\Omega^\bullet_{X/S})$$

The induced filtration F_{con} on $R^{p+q}\pi_*(\Omega^\bullet_{X/S})$ is called the conjugate filtration.

We will see later why this is some sort of analogue of the complex conjugate of the Hodge filtration in the complex-analytic setting.

13.2 Gauß–Manin Connection

In the complex analytic setting, given a smooth map $\pi\colon X \to S$ and $(\mathcal{E}, \nabla) \in \mathrm{MIC}^{\mathrm{an}}(X)$, there is a connection on $R^i\pi_*\mathcal{E}$ called the Gauß–Manin connection. We describe an algebraic analogue of the Gauß–Manin connection and prove that it satisfies Griffiths transversality.

Suppose we have smooth maps

$$\begin{tikzcd} X \arrow[dr, "\pi"] \arrow[dd, "\pi\circ f"'] & \\ & S \arrow[dl, "f"] \\ T & \end{tikzcd}$$

The Gauß–Manin connection ∇_{GM} on $R^i\pi_*\mathcal{E}$ will give a commutative diagram of functors

$$\begin{tikzcd} D\,\mathrm{MIC}(X/S) \arrow[d] \arrow[dr, "{(R\pi_*, \nabla_{\mathrm{GM}})}"] & \\ D\,\mathrm{MIC}(X/T) \arrow[d, "R(\pi\circ f)_*"'] & D\,\mathrm{MIC}(S/T) \arrow[dl, "Rf_*"] \\ D\,\mathrm{MIC}(T/T) & \end{tikzcd}$$

where $D\mathcal{A}$ denotes the derived category of $\mathcal{A}$. As a special case, we obtain a "Leray" spectral sequence

$$E_1^{p,q} = H^p_{\mathrm{dR}}(S/T, (H^q_{\mathrm{dR}}(X/S), \nabla_{\mathrm{GM}})) \implies H^{p+q}_{\mathrm{dR}}(X/T).$$

To construct ∇_{GM}, recall the exact sequence

$$0 \to \pi^*\Omega^1_{S/T} \to \Omega^1_{X/T} \to \Omega^1_{X/S} \to 0. \tag{3}$$

Exactness on the left follows from smoothness of $X \to S$ ([Sta24, Tag 06B6]). We define the filtration $G^i\Omega^n_{X/T}$ to be the image of

$$\pi^*\Omega^i_{S/T} \otimes_{\mathcal{O}_X} \Omega^{n-i}_{X/T} \to \Omega^n_{X/T}$$

Using (3), one can show that graded piece of this filtration is

$$\mathrm{gr}^i_G\, \Omega^n_{X/T} = \pi^*\Omega^i_{S/T} \otimes_{\mathcal{O}_X} \Omega^{n-i}_{X/S}.$$

Now, the spectral sequence induced via Lemma 13.4 by this filtration has E_1-page

$$E_1^{p,q} = R^{p+q}\pi_*\, \mathrm{gr}^p_G\, \Omega^\bullet_{X/T} = R^{p+q}\pi_*(\pi^*\Omega^p_{S/T} \otimes_{\mathcal{O}_X} \Omega^{\bullet-p}_{X/S}) \cong \Omega^p_{S/T} \otimes_{\mathcal{O}_S} R^q\pi_*(\Omega^\bullet_{X/S}) = \Omega^p_{S/T} \otimes_{\mathcal{O}_S} H^q_{\mathrm{dR}}(X/S),$$

where the middle isomorphism is the projection formula (which is allowed because $\pi^*\Omega^p_{S/T}$ is locally free). The Gauß–Manin connection ∇_{GM} is defined to be the map

$$H^q_{\mathrm{dR}}(X/S) = E_1^{0,q} \xrightarrow{d} E_1^{1,q} = \Omega^1_{S/T} \otimes_{\mathcal{O}_S} H^q_{\mathrm{dR}}(X/S).$$

In fact, the associated de Rham complex $\Omega^\bullet_{S/T} \otimes_{\mathcal{O}_S} H^q_{\mathrm{dR}}(X/S)$ is equal to the complex $E_1^{\bullet,q}$.

Theorem 13.7 (Griffiths transversality). *We have* $\nabla_{\mathrm{GM}}(F^iH^q_{\mathrm{dR}}(X/S)) \subseteq \Omega^1_{S/T} \otimes_{\mathcal{O}_S} F^{i-1}H^q_{\mathrm{dR}}(X/S)$.

Proof. Recall as stated in Lemma 13.4 that the differential $\nabla_{\mathrm{GM}} = d\colon E_1^{0,q} \to E_1^{1,q}$ is obtained by taking the boundary map in the long exact sequence coming from applying $R\pi^q_*$ to

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{gr}^1_G(K^\bullet) \arrow[r] \arrow[d, equal] & G^0K^\bullet/G^2K^\bullet \arrow[r] \arrow[d, equal] & \mathrm{gr}^0_G(K^\bullet) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \pi^*\Omega^1_{S/T} \otimes_{\mathcal{O}_X} \Omega^{\bullet-1}_{X/S} \arrow[r] & G^0\Omega^\bullet_{X/T}/G^2\Omega^\bullet_{X/T} \arrow[r] & \Omega^\bullet_{X/S} \arrow[r] & 0 \end{tikzcd}$$

We have the following subcomplex of the above complex:

$$0 \to \pi^*\Omega^1_{S/T} \otimes_{\mathcal{O}_X} F^{i-1}\Omega^{\bullet-1}_{X/S} \to F^i(G^0\Omega^\bullet_{X/T}/G^2\Omega^\bullet_{X/T}) \to F^i\Omega^\bullet_{X/S} \to 0.$$

Applying $R^q\pi_*$ and looking at the connecting homomorphisms, we get the commutative diagram

$$\begin{tikzcd} F^iH^q_{\mathrm{dR}}(X/S) \arrow[r, "\partial"] \arrow[d] & \Omega^1_{S/T} \otimes_{\mathcal{O}_S} F^{i-1}H^q_{\mathrm{dR}}(X/S) \arrow[d] \\ H^q_{\mathrm{dR}}(X/S) \arrow[r, "\nabla_{\mathrm{GM}}"] & \Omega^1_{S/T} \otimes_{\mathcal{O}_S} H^q_{\mathrm{dR}}(X/S). \end{tikzcd}$$

This gives the desired result. ∎

13.3 Characteristic p

We now work exclusively in characteristic p. Suppose S is a characteristic-p scheme, i.e. $p\mathcal{O}_S = 0$, and let $\pi\colon X \to S$ be smooth and proper. Recall that for any characteristic-p scheme T, we have an absolute Frobenius morphism $F_T\colon T \to T$, defined by taking the underlying map of topological spaces to be the identity, and defining the map on $\mathcal{O}_T$ to be $a \mapsto a^p$.

Define $X^{(p)}, \sigma, \pi^{(p)}$ to be the fiber product

$$\begin{tikzcd} X^{(p)} \arrow[r, dashed, "\sigma"] \arrow[d, dashed, "\pi^{(p)}"'] & X \arrow[d, "\pi"] \\ S \arrow[r, "F_S"] & S. \end{tikzcd}$$

Using that $\pi \circ F_X = F_S \circ \pi$, we get an induced map $F\colon X \to X^{(p)}$, the relative Frobenius morphism, making

$$\begin{tikzcd} X \arrow[dr, dashed, "F"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right, "\pi"'] & & \\ & X^{(p)} \arrow[r, "\sigma"] \arrow[d, "\pi^{(p)}"'] & X \arrow[d, "\pi"] \\ & S \arrow[r, "F_S"] & S. \end{tikzcd}$$

commute. When writing X/S, we mean the morphism $\pi\colon X \to S$, and when writing $X^{(p)}/S$, we mean the morphism $\pi^{(p)}\colon X^{(p)} \to S$.

While the complex $F_*\Omega^\bullet_{X/S}$ a priori only has $\mathcal{O}_S$-linear maps, note that for $f \in \mathcal{O}_X$, we have

$$d(\sigma^\#(f)\omega) = d(f^p\omega) = d(f^p) \wedge \omega + f^p d\omega = f^p d\omega = \sigma^\#(f) d\omega,$$

and so $F_*\Omega^\bullet_{X/S}$ is a complex of $\mathcal{O}_{X^{(p)}}$-modules. Moreover, we can explicitly describe $\mathcal{H}^i(F_*\Omega^\bullet_{X/S})$ via the Cartier isomorphism:

Theorem 13.8 ([Kat72, Proposition 2.1.1])**.** *For each i, there is a unique isomorphism*

$$\mathcal{C}^{-1} : \Omega^n_{X^{(p)}/S} \to \mathcal{H}^n(F_*\Omega^\bullet_{X/S})$$

called the Cartier isomorphism, such that

$$\mathcal{C}^{-1}(1) = 1$$
$$\mathcal{C}^{-1}(\omega \wedge \tau) = \mathcal{C}^{-1}(\omega) \wedge \mathcal{C}^{-1}(\tau)$$
$$\mathcal{C}^{-1}(d\sigma^{-1}(x)) = [x^{p-1}dx]$$

Using this, we are able to give a different form of the E_2-page of the conjugate filtration:

Corollary 13.9. *The Cartier isomorphism $\mathcal{C}$ induces an isomorphism of $\mathcal{O}_S$ modules*

$${}_{\mathrm{con}}E_2^{a,b} = R^a\pi_*\mathcal{H}^b(\Omega^\bullet_{X/S}) \xrightarrow{\sim} R^a\pi^{(p)}_*(\sigma^*\Omega^b_{X/S})$$

Proof. Using that F is a homeomorphism, we have the following isomorphisms:

$$\begin{aligned} R^a\pi_*\mathcal{H}^b(\Omega^\bullet_{X/S}) &= R^a(\pi^{(p)}_* F_*)\mathcal{H}^b(\Omega^\bullet_{X/S}) \\ &\cong R^a\pi^{(p)}_*\mathcal{H}^b(F_*\Omega^\bullet_{X/S}) \\ &\overset{\mathcal{C}}{\cong} R^a\pi^{(p)}_*\Omega^b_{X^{(p)}/S} \\ &\cong R^a\pi^{(p)}_*\sigma^*\Omega^\bullet_{X/S}. \end{aligned}$$

□

If F_S is flat or the $R^a\pi_*(\Omega^b_{X/S})$ are flat, flat base change and the above give

$$ {}_{\mathrm{con}}E_2^{a,b} \cong R^a\pi_*^{(p)}\sigma^*\Omega^b_{X/S} \cong F_S^* R^a\pi_*\Omega^b_{X/S}. $$

Theorem 13.10 ([DI87, Corollaire 3.7])**.** *Assume S lifts to $\widetilde{S}$ flat over $\mathbb{Z}/p^2$ and $X^{(p)}$ lifts to a smooth morphism over $\widetilde{S}$. Then $\tau_{<p}F_*\Omega^\bullet_{X/S}$ is decomposible. Each choice of $\widetilde{X}^{(p)} \to S$ induces a quasi-isomorphism*

$$ \bigoplus_{i<p} \Omega^i_{X^{(p)}/S}[-i] \overset{\mathcal{C}}{\cong} \bigoplus_{i<p} \mathcal{H}^i(F_*\Omega^\bullet_{X/S})[-i] \to F_*\Omega^\bullet_{X/S}. $$

In the statement of the Theorem, the $\tau_{<p}$ refers to the truncation of complexes, defined for a complex $K^\bullet$ with differentials $d^\bullet$ by

$$ \tau_{<n}(K^\bullet)^i = \begin{cases} K^i & i<n \\ \ker(d^n) & i=n \\ 0 & i>n. \end{cases} $$

From this result, one obtains the following on degeneration of the Hodge-to-de Rham spectral sequence in characteristic p:

Corollary 13.11. *If $S = \mathrm{Spec}(k)$ with k perfect, $\dim(X) < p$, and X lifts to $W_2(k)$, then the Hodge-to-de Rham spectral sequence $E_1^{p,q} = H^q(X, \Omega^p_{X/k}) \implies H^{p+q}(X, \Omega^\bullet_{X/k})$ degenerates at the E_1 page.*

We also have a theorem on the degeneration of the conjugate spectral sequence:

Theorem 13.12 ([Kat72, Proposition 2.3.2])**.** *If $R^af_*(\Omega_{X/S})$ is locally free of finite rank and the Hodge-to-de Rham spectral sequence degenerates at the E_1-page, then the conjugate spectral sequence ${}_{\mathrm{con}}E_2^{a,b} \implies R^{a+b}\pi_*(\Omega^\bullet_{X/S})$ degenerates at the E_2-page.*

Assume that $S = \mathrm{Spec}(k)$, k a field, and that both spectral sequences degenerate. The degeneration of the Hodge spectral sequence gives

$$ \mathrm{gr}^a(H^n(X, \Omega^\bullet_{X/S})) \cong H^{n-a}(X, \Omega^a) $$

and degeneration of the conjugate spectral sequence gives

$$ \mathrm{gr}^a_{\mathrm{con}}(H^n(X, \Omega^\bullet_{X/S})) \cong F_k^* H^a(X, \Omega^{n-a}_{X/k}). $$

Hence

$$ \mathrm{gr}^a_{\mathrm{con}} H^n(X, \Omega^\bullet_{X/k}) \cong F_k^* \,\mathrm{gr}^{n-a} H^n(X, \Omega^\bullet_{X/k}). $$

Viewing F_k as an analogue of complex conjugation, we see that the conjugate filtration in this setting behaves like the conjugate of the Hodge filtration in the complex analytic setting.

One can also use the degeneration of the Hodge-to-de Rham spectral sequence in characteristic p to show degeneration in characteristic 0:

Theorem 13.13. *Suppose S has characteristic 0 and $\pi\colon X \to S$ is smooth and proper. Then the Hodge-to-de Rham spectral sequence $E_1^{p,q} = H^q(X, \Omega^p_{X/S}) \implies H^{p+q}(X, \Omega^\bullet_{X/S})$ degenerates at page 1.*

This was originally proven using complex analytic techniques.

13.4 p-curvature

Suppose $f\colon S \to T$ is smooth and $(\mathcal{E}, \nabla) \in \mathrm{MIC}(S/T)$. A connection $\nabla\colon \mathcal{E} \to \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E}$ can equivalently be thought of as a map $\nabla\colon \mathrm{Der}(S/T) \to \mathrm{End}_{f^{-1}\mathcal{O}_T}(\mathcal{E})$ satisfying

$$ \nabla(D)(fe) = D(f)e + \nabla(D)e $$

for $f \in \mathcal{O}_S$, $e \in \mathcal{E}$ and

$$ \nabla(gD) = g\nabla(D), $$

for $g \in \mathcal{O}_S$. This is done by setting $\nabla(D)$ equal to the composition

$$ \mathcal{E} \xrightarrow{\nabla} \Omega^1_{S/T} \otimes_{\mathcal{O}_S} \mathcal{E} \xrightarrow{D \otimes 1} \mathcal{E}. $$

Definition 13.14. Assume $p\mathcal{O}_T = 0$. We define the <u>p-curvature</u> associated to a connection ∇ as the map

$$\psi_\nabla \colon \mathrm{Der}(S/T) \to \mathrm{End}_{\mathcal{O}_S}(\mathcal{E})$$

given by $D \mapsto \nabla(D)^p - \nabla(D^p)$.

Note that $\psi_\nabla(D)$ is indeed $\mathcal{O}_S$-linear as

$$\psi_\nabla(D)(fe) = \nabla(D)^p(fe) - \nabla(D^p)(fe) = D^p(f)e + f\nabla(D)^p(e) - (D^p(f)e + f\nabla(D^p)(e)) = f\psi_\nabla(D)(e).$$

Remark 13.15. The map ψ_∇ itself is not $\mathcal{O}_S$-linear, but one can compute that

$$\psi_\nabla(gD) = \nabla(gD)^p - \nabla(g^pD^p) = g^p\psi_\nabla(D).$$

Thus, the p-curvature can alternatively be thought of as a map $\psi_\nabla : F_S^*Der(S/T) \to \mathrm{End}_{\mathcal{O}_S}(\mathcal{E})$ or $\psi_\nabla \colon \mathcal{E} \to F_S^*(\Omega^1_{S/T}) \otimes \mathcal{E}$.

We say that the p curvature of $(\mathcal{E}, \nabla)$ is nilpotent if there exists a filtration of $\mathcal{E}$ such that the p-curvature is 0 on the graded pieces. It is a fact that the Gauß–Manin connection ∇_{GM} on ${}_{\mathrm{con}}E_2^{a,b} = R^a\pi_*(\mathcal{H}^b(\Omega^\bullet_{X/S})) \implies R^{a+b}\pi_*\Omega^\bullet_{X/S}$ has p-curvature 0 [Kat70, Theorem 7.4]. When the conjugate spectral sequence degenerates, such as in the hypothesis of Theorem 13.12, this then says that the Gauß–Manin connection has p-curvature 0 on the graded pieces of the conjugate filtration, and is therefore nilpotent.

Now in this same setting, because

$$\psi_{\nabla_{\mathrm{GM}}}(F^a_{\mathrm{con}}R^n\pi_*\Omega^\bullet_{X/S}) \subseteq F_S^*(\Omega^1_{S/T}) \otimes F^{a+1}_{\mathrm{con}}R^n\pi_*\Omega_{X/S},$$

we get an induced map

$$\mathrm{gr}^a_{\mathrm{con}} R^n\pi_*(\Omega^\bullet_{X/S}) \xrightarrow{\psi_{\nabla_{\mathrm{GM}}}} F_S^*(\Omega^1_{S/T}) \otimes \mathrm{gr}^{a+1}_{\mathrm{con}} R^n\pi_*(\Omega_{X/S}).$$

Theorem 13.16 ([Kat72, Theorem 3.2])**.** *Assuming the hypothesis of Theorem 13.12, one has a commutative diagram*

$$\begin{tikzcd}
\mathrm{gr}^a_{\mathrm{con}} R^n\pi_*(\Omega^\bullet_{X/S}) \arrow[r, "\psi_{\nabla_{\mathrm{GM}}}"] & F_S^*(\Omega^1_{S/T}) \otimes \mathrm{gr}^{a+1}_{\mathrm{con}} R^n\pi_*(\Omega^\bullet_{X/S}) \\
F_S^*(R^a\pi_*\Omega^{n-a}_{X/S}) \arrow[u, "\sim"] \arrow[r, "-F_S^*(\rho)"] & F_S(\Omega^1_{S/T}) \otimes F_S^*(R^{a+1}\pi_*\Omega^{n-a-1}_{X/S}) \arrow[u, "\sim"']
\end{tikzcd}$$

where ρ is the Kodaira-Spencer mapping, and the vertical isomorphisms are induced by the Cartier isomorphism.[24]

[24]Note that in [Kat72, Theorem 3.2], the bottom arrow is $(-1)^{n-a+1}F_S^*(\rho)$, but there seems to be a sign error.

14 Nonabelian cohomology and applications, lecture 4. Speaker: Daniel Litt. Notes by William C. Newman.

The following is the analogue of the Hodge conjecture/Tate conjecture for algebraic de Rham cohomology:

Conjecture 14.1 (Ogus Conjecture)**.** Suppose $R \subseteq \mathbb{C}$ is finitely generated over $\mathbb{Z}$. Let $K = \mathrm{Frac}(R)$. For X/R a smooth proper scheme, we have that the image of the cycle class map $Z^r(X_K) \otimes_{\mathbb{Z}} K \to H^{2r}_{\mathrm{dR}}(X_K/K)$ is

$$\mathrm{span}_K(\xi \in H^{2r}_{\mathrm{dR}}(X/R) | \xi \bmod \mathfrak{p} \in F^r_{\mathrm{con}} H^{2r}_{\mathrm{dR}}(X_{\mathfrak{p}}/\kappa(\mathfrak{p})), \mathfrak{p} \in U^{\mathrm{cl}}, U \subseteq \mathrm{Spec}(R) \text{ a dense open}\}$$

Now, we set up a nonableian version of the Ogus conjecture, due to André. Suppose k has characteristic p and X/k is smooth. For a vector bundle $\mathcal{E}$ on X with connection ∇, the condition that ∇ is flat is equivalent to the associated map

$$\nabla : T_X \to \mathrm{End}_k(\mathcal{E})$$

respecting the Lie bracket, where the Lie bracket on the right is the commutator of composition. In characteristic p, raising a derivation to the p-th power gives another derivation. As p-curvature is defined to be

$$\psi_\nabla : F_X^* T_X \to \mathrm{End}_{\mathcal{O}_X}(\mathcal{E})$$
$$v \mapsto \nabla(v)^p - \nabla(v^p),$$

(see Remark 13.15) we can view p-curvature as a measure of the failure of ∇ to respect the p-th power operation.

Conjecture 14.2 (Non-abelian Ogus Conjecture, André)**.** Suppose $R \subseteq \mathbb{C}$ is finitely generated over $\mathbb{Z}$. For X/R a smooth proper scheme, and $(\mathcal{E}, \nabla) \in \mathrm{MIC}(X/R)$, we have that $(\mathcal{E}, \nabla)$ is of geometric origin if and only if for all $\mathfrak{p} \in U^{\mathrm{cl}} \subseteq \mathrm{Spec}(R)$ a dense open, the $\mathfrak{p}$-curvature of $(\mathcal{E}, \nabla) \bmod \mathfrak{p}$ is nilpotent.

Remark 14.3. The forward implication is true because the p-curvature of ∇_{GM} vanishes on $\mathrm{gr}_{F_{\mathrm{con}}}(\mathcal{E}, \nabla)$.

Remark 14.4. As in the case of the Non-abelian Hodge conjecture and the Non-abelian Tate conjecture, one can give an equivalent statement of the Ogus conjecture that makes it look very similar to the non-abelian version.

The following conjecture describes when $(\mathcal{E}, \nabla) \in \mathrm{MIC}(X/R)$ is not just of geometric origin, but when it trivialized on an a finite étale cover, i.e. when it is a summand of $\pi_* \mathcal{O}_Y$, for finite étale $\pi : Y \to X$.

Conjecture 14.5 (Grothendieck-Katz p-Curvature Conjecture)**.** Suppose X/R is smooth, for $R \subseteq \mathbb{C}$ is a finitely generated subring, and $(\mathcal{E}, \nabla) \in \mathrm{MIC}(X/R)$. Then $(\mathcal{E}, \nabla)_{\mathbb{C}}$ is trivialized over a finite étale cover if and only if for all $\mathfrak{p} \in U^{\mathrm{cl}}, U \subseteq R$ a dense open, the pullback of $(\mathcal{E}, \nabla)$ along $\mathrm{Spec}\, \kappa(\mathfrak{p}) \to \mathrm{Spec}(R)$, $(\mathcal{E}, \nabla)_{\mathfrak{p}}$, has zero p-curvature.

Remark 14.6. We have that $(\mathcal{E}, \nabla) \in \mathrm{MIC}(X/\mathbb{C})$ is trivialized on a finite cover exactly when the solutions to the corresponding differential equation have finitely many branches, i.e. are algebraic functions. Thus, this conjecture predicts exactly when the solutions to linear differential equations over $\mathbb{C}$ are algebraic.

Example 14.7. We consider the differential equation $(\frac{d}{dz} - \frac{a}{z})f = 0$ on $\mathbb{A}^1_{\mathbb{C}} \setminus 0$, for $a \in \mathbb{C}$.

Take $a \in \mathbb{C}$, and set $R = \mathbb{Z}[a]$, $X = \mathbb{A}^1_R \setminus \{0\}$, and consider $\mathcal{E} = \mathcal{O}_X$, $\nabla = d - \frac{a dz}{z}$. The complex solutions to $\nabla = 0$ are constant multiples of z^a, so $(\mathcal{E}, \nabla)_{\mathbb{C}}$ is trivialized over a finite étale cover if and only if $a \in \mathbb{Q}$.

Next, we compute the p-curvature. Take $\mathfrak{p} \in \mathrm{Spec}(R)^{\mathrm{cl}}$ lying over $p \in \mathrm{Spec}(\mathbb{Z})$. Using that $(\frac{d}{dz})^p = 0$ in characteristic p, we have

$$\psi_(\frac{d}{dz})(f) = \nabla(\frac{d}{dz})^p(f) - 0 = (\frac{d}{dz} - \frac{a}{z})^p(f)$$

Evaluating at $f = z^n$, we get

$$\psi_p(\frac{d}{dz})(z^n) = (n-a)(n-a-1)\dots(n-a-p+1)z^{n-p}$$

For this to be 0, we must have $n - a - k \bmod \mathfrak{p}$ to be zero, for some $k \in \{0, \ldots, p-1\}$. In other words, since a generates $\mathbb{F}_{\mathfrak{p}}/\mathbb{F}_p$, this is zero if and only if $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_p$.

If a were transcendental over $\mathbb{Q}$, the only primes $\mathfrak{p} \in \mathrm{Spec}(R)^{\mathrm{cl}}$ where $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_p$ are those of the form $(a - i, p)$ for $i \in \mathbb{Z}$, so no such open dense $U \subseteq \mathrm{Spec}(R)$ exists. For a algebraic over $\mathbb{Q}$, we can assume a is integral by multiplying it by an appropriate integer $N \in \mathbb{Z}$. Note the Chebatarov density theorem restricts the density of primes for which $\mathbb{F}_{\mathfrak{p}} = \mathbb{F}_p$ for all primes $\mathfrak{p}$ over p for $\deg(a) > 1$. Hence, if the p-curvature is zero at closed points of a dense open $U \subseteq \mathrm{Spec}(R)$, we must have $\deg(a) = 1$, i.e. $a \in \mathbb{Q}$.

Here are some known cases of this conjecture:

- (Katz) In the geometric setting, i.e. for $(\mathcal{E}, \nabla) = (R^i\pi_*\Omega^{\bullet}_{Y/X}, \nabla_{\mathrm{GM}})$, for $\pi : Y \to X$ smooth and projective (though this is still open for *summands* of $(R^i\pi_*\Omega_{Y/X}, \nabla_{\mathrm{GM}})$).
- (André, Bost, Chudnovsky-Chudnovsky) When $(\mathcal{E}, \nabla)$ has solvable monodromy (think $\nabla = (\frac{d}{dz} - A)f(z)$, where A is upper triangular.
- (Esnault–Groechenig) Rigid $\mathbb{Z}$-local systems
- (Farb-Kisin) True for certain locally symmetric varieties in the superrigid regime.

14.1 (Non-abelian) GM Connections

We discuss Katz's aforementioned proof of the geometric case, as it will help motivate what follow.

Theorem 14.8 (Katz)**.** *Suppose $R \subseteq \mathbb{C}$ is finitely generated over $\mathbb{Z}$, X/R is smooth, and $\pi : Y \to X$ is smooth and proper. Then the p-curvature conjecture holds for $(R^i\pi_*\Omega_{Y/X}, \nabla_{\mathrm{GM}})$.*

Proof Idea. Write $(R^i\pi_*\Omega_{Y/X}, \nabla_{\mathrm{GM}}) = (\mathcal{E}, \nabla)$. This comes equipped with the Hodge filtration, F. We consider the induced map on the associated graded

$$\mathrm{gr}^i \nabla : \mathrm{gr}^i \mathcal{E} \to \mathrm{gr}^i \mathcal{E} \otimes_{\mathcal{O}_X} \Omega^1_X.$$

We claim that it is enough to show that this map is zero. Using either non-abelian Hodge theory or using polarization with the fact that ∇_{GM} preserves the Hodge filtration under these hypotheses, one sees that $(\mathcal{E}, \nabla)$ has unitary monodromy. So the corresponding representation lands in a compact group. We also know that the monodromy is discrete: it factors through $\mathrm{GL}_n(\mathbb{Z})$ because it has the structure of a $\mathbb{Z}$-local system. Thus, the image of the monodromy representation is finite.

To see that $\mathrm{gr}_i \nabla = 0$, recall that, mod $\mathfrak{p}$ we have the diagram

$$\begin{tikzcd}
\mathrm{gr}^a_{\mathrm{con}}(\mathcal{E}_{\mathfrak{p}}) \arrow[r, "\psi_{\nabla_{\mathrm{GM}}}"] & F_S^*(\Omega^1_{X_{\mathfrak{p}}/\kappa(\mathfrak{p})}) \otimes \mathrm{gr}^{a+1}_{\mathrm{con}}(E_{\mathfrak{p}}) \\
F^*_{\mathrm{abs}}(\mathcal{E}_{\mathfrak{p}}) \arrow[u, "\sim"] \arrow[r, "F^*_{\mathrm{abs}}(\rho)"] & F_{\mathrm{abs}}(\Omega^1_{X_{\mathfrak{p}}/\kappa(\mathfrak{p})}) \otimes F^*_{\mathrm{abs}}(\mathcal{E}_{\mathfrak{p}}) \arrow[u, "\sim"']
\end{tikzcd}$$

By hypothesis, the top map is zero, and so the bottom map must be zero as well. This bottom map is essentially the pull back of $\mathrm{gr}_i \nabla_{\mathrm{GM}}$ mod $\mathfrak{p}$. Since we have that $\mathrm{gr}_i \nabla_{\mathrm{GM}}$ is zero mod $\mathfrak{p}$ on an open dense subset, it must be identically zero. $\square$

We now define relative versions of moduli spaces of local systens. Given $\pi : X \to S$ a smooth and proper map of complex manifolds, $s \in S$, and a universal cover $\widetilde{S} \to S$, define

$$M_B(X/S) := (M_B(X_s) \times \widetilde{S})/\pi_1(S, s).$$

Note this is independent of $s \in S$, because all fibers of π are diffeomorphic.

Definition 14.9. Let $Y \to W$ be a smooth map (either in the algebraic or analytic setting). A horizontal foliation on Y/T is a subbundle $\mathcal{F} \subseteq T_Y$ closed under the Lie bracket, such that $\mathcal{F} \to \pi^* T_W$ is an isomorphism. A leaf of $\mathcal{F}$ is a map $f : Z \to Y$ such that the induced map $df : T_Z \to f^* T_Y$ factors through $f^*\mathcal{F}$, with $T_Z \to f^*\mathcal{F}$ an isomorphism.

Remark 14.10. Horizontal foliations should be thought of as non-linear differential equations, and leaves should be thought of as solutions. If Y/W is a vector bundle, a foliation satisfying some mild hypothesis is the same as the data of a connection.

If $Y = W \times Y'$, we can view $\pi^* T_W$ as a sub-bundle of T_Y, giving a horizontal foliation. Thus, we have a foliation on $M_B(X_s) \times \widetilde{S} \to \widetilde{S}$. This foliation descends when quotienting by $\pi(S, s)$, giving a horizontal foliation on (the smooth locus of) $M_B(X/S) \to S$.

We also have a relative $M_{\mathrm{dR}}(X/S) \to S$, which works in either the algebraic or analytic setting. Its fiber over $s \in S$ parameterizes $(\mathcal{E}, \nabla) \in \mathrm{MIC}(X_s)$. By the Riemann-Hilbert correspondence, in the complex analytic setting, this space should be isomorphic to $M_B(X/S) \to S$, giving rise to a horizontal foliation (on the smooth locus). It is a fact that this horizontal foliation always exists on $M_{\mathrm{dR}}(X/S) \to S$ in the algebraic setting as well. It is called the Gauss-Manin foliation, denote by $\mathcal{F}_{\mathrm{GM}}$.

Remark 14.11. The more correct way of saying the above is that, viewing $M_{\mathrm{dR}}(X/S) \to S$ as a stack, it is a crystal.

The p-curvature conjecture predicts when the solutions to a linear differential equation are algebraic. We can ask the analogous question in this setting:

Question 14.12. When are the leaves of a foliation algebraic?

Example 14.13. Consider $\mathbb{P}^1_{\mathrm{Conf}^n(\mathbb{P}^1)} \setminus D \to \mathrm{Conf}^n(\mathbb{P}^1)$. Inside of the associated M_{dR}, we have the locus of Fuchsian ODES, which are those of the form $(\mathcal{O}^n, \nabla)$ with $\nabla = d + \sum_i \frac{B_i}{z - x_i} d$. On this locus, $\mathcal{F}_{\mathrm{GM}}$ is given by the Schlesinger equations (see Answer 2.6).

Conjecture 14.14 (Ekedahl-Shepherd-Barron-Taylor, [ENSBT99])**.** All leaves of a horizontal foliation are algebraic if and only if the foliation is closed under taking p-th powers mod p for almost all p.

Remark 14.15. This conjecture has been studied by Menzies and Papaioannou.

Instead of asking for all of the leaves of a horizontal foliation to be algebraic, one could instead ask when the leaf through a particular point of a foliation is algebraic. The following conjecture addresses this question for the foliation on $M_{\mathrm{dR}}(X/S)$.

Conjecture 14.16. Let $R \subseteq \mathbb{C}$ be finitely generated over $\mathbb{Z}$, S/R smooth, $s \in S(R)$, and $\pi : X \to S$ smooth and proper. Take $(X_s, \mathcal{E}, \nabla) \in M_{\mathrm{dR}}(X/S)(R)$. The leaf through $(X_s, \mathcal{E}, \nabla)_{\mathbb{C}}$ is algebraic if and only if the formal leaf through $(X_s, \mathcal{E}, \nabla)$ is p-integral to order $\omega(p^2)$ (i.e. if the formal leaf has the form $(\sum_{i=0}^{\infty} a_{1,i} t^i, \ldots, \sum_{i=0}^{\infty} a_{n,i} t^i)$ and the function $f(p)$ is defined to be the smallest i such that $a_{j,i}$ has a p in the denominator for some j, then $f(p)$ is eventually greater than εp^2 for all $\varepsilon > 0$).

Theorem 14.17 (Lam-L)**.** *This is true if $(\mathcal{E}, \nabla) = (R^i \pi_* \Omega_{Y/X_s}, \nabla)$ for some $Y \to X$ smooth and proper.*

Proof Idea. We have $(\mathcal{E}, \nabla) \in \mathrm{MIC}(X_s)$, which we know has a $\mathbb{Z}$-variation of Hodge structure. We want to show that $(\mathcal{E}, \nabla)$ extends to a $\mathbb{Z}$-variation of Hodge structure on a general fiber of π. To do this, we show that the filtrations F, F_{con} extend to a formal neighborhood. This is done iteratively, using the characteristic p version of non-abelian Hodge theory due to Ogus-Vologodsky. $\square$

15 More on variations of Hodge structures, period maps. Speaker: Andy Jiang. Notes by Kyle Binder

15.1 Polarized Variation of Hodge Structure

Definition 15.1. A $\underline{\text{ℂ-Variation of Hodge Structure (ℂ-VHS) of weight } n}$ on a complex manifold S is a $\mathbb{C}$-local system H whose associated vector bundle $E := H \otimes_{\mathbb{C}} \mathcal{C}^\infty$ has a Hodge decomposition of smooth bundles

$$E = \bigoplus_{p+q=n} E^{p,q}$$

satisfying the following: Writing the associated integrable connection

$$\nabla \colon E \longrightarrow E \otimes \mathcal{A}^1$$

into its holomorphic and anti-holomorphic parts

$$\nabla = \partial + \overline{\partial},$$

1. The filtration
$$F^p E := \bigoplus_{a \geq p} E^{a,b}$$
must be holomorphic (i.e., it is $\overline{\partial}$-stable).
2. The filtration
$$\overline{F}^q E := \bigoplus_{b \geq q} E^{a,b}$$
must be anti-holomorphic (i.e., it is ∂-stable).
3. The following version of Griffiths Transversality holds:
$$\partial \colon F^p E \longrightarrow F^{p-1} E \otimes \mathcal{A}^{1,0}$$
$$\overline{\partial} \colon \overline{F}^q E \longrightarrow \overline{F}^{q-1} E \otimes \mathcal{A}^{0,1}.$$

Remark 15.2. If H is an irreducible local system which underlies a $\mathbb{C}$-VHS, then the integers

$$\{a : E^{a,b} \neq 0\}$$

must be a set of consecutive integers.

Proof. Suppose for contradiction we have a $\mathbb{C}$-VHS coming from H with $E^{a,b} = 0$ even though $E^{a-j,b+j} \neq 0$ and $E^{a+k,b-k} \neq 0$ for some $j, k \in \mathbb{N}$. This means we have the decomposition

$$\cdots \oplus E^{a-1,b+1} \oplus 0 \oplus E^{a+1,b-1} \oplus \cdots$$

Then the subbundle

$$E_1 := \bigoplus_{p \geq a+1} E^{p,q}$$

is $\overline{\partial}$-stable and moreover is ∂-stable because of Griffiths Transversality and the vanishing of $E^{a,b}$. Therefore $(E_1, \nabla_{|E_1})$ is a submodule with integrable connection of (E, ∇).

The same proof shows for

$$E_2 := \bigoplus_{p \leq a-1} E^{p,q}$$

that $\left(E_2, \nabla_{|E_2}\right)$ is a submodule with integrable connection of (E, ∇).

But this shows (E, ∇) decomposes non-trivially as the direct sum of $\left(E_1, \nabla_{|E_1}\right)$ and $\left(E_2, \nabla_{|E_2}\right)$ which contradicts the irreducibility of H. $\square$

Definition 15.3. A polarization of a $\mathbb{C}$-VHS is a map of $\mathbb{C}$-VHS

$$\psi \colon H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}(-n)$$

such that

$$\sum i^{p-q} \psi \otimes \mathcal{C}^{\infty}_{|E^{p,q}}$$

is a hermitian metric.

Here we recall $\mathbb{C}(-n)$ is a weight $2n$ VHS concentrated in the (n,n)-th component where it is $(2\pi i)^n\mathbb{C}$. The fact that ψ is a map of $\mathbb{C}$-VHS implies the smooth bundles $E^{p,q}$ are pairwise orthogonal with respect to this metric:

$$E^{p,q} \perp_{\psi \otimes \mathcal{C}^{\infty}} E^{p',q'} \quad \text{for } p \neq p'.$$

Note that on each fibre ψ induces a non-degenerate hermitian form; this is not necessarily positive definite.

Real variations of Hodge structures ($\mathbb{R}$-VHS) are defined similarly. They are the $\mathbb{C}$-VHS which are invariant under conjugation. Then a polarization of an $\mathbb{R}$-VHS is a polarization of $\mathbb{C}$-VHS which respects this conjugation.

Remark 15.4. Polarizations come up in the geometric case by way of the Hard Lefschetz Theorem: multiply the two things together and multiply by enough powers of the Kähler form to integrate. Specifically the Hard Lefschetz Theorem allows one to define the primitive cohomology; then the Hodge–Riemann relations on the primitive cohomology shows how this integration yields a hermitian metric.

15.2 Theorem of the Fixed Part

For the following theorem, we restrict to the case of quasi-projective varieties even though the theorem holds in the slightly more general context of a compact complex analytic space with some closed complex analytic subspaces removed.

Theorem 15.5 ([CMSP17, Theorem 13.1.10])**.** *Let S be quasi-projective and $(H, E^{p,q}, \psi)$ a polarized $\mathbb{C}$-VHS on S. Suppose $s \in H(S)$ is a global flat section of E. Then writing $s = \sum s^{p,q}$, where $s^{p,q} = s_{|E^{p,q}}$, each $s^{p,q}$ is flat.*

We omit the proof, but the idea is to verify it using induction on p by making an analytic argument using plurisubharmonic functions.

Corollary 15.6. *On a quasi-projective S, any irreducible $\mathbb{C}$-local system H may be enhanced into a polarized $\mathbb{C}$-VHS in at most one way (up to tensoring with a $\mathbb{C}$-VHS on a trivial local system).*

Proof. If H underlies two different $\mathbb{C}$-VHS, we will construct the internal hom between them and produce a flat section of this which is in a pure Hodge component.

Suppose $H_1 := (H, E^{p,q}, \psi)$ and $H_2 := (H, E'^{p,q}, \psi')$ are two different polarized $\mathbb{C}$-VHS. Consider the internal hom of polarized $\mathbb{C}$-VHS

$$\widetilde{\mathrm{Hom}}\,(H_1, H_2) := \left(\widetilde{\mathrm{Hom}_{\mathbb{C}}}\,(H, H)\,, \widetilde{\mathrm{Hom}}\left(\bigoplus E^{p,q}, \bigoplus E'^{p,q}\right)^{p'',q''}, \psi^{\vee} \otimes \psi'\right).$$

This internal hom has an obvious non-zero flat section, id, given by the identity map on the underlying local system H. Using the Theorem of the Fixed Part, we have

$$\mathrm{id} = \sum (\mathrm{id})^{p,q}$$

with $(\mathrm{id})^{p,q} \in \mathrm{Hom}\left(\bigoplus E, \bigoplus E'\right)^{p,q}$ flat. Flatness and irreducibility ensures there is only one non-zero component $\mathrm{id}^{p,q}$. This yields a polarized $\mathbb{C}$-VHS on the trivial local system $\mathbb{C}$ and a map

$$\mathbb{C} \longrightarrow \widetilde{\mathrm{Hom}}\,(H_1, H_2)$$

of $\mathbb{C}$-VHS, because id is concentrated in a single Hodge component. This gives a non-zero map

$$\mathbb{C} \otimes H_1 \longrightarrow H_2$$

which is an isomorphism due to the irreducibility of local systems. $\square$

15.3 Period Domains

A period domain is roughly the moduli space of (polarized) Hodge structures on a fixed $\mathbb{C}$-vector space with fixed weight and Hodge numbers (and polarization). For details of the following, see [CMSP17, Section 4.4].

We restrict our discussion to the case of period domains for polarized $\mathbb{C}$-VHS. For a vector space H, weight n, polarization

$$\psi \colon H \otimes \overline{H} \longrightarrow \mathbb{C},$$

and Hodge numbers, there is a complex analytic space $\mathcal{D}$ such that a map

$$S \longrightarrow \mathcal{D}$$

from S a complex manifold is the data of a polarized $\mathbb{C}$-VHS (minus the condition of Griffiths Transversality) with underlying trivial local system H with prescribed weight n, polarization ψ, and Hodge numbers.

For a rough idea of why such a space exists, once we fix ψ, the $E^{p,q}$ of a VHS must be orthogonal to one another, and this says the F^pE and $\overline{F}^{n-p}E$ are orthogonal complements to one another. Therefore the data of the $\overline{F}^{n-p}E$ is determined by that of F^pE. This also determines the data of the $E^{p,q}$. So the data of the $\mathbb{C}$-VHS is completely determined by the data of the flag $\{F^pE\}_p$ with the correct rank given by the Hodge numbers. For an open subset of flags, this will give rise to a polarized $\mathbb{C}$-VHS. The only condition such a flag may break is the condition for the Hermitian metric, but its failure is a closed condition. Therefore this period domain $\mathcal{D}$ will be an open subset of a flag variety $\mathcal{D}^\vee$. This gives $\mathcal{D}$ a complex structure.

For the case of $\mathbb{R}$-VHS, we can also define a period domain. After complexification, we are in the case of a $\mathbb{C}$-VHS invariant under conjugation. So for the associated complex period domain $\mathcal{D}$ there is a real analytic automorphism whose fixed points give the real period domain. Hence for the real period domain we have to impose a closed condition (for being fixed under the automorphism) and then an open condition (for satisfying the bilinear metric), while for the complex period domain we only have to impose the open condition on the flag variety.

Note that the flag variety $\mathcal{D}^\vee$ is the quotient of $\mathrm{GL}_n(\mathbb{C})$ by a parabolic subgroup. Moreover, the unitary group corresponding to the Hermitian form ψ acts transitively on the space of flags in $\mathcal{D}$ that gives rise to a polarized Hodge structure. Therefore the period domain is a unitary group modulo the stabilizer of this action.

Remark 15.7. This definition of period domain is only for trivial local systems. In the general case, even the moduli space of local systems involves some stackiness. If we have a non-trivial local system, we can pass to a cover where it is trivialized. Then this gives a map to the period domain. To remove ambiguity, we can then get a map to the period domain modulo the image of $\pi_1(S)$.

15.4 Griffiths Transversality

Suppose $x \in \mathcal{D} \subseteq \mathcal{D}^\vee = \mathrm{GL}_n/P$ for some parabolic subgroup P. At this x, the tangent space of GL_n, which is the Lie algebra $\mathfrak{gl}_n$, maps to the tangent space $T_x\mathcal{D}^\vee$, and the kernel of this map is the Lie algebra $\mathfrak{P}_x$ of the parabolic subgroup P. Then because $\mathcal{D}$ is an open in the flag variety $\mathcal{D}^\vee$,

$$T_x\mathcal{D} = \mathfrak{gl}(H)/\mathfrak{P}_x = \mathrm{End}(H)/F^0\,\mathrm{End}(H),$$

where $F^0\,\mathrm{End}(H)$ is coming from the Hodge structure given by $x \in \mathcal{D}$.

To satisfy Griffiths Transversality, a map

$$S \longrightarrow \mathcal{D}$$

must have derivative

$$TS \longrightarrow F^{-1}\operatorname{End}(H)/F^0\operatorname{End}(H).$$

From the data of the holomorphic part ∂ of the connection ∇ on E, we get a Higgs field by restricting to the associated gradeds of the Hodge structure, and the Higgs field is equivalent to the data of the map of tangent bundles if you have a variation of Hodge structures.

Recall $\nabla = \partial + \overline{\partial}$. Then

$$\partial\colon E^{p,q} \longrightarrow \left(E^{p-1,q+1} \oplus E^{p,q}\right) \otimes \mathcal{A}^{1,0}.$$

This implies the projection

$$\sigma\colon E^{p,q} \longrightarrow E^{p-1,q+1} \otimes \mathcal{A}^{1,0}$$

is a Higgs field on $(\operatorname{gr} F)(E)$.

For a map $S \longrightarrow \mathcal{D}$, this Higgs field encodes the data of the derivative of the period map. For a precise statement, see [CMSP17, Lemma 5.3.2].

15.5 Finitude of Local Systems

Theorem 15.8 ([Del87, Théorème 0.5])**.** *For a fixed smooth variety S and integer N, the number of isomorphism classes of dimension N $\mathbb{Q}$-local systems which admit a polarized $\mathbb{Z}$-VHS is finite.*

Proof. For fixed Hodge numbers, polarization, and $\sigma \in \pi_1(S)$, there is a bound on coefficients of the matrix of the monodromy action of σ which is uniform across the variation of Hodge structures ([Del87, Corollaire 1.8]). To see this, take a universal cover of S endowed with a metric so the map of the period domain decreases distance. This yields the bound on coefficients.

Then one needs to show that the period domain (i.e., the choice of Hodge numbers) doesn't matter for this uniform bound ([Del87, Corollaire 1.9])[25].

These bounds on entries of the matrices yield a bound on the trace ([Del87, Corollaire 1.10]).

Then we show we can choose finitely many $\sigma \in \pi_1(S)$ that determine a bound on the traces for all elements of $\pi_1(S)$ ([Del87, Théorème 2.1]). As the traces are integral, this gives a finite number of traces which can occur. For each choice, as these local systems are semi-simple, there is a unique local system which has these traces.

□

Remark 15.9. The bounds this theorem gives depends on the variety S, but for fixed S, the proof gives an effective way to get the bound.

[25]For given N, restrict to irreducibles, and use Remark 15.2 to show the non-zero Hodge numbers must be a consecutive interval of integers. Then twist the weight so this interval begins at 0, whence it is obvious there are only finitely many choices of Hodge numbers, so the bound can be made uniform across period domains.

16 The p-adic Riemann-Hilbert correspondence, lecture 4. Speaker: Alexander Petrov. Notes by Min Shi.

This talk will cover some relative p-adic Hodge theory (Faltings, Brinon, Scholze, Liu-Zhu, Diao). In the following, K will denote a complete discrete valued field of characteristic 0 with a perfect residue field with characteristic p.

16.1 Description of the relative de Rham functor

Let X be a smooth variety over K. We will sketch some ingradients in the construction of the relative de Rham functor.

$$D_{\mathrm{dR}} : \{\mathbb{Q}_p - \text{local system on } X\} \to \{\text{filtered vector bundle with flat connection on } X\}.$$

Assume X is proper, with rigid analytification $X^{\mathrm{an}} = \cup \mathrm{Sp}(R_i)$. Then there are equivalences of categories:

$$\{\text{vector bundles on } X\} \cong \{\text{vector bundles on } X^{\mathrm{an}}\} \cong \{\text{vector bundles on } \mathrm{Sp}(R_i) \text{ and some gluing data}\},$$

where the first isomorphism is rigid GAGA (using that X is proper).

Remark 16.1. The vector bundles on $\mathrm{Sp}(R_i)$ are just projective modules over R_i.

Example 16.2. $X = \mathbb{P}^1_K = \mathrm{Spec}(K[X]) \cup \mathrm{Spec}(K[X^{-1}])$, and the analytification $X^{\mathrm{an}} = \mathrm{Sp}(K\langle x\rangle) \cup \mathrm{Sp}(K\langle x^{-1}\rangle)$.

Let $\mathbb{L}$ be a $\mathbb{Q}_p$ local system. Then $\mathbb{L}|_{\mathrm{Sp}(R)}$ is a representation of $\pi_1^{\acute{e}t}(R)$. Let $\overline{R}$ be the colimit of R_i where R_i is a connected finite étale cover of R, and let $\hat{\overline{R}}$ be the completion of $\overline{R}$. The relative de Rham functor is defined as

$$D_{\mathrm{dR}}(\mathbb{L})_R := (\mathbb{L} \bigotimes \mathcal{O}\mathbb{B}_{\mathrm{dR}}(\hat{\overline{R}}))^{\pi_1^{\acute{e}t}(R)}$$

for a suitable "period sheaf" $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$. $\mathcal{O}\mathbb{B}_{\mathrm{dR}}(\hat{\overline{R}})$ carries a $\pi_1^{\acute{e}t}(R)$ action.

To understand D_{dR}, we will roughly sketch some properties of $\mathcal{O}\mathbb{B}_{\mathrm{dR}}(\hat{\overline{R}})$. The first property is that $\mathcal{O}\mathbb{B}_{\mathrm{dR}}(\hat{\overline{R}})$ has a filtration with $\mathrm{gr}^i \cong \mathcal{O}\mathbb{C}(i)$ for another period sheaf $\mathcal{O}\mathbb{C}$, and this filtration is compatible with the action of $\pi_1^{\acute{e}t}(R)$.

Remark 16.3. If $R = K$, so $\hat{\overline{R}} = C$, then $\mathcal{O}\mathbb{C} = C$.

First guess: $\mathcal{O}\mathbb{C}(\hat{\overline{R}}) = \hat{\overline{R}}$, but it is not suitable, since it will not capture the following periods:

Example 16.4. Consider Kummer local systems over $R = K\langle x_1^{\pm 1}, ..., x_d^{\pm 1}\rangle$, i.e. extensions:

$$0 \to \mathbb{Q}_p(1) \to \mathbb{L} \to \mathbb{Q}_p \to 0 \tag{4}$$

They are classified by $\mathrm{Ext}^1_{\pi_1^{\acute{e}t}(R)}(\mathbb{Q}_p, \mathbb{Q}_p(1)) = H^1_{\acute{e}t}(R, \mathbb{Q}_p(1))$. We recall this identification: suppose $\langle e_1, e_0\rangle$ is a basis for $\mathbb{L}$, where e_1 is the image of a base element in $\mathbb{Q}_p(1)$, and the image of e_0 is a base element for $\mathbb{Q}_p$. Then $g.e_1 = \chi(g)e_1$, with χ the cyclotomic character and $g.e_0 = ae_1 + e_0$ for some $a \in \mathbb{Q}_p$. Then $g \to a$ is a 1-cocycle in $H^1_{\acute{e}t}(\pi_1(R), \mathbb{Q}_p(1)) \cong H^1_{\acute{e}t}(R, \mathbb{Q}_p(1))$. Denote the boundary map in the long exact sequence induced from the Kummer sequence by $\kappa : R^\times \to H^1_{\acute{e}t}(R, \mathbb{Q}_p(1))$. Denote by $\mathbb{L}_i$ one representative of the class $[\mathbb{L}_i]$ such that $[\mathbb{L}_i] = \kappa(x_i)$.

Remark 16.5. These are of geometric origin, but they are not seen by our "first guess" for $\mathcal{O}$.

Correct answer: $\mathcal{O}\mathbb{C}(\hat{\overline{R}}) = \hat{\overline{R}}[v_1, ..., v_d]$, where the action of $\pi_1^{\acute{e}t}(R)$ on v_i is defined such that $(e_1 \bigotimes v_i + e_0 \bigotimes 1) \in (\mathbb{L}_i \bigotimes \mathcal{O}\mathbb{C}(\hat{\overline{R}}))^{\pi_1^{\acute{e}t}(R)}$, where $\langle e_0, e_1 \rangle$ is a chosen basis for $\mathbb{L}_i$ as in the discussion below (4).

Remark 16.6. This is the construction of $\mathcal{O}\mathbb{C}$ in local coordinates. It glues and can be made functorial and independent of coordinates. Then $\mathrm{gr}\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ sees the periods of all local systems of geometric origin. For more details on $\mathcal{O}\mathbb{C}$, see [LZ17, Remark 2.1] and [Sch13, §6].

We do not construct $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$.

16.2 Automatic de Rhamness of p-adic local systems

We return to the disccussion of the first lecture.

Conjecture 16.7 (The relative Fontaine-Mazur conjecture)**.** Let S be a smooth variety over $\mathbb{C}$. For a semisimple $\overline{\mathbb{Q}_p}$-local system $\mathbb{L}$ on $S(\mathbb{C})$, $\mathbb{L}$ is arithmetic, i.e., there is some $\widetilde{\mathbb{L}}$ on S_0/F, where F is some finitely generated subfield of $\mathbb{C}$, extending $\mathbb{L}$, if and only if $\mathbb{L}$ is of geometric origin.

A basic question here is for arithmetic $\mathbb{L}$ to try to recover an integral variation of Hodge structure (a $\mathbb{Z}$-VHS). This question is wide open.

Remark 16.8. If $\mathbb{L}$ is irreducible, then for all other $\widetilde{\mathbb{L}}'$ on S_0 such that $\widetilde{\mathbb{L}}|_{S,\overline{F}} = \widetilde{\mathbb{L}}'|_{S,\overline{F}}$, $\widetilde{\mathbb{L}}' = \widetilde{\mathbb{L}} \bigotimes \chi$ for some $\chi : G_F \to \overline{\mathbb{Q}_p}^{\times}$, since we can look at the G_F action on $\mathrm{Hom}_{\pi_1(S_{\overline{F}})}(\widetilde{\mathbb{L}}, \widetilde{\mathbb{L}}') \cong \overline{\mathbb{Q}_p}$. Thus the topological monodromy group mostly sees all Galois representation obtained by specializing a descent to F-points of S_0.

Theorem 16.9. [*Pet23, Theorem 1*] *Let X/K be a smooth variety and let $\mathbb{L}$ be a $\overline{\mathbb{Q}_p}$ local system on X such that $\mathbb{L}|_{X_{\overline{K}}}$ is irreducible. Then there exists a character of the Galois group $\chi : G_K \to \overline{\mathbb{Q}_p}^{\times}$ such that $\mathbb{L} \bigotimes \chi$ is de Rham.*

Let $\mathbb{L}$ be a de Rham local system on X. Then $D_{\mathrm{dR}}(\mathbb{L})$ is a vector bundle with filtration and flat connection. To approach finding a $\mathbb{Z}$-VHS in the setting of conjecture 16.7, we face a question of compatibility between $\mathbb{C}$-Riemann-Hilbert correspondence and p-adic Riemann-Hilbert correspondence.

Question 16.10 (Diao-Lan-Liu-Zhu)**.** Choose $K \hookrightarrow \mathbb{C}$. Is $(D_{\mathrm{dR}}(\mathbb{L})^{\mathrm{an}}_{\mathbb{C}})^{\nabla=0} \cong \mathbb{L}|_{X_{\mathbb{C}}} \bigotimes_{\mathbb{Q}_p} \mathbb{C}$?

This question has a positive answer when $\mathbb{L}$ is of the form $R^i f_* \mathbb{Q}_p$ for some i, where $f : X \to S$ is a smooth proper family of varieties. However, for general local systems $\mathbb{L}$, this question is open, and it is unclear whether this should hold for arbitrary de Rham $\mathbb{L}$.

16.3 Some ideas in the proof

Let $\mathbb{L}$ be a $\mathbb{Q}_p$-local system on a smooth variety X over K. In lecture 3 we discussed the functors:

$$H : \{\mathbb{Q}_p\text{-local systems}\} \to \left\{ \begin{array}{l} \text{Higgs bundles } M \text{ on } X_{K_\infty} \text{ with} \\ \text{Higgs field } \theta : M \to M \bigotimes \Omega^1_{X_{K_\infty}} \\ \text{and an operator } \phi \text{ on } M \text{ such that} \\ \theta \circ \phi = (\phi - 1) \circ \theta \end{array} \right\}$$

$$RH^+ : \{\mathbb{Q}_p\text{-local systems}\} \to \left\{ \begin{array}{l} \text{vector bundles } \mathcal{E} \text{ on } X_{K_\infty}[[t]] \\ \text{with } \nabla^{\mathrm{geom}} : \mathcal{E} \to \mathcal{E} \bigotimes \frac{1}{t}\Omega^1_{X_{K_\infty}}[[t]] \\ \text{and an } \mathcal{O}_{X_{K_\infty}}\text{-linear operator } \phi : \mathcal{E} \to \mathcal{E} \text{ commuting with } \nabla^{\mathrm{geom}} \\ \text{and satisfying: } \phi(am) = a\phi(m) + t\partial_t(a)m,\ a \in \mathcal{O}_{X_{K_\infty}}[[t]],\ m \in \mathcal{E} \end{array} \right\}$$

These two functors are related to each other by $RH^+(\mathbb{L})/t \cong H(\mathbb{L})$.

Now, $\mathbb{L}$ is de Rham if and only if $\mathrm{rk}_{\mathcal{O}_{X_{K_\infty}}} RH^+\mathbb{L}[1/t]^{\phi=0} = \mathrm{rk}_{\mathbb{Q}_p}\mathbb{L}$. If this holds, then the following condition holds:

$$\text{the action of } \phi \text{ on } H(\mathbb{L}) \text{ is semi-simple and has integer eignenvalues.} \tag{5}$$

This condition is slightly weaker than than the condition that $\mathbb{L}$ is de Rham, because even in classical p-adic Hodge theory, condition 5 is the Hodge-Tate condition, which is weaker than the de Rham condition.

Theorem 16.11 (Shimizu)**.** $\det(\mathrm{Id}_{H(\mathbb{L})} - \phi \cdot T) \in H^0(X_{K_\infty}, \mathcal{O}_{X_{K_\infty}})[T]$ *has constant coefficients, i.e., it is an element of* $K[T]$.

This implies that $H(\mathbb{L})$ admits a generalized eigenspace decomposition: $H(\mathbb{L}) = \bigoplus_{\lambda \in \overline{K}} H(\mathbb{L})_\lambda$. By the condition on θ and ϕ, θ maps $H(\mathbb{L})_\lambda$ to $H(\mathbb{L})_{\lambda+1} \bigotimes \Omega^1_{X_{K_\infty}}$.

For the rest of this notes, we explain how a local system can be twisted so that the condition (2) holds. For simplicity, assume that X/K is a proper variety. Let $\mathbb{L}_1, \mathbb{L}_2$ be two $\mathbb{Q}_p$ local systems. Then

$$\mathrm{Hom}_{X_{\overline{K}}}(\mathbb{L}_1, \mathbb{L}_2) \bigotimes_{\mathbb{Q}_p} C \cong \mathrm{Hom}_{X_{K_\infty}, \theta}(H(\mathbb{L}_1), H(\mathbb{L}_2)) \bigotimes_{K_\infty} C$$

naturally. Assume $\mathbb{L}$ is a $\mathbb{Q}_p$ local system such that $\mathbb{L} \bigotimes_{\mathbb{Q}_p} \overline{\mathbb{Q}_p}|_{X_{\overline{K}}}$ is irreducible. Then by Schur's lemma, $\mathrm{End}_{X_{\overline{K}}}\mathbb{L} = \mathbb{Q}_p \cdot \mathrm{Id}$. By the natural isomorphism above, $\mathrm{End}_{X_{K_\infty}, \theta} H(\mathbb{L}) = K_\infty \cdot \mathrm{Id}$. Therefore the eigenvalues of ϕ belong to a single coset of $\overline{K}/\mathbb{Z}$, because otherwise this will contradict how θ interacts with the generalized eigenspace decomposition. Denote this single coset by $a + \mathbb{Z}$, for some $a \in \overline{K}$. The next step is to find a character $\chi : G_K \to \overline{\mathbb{Q}_p}^\times$ such that the associated Higgs bundle $H(\chi)$ bears a ϕ-action such that ϕ acts by $-a$. χ can be a suitable power of χ_{cyc}. A little more linear algebra shows ϕ is also semisimple and has integer eigenvalues.

Remark 16.12. Compare the above discussion with that in Andy's talk. A refined version of this argument also gives the de Rham result.

References

[Bas80] Hyman Bass, Groups of integral representation type, Pacific J. Math. **86** (1980), no. 1, 15–51. MR 586867 39

[BC] Olivier Brinon and Brian Conrad, CMI summer school notes on p-adic Hodge theory, `https://math.stanford.edu/~conrad/papers/notes.pdf`. 24

[Bha] Bhargav Bhatt, The Hodge-Tate decomposition via perfectoid space, Arizona Winter School notes, `https://swc-math.github.io/aws/2017/2017BhattNotes.pdf`. 24

[BMS67] H. Bass, J. Milnor, and J.-P. Serre, Solution of the congruence subgroup problem for SL_n $(n \geq 3)$ and Sp_{2n} $(n \geq 2)$, Inst. Hautes Études Sci. Publ. Math. (1967), no. 33, 59–137. MR 244257 34

[Bos14] Siegfried Bosch, Lectures on formal and rigid geometry, Lecture Notes in Mathematics, vol. 2105, Springer, Cham, 2014. MR 3309387 42, 43, 44

[CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters, Period mappings and period domains, second ed., Cambridge Studies in Advanced Mathematics, vol. 168, Cambridge University Press, Cambridge, 2017. MR 3727160 59, 60, 61

[Con] Brian Conrad, Several approaches to non-archimedean geometry, `https://math.stanford.edu/~conrad/papers/aws.pdf`. 42, 44

[CS08] Kevin Corlette and Carlos Simpson, On the classification of rank-two representations of quasiprojective fundamental groups, Compos. Math. **144** (2008), no. 5, 1271–1331. MR 2457528 20, 21

[Del71] Pierre Deligne, Théorie de Hodge. II, Inst. Hautes Études Sci. Publ. Math. (1971), no. 40, 5–57. MR 498551 31

[Del87] P. Deligne, Un théorème de finitude pour la monodromie, Discrete groups in geometry and analysis (New Haven, Conn., 1984), Progr. Math., vol. 67, Birkhäuser Boston, Boston, MA, 1987, pp. 1–19. MR 900821 35, 61

[DI87] Pierre Deligne and Luc Illusie, Rel'evements modulo p^2 et d´ecomposition du complexe de de rham, Inventiones Mathematicae **89** (1987), 247–270. 53

[dJEG22] Johan de Jong, Hélène Esnault, and Michael Groechenig, Rigid non-cohomologically rigid local systems, 2022. 36

[Dri12] Vladimir Drinfeld, On a conjecture of Deligne, Mosc. Math. J. **12** (2012), no. 3, 515–542, 668. MR 3024821 37

[EG18] Hélène Esnault and Michael Groechenig, Cohomologically rigid local systems and integrality, Selecta Math. (N.S.) **24** (2018), no. 5, 4279–4292. MR 3874695 32, 33, 36, 38, 40

[ENSBT99] T. Ekedahl, N.I. N.I. Sheppard-Barron, and R. Taylor, A conjecture on the existence of compact leaves of algebraic foliations, 1999. 57

[Fal88] Gerd Faltings, p-adic Hodge theory, J. Amer. Math. Soc. **1** (1988), no. 1, 255–299. MR 924705 24

[Fal89] ———, Crystalline cohomology and p-adic Galois-representations, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 25–80. MR 1463696 25

[Fon82] Jean-Marc Fontaine, Sur certains types de représentations p-adiques du groupe de Galois d'un corps local; construction d'un anneau de Barsotti-Tate, Ann. of Math. (2) **115** (1982), no. 3, 529–577. MR 657238 25

[Gro68] Alexander Grothendieck, Crystals and the de rham cohomology of schemes, Dix Exposes sur la Cohomologie des Sch ´ emas (1968), 306–358. 49

[hl] Daniel Litt (https://mathoverflow.net/users/6950/daniel litt), Equivalence between vector bundles with integrable connections to isocrystals, MathOverflow, URL:https://mathoverflow.net/q/466276 (version: 2024-03-02). 5

[JZ97] Jürgen Jost and Kang Zuo, Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasiprojective varieties, J. Differential Geom. **47** (1997), no. 3, 469–503. MR 1617644 21

[JZ01] ———, Representations of fundamental groups of algebraic manifolds and their restrictions to fibers of a fibration, Math. Res. Lett. **8** (2001), no. 4, 569–575. MR 1851272 37

[Kat70] Nicholas M. Katz, Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin, Inst. Hautes Études Sci. Publ. Math. (1970), no. 39, 175–232. MR 291177 5, 54

[Kat72] ———, Algebraic solutions of differential equations (p-curvature and the hodge filtration), Inventiones Mathematicae **18** (1972), 1–118. 52, 53, 54

[Kat96] ———, Rigid local systems, Annals of Mathematics Studies, vol. 139, Princeton University Press, Princeton, NJ, 1996. MR 1366651 7, 32

[KP02] Ludmil Katzarkov and Tony Pantev, Nonabelian (p, p) classes, Motives, polylogarithms and Hodge theory, Part II (Irvine, CA, 1998), Int. Press Lect. Ser., vol. 3, II, Int. Press, Somerville, MA, 2002, pp. 625–715. MR 1978715 37

[KP22] Christian Klevdal and Stefan Patrikis, G-cohomologically rigid local systems are integral, Trans. Amer. Math. Soc. **375** (2022), no. 6, 4153–4175. MR 4419055 30, 36

[Laf02] Laurent Lafforgue, Chtoucas de Drinfeld et correspondance de Langlands, Invent. Math. **147** (2002), no. 1, 1–241. MR 1875184 36, 37

[Lit13] Daniel Litt, Online note, https://virtualmath1.stanford.edu/ conrad/shimsem/2013Notes/Littvhs.pdf (2013). 16

[LL24] Aaron Landesman and Daniel Litt, Canonical representations of surface groups, Ann. of Math. (2) **199** (2024), no. 2, 823–897. MR 4717077 22

[LZ17] Ruochuan Liu and Xinwen Zhu, Rigidity and a Riemann-Hilbert correspondence for p-adic local systems, Invent. Math. **207** (2017), no. 1, 291–343. MR 3592758 47, 63

[Mar91] G. A. Margulis, Discrete subgroups of semisimple Lie groups, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991. MR 1090825 34

[Moc06] Takuro Mochizuki, Kobayashi-Hitchin correspondence for tame harmonic bundles and an application, Astérisque (2006), no. 309, viii+117. MR 2310103 32, 37

[Mor78] John W. Morgan, The algebraic topology of smooth algebraic varieties, Inst. Hautes Études Sci. Publ. Math. (1978), no. 48, 137–204. MR 516917 30

[Pet23] Alexander Petrov, Geometrically irreducible p-adic local systems are de Rham up to a twist, Duke Math. J. **172** (2023), no. 5, 963–994. MR 4568051 63

[Put] Andrew Putman, The representation theory of $\mathrm{SL}_n(\mathbb{Z})$, https://www3.nd.edu/~andyp/notes/RepTheorySLnZ.pdf. 34

[Ric88] R. W. Richardson, Conjugacy classes of n-tuples in Lie algebras and algebraic groups, Duke Math. J. **57** (1988), no. 1, 1–35. MR 952224 31

[Sch13] Peter Scholze, p-adic Hodge theory for rigid-analytic varieties, Forum Math. Pi **1** (2013), e1, 77. MR 3090230 63

[Ser65] Jean-Pierre Serre, Classification des variétés analytiques p-adiques compactes, Topology **3** (1965), 409–412. MR 179170 41

[sga72] Groupes de monodromie en géométrie algébrique. I, Lecture Notes in Mathematics, vol. Vol. 288, Springer-Verlag, Berlin-New York, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. MR 354656 31

[Sim91] Carlos T. Simpson, The ubiquity of variations of Hodge structure, Complex geometry and Lie theory (Sundance, UT, 1989), Proc. Sympos. Pure Math., vol. 53, Amer. Math. Soc., Providence, RI, 1991, pp. 329–348. MR 1141208 19, 20, 21

[Sim92] ———, Higgs bundles and local systems, Inst. Hautes Études Sci. Publ. Math. (1992), no. 75, 5–95. MR 1179076 20, 31, 32

[Sta24] The Stacks project authors, The stacks project, https://stacks.math.columbia.edu, 2024. 44, 49, 51

[Tat67] J. T. Tate, p-divisible groups, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin-New York, 1967, pp. 158–183. MR 231827 23