Lecture Notes for Étale Cohomology, Fall 2022

These are notes from a course taught by Stefan Patrikis at Ohio State University in Fall 2022. Yifei Zhang revised them to the present state based on notes originally typed by the students in the class (Kacey Aurum, Mehmet Basaran, Kyle Binder, Suxuan Chen, Juan Pablo De Rasis, Deniz Genlik, Jake Huryn, Jonghoo Lee, Will Newman, Stefan Nikoloski, Min Shi, Luke Wiljanen, and Yifei Zhang) on a rotating basis. They may be polished more in the future, but in the meantime comments and corrections are most welcome (please email `patrikis.1@osu.edu`).

Contents

1 Motivation on zeta functions and the Weil conjectures

1.1 Lecture 1, 8/24

A natural motivation for our course is given by *Zeta functions*. The first Zeta function one always comes up with is the function given by the series

$$\zeta(s) := \sum_{n=1}^{+\infty} n^{-s} = \prod_{p \in \mathbb{N} \text{ prime}} \left(1 - p^{-s}\right)^{-1}, \tag{1}$$

which converges absolutely and uniformly on compact subsets of $\{s \in \mathbb{C} : \operatorname{Re}(s) > 1\}$ (this is easily seen by the identity $|n^{-s}| = n^{-\operatorname{Re}(s)}$) and thus defines an analytic function over that set. The equality $\sum_{n=1}^{+\infty} n^{-s} = \prod_{p \in \mathbb{N} \text{ prime}} \left(1 - p^{-s}\right)^{-1}$ is given by unique factorization and the identity $\left(1 - p^{-s}\right)^{-1} = \sum_{i=0}^{+\infty} \left(p^{-s}\right)^{i} = \sum_{i=0}^{+\infty} \left(\frac{1}{p^i}\right)^{s}$.

This function has some remarkable properties. We define $\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$, where Γ is the *Gamma function*

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t} \, \mathrm{d}t \text{ for all } z \in \mathbb{C} \text{ such that } \operatorname{Re}(z) > 0.$$

Recall that Γ has a meromorphic continuation to an analytic function whose poles are exactly $\mathbb{Z}_{\leqslant 0}$, and these are all simple. Moreover, Γ has no zeros, thus $\frac{1}{\Gamma}$ is an entire function.

Theorem 1.1 (Riemann) *ξ has a meromorphic continuation to $\mathbb{C}$ with only two poles: one at 0 and the other at 1, both simple. Moreover, for all $s \in \mathbb{C} \setminus \{0, 1\}$ we have $\xi(s) = \xi(1 - s)$.*

Proof. See https://people.reed.edu/~jerry/311/zeta.pdf. ■

From the product formula in (1) we see that $\zeta(s) \neq 0$ for all $s \in \mathbb{C}$ such that $\operatorname{Re}(s) > 1$. But if $\operatorname{Re}(s) < 0$ then $\operatorname{Re}(1 - s) > 1$, thus $\zeta(1 - s) \neq 0$ and therefore Theorem 1.1 and the fact that Γ has no zeros give

$$\zeta(s) = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \underbrace{\Gamma\left(\frac{1-s}{2}\right) \pi^{\frac{s}{2}} \pi^{-\frac{1-s}{2}} \zeta(1 - s)}_{\neq 0}.$$

We know that $\Gamma\left(\frac{1-s}{2}\right) \in \mathbb{C}^{\times}$ because $\operatorname{Re}\left(\frac{1-s}{2}\right) > \frac{1}{2}$, so $\frac{1-s}{2}$ is not a pole of Γ. Thus, since Γ has all its poles in $\mathbb{Z}_{\leqslant 0}$ and they are all simple, then $\zeta(s) = 0$ if and only if $\frac{s}{2} \in \mathbb{Z}_{\leqslant 0}$, and since $s \neq 0$ is excluded from our analysis, then $s = -2k$ with $k \in \mathbb{N} = \mathbb{Z}_{\geqslant 1}$.

We have found all zeros of ζ in the set $\{s \in \mathbb{C} : \operatorname{Re}(s) > 1\}$ (there are none) and in the set $\{s \in \mathbb{C} : \operatorname{Re}(s) < 0\}$ (strictly negative even integers). These are called *trivial zeros of ζ*. It remains to study zeros over the set $\{s \in \mathbb{C} : 0 \leqslant \operatorname{Re}(s) \leqslant 1\}$. These are called the *nontrivial zeros of ζ*.

Conjecture 1.2 (Riemann) *If $s \in \mathbb{C}$ is a nontrivial zero of ζ then $\operatorname{Re}(s) = \frac{1}{2}$.*

Analogously, given a number field K, we can define the *Dedekind Zeta function*

$$\zeta_K(s) = \sum_{\substack{\text{nonzero} \\ \text{ideals } I \trianglelefteq \mathscr{O}_K}} [\mathscr{O}_K : I]^{-s},$$

where for each nonzero prime ideal I of $\mathscr{O}_K$, $[\mathscr{O}_K : I]$ is the group-theoretic index of I in $\mathscr{O}_K$ (which is always finite; see [Mar77, Theorem 14, p. 56]). We denote this by $N(I)$. Since N is multiplicative (see [Mar77, Theorem 22a, p. 66]) and nonzero ideals factor uniquely as the product of prime ideals in $\mathscr{O}_K$ (see [Mar77, Theorem 16, p. 59]), then the identity

$$\left(1 - N(\mathfrak{p})^{-s}\right)^{-1} = \sum_{i=0}^{+\infty} \left(\frac{1}{N(\mathfrak{p})^i}\right)^s \text{ for all } s \in \mathbb{C} \text{ such that } \operatorname{Re}(s) > 1$$

(where $\mathfrak{p}$ is a nonzero prime ideal of $\mathscr{O}_K$) immediately gives the product formula

$$\zeta_F(\mathfrak{p}) = \prod_{\substack{\text{nonzero prime} \\ \text{ideals } \mathfrak{p} \trianglelefteq \mathscr{O}_K}} \left(1 - N(\mathfrak{p})^{-s}\right)^{-1}.$$

We can then formulate an analogous theorem and conjecture for this more general case.

Remark 1.3 *Let K be a number field. Since every nonzero prime ideal of $\mathscr{O}_K$ is maximal (see [Mar77, Theorem 14, p. 56]), then we can rewrite*

$$\zeta_F(s) = \sum_{x \in \operatorname{Spec}(\mathscr{O}_K)_{\mathrm{cl}}} \left(1 - N(x)^{-s}\right)^{-1},$$

where for each scheme X we denote by X_{cl} the set of its closed points. Observe that $N(x) = [\mathscr{O}_K : \mathfrak{p}_x]$, where $\mathfrak{p}_x$ is the maximal ideal corresponding to $x \in \operatorname{Spec}(\mathscr{O}_K)$ where $[\mathscr{O}_K : \mathfrak{p}_x] = |\mathscr{O}_K/\mathfrak{p}_x|$ is the cardinality of the residue field at x.

We prove the following lemma.

Lemma 1.4 *Let X be a scheme of finite type over $\mathbb{Z}$. If $x \in X$, then $x \in X_{\mathrm{cl}}$ if and only if $\sharp\kappa(x) < \infty$, where $\kappa(x)$ is the residue field at x.*

Proof. Remember that X being of finite type over $\mathbb{Z}$ means that it is quasi-compact and locally of finite type over $\mathbb{Z}$ (this last property is local). Fixing $x \in X$ and taking an affine open cover of X, we may consider an open affine subscheme where x belongs and assume that X is affine (affine schemes are quasi-compact). Thus $X = \operatorname{Spec}(A)$ for some unital commutative algebra A of finite type over $\mathbb{Z}$. The morphism $X \to \operatorname{Spec}(\mathbb{Z})$ then corresponds to a ring homomorphism $\mathbb{Z} \to A$ making A into a finitely generated $\mathbb{Z}$-algebra.

If $x \in X_{\mathrm{cl}}$ then x corresponds to a maximal ideal $\mathfrak{m}$ of A, thus $\kappa(x) = \frac{A_{\mathfrak{m}}}{\mathfrak{m}A_{\mathfrak{m}}} \cong A/\mathfrak{m}$ is a field which is finitely generated as a $\mathbb{Z}$-algebra. It is a well-known algebraic fact that such fields are finite (see https://math.stackexchange.com/questions/148745/fields-finitely-generated-as-mathbb-z-algebras-are-finite). Conversely, if $\kappa(x)$ is finite, let $\mathfrak{p}$ be the prime ideal of A corresponding to x. Then there exists a prime number $p \in \mathbb{N}$ such that $\mathbb{F}_p \hookrightarrow A/\mathfrak{p} \hookrightarrow \kappa(x)$. Since $\kappa(x)$ is finite then $\kappa(x)$ is an algebraic extension of $\mathbb{F}_p$. Since every subring of an algebraic field extension containing the base field is again a field, then $\mathfrak{p}$ must be a maximal ideal, meaning that x is a closed point. ■

Remark 1.3 and Lemma 1.4 allow us to give the following generalization of Dedekind Zeta functions:

Definition 1.5 *Let X be a scheme of finite type over $\mathbb{Z}$. We define*

$$\zeta(X,s) := \prod_{x \in X_{\mathrm{cl}}} \left(1 - N(x)^{-s}\right)^{-1},$$

where for each $x \in X_{\mathrm{cl}}$ we let $N(x)$ be the number of elements in the (finite) residue field of X at x.

Given a scheme X of finite type over $\mathbb{Z}$, the set $\{x \in X_{\mathrm{cl}} : N(x) \leqslant M\}$ is finite for all $M \in \mathbb{R}$. This can be seen by reducing to the case where X is of finite type over $\mathbb{F}_p$ where $p \in \mathbb{N}$ is prime, where we will see that the result follows from the finiteness of $X(\mathbb{F}_{p^r})$ for all $r \in \mathbb{N}$.

Thus, we can formally expand $\zeta(X,s)$ to a *Dirichlet series*

$$\sum_{n=1}^{+\infty} a_n n^{-s}$$

(see [Ser73, Chapter VI] for a quick introduction to Dirichlet series).

Lemma 1.6 *$\zeta(X,s)$ converges absolutely and uniformly over compact subsets of $\{s \in \mathbb{C} : \mathrm{Re}(s) > \dim(X)\}$.*

This can be easily verified by reducing to the affine case where $X = \mathbb{Z}[T_1, \cdots, T_n]$ or $X = \mathbb{F}_p[T_1, \cdots, T_n]$, for some $n \in \mathbb{N}$ and a prime number $p \in \mathbb{N}$, and then reducing again to the classic zeta function ζ.

Question 1.7 *Given a scheme X of finite type over $\mathbb{Z}$, it is natural to ask whether we can find a meromorphic continuation and a functional equation for $\zeta(X,s)$, as well as giving an analysis of the resulting zeros and poles.*

When $X = \mathscr{O}_K$ for a number field K, for each prime number $p \in \mathbb{N}$ we can consider the base-change

$$X_{\mathbb{F}_p} = \mathrm{Spec}(\mathscr{O}_K) \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{F}_p) = \mathrm{Spec}(\mathscr{O}_K \otimes_{\mathbb{Z}} \mathbb{F}_p) \cong \coprod_{\substack{\mathfrak{p} \in \mathrm{MaxSpec}(\mathscr{O}_K) \\ \mathfrak{p} | p}} \mathrm{Spec}\left(\frac{\mathscr{O}_K}{\mathfrak{p}^{e(\mathfrak{p}|p)}}\right),$$

where the last isomorphism follows from the Chinese Remainder Theorem isomorphism

$$\frac{\mathscr{O}_K}{p\mathscr{O}_K} = \frac{\mathscr{O}_K}{\prod_{\substack{\mathfrak{p} \in \mathrm{MaxSpec}(\mathscr{O}_K) \\ \mathfrak{p} | p}} \mathfrak{p}^{e(\mathfrak{p}|p)}} \cong \bigoplus_{\substack{\mathfrak{p} \in \mathrm{MaxSpec}(\mathscr{O}_K) \\ \mathfrak{p} | p}} \frac{\mathscr{O}_K}{\mathfrak{p}^{e(\mathfrak{p}|p)}}.$$

We then get

$$\zeta\left(X_{\mathbb{F}_p}, s\right) = \prod_{\substack{\mathfrak{p} \in \mathrm{MaxSpec}(\mathscr{O}_K) \\ \mathfrak{p} | p}} \left(1 - N(\mathfrak{p})^{-s}\right)^{-1},$$

which is a finite product and therefore induces a rational function in p^{-s}. Using this, it is easy to answer Question 1.7.

But for a general finite type scheme X over $\mathbb{F}_p$, where $p \in \mathbb{N}$ is prime, $\zeta(X, s)$ is already very interesting and will take the whole course to say much about.

The reinterpretation in this case is the following: If X is a finite type scheme over $\mathbb{F}_q$, where $q = p^f$, $f \in \mathbb{N}$ and $p \in \mathbb{N}$ is prime, then for all $x \in X_{\mathrm{cl}}$ we set $\deg(x) := [\kappa(x) : \mathbb{F}_q]$, and define

$$Z(X/\mathbb{F}_q, t) = \prod_{x \in X_{\mathrm{cl}}} \left(1 - t^{\deg(x)}\right)^{-1} \in 1 + t\mathbb{Z}[\![t]\!],$$

so that $\zeta(X, s) = Z(X/\mathbb{F}_q, q^{-s})$.

For any $n \in \mathbb{N}$ consider $X(\mathbb{F}_{q^n}) = \mathrm{Hom}_{\mathrm{Sch}/\mathbb{F}_q}(\mathrm{Spec}(\mathbb{F}_{q^n}, X))$, which is in bijection with the set of pairs (x, ι) where $x \in X_{\mathrm{cl}}$ and ι is a field-embedding of $\kappa(x)$ into $\mathbb{F}_{q^n}$ fixing $\mathbb{F}_q$. The bijection is given by sending each $\alpha \in X(\mathbb{F}_{q^n})$ to the pair formed by $\alpha(\{0\}) \in X_{\mathrm{cl}}$ and the corresponding map on residue fields.

From field theory, we know that there are $\deg(x)$ many embeddings of $\kappa(x)$ into $\mathbb{F}_{q^n}$ fixing $\mathbb{F}_q$ when $\deg(x) \mid n$, and none otherwise, thus we have shown:

Lemma 1.8 *Let X be a finite-type scheme over $\mathbb{F}_q$. Then, for all $n \in \mathbb{N}$,*

$$\sharp X(\mathbb{F}_{q^n}) = \sum_{d \mid n} d \cdot \sharp \{x \in X_{\mathrm{cl}} : \deg(x) = d\}.$$

Lemma 1.8 allows us to prove the following result, which makes use of the *formal logarithm* $\log : 1 + t\mathbb{Q}[\![t]\!] \to \mathbb{Q}[\![t]\!]$ (see [Sil09, Ch. IV.2-IV.5] for a quick introduction).

Lemma 1.9 *Let X be a finite type scheme over $\mathbb{F}_q$, where $q \in \mathbb{N}$ is a prime power. For each $n \in \mathbb{N}$ let $N_n := \sharp X(\mathbb{F}_{q^n})$. Then*

$$\log(Z(X/\mathbb{F}_q, t)) = \sum_{n \in \mathbb{N}} N_n \frac{t^n}{n}.$$

Proof. For each $d \in \mathbb{N}$ call $C_d := \sharp\{x \in X_{\mathrm{cl}} : \deg(x) = d\}$. We have

$$\log(Z(X/\mathbb{F}_q, t)) = \log\left(\prod_{x \in X_{\mathrm{cl}}} \left(1 - t^{\deg(x)}\right)^{-1}\right) = \sum_{x \in X_{\mathrm{cl}}} -\log\left(1 - t^{\deg(x)}\right) =$$

$$= \sum_{d \in \mathbb{N}} \sum_{\substack{x \in X_{\mathrm{cl}} \\ \deg(x) = d}} -\log\left(1 - t^{\deg(x)}\right) = \sum_{d \in \mathbb{N}} \sum_{\substack{x \in X_{\mathrm{cl}} \\ \deg(x) = d}} \sum_{m=1}^{+\infty} \frac{t^{dm}}{m} =$$

$$= \sum_{d \in \mathbb{N}} \sum_{m=1}^{+\infty} C_d \frac{t^{dm}}{m} = \underbrace{\sum_{n \in \mathbb{N}} \sum_{d \mid n} C_d d \frac{t^n}{n} = \sum_{n \in \mathbb{N}} \sharp X(\mathbb{F}_{q^n}) \frac{t^n}{n}}_{\text{Lemma 1.8}}.$$

This completes the proof. ■

Applying exponentials to the identity in Lemma 1.9 we get $Z(X/\mathbb{F}_q, t) = \exp\left(\sum_{n \in \mathbb{N}} N_n \frac{t^n}{n}\right)$.

Remark 1.10 *Let X be a finite type scheme over $\mathbb{F}_{q^n}$, where $n \in \mathbb{N}$ and $q \in \mathbb{N}$ is a prime power. The canonical embedding $\mathbb{F}_q \hookrightarrow \mathbb{F}_{q^n}$ gives a scheme-theoretic sequence $X \to \operatorname{Spec}(\mathbb{F}_{q^n}) \to \operatorname{Spec}(\mathbb{F}_q)$. Hence we get $Z(X/\mathbb{F}_{q^n}, t) = Z(X/\mathbb{F}_q, t^n)$.*

As an example of an application of Lemma 1.9, fix $n \in \mathbb{N}_0$ and a prime power $q \in \mathbb{N}$. We know that $\mathbb{A}^n(\mathbb{F}_{q^m})$ has $(q^m)^n = q^{mn}$ elements for any $m \in \mathbb{N}$, thus

$$Z(\mathbb{A}^n/\mathbb{F}_q, t) = \exp\left(\sum_{m=1}^{+\infty} \sharp\mathbb{A}^n(\mathbb{F}_{q^m})\frac{t^m}{m}\right) = \exp\left(\sum_{m=1}^{+\infty} q^{mn}\frac{t^m}{m}\right) =$$

$$= \exp(-\log(1-q^n t)) = \frac{1}{1-q^n t} \in \mathbb{Q}(t).$$

Now, assume X and Y are finite-type schemes over $\mathbb{F}_q$. By the universal property of the categorical product, $\sharp\left(X \times_{\mathbb{F}_q} Y\right)(\mathbb{F}_{q^n}) = \sharp X(\mathbb{F}_{q^n}) \cdot \sharp Y(\mathbb{F}_{q^n})$. In particular, setting $Y := \mathbb{A}^n/\mathbb{F}_q$ and using the above, we get

$$Z\left(X \times_{\mathbb{F}_q} \mathbb{A}^n/\mathbb{F}_q\right) = \exp\left(\sum_{m\in\mathbb{N}} \sharp X(\mathbb{F}_{q^m})\, q^{mn}\frac{t^m}{m}\right) = Z(X/\mathbb{F}_q, q^n t).$$

Again, assume X and Y are finite-type schemes over $\mathbb{F}_q$, and assume that there is a closed embedding $Y \hookrightarrow X$. If $U := X \backslash Y$, then $\sharp X(\mathbb{F}_{q^n}) = \sharp Y(\mathbb{F}_{q^n}) + \sharp U(\mathbb{F}_{q^n})$, and therefore

$$Z(X/\mathbb{F}_q, t) = Z(Y/\mathbb{F}_q, t)\, Z(U/\mathbb{F}_q, t),$$

which follows from the fact that exp transforms sums into products. Applying this recursively to $\mathbb{P}^n/\mathbb{F}_q = \bigcup_{i=0}^n \mathbb{A}^i/\mathbb{F}_q$ and using our first example, we get

$$Z(\mathbb{P}^n/\mathbb{F}_q, t) = \prod_{i=0}^{n}\left(1-q^i t\right)^{-1}.$$

1.1.1 Weil Conjectures (1949)

Recall that if X is a finite type scheme over $\mathbb{F}_q$, where $q \in \mathbb{N}$ is a prime power, then $Z(X/\mathbb{F}_q, t) \in 1 + t\mathbb{Q}[\![t]\!]$. The first Weil Conjecture says something more.

Conjecture 1.11 *Let X be a finite type scheme over $\mathbb{F}_q$, where $q \in \mathbb{N}$ is a prime power. Then $Z(X/\mathbb{F}_q, t) \in \mathbb{Q}(t)$.*

Observe that Conjecture 1.11 implies that $\zeta(X, s) = Z(X/\mathbb{F}_q, q^{-s})$ has a meromorphic continuation.

Conjecture 1.12 *Let X be a finite type scheme over $\mathbb{F}_q$, where $q \in \mathbb{N}$ is a prime power, and assume $X/\mathbb{F}_q$ is smooth, projective, and geometrically connected. If $d := \dim(X)$, then*

$$Z\left(X/\mathbb{F}_q, \frac{1}{q^d t}\right) = \pm q^{\frac{d\chi}{2}} t^{\chi} Z(X/\mathbb{F}_q, t),$$

where χ is the Euler characteristic *of X (which is defined as the number of self-intersection of Δ, where $\Delta \hookrightarrow X \times X$ is the diagonal).*

Thus, $\zeta(X, d-s) = \pm q^{\frac{d\chi}{2}} q^{-s\chi}\zeta(X, s)$. (this is a reflection on Poincaré's duality.)

Finally, the third Weil Conjecture is the Riemann Hypothesis in this context.

Conjecture 1.13 (Riemann Hypothesis) *Let X be a finite type scheme over $\mathbb{F}_q$, where $q \in \mathbb{N}$ is a prime power, and assume $X/\mathbb{F}_q$ is smooth, projective, and geometrically connected. If $d := \dim(X)$, then for each $i \in [0, 2d] \cap \mathbb{Z}$ there exists $p_i \in \mathbb{Z}[t]$ such that:*

- $p_0(t) = 1 - t$ *and* $p_{2d}(t) = 1 - q^d t$.
- *If $i \in [0, 2d] \cap \mathbb{Z}$, then for each $j \in \mathbb{N}_{\leqslant \deg(p_i)}$ there exists an algebraic integer α_{ij} such that $|\alpha_{ij}| = q^{\frac{i}{2}}$ and $p_i(t) = \prod_{j \in \mathbb{N}_{\leqslant \deg(p_i)}} (1 - \alpha_{ij} t)$.*
- $Z(X/\mathbb{F}_q, t) = \prod_{i=0}^{2d} p_i(t)^{(-1)^{i+1}}$.

Under the appropriate translation of $\zeta(X, s)$ we can see that Conjecture 1.13 is equivalent to asking that the zeros and poles lie on certain vertical lines.

Exercise 1.14 *Verify the Weil Conjectures for $\mathbb{P}^n$ (here $\chi = d + 1$).*

In general, for each $r \in [0, 2d] \cap \mathbb{Z}$ we set $b_r = \deg(p_r)$, and then

$$\chi = \sum_{r=0}^{2d} (-1)^r b_r.$$

Weil also conjectured that if X is the reduction modulo a prime ideal $\mathfrak{p}$ of a smooth projective scheme Y over a number field with good reduction at $\mathfrak{p}$, then

$$\deg(p_r) = \dim_{\mathbb{Q}}\left(H^r_{\text{singular}}\left(Y(\mathbb{C})^{\text{an}}, \mathbb{Q}\right)\right),$$

where $Y(\mathbb{C})^{\text{an}}$ is a classical topological space.

Weil's (presumed) heuristic was the following: a "cohomology space theory" for varieties over finite fields governs these conjectures with the intimate relation between Conjecture 1.12 and Poincaré duality, the one between the given decomposition in Conjecture 1.13 and *Lefschetz fixed point*, and between Conjecture 1.13 itself with Serre's "Kahler" analogous of Riemann Hypothesis (which was proved by Serre after Weil's statement of his conjectures).

Conjecture 1.11 was proven by Dwork and redone by Grothendieck et al. Conjecture 1.12 was proven by Grothendieck et al. in the 1960's, and Conjecture 1.13 was proven by Deligne et al. in the 1970's. Weil had already proven these for abelian varieties, while Deligne had previously proved these for $K3$ surfaces.

1.2 Lecture 2, 8/29

Recall: For all $X \to \text{Spec}(\mathbb{Z})$ finite type, we defined $\zeta(X, s) = \prod_{x \in X_{cl}} (1 - \#\kappa(x)^{-s})^{-1}$. For $X/\mathbb{F}_q$, set $Z(X/\mathbb{F}_q, t) = \prod_{x \in X_{cl}} (1 - t^{\deg(x)})^{-1} \in \mathbb{Z}[\![t]\!]$. We found that $Z(X/\mathbb{F}_q, t) = \exp(\sum_{m \geqslant 1} \#X(\mathbb{F}_{q^m}) \frac{t^m}{m})$, and we stated the Weil conjectures for $X/\mathbb{F}_q$ smooth projective.

1.2.1 Comments on the Global Case

Let X/F be smooth projective for a number field F. Then, X spreads out to a smooth projective model

$$\begin{array}{ccc} \mathfrak{X} & \longleftarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathscr{O}_F[\frac{1}{N}] & \longleftarrow & \operatorname{Spec} F \end{array}$$

for some $N \in \mathbb{Z}$.

We get a partial zeta function $\zeta^S(\mathfrak{X}, s) = \prod_{x \in \mathfrak{X}_{Cl}} (1 - \#\kappa(x)^{-s})^{-1}$ where $S = \{p \mid N\}$ which converges to a holomorphic function for $\operatorname{Re}(s) \gg 0$. We have a similar conjecture as before: there should be a meromorphic continuation satisfying a suitable functional equation which relates s with $\dim \mathfrak{X} - s = \dim(X) + 1 - s$. The structure of this conjecture is clearer "one degree of cohomology at a time" (using étale cohomology in the formulation).

For example, let $E/\mathbb{Q}$ be an elliptic curve with good reduction outside a finite set S of primes. Then,

$$\zeta^S(\mathscr{E}/\mathbb{Z}[1/S], s) = \prod_{p \notin S} \zeta(\mathscr{E}/\mathbb{F}_p, s) = \prod_{p \notin S} \frac{1 - a_p p^{-s} + p^{1-2s}}{(1 - p^{-s})(1 - p^{-1})}.$$

The $1 - a_p p^{-s} + p^{1-2s}$ will come from H^1 while the $(1 - p^{-s})$ term will come from H^0 and the $(1 - p^{1-s})$ term will come from H^2. We denote $L^S(E/\mathbb{Q}, s)^{-1} = \prod_{p \notin S} (1 - a_p p^{-s} + p^{1-2s})$. This form suggests that we should replace the away from S term with the usual completed L-function. When you do this, you get a completed zeta function of $E/\mathbb{Q}$ given by

$$\mathfrak{Z}(E/\mathbb{Q}, s) = \frac{\xi(s)\xi(s-1)}{\Lambda(E/\mathbb{Q}, s)}$$

where $\xi(s) = \zeta(s) \cdot \pi^{-s/2} \Gamma(s/2)$ and $\Lambda(E/\mathbb{Q}, s) = L^S(E/\mathbb{Q}, s) \cdot \prod_{p \in S} (1 - a_p p^{-s})^{-1} \cdot (2\pi)^{-s} \Gamma(s)$. For $p \in S$, $a_p = \begin{cases} \pm 1 & \text{(split) mult. reduction} \\ 0 & \text{additive...} \end{cases}$. Note that $\xi(s) = \xi(1-s)$ and (granted the Shimura-Taniyama conjecture) $\Lambda(E/\mathbb{Q}, s)$ has analytic continuation and $\Lambda(E/\mathbb{Q}, s) = \pm N_E^{1-s} \Lambda(E/\mathbb{Q}, 2-s)$ where N_E is the conductor (note $2 = \dim E + 1$). Therefore, $\mathfrak{Z}(E/\mathbb{Q}, s) = \pm N_E^{s-1} \mathfrak{Z}(E/\mathbb{Q}, 2-s)$. (Note that $\xi(2-s) = \xi(1-(s-1)) = \xi(s-1)$.)

A very general hope (the Langland's conjecture) is that general zeta functions of (smooth projective) varieties over number fields can be written as some alternating product of "automorphic L-functions".

2 Motivation on Cohomological Formalism

2.1 Lecture 2, 8/29 (cont)

To any smooth projective variety $X/\mathbb{C} \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ we obtain a compact complex manifold

$$X^{an} \hookrightarrow \mathbb{P}^n(\mathbb{C})(= \mathbb{CP}^n).$$

This has various cohomological realizations:

- singular cohomology, $H^*_{\text{sing}}(X^{an}, \mathbb{Z} \text{ or } \mathbb{Q} \text{ or} \ldots)$,

- sheaf cohomology, $H^*(X^{an}, \underline{\mathbb{Z}} \text{ or } \underline{\mathbb{Q}} \text{ or} \ldots)$ where $\underline{\mathbb{Z}}, \underline{\mathbb{Q}}$ are constant sheaves,
- de Rham cohomology, $H^*_{dR}(X^{an}, \mathbb{R} \text{ or } \mathbb{C})$.

For varying X, these come with natural (in X) comparison isomorphisms: for example,

$$H^*(X^{an}, \mathbb{C}) \simeq H^*_{dR}(X^{an}, \mathbb{C}).$$

Key structures (for our purposes) that they share include

- Künneth isomorphisms
- Poincaré duality
- "cycle class maps" associating to suitable subvarieties/manifolds cohomology classes (think fundamental class in homology).

Our goal is then to explain how an analogous formalism for smooth projective varieties over $\mathbb{F}_q$ implies much of the Weil conjectures (rationality and functional equation). To precisely define the cycle class map, we need to digress to the discussion of Chow ring a little.

We will introduce Chow Ring here as concisely as possible. To do so we first need the notion of k-cycles on a variety X. We use variety to mean a finite-type separated integral scheme over a field, and use subvariety to mean an integral closed subscheme.

Definition 2.1 *The group of k-cycles on variety X, denoted by $Z_k(X)$, is the free abelian group on symbols $\{[Z] | Z$ is a subvariety of dimension k in $X\}$.*

The *Chow Group of k-cycles on X $CH_k(X)$* is the quotient of $Z_k(X)$ by the rational equivalence. One can define this rational equivalence in the following way. For a subvariety W of dimension $k+1$ and a further subvariety Z of codimension 1 in W, let f be in $\mathscr{O}_{W,Z}$ ($\mathscr{O}_{W,Z}$ is the stalk of $\mathscr{O}_W$ at the generic point of Z), we decree

$$\mathrm{ord}_{W,Z}(f) := \mathrm{length}_{\mathscr{O}_{W,Z}} \mathscr{O}_{W,Z}/f\mathscr{O}_{W,Z}$$

. For a non-zero rational function f on W, we can write $f = g/h$ for $g, h \in \mathscr{O}_{W,Z}$, and we define

$$\mathrm{ord}_{W,Z}(f) := \mathrm{ord}_{W,Z}(g) - \mathrm{ord}_{W,Z}(h)$$

. The *principal divisor associated to a rational function f on W* is

$$\mathrm{div}_W(f) := \sum_Z \mathrm{ord}_{W,Z}(f)[Z]$$

where Z runs through all codimension 1 (equivalently, dimension k) subvarieties of W. General theory of commutative algebra guarantees that $\mathrm{ord}_{W,Z}(f)$ is finite for all codimension 1 subvarieties Z and for a fixed f there only exists finitely many such Z such that $\mathrm{ord}_{W,Z}(f) \neq 0$. In other words, we can view $\mathrm{div}_W(f) \in Z_k(X)$.

Remark 2.2 *In addition, in the same setting, if W is regular at the generic point of Z, then $\mathscr{O}_{W,Z}$ is a valuation ring and $\mathrm{ord}_{W,Z}(f) = val(f)$. Therefore if W is normal, the divisor we just defined coincides with the Weil divisor on W. Alternatively, we can write $Z_k(W) = div(W)$.*

Definition 2.3 *For a variety X and a closed subscheme Z in X, we also define the k-cycle associated to a closed subscheme Z $[Z]_k$ as follow. Let Z_i run through the irreducible components of Z of dimension k, and we define*

$$[Z]_k := \sum_i (length_{\mathcal{O}_{X,Z_i}} \mathcal{O}_{Z,Z_i})[Z_i] \in Z_k(X)$$

Definition 2.4 *Two k-cycles α, α' are rationally equivalent if $\alpha - \alpha'$ is in the subgroup generated by $\{div_W(f) | W\ k+1$-dimensional subvariety and $f \in K(W)^\times\}$, in which case we write $\alpha \sim_{rat} \alpha'$. We define Chow group of k-cycles $CH_k(X) := Z_k(X)/\sim_{rat}$.*

We proceed to define Chow ring out of Chow group on a smooth projective X.

Given $\alpha = \sum_i n_i[W_i] \in Z_s(X)$ and $\beta = \sum_j m_j[V_j] \in Z_r(X)$, we say this two cycles *intersect properly* if, for each i, j, W_i, V_j intersect properly ($W_i \cap V_j$ is equidimensional with codimension $\mathrm{codim}_X(W_i) + \mathrm{codim}_X(V)$). For such two cycles, we define

$$[W_i] \cdot [V_j] := \sum_k i(W_i.V_j, Y_k; X)[Y_k] \in Z_{s+r-\mathrm{dim}X}$$

where $[Y_k]$ runs through the irreducible components of $W_i.V_j$ and the coefficients are the intersection multiplicities, and we define $\alpha.\beta$ from "distributivity". Not every pair of cycles intersects properly, but luckily we have the following lemma.

Lemma 2.5 ***Chow's moving lemma*** *For cycles α, β on quasi-projective smooth variety X, there exists $\alpha' \sim_{rat} \alpha$ such that α' and β intersect properly.*

This lemma gives rise to the group morphism

$$CH_s(X) \times CH_r(X) \longrightarrow CH_{s+r-\mathrm{dim}X}(X)$$

where (class of α, class of β) is sent to class of $\alpha'.\beta$ such that α' and β intersect properly and $\alpha' \sim_{rat} \alpha$. Should we define $CH^c(X) := CH_{\mathrm{dim}X-c}(X)$, the above morphism is rewritten as

$$CH^k(X) \times CH^l(X) \longrightarrow CH^{k+l}(X)$$

. Let $CH^*(X)$ denote $\bigoplus_{i=0}^{\mathrm{dim}X} CH^i(X)$, the collection of morphisms $\{CH^k(X) \times CH^l(X) \longrightarrow CH^{k+l}(X)\}_{k,l}$, extended by "distributivity", gives a binary operation on $CH^*(X)$. One can check its commutativity, distributivity and associativity (the only non-trivial one), and it clearly has an identity which is the class of $[X] \in CH^0(X)$; hence we upgraded $CH^*(X)$ to a graded commutative ring, which is the *Chow ring* of X.

We now discuss the functoriality of Chow rings: for morphism $f : X \longrightarrow Y$, do we get a reasonable graded ring morphism $f^* : CH^*(Y) \longrightarrow CH^*(X)$?

For $f : X \longrightarrow Y$ proper morphism of varieties, and let $Z \hookrightarrow X$ be a subvariety of dimension k. We let $f_*([Z]) = 0$ if dim $f(Z)$<k, otherwise we set $f_*([Z]) = d[f(Z)]$ where $d = [K(Z) : K(f(Z))]$ is the degree of the (generically finite) induced morphism

$$f : Z \longrightarrow f(Z)$$

. Extending it by linearity we get the *proper pushforward*

$$f_* : Z_k(X) \longrightarrow Z_k(Y)$$

which is compatible with rational equivalence; hence we get

$$f_* : CH_k(X) \longrightarrow CH_k(Y)$$

.

For $f : X \longrightarrow Y$ flat morphism of varieties and let $r = \dim X - \dim Y$ and Z be a subvariety of dimension k. We define $f^*([Z]) = [Z \times_Y X]_{k+r}$ where $Z \times_Y X$ is considered as a closed subscheme of X and use definition 2.3. Extending it by linearity we have the *flat pullback*

$$f^* : Z_k(Y) \longrightarrow Z_{k+r}(X)$$

. Some work can show that it's also compatible with rational equivalence; hence we have

$$f^* : CH_k(Y) \longrightarrow CH_k(X)$$

or

$$f^* : CH^k(Y) \longrightarrow CH^k(X)$$

. Now we can define the pullback for general morphism of smooth projective varieties $f : X \longrightarrow Y$. For $\alpha \in CH^k(Y)$ (note that now α is not a cycle but a cycle class), we set $f^*(\alpha) = pr_{X*}([\Gamma_f] \cdot pr_Y^*(\alpha))$ to get $f^* : CH^k(Y) \longrightarrow CH^k(X)$ where pr_X and pr_Y are the projection from $X \times Y$ to X and Y respectively and $[\Gamma_f]$ is the cycle associated to Γ_f. Note that pr_x is proper because being proper is stable under base change and $Y \longrightarrow \text{Spec}(k)$ is proper; similarly pr_Y is flat because flatness is stable under base change and $X \longrightarrow \text{Spec}(k)$ is flat. Therefore we have defined pr_{X*} and pr_Y^* as the proper pushforward and flat pullback respectively in the previous discussion.

As usual we upgrade f^* to morphism of graded abelian group $f^* : CH^*(Y) \longrightarrow CH^*(X)$, and about it we have the following lemma which establishes the functoriality of Chow ring and more.

Lemma 2.6 *let $f : X \longrightarrow Y$ be a morphism of smooth projective varieties, then the followings hold.*

(1) $f^* : CH^*(Y) \longrightarrow CH^*(X)$ is a graded ring morpism.

1. $(f \circ g)^* = g^* \circ f^*$
2. *The projection formula: $f_*(\alpha) \cdot \beta = f_*(\alpha . f^*(\beta))$ holds (note that f proper so f_* is defined).*
3. *If f is also flat then f^* agrees with the flat pullback previously defined.*

This concludes the discussion of Chow ring for our cohomology formalism.

Let $k = \mathbb{F}_q$, $\overline{k} = \overline{\mathbb{F}}_q$. Let $\mathcal{V}$ be the category of smooth projective varieties over $\overline{k}$. **Assume** there exists a characteristic zero field E and a functor

$$\mathcal{V}^{op} \rightarrow \{\mathbb{Z}_{\geqslant 0} - \text{graded commutative } E - \text{algebras of finite dimension}\}$$

where "graded commutative algebra" means A can be decomposed into $A = \bigoplus_{n \geqslant 0} A_n$ and has multiplication which plays nicely with the grading $A_n \times A_m \rightarrow A_{n+m}$ satisfying

$\alpha \cdot \beta = (-1)^{mn} \beta \cdot \alpha$ if $\alpha \in A_m$, $\beta \in A_n$. So $H^*(X)$ is an E-algebra, and we'll denote the product $H^i(X) \times H^j(X) \to H^{i+j}(X)$ by $\smile$, $(\alpha, \beta) \mapsto \alpha \smile \beta$. This must satisfy three properties:

I. Künneth. For $X, Y \in \mathcal{V}$, the graded E-algebra homomorphism

$$H^*(X) \otimes_E H^*(Y) \to H^*(X \times Y)$$

given by $\alpha \otimes \beta \mapsto p^*\alpha \cup q^*\beta$ is an isomorphism where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the natural maps.

II. Poincaré duality. Let $X \in \mathcal{V}$ be connected of dimension d. Then there are trace isomorphisms $\mathrm{tr}_X : H^{2d}(X) \xrightarrow{\sim} E$ and for all $i \in \mathbb{Z}$, the E-bilinear pairing given by the composition

$$H^i(X) \times H^{2d-i}(X) \xrightarrow{\smile} H^{2d}(X) \xrightarrow{\mathrm{tr}_X} E$$

is non-degenerate (so, in particular for $i > 2d$, $H^i(X) \simeq H^{2d-i}(X) = 0$ since $H^{<0}(X) = 0$). Moreover, we require for X, Y connected, $p : X \times Y \to X$, $q : X \times Y \to Y$ the natural maps that $\mathrm{tr}_{X \times Y}(p^*\alpha \smile q^*\beta) = \mathrm{tr}_X(\alpha)\mathrm{tr}_Y(\beta)$ for $\alpha \in H^{2\dim(X)}(X)$, $\beta \in H^{2\dim(Y)}(Y)$.

Note that **I.** and **II.** together give a covariant functoriality for $H^*(-)$. Let X, Y connected be in $\mathcal{V}$, and let $f : Y \to X$ be a morphism. Then, for all $r \in \mathbb{Z}$, we have

$$f^* : H^r(X) \to H^r(Y).$$

Since $H^r(X) \simeq H^{2\dim(X)-r}(X)^*$ and $H^r(Y) \simeq H^{2\dim(Y)-r}(Y)^*$, we get a map

$$H^{2\dim(X)-r}(Y) \to H^{2\dim(X)-r}(X),$$

i.e., for $c = \dim(X) - \dim(Y)$, we get a map $f_* : H^r(Y) \to H^{r+2c}(X)$ which is usually called the Gysin map. The map f_* is characterized by $\mathrm{tr}_X(f_*\alpha \smile \beta) = \mathrm{tr}_Y(\alpha \smile f^*\beta)$ for $\alpha \in H^r(Y)$, $\beta \in H^{2\dim(X)-r-2c}(X)$.

Exercise 2.7 *We have $(f \circ g)_* = f_* \circ g_*$ and for all $f : Y \to X$, $f_*(\alpha \smile f^*\beta) = f_*\alpha \smile \beta$ ("adjunction").*

A key example of f_* lets us construct "cycle classes".

Definition 2.8 *Let $\iota : Z \hookrightarrow X$ be a closed immersion with Z, X connected in $\mathcal{V}$ and let $c = \mathrm{codim}_X Z$. Then, we get $\iota_* : H^r(Z) \to H^{r+2c}(X)$, and so in particular, we get a map $H^0(Z) \to H^{2c}(X)$. Thus, since $H^0(Z) \simeq H^{2\dim(Z)}(Z)^*$ is a 1 dimensional E vector space, the E-algebra structure map $E \to H^0(Z)$ is an isomorphism. The image of 1 gives $\iota_*(1) \in H^{2c}(X)$ which is by definition $cl_X(Z) = \iota_*(1) \in H^{2c}(X)$, the cycle class of $Z \hookrightarrow X$.*

If Z is reduced with irreducible components $Z_1, \ldots, Z_n$, each of which are smooth, and $\iota : Z \hookrightarrow X$ is a closed immersion, we set

$$cl_X(Z) = \sum_{i=1}^{n} cl_X(Z_i)$$

III. We will need cycle classes compatible with intersections, i.e. cl_X induces a (graded) ring homomorphism $CH^*(X) \to H^{2*}(X)$ where, for a subvariety Z, $[Z] \in CH^c(X)$ is sent to $cl_X(Z) \in H^{2c}(X)$ which is defined in the definition 2.8; by abusing of notation, we also call this ring morphism cl_X. In addition, we also need the following 3 conditions:

1. (functoriality) If $f : X \longrightarrow Y$ is a morphism of projective smooth varieties, then the following holds
$$f^* cl_Y = cl_X f^*$$
; notice that f^* on the left and f^* on the right are different things!

2. (multiplicativity) $cl_{X\times Y}(Z \times W) = cl_X(Z) \otimes cl_Y(W)$ if we identify $H^*(X) \otimes H^*(Y)$ with $H^*(X \times Y)$ by Künneth formula.

3. (non-triviality) If P is a point, then
$$cl_P : CH^*(P) \cong \mathbb{Z} \longrightarrow H^*(P) \cong E$$
is the usual ring homomorphism.

All we'll need for now is that if $Y, Z \hookrightarrow X$ intersects properly ($\text{codim}Y \cap Z = \text{codim}Y + \text{codim}Z$) and transversally, and $Y \cap Z$ is still smooth, then $cl_X(Y \times_X Z) = cl_X(Y \cap Z) = cl_X(Y) \smile cl_X(Z)$. We'll only use the case now where Y, Z are half-dimensional and their intersection is a finite collection of reduced points.

Remark 2.9 *A nice and more fully-developed discussion of the formalism of a "Weil cohomology theory" appears in Yves André's (excellent) book Une introduction aux motifs. There is also an old article of Kleiman, Algebraic cycles and the Weil conjectures, as well as a discussion in the Stacks Project. (Different references present the theory slightly differently.)*

2.2 Lecture 3 8/31

Recall from the last time we assume there is a "nice" cohomology theory: We recall its definition and properties briefly.

Let $\mathcal{V}$ be the category of smooth projective varieties over an algebraically closed field $\overline{k}$, E a field of characteristic 0. By a "nice cohomology theory", we mean a functor
$$H^* : \mathcal{V}^{op} \to \{\mathbb{Z}_{\geqslant 0}\text{-graded commutative } E\text{-algebras of finite dimension}\}$$
satisfying the following properties:

I. We have Kunneth isomorphism: For $X, Y \in \mathcal{V}$, there is an graded E-algebra isomorphism
$$H^*(X) \otimes_E H^*(Y) \to H^*(X \times Y), \quad \alpha \otimes \beta \mapsto p^*\alpha \smile q^*\beta$$
where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are projections.

II. For a connected $X \in \mathcal{V}$ with $\dim X = d$, there is an isomorphism of E-vector spaces
$$\text{tr}_X : H^{2d}(X) \xrightarrow{\sim} E,$$
called the **trace map**, inducing Poincaré duality: For all $i \in \mathbb{Z}$, the E-bilinear pairing
$$H^i(X) \times H^{2d-i}(X) \xrightarrow{\smile} H^{2d}(X) \xrightarrow{\text{tr}_X} E$$
is non-degenerate. (Reminder: we also stated the product compatibility $\text{tr}_{X\times Y}(p^*\alpha \smile q^*\beta) = \text{tr}_X(\alpha)\text{tr}_Y(\beta)$ last time, and we do require this.)

Using Poincaré duality, we can define the **Gysin map** f_* associated to any morphism $f : Y \to X$ in $\mathcal{V}$ which is covariant in spaces. More precisely, a morphism $f : Y \to X$ induces a map

$$f_* : H^r(Y) \to H^{r+2c}(X)$$

where $c = \dim X - \dim Y$, and $r \in \mathbb{Z}$. In particular, if f is a closed immersion, then we obtain a map

$$f_* : H^0(Y) \to H^{2\mathrm{codim}_X Y}(X).$$

Note $H^0(Y) \cong (H^{2\dim Y}(Y))^* \cong E$. We define the cycle class of Y in X to be the image of the image of $1 \in E$ in $H^0(Y)$ under f_*:

$$\mathrm{cl}_X(Y) := f_*(1_Y) \in H^{2\mathrm{codim}_X Y}(X).$$

III. We need the cycle class map $cl_X : CH^*(X) \longrightarrow H^{2*}(X)$ with 3 prescribed condition. Details are in the last lecture.

We make one more formal observation: For any $X \in \mathcal{V}$ and $p \in X_{cl}$ with $\dim X = d$, the cycle class of p in X has degree $2d$ in $H^*(X)$ because $\mathrm{codim}_X p = d$. In particular, $\mathrm{tr}_X(\mathrm{cl}_X(p))$ is defined. One can show that

$$\mathrm{tr}_X(\mathrm{cl}_X(p)) = 1.$$

To see this, we note that $1_X \in H^0(X)$ is the multiplicative identity in $H^*(X)$, and that

$$\mathrm{tr}_X(\mathrm{cl}_X(p)) = \mathrm{tr}_X(\mathrm{cl}_X(p) \smile 1_X) = \mathrm{tr}_X(i_*(1_p) \smile 1_X) = \mathrm{tr}_p(1_p \smile i^*(1_X)) = \mathrm{tr}_p(1_p) = 1.$$

Frobenius Morphism Let $k = \mathbb{F}_q$ and $\overline{k} = \overline{\mathbb{F}_q}$. For a smooth projective variety X over k, we write $\overline{X}$ for $X \times_{\mathrm{Spec} k} \mathrm{Spec}\overline{k}$. We define the **Frobenius morphism** $F : \overline{X} \to \overline{X}$ as follows:

First, we describe this map locally on X. By replacing it with an affine open subscheme, assume $X = \mathrm{Spec}A$ where A is a finitely generated k-algebra. Then, we define the absolute q-Frobenius to be the map corresponding to the k-algebra homomorphism

$$\mathrm{Frob}_{A,q} : A \to A, \quad a \mapsto a^q.$$

Define $\mathrm{Frob}_{\overline{A},q} = \mathrm{Frob}_{A,q} \otimes_k \mathrm{id}_{\overline{k}}$. We can glue this morphism to a $\overline{k}$-morphism $F = \mathrm{Frob}_{\overline{X},q} : \overline{X} \to \overline{X}$. In terms of coordinates (once we choose coordinates by choosing a closed immersion $\mathrm{Spec}A \hookrightarrow \mathbb{A}^n_k$), it is given by $(a_1, \ldots, a_n) \mapsto (a_1^q, \ldots, a_n^q)$. Therefore, for all $r \in \mathbb{Z}_{\geqslant 1}$,

$$X(\mathbb{F}_{q^r}) \subseteq X(\overline{k}) = \overline{X}(\overline{k}) = \overline{X}_{cl}$$

is identified with the set $\overline{X}_{cl}^{F^r}$ fixed points of $\overline{X}_{cl}$ under F^r. This gives a geometric interpretation of $\mathbb{F}_{q^r}$-points of X.

The scheme-theoretic version of the set of fixed points of F is given by the closed immersion $\Gamma_F \cap \Delta \to \overline{X} \times \overline{X}$. Since $X(\mathbb{F}_q) = \overline{X}^F$ is finite, the underlying set of points of $\Gamma_F \cap \Delta$ is finite as well.

Lemma 2.10 *If $\overline{X}$ is smooth, then $\Gamma_F \cap \Delta$ is reduced.*

Proof. See the proof of Lemma 2.13.

Recall that for a closed point p, $\mathrm{tr}_X(\mathrm{cl}_X(p)) = 1$. As $\Gamma_F \cap \Delta$ consists of a finite number of closed points, we conclude that

$$\mathrm{tr}_{\overline{X}\times\overline{X}}(\mathrm{cl}_{\overline{X}\times\overline{X}}(\Gamma_F \cap \Delta)) = \#(\Gamma_F \cap \Delta) = \#X(\mathbb{F}_q).$$

Lefschetz Fixed Point Formula Assume we have a nice cohomology theory H^* on $\mathcal{V}$ as described above where $\mathcal{V}$ is the category of smooth projective varieties over an algebraically closed field $\bar{k}$. For $X \in \mathcal{V}$, ($d = \dim X$), and a morphism $f : X \to X$, we will compute the intersection number $(\Gamma_f \cdot \Delta)$. The intersection number is yet to be defined precisely. But, when $\Gamma_f \cap \Delta$ is a finite set of reduced points, we have

$$(\Gamma_F \cdot \Delta) = \#(\Gamma_f \cap \Delta) = \operatorname{tr}_{X \times X}(\operatorname{cl}_X(\Gamma_X \cap \Delta)).$$

In general, the precise formulation is

$$(\Gamma_f \cdot \Delta) = \deg([\Gamma_f] \smile [\Delta]) = \operatorname{tr}_{X \times X}(\operatorname{cl}_X([\Gamma_f] \smile [\Delta]))$$

where $[\Gamma_f] \smile [\Delta]$ is an element of degree 0 cycles on $X \times X$ modulo rational equivalence.

Theorem 2.11 *Keeping the same notations as above, we have*

$$(\Gamma_f \cdot \Delta) = \sum_{r=0}^{2 \dim X} (-1)^r \operatorname{Tr}(f^*|_{H^r X})$$

Proof. Let $\{e_i^r\}$ ($r = 0, \ldots, 2d, i = 1, \ldots, \dim H^r(X)$) be a basis of $H^* X$ and let $\{f_j^{2d-r}\}$ be the Poincaré-dual basis so that

$$\operatorname{tr}_X(f_j^{2d-s} \smile e_i^r) = \begin{cases} 1 & \text{if } r = s, i = j \\ 0 & \text{else} \end{cases}.$$

Equivalently, if we let $e^{2d} \in H^{2d}(X)$ be $\operatorname{tr}_X^{-1}(1)$, then we have

$$f_j^{2d-s} \smile e_i^r = \begin{cases} e^{2d} & \text{if } r = s, i = j \\ 0 & \text{else} \end{cases}.$$

We compute $\operatorname{cl}_{X \times X}(\Gamma_f)$ as follows: Note that $\operatorname{cl}_{X \times X}(\Gamma_f) \in H^*(X \times X)$. Using **I.** Kunneth, we can write it as

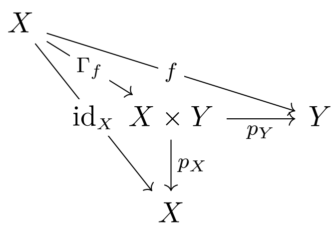
$$\operatorname{cl}_{X \times X}(\Gamma_f) = \sum_{i,r} p^* a_i^r \smile q^* f_i^r \quad \text{where } a_i^r \in H^*(X).$$

Now we first prove the following lemma.

Lemma 2.12 *For any $f : X \to Y$ and $\beta \in H^*(Y)$, we have*

$$f^*(\beta) = p_{X*}\big(\operatorname{cl}_{X \times Y}(\Gamma_f) \smile p_{Y*}(\beta)\big).$$

Proof. The lemma follows from the commutative diagram



Indeed, the projection formula applied to

$$\mathrm{cl}_{X\times Y}(\Gamma_f) \smile p_{Y*}(\beta) = \Gamma_{f*}(1_X) \smile p_{Y*}(\beta)$$

yields

$$\mathrm{cl}_{X\times Y}(\Gamma_f) \smile p_{Y*}(\beta) = \Gamma_{f*}\big(1_X \smile \Gamma_f^*\big(p_Y^*(\beta)\big)\big) = \Gamma_{f*}(1_X \smile f^*(\beta)).$$

Therefore,

$$p_{X*}\big(\,\mathrm{cl}_{X\times Y}(\Gamma_f) \smile p_{Y*}(\beta)\big) = p_{X*}\big(\Gamma_{f*}(1_X \smile f^*(\beta))\big) = \mathrm{id}_{X*}(f^*(\beta)) = f^*(\beta).$$

By Lemma, we have

$$\begin{aligned} f^*(e_j^s) &= p_*(\mathrm{cl}_{X\times X}(\Gamma_f) \smile q^* e_j^s) \\ &= p_*\Big(\big(\sum_{i,r} p^* a_i^r \smile q^* f_i^r\big) \smile q^* e_j^s\Big) \\ &= p_*(p^* a_j^{2d-s} \smile q^* e^{2d}) = a_j^{2d-s} \end{aligned}$$

where the last equality is due to **II.** Therefore,

$$\mathrm{cl}_{X\times X}(\Gamma_f) = \sum_{i,r} p^* f^*(e_i^r) \smile q^* f_i^{2d-r}.$$

Since $\Delta = \Gamma_{\mathrm{id}_X}$, we also obtain the following by taking $f = \mathrm{id}_X$:

$$\mathrm{cl}_{X\times X}(\Delta) = \sum_{i,r} p^* e_i^r \smile q^* f_i^{2d-r} = \sum_{i,r} (-1)^r p^* f_i^{2d-r} \smile q^* e_i^r$$

where the last equality follows by applying the formula using instead the dual bases $\{(-1)^r f_i^{2d-r}\}$ and $\{e_j^r\}$. Therefore,

$$\begin{aligned} &\mathrm{cl}_{X\times X}([\Gamma_f] \smile [\Delta]) = \mathrm{cl}_{X\times X}([\Gamma_f]) \smile \mathrm{cl}_{X\times X}([\Delta]) \\ &= \big(\sum_{i,r} p^* f^*(e_i^r) \smile q^* f_i^{2d-r}\big) \smile \big(\sum_{j,s} (-1)^s p^* f_j^{2d-s} \smile q^* e_j^s\big) \\ &= \sum_{i,r} (-1)^{r+(2d-r)(2d-r)} p^*(f^* e_i^r \smile f_i^{2d-r}) \smile q^* e^{2d}. \end{aligned}$$

Write $f^* e_i^r = \sum_j b_{ij}^r e_j^r$. Then, $f^* e_i^r \smile f_i^{2d-r} = b_{ii}^r (-1)^r e^{2d}$ and thus

$$\sum_i f^* e_i^r \smile f_i^{2d-r} = (-1)^r \operatorname{Tr}(f^*|_{H^r X}) e^{2d}.$$

Therefore, the last expression in the above computation is equal to

$$\sum_r (-1)^r \operatorname{Tr}(f^*|_{H^r X}) p^* e^{2d} \smile q^* e^{2d}.$$

Taking $\mathrm{tr}_{X\times X}$, we conclude that

$$(\Gamma_f \cdot \Delta) = \mathrm{tr}_{X\times X}(\mathrm{cl}_{X\times X}([\Gamma_f] \smile [\Delta])) = \sum_r (-1)^r \mathrm{tr}(f^*|_{H^r X}).$$

For the last equality, recall that tr is multiplicative in spaces.

2.3 Lecture 4, 9/7

Continue to assume that we are in the presence of a nice ("Weil") cohomology theory H^* on $\mathscr{V} := \mathsf{SmProj}_{/\overline{k}}$. Recall that for $X \in \mathscr{V}$ connected of dimension d and $f \colon X \to X$, we have

$$(\Gamma_f \cdot \Delta) = \sum_{r=0}^{2d} (-1)^r \operatorname{Tr} f^*|_{H^r(X)}$$

by Lefschetz's trace formula.

We wish to apply this back to the Weil conjectures, and to do so we will henceforth assume $k = \mathbb{F}_q$. For a k-scheme X, we will write $\overline{X} := X \times_k \overline{k}$. We defined the "relative q-Frobenius" $F \colon \overline{X} \to \overline{X}$ by base-changing the "absolute q-Frobenius" on X (given affine-locally by $a \mapsto a^q$) to $\overline{k}$. We stated (in slightly less generality) but left as an exercise the following lemma, which we will now explain.

Lemma 2.13 *Assume X is of finite type over k. Then $\Gamma_F \cap \Delta$ is reduced and is a finite collection of closed points. Consequently, the intersection number $(\Gamma_F \cdot \Delta)$ is the number of closed points of the intersection.*

Proof. We first reduce to the case of affine X. Recall that $\Gamma_F \cap \Delta$ is characterized by the pullback square

$$\begin{array}{ccc} \Gamma_F \cap \Delta & \longrightarrow & \overline{X} \\ \downarrow & & \downarrow \Delta \\ \overline{X} & \xrightarrow{\Gamma_F} & \overline{X} \times \overline{X}. \end{array} \tag{2}$$

For $U \subseteq X$ affine open, clearly $\overline{U} \subseteq \overline{X}$ is Γ_F-stable, so taking the preimage of $\overline{U} \times \overline{U}$ in the above square shows that $(\Gamma_F \cap \Delta) \cap (\overline{U} \times \overline{U}) = \Gamma_{F|_{\overline{U}}} \cap \Delta_{\overline{U}}$. Now since $\overline{X}$ is covered by such opens $\overline{U}$, the diagonal Δ is covered by the $\overline{U} \times \overline{U}$. Therefore, since the desired property of $\Gamma_F \cap \Delta$ can be checked on an open cover, if we know the claim for the affine opens of X then it follow for X itself.

Thus we assume $X = \operatorname{Spec}(A)$, so $\overline{X} = \operatorname{Spec}(\overline{A})$, where $\overline{A} = A \otimes_k \overline{k}$. By (2), we have $\Gamma_F \cap \Delta = \operatorname{Spec}(\overline{A} \otimes_{\overline{A} \otimes \overline{A}} \overline{A})$, where the maps $\overline{A} \otimes \overline{A} \to \overline{A}$ are the multiplication map $\mathrm{id} \otimes \mathrm{id}$ (corresponding to Δ) and $\mathrm{id} \otimes F$. Some straightforward (but slightly nasty) reasoning with tensors verifies that $\operatorname{Ker}(\mathrm{id} \otimes \mathrm{id})$ is generated by elements of the form $a \otimes 1 - 1 \otimes a$ and that $\overline{A} \otimes_{\overline{A} \otimes \overline{A}} \overline{A} \cong \overline{A}/(\mathrm{id} \otimes F)(\operatorname{Ker}(\mathrm{id} \otimes \mathrm{id}))\overline{A}$. Let us now write down a presentation $A \cong k[t_1, \ldots, t_n]/I$. Then

$$\overline{A} \cong \frac{\overline{k}[t_1, \ldots, t_n]}{\overline{k}I + (\mathrm{id} \otimes F)(\operatorname{Ker}(\mathrm{id} \otimes \mathrm{id}))},$$

and by our remark above, $\operatorname{Ker}(\mathrm{id} \otimes \mathrm{id})$ is generated by $t_i \otimes 1 - 1 \otimes t_i$. Thus $\overline{A}$ is isomorphic to a quotient of $\overline{k}[t_1, \ldots, t_n]/(t_1 - t_1^q, \ldots, t_n - t_n^q)$. Since each polynomial $t_i - t_i^q$ is separable over $\overline{k}$, this shows that $\overline{A}$ is a product of copies of $\overline{k}$. □

The previous lemma allows us to apply Lefschetz's trace formula to $\Gamma_F \cap \Delta$ in a useful way, but before doing this, we need one more fact about the morphism F.

Lemma 2.14 *Assume X is smooth projective over k and is geometrically connected of dimension d. Then $F \colon \overline{X} \to \overline{X}$ is finite flat of degree q^d.*

Proof sketch. (F finite). This easy and only uses that X is of finite type over k.

(F flat). This uses the smoothness of X (i.e. that $\overline{X}$ is regular). Then the flatness of F is an immediate consequence of "miracle flatness", which says that a map of smooth varieties with equidimensional fibers is flat. Alternatively, one can proceed directly by showing that for any ring A containing $\mathbb{F}_p$, the absolute Frobenius on A is flat (reduce to the complete case and hence, by Cohen's structure theorem, assume $A = \ell[\![t]\!]$ for some finite $\ell | k$).

(Degree of F). One can compute the degree of F on function fields (i.e. on the generic fiber). Here we note that $k(X)$ is a finite separable extension of $k(T_1, \dots, T_d)$. ◻

Actually, what we really wanted is to know what F is doing on H^{2d}. First, let us briefly recall that for a closed subscheme $W \subseteq X$, we write $[W]$ for the cycle class associated to W, which is defined to be

$$[W] := \sum_{Z \subseteq W} m_Z [Z_{\mathrm{red}}], \qquad m_Z := \mathrm{length}_{\mathscr{O}_{X,\eta_Z}} \mathscr{O}_{W,\eta_Z},$$

where Z ranges over the irreducible components of W and η_Z is the generic point of Z.

Corollary 2.15 *Let X be as in the previous lemma. Then F acts on $H^{2d}(\overline{X})$ as multiplication by q^d.*

Proof. We know that $H^{2d}(\overline{X})$ is spanned over E by $\mathrm{cl}_{\overline{X}}[P]$ for any $P \in \overline{X}_{\mathrm{cl}}$. By one of the axioms of H^*, we have $F^* \mathrm{cl}_{\overline{X}}[P] = \mathrm{cl}_{\overline{X}} F^*[P]$, which, since F is flat, equals $\mathrm{cl}_{\overline{X}}[F^{-1}P]$. But $[F^{-1}P] = q^d[P']$, where P' is the unique closed point mapping to P under F (filling in the details is left as an exercise). ◻

On to the Weil conjectures. We are now able to derive rationality and the functional equation from the formalism:

Theorem 2.16 *Assume X is smooth projective over k and geometrically connected of dimension d.*

1. *We have $Z(X,t) \in \mathbb{Q}(t)$. We moreover have*

$$Z(X,t) = \frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\cdots P_{2d}(t)},$$

where $P_r = \det(1 - F^*t|_{H^r(\overline{X})})$ (which lies in $E[t]$). Also $P_0(t) = 1 - t$ and $P_{2d}(t) = 1 - q^d t$.

2. *We have $Z(X, 1/(q^d t)) = \pm q^{d\chi/2} t^{\chi} Z(X,t)$, where $\chi := \sum_{r=0}^{2d} (-1)^r \dim_E H^r(\overline{X})$ (the Euler characteristic of X, which equals $(\Delta \cdot \Delta)$ by Lefschetz's trace formula).*

Proof. (*1*). Write $N_m := \#X(\mathbb{F}_{q^m}) = \#\overline{X}^{F^m}$. By the lemma about $\Gamma_F \cap \Delta$ and Lefschetz's trace formula,

$$N_m = (\Gamma_{F^m} \cdot \Delta) = \sum_{r=0}^{2d} (-1)^r \,\mathrm{Tr}\,(F^m)^*|_{H^r(\overline{X})}.$$

Letting $\alpha_1, \dots, \alpha_s$ denote the eigenvalues of F^* on $H^r(\overline{X})$ listed with the appropriate multiplicity, we have $\operatorname{Tr}(F^m)^*|_{H^r(\overline{X})} = \operatorname{Tr}(F^*)^m|_{H^r(\overline{X})} = \sum_i \alpha_i^m$ for each m. Then

$$\log \frac{1}{P_r(t)} = \log \frac{1}{\prod_i (1 - t\alpha_i)} = \sum_{i=1}^{s} -\log(1 - t\alpha_i) = \sum_{i=1}^{s} \sum_{m \geqslant 1} \frac{t^m \alpha_i^m}{m}$$
$$= \sum_{m \geqslant 1} \left(\operatorname{Tr}(F^m)^*|_{H^r(\overline{X})} \right) \frac{t^m}{m}.$$

So we get

$$\log Z(X,t) = \sum_{m \geqslant 1} N_m \frac{t^m}{m} = \sum_{m \geqslant 1} \sum_{r=0}^{2d} (-1)^r \left(\operatorname{Tr}(F^m)^*|_{H^r(\overline{X})} \right) \frac{t^m}{m} = \sum_{r=0}^{2d} (-1)^r \log \frac{1}{P_r(t)},$$

which is the desired formula for $Z(X,t)$. To obtain rationality, we observe that $Z(X,t)$ lies in both $\mathbb{Q}[\![t]\!]$ (by definition) and $E(t)$ (since each $P_r(t)$ is in $E[t]$), and use that $\mathbb{Q}[\![t]\!] \cap E(t) = \mathbb{Q}(t)$—a proof is given below. Finally, the formulas for $P_0(t)$ and $P_{2d}(t)$ are obtained in *(2)* below.

(2). Under the Poincaré duality map $H^r(\overline{X}) \times H^{2d-r}(\overline{X}) \to H^{2d}(\overline{X}) \xrightarrow{\sim} E$, we have

$$(F^*\alpha, F^*\beta) \mapsto \operatorname{Tr}_{\overline{X}}(F^*\alpha \cup F^*\beta) = \operatorname{Tr}_{\overline{X}} F^*(\alpha \cup \beta) = q^d \operatorname{Tr}_{\overline{X}}(\alpha \cup \beta)$$

by the properties of the cup product and the description of F acting on $H^{2d}(\overline{X})$. It follows that if $\alpha_{r,1}, \dots, \alpha_{r,b_r}$ are the eigenvalues of F^* acting on $H^r(\overline{X})$ (with multiplicity, so that $b_r = \dim_E H^r(\overline{X})$), then there is an equality of multisets between $\alpha_{r,1}, \dots, \alpha_{r,b_r}$ and $q^d/\alpha_{2d-r,1}, \dots, q^d/\alpha_{2d-r,b_{2d-r}}$. Thus

$$P_r(t) = \prod_{i=1}^{b_r} (1 - \alpha_{r,i} t), \qquad P_{2d-r}(t) = \prod_{i=1}^{b_r} \left(1 - \frac{q^d}{\alpha_{r,i}} t \right),$$

from which we compute that

$$P_r\left(\frac{1}{q^d t}\right) = \prod_{i=1}^{b_r} \left(1 - \frac{\alpha_{r,i}}{q^d t} \right) = \prod_{i=1}^{b_r} \frac{\alpha_{r,i}}{-q^d t} \left(1 - \frac{q^d}{\alpha_{r,i}} t \right) = (-q^d t)^{-b_r} P_{2d-r}(t) \prod_{i=1}^{b_r} \alpha_{r,i}.$$

Using (a), this gives, for the zeta-function as a whole, that

$$Z\left(X, \frac{1}{q^d t}\right) = \prod_{r=0}^{2d} P_r\left(\frac{1}{q^d t}\right)^{(-1)^{r+1}} = (-q^d t)^{\chi} Z(X,t) \prod_{r,i} \alpha_{r,i}^{(-1)^{r+1}}.$$

To take care of the product of eigenvalues, write $\xi_r := \prod_i \alpha_{r,i}$, so that the quantity of interest is $\Xi := \prod_r \xi_r^{(-1)^{r+1}}$. Due to the pairing between the eigenvalues in degree r and in degree $2d-r$, we have $\xi_r \xi_{2d-r} = (q^d)^{b_r}$. The factor ξ_d is more subtle. Some eigenvalues in degree d might pair with themselves, in which case we can only say that they equal $\pm q^{d/2}$. Thus $\xi_d = \pm (q^{d/2})^m (q^d)^n$, where m is the number of degree-d eigenvalues which pair with themselves; since $m + 2n = b_d$, we find that $\xi_d = \pm q^{db_d/2}$. Consequently,

$$\Xi = \frac{\prod_{\substack{r<d \\ \text{odd}}} \xi_r \xi_{2d-r}}{\prod_{\substack{r<d \\ \text{even}}} \xi_r \xi_{2d-r}} \xi_d^{(-1)^{d+1}} = \frac{\prod_{\substack{r<d \\ \text{odd}}} q^{db_r}}{\prod_{\substack{r<d \\ \text{even}}} q^{db_r}} \cdot \pm q^{d \cdot (-1)^{d+1} b_d/2} = \pm q^{-d\chi/2}.$$

Plugging this in above completes the proof. □

The rationality of $Z(X,t)$ used the following fact.

Lemma 2.17 $\mathbb{Q}[\![t]\!] \cap E(t) = \mathbb{Q}(t)$.

Proof. Let $f(t) := \sum_i a_i t^i \in \mathbb{Q}[\![t]\!] \cap E(t)$. Then there exists an identity of the form

$$(x_0 + x_1 t + \cdots + x_m t^m) f(t) = y_0 + y_1 t + \cdots + y_n t^n$$

for $x_0, \ldots, x_m \in E$ not all zero. Consider now, putting $a_i := 0$ for $i < 0$, the matrices

$$A_r := \begin{pmatrix} a_{n+1} & \cdots & a_{n+1-m} \\ a_{n+2} & \cdots & a_{n+2-m} \\ \vdots & \ddots & \vdots \\ a_{n+r} & \cdots & a_{n+r-m} \end{pmatrix}.$$

Viewed as a linear transformation $A_{r,E} \colon E^{m+1} \to E^r$, there exists a common nonzero vector each $\operatorname{Ker} A_{r,E}$, namely $(x_0, \ldots, x_m)^{\mathrm{t}}$. Conversely, the existence of such a vector implies $f \in E(t)$ (and moreover that f can be written with numerator of degree $\leqslant n$ and denominator of degree $\leqslant m$).

We can also view A_r as a linear transformation $A_{r,\mathbb{Q}} \colon \mathbb{Q}^{m+1} \to \mathbb{Q}^r$. This cannot be injective, since otherwise $A_{r,E} = A_{r,\mathbb{Q}} \otimes_{\mathbb{Q}} \mathrm{id}_E$ would be injective by the flatness of E over $\mathbb{Q}$. Thus each $\operatorname{Ker} A_{r,\mathbb{Q}}$ is nonzero. Clearly $\operatorname{Ker} A_{1,\mathbb{Q}} \supseteq \operatorname{Ker} A_{2,\mathbb{Q}} \supseteq \cdots$, hence there exists a common nonzero vector lying in each $\operatorname{Ker} A_{r,\mathbb{Q}}$. As noted in the previous paragraph, this implies that $f \in \mathbb{Q}(t)$. □

Thus we find ourselves amply motivated to construct such an H^*!

Remark 2.18 *The Riemann hypothesis does not follow formally from what we've discussed, but there does exist a still-conjectural formalism inspired by Hodge theory, known as Grothendieck's standard conjectures, which would imply the Riemann hypothesis. (More on this later.)*

As our very first step toward finding a Weil cohomology theory, we ask: what should we take as the field E? Over $\mathbb{C}$, singular cohomology can use $\mathbb{Q}$-coefficients, but this won't work for varieties over finite fields, due to the following example of Serre:

Example 2.19 *Let X be a supersingular elliptic curve over $\mathbb{F}_q$, i.e.* $\operatorname{End} X$ *is an order in a nonsplit quaternion algebra ramified exactly at p and ∞ (after possibly enlarging q). If H^* is a $\mathbb{Q}$-valued Weil cohomology theory, then by functoriality we get an algebra homomorphism* $(\operatorname{End} X) \otimes_{\mathbb{Z}} \mathbb{Q} \to \operatorname{End}_{\mathbb{Q}} H^1(\overline{X})$*, the latter isomorphic to $M_2(\mathbb{Q})$ by Lefschetz's fixed point theorem* ($0 = (\Delta \cdot \Delta) = 2 - \dim_{\mathbb{Q}} H^1(\overline{X})$)*. But there is no such algebra homomorphism!*

In fact, the same argument shows that there is no $\mathbb{Q}_p$- or $\mathbb{R}$-valued theory. But, as we will see, there is a $\mathbb{Q}_\ell$-valued theory for all primes $\ell \neq p$.

3 Zeta function of a curve

3.1 Lecture 5, 9/12

See for example [Lor21], or lecture notes on Zeta functions in algebraic geometry by Mircea Mustaţă found here: https://dept.math.lsa.umich.edu/~mmustata/zeta_book.pdf

Remark 3.1 *(1) For a projective smooth geometrically connected curve $X/\mathbb{F}_q$ of genus g, one can show using just Riemann-Roch, that*

$$Z\left(X/\mathbb{F}_q, t\right) = \frac{P_1(t)}{(1-t)(1-qt)},$$

where $P_1(t) \in \mathbb{Z}[t]$ with degree $2g$. The Riemann hypothesis takes more, but can be proven using intersection theory on $X \times X$.

(2) The Riemann hypothesis says

$$P_1(t) = \prod_{j=1}^{2g} (1 - \alpha_j t) \in \overline{\mathbb{Q}}[t]$$

where each α_j is an algebraic integer such that for all embeddings $i\colon \overline{\mathbb{Q}} \to \mathbb{C}$, it holds $|i(\alpha_j)| = \sqrt{q}$ for all j. This implies, since

$$Z\left(X/\mathbb{F}_q, t\right) = \exp\left(\sum_{m\geqslant 1} \#X\left(\mathbb{F}_q\right) \frac{t^m}{m}\right),$$

that $\#X\left(\mathbb{F}_q\right) = 1 + q^m - \sum_{j=1}^{2g} \alpha_j^m$ and so

$$\left|\#X\left(\mathbb{F}_q\right) - (q^m + 1)\right| \leqslant 2g\sqrt{q^m}.$$

Proof. We give a direct proof of item (1) here using Riemann-Roch. First we have the degree map

$$\operatorname{Pic}(X) \to \mathbb{Z}.$$

with image $e\mathbb{Z}$ (we will see $e = 1$ later). Denote degree d line bundle in $\operatorname{Pic}(X)$ by $\operatorname{Pic}^d(X)$. For each d, $\# \operatorname{Pic}^d(X) = \# \operatorname{Pic}^0(X)$ for $e|d$, which is the size of the kernel of the degree map, and we want to show this quantity is finite. Indeed, by Riemann-Roch, for $d >> 0$, every divisor is equivalent to an effective divisor, but there are only finitely many effectivve divisor of degree d (notice that for P a closed point, the degree of P as a divisor is exactly $[\kappa(P) : \mathbb{F}_q]$, combine this with finitness of $\mathbb{F}_{q^n}$-valued points). Given this finiteness of $\operatorname{Pic}^d(X)$, we can view $Z(X/\mathbb{F}_q, t)$ as generating function of effective divisors

$$Z(X/\mathbb{F}_q, t) = \sum_{D\geqslant 0} t^{\deg(D)}.$$

We further split the sum

$$\sum_{D\geqslant 0} t^{\deg(D)} = \sum_{\deg(D)\leqslant 2g-2} t^{\deg(D)} + \sum_{\deg(D)\geqslant 2g-1} t^{\deg(D)}.$$

We now do the following calculation for $\mathscr{L} \in \operatorname{Pic}^d(X)$ for $d \geqslant 2g-1$

$$\#\{D \geqslant 0 : \mathscr{O}_X(D) \simeq \mathscr{L}\} = \#\mathbb{P}^{h^0(X,\mathscr{L})}_{\mathbb{F}_q}(\mathbb{F}_q) = \frac{q^{h^0(X,\mathscr{L})}-1}{q-1} = \frac{q^{d-g+1)}}{q-1}$$

where $h^0(X,\mathscr{L})$ denote $\dim(H^0(X,\mathscr{L})) = d+g-1$. Let h denote $\#\operatorname{Pic}^d(X)$ for any $e|d$, and pick minimal d_0 such that $ed_0 \geqslant 2g-1$. This gives

$$\begin{aligned} Z(X/\mathbb{F}_q, t) &= \sum_{\deg(D)\leqslant 2g-2} t^{\deg(D)} + h\sum_{d\geqslant d_0} \frac{q^{ed-g+1}-1}{q-1} t^{ed} \\ &= \sum_{\deg(D)\leqslant 2g-2} t^{\deg(D)} + \frac{h}{q-1}\left(q^{1-g}\frac{(qt)^{ed_0}}{1-(qt)^e} - \frac{t^{ed_0}}{1-t^e}\right) \\ &= \frac{f(t^e)}{(1-t^e)(1-(qt)^e)} \end{aligned}$$

for some $f(t) \in \mathbb{Q}[t]$ with degree $\leqslant \max\{d_0+1, 2+(2g-2)/e\}$. Since $Z(X/\mathbb{F}_q, t)$ has integral coefficient, clearly so does $f(t)$. We see that $Z(X/\mathbb{F}_q, t)$ has a pole at $t=1$ of order e. For ξ a primitive e-th root of uninty,

$$Z(X_{\mathbb{F}_{q^e}}/\mathbb{F}_{q^e}, t^e) = \prod_{i=1}^{e} Z(X/\mathbb{F}_q, \xi^i t) = Z(X/\mathbb{F}_q, t)^e.$$

Same argument shows the LHS has a pole at $t=1$ of order 1, but RHS has a pole at $t=1$ of order e, hence $e=1$. We also claim that

$$Z(X/\mathbb{F}_q, \frac{1}{qt}) = q^{1-g}t^{2-2g}Z(X/\mathbb{F}_q, t)$$

. $g=0$ is easy, so we assume $g \geqslant 1$. Directly working with $\sum_{\deg(D)\leqslant 2g-2} t^{\deg(D)}$ and $\sum_{\deg(D)\geqslant 2g-1} t^{\deg(D)}$ yields the desired result. This also implies $\deg(f) = 2g$.

4 Review of smooth and étale morphisms

(1) In a "varieties" course, a finite type scheme over a field k is smooth, if $X_{\overline{k}}$ is regular.

Definition 4.1 (S1) *A morphism $f\colon X \to S$ of schemes is smooth, if f is locally of finite presentation, flat and for all $s \in S$, the fiber $X_s \to \operatorname{Spec}(\kappa(s))$ is smooth in the previous sense (geometrically regular). Here, locally of finite presentation means that for all open affine $U \subset X$ mapping to an open affine $V \subset S$, the ring homomorphism $\mathscr{O}_S(V) \to \mathscr{O}_X(U)$ is finitely presented.*

(2) Another source of intuition from manifold theory: If $f\colon X \to S$ is a morphism of C^∞-manifolds, "smoothness" means f is a submersion ($df_x\colon T_xX \to T_{f(x)}S$ is surjective). Locally on X, there are trivial fibrations:

for all $x \in X$ there are neighborhoods $x \in U \subset X$ and $f(x) \in V \subset S$ such that $f(U) \subset V$ and

$$\begin{array}{ccccc} X & \supset & U & \simeq & V \times \mathbb{R}^d \\ \downarrow & & \downarrow f & \swarrow \text{proj} & \\ S & \supset & V & & \end{array}$$

This is too strong for the Zariski topology, but we replace with

Definition 4.2 (S2) *Let $f\colon X \to S$ be a morphism of schemes. Let $x \in X$. Then f is smooth at x (and f is smooth if it is smooth at every $x \in X$), if there are affine open neighborhoods*

$$\begin{array}{ccccc} x & \in & U & \subset & X \\ \downarrow & & \downarrow f & & \\ f(x) & \in & V & \subset & S \end{array}$$

and $d \in \mathbb{Z}_{\geqslant 0}$ and a map

$$\begin{array}{ccc} U & \xrightarrow{\pi} & \mathbb{A}^d_V \\ \downarrow f & \swarrow \text{proj} & \\ V & & \end{array}$$

such that the diagram commutes and π is étale.

So now we need what an étale morphism is. We find our way to a definition by thinking of what Definition 4.1 (S1) becomes if we require fibers to be zero-dimensional.

Lemma 4.3 *If $X \to \mathrm{Spec}(k)$ is zero-dimensional and smooth in the sense of Definition 4.1 (S1), what does X look like?*

Locally, X looks like $\mathrm{Spec}(A)$ for a finite type k-algebra A. In fact, A is a finite k-algebra (A is Artinian), and the smoothness in sense of S1 amounts to requiring $A_{\overline{k}} = A \otimes_k \overline{k}$ is reduced.

Lemma 4.4 *If $k \to A$ is finite and $A_{\overline{k}}$ is reduced, then $A \simeq \prod_{i=1}^n k_i$, where each k_i/k is a finite separable extension.*

Proof. Exercise.

This gives a first definition of étale morphisms.

Definition 4.5 (E1) *A morphism $f\colon X \to S$ of schemes is étale, if it is locally finitely presented, flat, and all fibers $X_S \to \mathrm{Spec}(\kappa)(s)$ for $s \in S$ are isomorphic to $\coprod_{i\in I} \mathrm{Spec}(k)_i$, with $k_i/\kappa(s)$ a finite separable field extension. (It can be infinite, we only require local finite presentation.)*

(Put another way, for all $x \in X$, $\mathscr{O}_{X,x}/\mathfrak{m}_{s,f(x)}\mathscr{O}_{X,x} \simeq \mathscr{O}_{X_{f(x)},x}$ is a finite separable field extension at $\kappa(f(x))$.)

Via Definition 4.5 E1, the Definition 4.2 S2 now makes sense. We will see how S2 easily gives a differential criterion for smoothness once we think about étale maps.

Proposition 4.6 *Let $f\colon X \to S$ be locally finite type. TFAE:*

(1) for all $s \in S$, $X_s \simeq \coprod \operatorname{Spec}(k)_i$, with finite separable field extensions $k_i/\kappa(s)$.

(2) $\Omega^1_{X/S} = 0$.

(3) $\Delta_{X/S}\colon X \to X \times_S X$ is an open immersion.

Proof. Suppose (1) holds and let $x \in X$, $s = f(x) \in S$. Then

$$\Omega^1_{X/S} \otimes_{\mathscr{O}_S} \kappa(s) \simeq \Omega^1_{X_s/\kappa(s)} = 0,$$

since $X_s \simeq \coprod \operatorname{Spec}(k)_i$ by assumption (exercise). Thus the finite type $\mathscr{O}_{X,x}$-module

$$\left(\Omega^1_{X/S}\right)_x \simeq \Omega^1_{\left(\mathscr{O}_{X,x}/\mathscr{O}_{S,f(x)}\right)} \subset \mathfrak{m}_{S,f(x)} \cdot \left(\Omega^1_{X/S}\right)_x,$$

so by Nakayama's lemma we obtain $\left(\Omega^1_{X/S}\right)_x = 0$ for all x, which implies (2).

Now suppose (2), i.e. $\Omega^1_{X/S} = 0$. Then

$$\begin{array}{ccc} \Delta_{X/S}\colon X & \hookrightarrow & X \times_S X \\ & \searrow^{i,\ \text{closed}} & \uparrow \text{open} \\ & & U \end{array}$$

One definition of $\Omega^1_{X/S}$ is: Let $\mathscr{I}_X$ be the quasi-coherent ideal sheaf on U defining $i_X\colon X \hookrightarrow U$. Then $\Omega^1_{X/S} = i_X^*\left(\mathscr{I}_X/\mathscr{I}_X^2\right)$.

There is an open neighborhood of $\Delta(X)$ in U of the form $\operatorname{Spec}(A \otimes A)$, where

$$\begin{array}{ccc} \operatorname{Spec}(A) & \subset & U \\ \downarrow & & \downarrow f \\ \operatorname{Spec}(R) & \subset & S \end{array}$$

are open affines, and for the multiplication map $\text{mult}\colon A \otimes A \to A$, $I = \ker(\text{mult})$, we are looking at the module I/I^2. We want to show that $I/I^2 = 0$ implies that $\Delta_{X/S}$ is an open immersion. Since $R \to A$ is finite type, I is finitely generated by $(a_i \otimes 1 - 1 \otimes a_i)_i$, where the a_i generate A as an R-algebra. Since we are given $I/I^2 = 0$, we get $\mathscr{I}_x/\mathscr{I}_X^2 = 0$ for all $x \in \operatorname{Spec}(A)$ and hence $\mathscr{I}_x = 0$ by Nakayama's lemma. Since I is finitely generated, the quasi coherent sheaf $\tilde{\mathscr{I}}$ on $\operatorname{Spec}(A \otimes A)$ is zero in a neighborhood of x, so $\mathscr{I}_x|_V = 0$ for some open neighborhood V of $\Delta(X)$ in U.

Now suppose that $\Delta_{X/S}$ is an open immersion and let $s \in S$ and let $\overline{k}$ be an algebraic closure of $\kappa(s)$. We want to show that

$$X_{\overline{s}} = X_s \times \operatorname{Spec}(\overline{k}) \to \operatorname{Spec}(\overline{k})$$

is a disjoint union of copies of $\operatorname{Spec}(\overline{k})$. Let $x \in (X_{\overline{s}})_{\text{cl}}$ corresponding to $\operatorname{Spec}(\overline{k}) \xrightarrow{x} X_{\overline{s}}$. Then

$$\begin{array}{ccc} X_{\overline{s}} & \xrightarrow{\Delta_{X/S}} & X_{\overline{s}} \times_{\overline{k}} X_{\overline{s}} \\ x \uparrow & & \uparrow (x \circ f_{\overline{s}}, \text{id}) \\ \operatorname{Spec}(\overline{k}) & \xrightarrow{x} & X_{\overline{s}} \end{array}$$

is a fiber product diagram, so $\mathrm{Spec}(\overline{k}) \xrightarrow{x} X_{\overline{s}}$ is an open immersion, so $X_{\overline{s}}$ is a discrete topological space. $\mathscr{O}_{X_{\overline{s}},x}$ is an Artinian local ring with residue field $\overline{k}$, so

$$\mathrm{Spec}\left(\mathscr{O}_{X_{\overline{s}},x} \otimes_{\overline{k}} \mathscr{O}_{X_{\overline{s}},x}\right)$$

is a singleton. Since

$$\mathrm{Spec}(\mathscr{O}_{X_{\overline{s}},x}) \to \mathrm{Spec}\left(\mathscr{O}_{X_{\overline{s}},x} \otimes_{\overline{k}} \mathscr{O}_{X_{\overline{s}},x}\right)$$

is an open immersion, it is an isomorphism, and we win, since $\mathscr{O}_{X_{\overline{s}},x} \simeq \overline{k}$.

Definition 4.7 *A morphism $f\colon X \to S$ of schemes is called unramified, if it satisfies any of the three equivalent conditions of Proposition 4.6.*

So we have the definition

Definition 4.8 (E2) *An étale map $f\colon X \to S$ is a flat, locally finitely presented and unramified morphism (likewise, define "étale at x", "unramified at x").*

Example 4.9 *(1) open immersions are étale*

(2) any immersion is unramified

(3) Let A be a Dedekind domain, $K = \mathrm{Frac}(A)$, L/K a finite, separable extension and let B be the integral closure of A in L. Then B is a Dedekind domain and $A \to B$ is finite and flat. Let

$$\begin{array}{ccc} \mathfrak{q} & \in & \mathrm{Spec}(B) \\ \big\downarrow & & \big\downarrow f \\ \mathfrak{p} & \in & \mathrm{Spec}(A) \end{array}$$

be a map. To say that f is unramified at $\mathfrak{q}$ means $\mathfrak{p}B_{\mathfrak{q}} = \mathfrak{q}B_{\mathfrak{q}}$ and $\kappa(\mathfrak{q}) = B_{\mathfrak{q}}/\mathfrak{q}B_{\mathfrak{q}}$ is a finite separable extension of $\kappa(\mathfrak{p})$. (We recover the usual algebra/number theory notion.)

Note:

$$\Omega^1_{B/A} \otimes K \simeq \Omega^1_{B\otimes K/K} \simeq \Omega^1_{L/K} = 0,$$

since L/K is separable, and so $\Omega^1_{B/A}$ is a finite torsion B-module supported at a finite set $S \subset \mathrm{Spec}(B)$ of maximal ideals, and $\mathrm{Spec}(B)\backslash S \xrightarrow{f} \mathrm{Spec}(A)\backslash f(S)$ is now étale.

For $X \to S$ locally finite type of locally Noetherian,

$$\mathrm{supp}\left(\Omega^1_{X/S}\right) = V\left(\mathrm{Ann}_{\mathscr{O}_X}\left(\Omega^1_{X/S}\right)\right) \hookrightarrow X$$

is a closed subscheme and this support is called the "branch locus" of f.

4.1 Lecture 6, 9/14

Recall: A morphism $f : X \to S$ of schemes is called *étale* at $x \in X$ if f is locally of finite presentation at x, flat at x (i.e., the induced map $\mathscr{O}_{S,f(x)} \to \mathscr{O}_{X,x}$ is flat), and unramified at x (i.e., $\frac{\mathscr{O}_{X,x}}{\mathfrak{m}_{S,f(x)}\mathscr{O}_{X,x}}$ is a finite separable field extension of $\kappa(f(x))$).

Continuing with our list of examples:

Example 4.10 *If R is any unital commutative ring, a **standard étale** morphism $X \to \operatorname{Spec}(R)$ is one of the form $A = \frac{R[t]_h}{(g)}$, where $g, h \in R[t]$ with g monic and $g'(t)$ invertible in A (see Figure 1).*

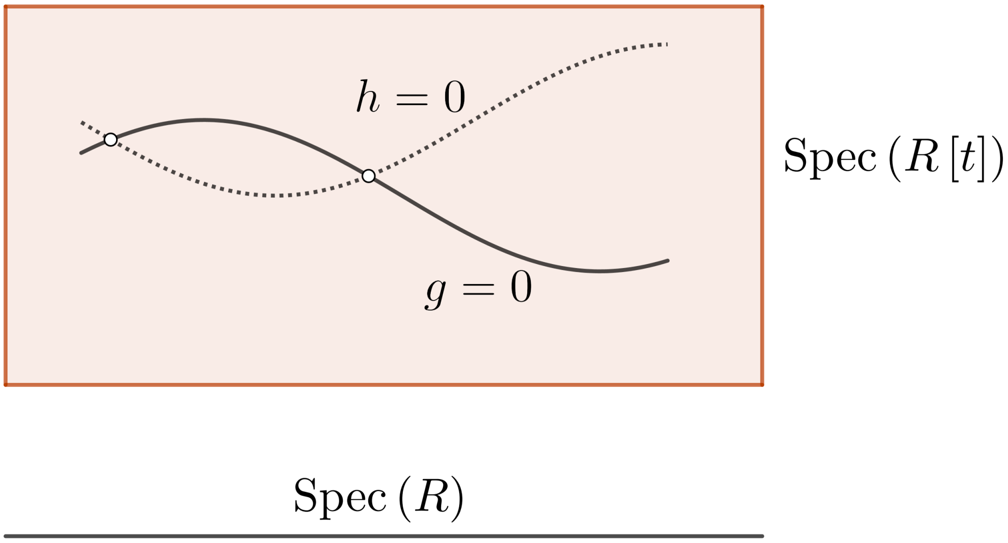


Figure 1: g' is invertible in A if and only if in all fibers (away from $h = 0$) g has simple roots.

Calculation: If f is standard étale, then it is étale (using that g is monic, so $R[t]/(g)$ is free and finite over R, so $R \to A$ is flat of finite presentation).

Remember: *For all R-algebra surjections $B \to B/I$ we have*

$$I/I^2 \xrightarrow{d} \Omega^1_{B/R} \otimes_B B/I \to \Omega^1_{B/I} \to 0.$$

In our case,

$$\Omega^1_{A/R} \cong \frac{\frac{R[t]_h}{(g)} dt}{\text{submodule generated by } dg},$$

and since g' is invertible in $\frac{R[t]_h}{(g)}$ then there exist $a, b \in R[t]_h$ such that $ag + bg' = 1$, thus $agdt + bdg = dt$ (using $dg = g'dt$). Thus dt belongs to te submodule of $\frac{R[t]_h}{(g)}$ spanned by dg.

Proposition 4.11 *Let $f : X \to S$, $x \in X$, $f(x) \in V \subseteq S$ (where V is an affine open subscheme of S). Then f is étale at x if and only if there is an affine open subscheme U of X containing x such that $f(U) \subseteq V$ and $f : U \to V$ is standard étale.*

In particular, $\{x \in X : f \text{ is étale at } x\}$ *is open.*

Proof. See [Mil80, Chapter I, §3, Theorem 3.14] or https://stacks.math.columbia.edu/tag/02GT. ∎

Example 4.12 • *Given* $X \xrightarrow{f} Y \xrightarrow{g} S$*, if* f *and* g *are étale, then so is* $g \circ f$*.*

- *Given* $X \xrightarrow{f} S$ *and* $S' \to S$*, if* f *is étale then so is* $X \times_S S' \to S'$*.*

$$\begin{array}{ccc} X \times_S S' & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f \text{ étale} \Rightarrow g \text{ étale}} \\ S' & \longrightarrow & S \end{array}$$

- *Given* $\begin{array}{ccc} X & \xrightarrow{h} & Y' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle g} & \\ & S & \end{array}$ *If* f *and* g *are étale, so is* h*.*

Example 4.13 $\mathbb{G}_{m,R} = \operatorname{Spec}\left(R\left[t, t^{-1}\right]\right) \to \mathbb{G}_{m,R}$ *given by* $z \mapsto z^n$ *is étale if and only if* $n \in R^{\times}$*.*

$\mathbb{A}^1_R \to \mathbb{A}^1_R$ *given by* $z \to z^n$ *is not étale for* $n > 1$ *at* 0*.*

Example 4.14 *Let* $f : X \to S$ *be locally of finite type with* S *locally Noetherian. Let* $x \in X$ *be such that* $\kappa(x) = \kappa(f(x))$*, and let* $\widehat{\mathscr{O}_{S,f(x)}} \xrightarrow{\widehat{f_x^\sharp}} \widehat{\mathscr{O}_{X,x}}$ *be the map on complete local rings. Then* f *is étale at* x *if and only if* $\widehat{f_x^\sharp}$ *is an isomorphism.*

Typical example: $X \to S$ *a map of varieties over an algebraically closed field, and* x *a closed point.*

Remark 4.15 *Proof not so hard but omitted. The Noetherian hypothesis is used in the form: for* ***Noetherian*** *local ring* A*, the map* $A \to \widehat{A} = \varprojlim A/\mathfrak{m}_A^n$ *is faithfully flat.*

Now a basic reformulation of what it means to be étale/smooth/unramified.

Definition 4.16 *A closed immersion* $S_0 \hookrightarrow S$ *is an nth order if* $\mathscr{I} = \ker(\mathscr{O}_S \to \iota_* \mathscr{O}_{S_0})$ *satisfies* $\mathscr{I}^{n+1} = 0$*.*

Example 4.17 $\operatorname{Spec} \begin{smallmatrix}(\mathbb{Z}/p\mathbb{Z}) \\ (\mathbb{C})\end{smallmatrix} \to \operatorname{Spec} \begin{smallmatrix}(\mathbb{Z}/p^2\mathbb{Z}) \\ (\mathbb{C}[t]/(t^2))\end{smallmatrix}$.

Definition 4.18 *A morphism* $f : X \to S$ *of schemes is formally smoth/unramified/étale, if for each* 1*st order* $i : T_0 \to T$ *thickening of affine schemes and any commutative diagram*

$$\begin{array}{ccc} T_0 & \longrightarrow & X \\ \big\downarrow & \overset{u}{\nearrow} & \downarrow{\scriptstyle f} \\ T & \longrightarrow & S \end{array}$$

there exists $\underbrace{\text{at least one lift}}_{\text{formally smooth}} / \underbrace{\text{at most one lift}}_{\text{formally unramified}} / \underbrace{\text{exactly one lift}}_{\text{formally étale}}$ $u : T \to X$*.*

Example 4.19 *Note that there are no finiteness hypotheses here, and* $\operatorname{Spec}(\mathbb{C}[t]) \xrightarrow{f} \operatorname{Spec}(\mathbb{C})$ *is formally smooth.*

(Exercise: see what this has to do with Hensel's lemma).

Theorem 4.20 *Let $f : X \to S$ be a morphism of schemes. Then f is smooth/unramified/étale if and only is f is formally smooth/unramified/étale and locally of finite preentation/locally of finite type/locally of finite presentation.*

Proof omitted, but here are a couple calculations that give the flavor of étale $\Rightarrow$ formally étale.

Let $R \to A$ be a ring homomorphism, and consider square-zero thickening

$$\begin{array}{ccc} \operatorname{Spec}(B/I) & \longrightarrow & \operatorname{Spec}(A) \\ \downarrow & & \downarrow \\ \operatorname{Spec}(B) & \longrightarrow & \operatorname{Spec}(R) \end{array}$$

which is dual to

$$\begin{array}{ccc} B/I & \longleftarrow & A \\ \uparrow & \swarrow_{u_1, u_2} & \uparrow \\ B & \longleftarrow & R \end{array}$$

Suppose we have two lifts u_1, u_2 as in the above diagram. Then $u_1 - u_2 : A \to I$ is an R-linear derivation (and adding an element of $\operatorname{Der}_R(A, I)$ to any lift gives another lift), so the set of all such lifts is a (possibly finite) torsor (principal homogeneous space) under $\operatorname{Der}_R(A/I) \cong \operatorname{Hom}_A\left(\Omega^1_{A/R}, I\right)$.

Translate this to:

Lemma 4.21 *$R \to A$ is formally unramified if and only if $\Omega^1_{A/R} = 0$, and for schemes $X \to S$ is formally unramified if and only if $\Omega^1_{X/S} = 0$ (details: exercise).*

To complete the proof of étale $\Rightarrow$ formally étale, must show:

Lemma 4.22 *If $I^2 = 0$ and $f : X \to S$ is étale, then there is a lift u making the diagram*

$$\begin{array}{ccc} \operatorname{Spec}(B/I) & \longrightarrow & X \\ \downarrow & \nearrow^{u} & \downarrow f \\ \operatorname{Spec}(B) & \longrightarrow & S \end{array}$$

commute.

Proof. Using that lifts are unique, reduce to case where everything is affine and $f : X \to S$ is even standard étale.

$$\begin{array}{ccc} B/I & \xleftarrow{\varphi} & \frac{R[t]_h}{(g)} \\ \uparrow & & \uparrow \\ B & \longleftarrow & R \end{array}$$

(here g is monic and g' is invertible in $\frac{R[t]_h}{(g)}$). Set $\varphi(t) = \bar{b} \in B/I$, lift to some $b \in B$. Maybe $g(b) \neq 0$, but we can look for $i \in I$ such that $g(b+i) = 0$. Indeed, writing $g(b+i) = g(b) + ig(b)$ (because $I^2 = 0$), then we want $i = -g(b) g'(b)^{-1}$. Does this make sense?

Yes: $g(b) \in I$ (use φ), $g'(t)$ is a unit in $\frac{R[t]_h}{(g)}$, so $g'(b)$ is in $(B/I)^\times$, so $g'(b) \in B^\times$. Taking $t \mapsto b - g(b) g'(b)^{-1}$ we get our lift (by "Newton's Method"). ■

To wrap up a loose end, state differential criterion of smoothness.

Definition 4.23 (S3) *$f : X \to S$ is smooth (in any of the previous senses) if and only if f is locally of finite type presentation, flat, and $\Omega^1_{X/S}$ is locally free, with rank at any $x \in X$ given by* $\dim_x X_{f(x)}$.[1]

Remark 4.24 *Can't omit the rank condition after requiring $\Omega^1_{X/S}$ locally free. Example: $\mathbb{F}_p[t^p] \subseteq \mathbb{F}_p[t]$ on* Spec() *is finite free and $\Omega^1_{\mathbb{F}_p[t]/\mathbb{F}_p[t^p]}$ is free of rank* 1, *but relative dimension is* 0.

5 Sites and sheaves

Motivation: Let X be a topological space, and let $\mathcal{C} = \mathrm{Open}(X)$ be its category of open sets, i.e., $\mathrm{Obj}(\mathcal{C}) = \{U \subset X \text{ open}\}$, $\mathrm{Hom}_{\mathcal{C}}(U,V) = \begin{cases} \text{singleton}, & U \subset V, \\ \varnothing, & U \not\subset V. \end{cases}$ A presehaf of sets (or valued in any category $\mathcal{A}$) on $\mathcal{C}$ is a functor $\mathscr{F} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ (or $\mathcal{A}$) and $\mathscr{F}$ is a sheaf if the natural map, for any $U = \bigcup_{i\in I} U_i$ open cver of $U \in \mathcal{C}$,

$$\mathscr{F}(U) \to \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i) : \forall i,j \in I \left(s_i|_{U_i \cap U_j} = s_j|_{U_i\cap U_j} \right) \right\}$$

is bijective.

From a categorial perspective, $U_i \cap U_j = U_i \times_U U_j$ are fiber products in $\mathcal{C}$, and we can reformulate as follows: for all open cover $U = \bigcup_{i\in I} U_i$, the diagram

$$\mathscr{F}(U) \longrightarrow \prod_{i\in I} \mathscr{F}(U_i) \underset{\mathscr{F}(\mathrm{pr}_2)}{\overset{\mathscr{F}(\mathrm{pr}_1)}{\rightrightarrows}} \prod_{i\in I} \mathscr{F}(U_j \times_U U_k)$$

is an equalizer in **Set**.

That is, $\mathscr{F}(U) \to \prod \mathscr{F}(U_i)$ is injective, and its image is the set of $(s_i)_{i\in I}$ with same image under $\mathscr{F}(\mathrm{pr}_1)$ and $\mathscr{F}(\mathrm{pr}_2)$. Here $\mathscr{F}(\mathrm{pr}_1)$ is the map whose (i,k)-component is

$$\prod_{i\in I} \mathscr{F}(U_i) \to \mathscr{F}(U_j) \xrightarrow{\mathscr{F}(\mathrm{pr}_1)} \mathscr{F}(U_j \times_U U_k).$$

We have used very few formal properties of $\mathrm{Open}(X)$ and its "covers" to formulate the sheaf condition.

[1]This number is the maximal dimension of an irreducible componen of $X_{f(x)}$ that passes through x.

Definition 5.1 *A* <u>*site*</u> *is a category* $\mathcal{C}$ *and a set (or may be sometimes proper class)* $\mathrm{Cov}(\mathcal{C})$ *of families of morphisms with fixed target* $\left\{U_i \xrightarrow{f_i} U\right\}_{i\in I}$.

An element of $\mathrm{Cov}(\mathcal{C})$ *is the data:*

(1) $U \in \mathrm{Obj}(\mathcal{C})$,

(2) a set I*, and*

(3) objects $U_i \in \mathcal{C}$ *and morphisms* $U_i \xrightarrow{f_i} U$ *in* $\mathcal{C}$,

such that:

- *If* $f : V \to U$ *is an isomorphism, then* $\left\{V \xrightarrow{f} U\right\} \in \mathrm{Cov}(\mathcal{C})$.
- *If* $\left\{U_i \xrightarrow{f_i} U\right\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ *and, for each* $i \in I$, $\left\{V_{ij} \xrightarrow{g_{ij}} U_i\right\}_{j\in J_i} \in \mathrm{Cov}(\mathcal{C})$ *then* $\left\{V_{ij} \xrightarrow{f_i\circ g_{ij}} U\right\}_{i\in I, j\in J_i} \in \mathrm{Cov}(\mathcal{C})$.
- *If* $\left\{U_i \xrightarrow{f_i} U\right\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ *and* $f : V \to U$ *any arrow, then the fiber products* $U_i \times_U V$ *exist, and* $\{U_i \times_U V \to V\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$.

Example 5.2 $\mathcal{C} = \mathrm{Open}(X)$*, where* X *is a topological space, and* $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ *if and only if* $\bigcup_{i\in I} U_i = U$*. This is a site.*

5.1 Lecture 7, 9/19

Example 5.3 *Take* X *to be a topological space, and let* $\mathcal{C}$ *be the category whose objects are open immersions* $U \to X$ *and whose morphisms are maps over* X*. We define a site* X_{top} *by taking the covers to be all families* $\left\{U_i \xrightarrow{f_i} U\right\}_{i\in I}$ *such that* $\bigcup_{i\in I} f_i(U_i) = U$.

Note that each of the f_i *is an open immersion because the structure morphisms to* X *are.*

Example 5.4 *For a slight variant of the example above, take* $\mathcal{C}$ *to be the category of* étale spaces over X*. By definition, objects are morphisms* $U \xrightarrow{f} X$ *such that* f *is locally a homeomorphism*[2]*, and morphisms are continuous maps over* X.

We then define a site $X_{\text{ét}}$ *by taking the covers to be all families* $\left\{U_i \xrightarrow{f_i} U\right\}_{i\in I}$ *such that* $\bigcup_{i\in I} f_i(U_i) = U$.

Remark 5.5 *Note that the sites* X_{top} *and* $X_{\text{ét}}$ *are not equivalent as categories. In particular, the category* $X_{\text{ét}}$ *has non-trivial automorphisms corresponding to* deck transformations *of covers.*

We may define sheaves on a site.

[2]Recall this is local on the source.

Definition 5.6 *Let $\mathcal{C}$ be a site. A* presheaf *(of sets, or in general an abelian category or category with arbitrary products) on $\mathcal{C}$ is a functor*

$$\mathscr{F} : \mathcal{C}^{\text{op}} \to \underline{\text{Set}}.$$

The class of presheaves of $\mathcal{C}$ forms a category by taking the morphisms to be the natural transformation between the two functors defining the presheaves.

A presheaf $\mathscr{F}$ is a sheaf *if moreover for every cover $\left\{U_i \xrightarrow{f_i} U\right\}_{i\in I} \in \text{Cov}(\mathcal{C})$, the diagram $\mathscr{F}(U) \xrightarrow{\mathscr{F}(f_i)} \prod_{i\in I} \mathscr{F}(U_i) \underset{\mathscr{F}(\text{pr}_2)}{\overset{\mathscr{F}(\text{pr}_1)}{\rightrightarrows}} \prod_{j,k\in I} \mathscr{F}\left(U_j \times_U U_k\right)$ is an equalizer.*

The category of sheaves on the site $\mathcal{C}$[3] *is defined as a full subcategory of presheaves on $\mathcal{C}$.*

Remark 5.7 *Even though the sites X_{top} and $X_{\text{ét}}$ are not equivalent, the categories of sheaves on X_{top} and sheaves on $X_{\text{ét}}$ are equivalent.*

When we come around to defining derived cohomology of sheaves on a site, this will imply the cohomologies are isomorphic.

Here are some examples of sites which are associated to a scheme X.

Example 5.8 *The* small Zariski site of X*, X_{zar}, is the site X_{top} when X is endowed with the Zariski topology.*

The classical definition of sheaves on a scheme is precisely the definition of sheaves on X_{zar}.

Example 5.9 *The* big *or* global Zariski site of X*, denoted $(\text{Sch}/X)_{\text{zar}}$ or X_{Zar}, is the site whose underlying category has as objects schemes over X and morphisms are morphisms of schemes over X. The covers are families $\left\{U_i \xrightarrow{f_i} U\right\}_{i\in I}$ such that the f_i are open immersions, and $\bigcup_{i\in I} f_i(U_i) = U$. This last condition is frequently referred to as the family being* jointly surjective.

Example 5.10 *The* small étale site*, denoted $X_{\text{ét}}$, is the site on the category of étale schemes over X whose covers are jointly surjective maps to a fixed target.*

Note that in the category of étale schemes over X, any morphism is étale by 4.12. Also, any flat morphism of local finite presentation is open, so any étale morphism is open, so an étale cover is an open cover.

Example 5.11 *The* big *or* global étale site over X*, denoted $(\text{Sch}/X)_{\text{ét}}$ or $X_{\text{Ét}}$, is the site whose underlying category is schemes over X. A family of morphisms $\left\{U_i \xrightarrow{f_i} U\right\}_{i\in I}$ is a cover if and only if each f_i is* étale *and the family is jointly surjective.*

Example 5.12 *The* fppf site over X*, denoted $(\text{Sch}/X)_{\text{fppf}}$, is the site whose underlying category is* Sch$/X$*, and a family $\left\{U_i \xrightarrow{f_i} U\right\}$ is a cover if and only if each f_i is* fppf *and the family is jointly surjective.*

Recall that fppf *is the French abbreviation of* fidèlement plat de présentation finite*—faithfully flat and of finite presentation. This means the morphism is surjective and flat (faithfully flat) and locally of finite presentation.*

[3] A category of sheaves of sets on a site is sometimes called a *topos*.

The final site we define is slightly more subtle due to the definition of a *fpqc* morphism.

Lemma 5.13 *Let $f\colon X \twoheadrightarrow Y$ be a surjective morphism of schemes. The following conditions are equivalent:*

1. *There exists an open affine cover $\{Y_i\}$ of Y such that each Y_i is the image $Y_i = f(X_i)$ where X_i is a quasi-compact open subset of X.*
2. *For every $x \in X$ there is a quasi-compact open $x \in W \subseteq X$ such that $f(W)$ is an open affine.*

Proof. The implication 2. $\implies$ 1. is immediate from the surjectivity of f.

For the reverse implication 1. $\implies$ 2., take $x \in X$. Then there is some Y_i an open affine of Y containing $f(x)$ such that $Y_i = f(X_i)$ for some X_i open and quasi-compact. Also, there is an open affine $U_x \subseteq f^{-1}(Y_i)$ containing x. Then we may take $W = X_i \cup U_x$.

Definition 5.14 *A morphism $f\colon X \to Y$ of schemes is an* fpqc morphism *if it is faithfully flat, and satisfies the equivalent conditions of Lemma 5.13.*

We say a family $\{U_i \to U\}$ is an fpqc cover *if the induced map*

$$\coprod U_i \to U$$

is an fpqc *morphism.*

Definition 5.15 *The* fpqc site, *denoted $(\mathrm{Sch}/X)_{\mathrm{fpqc}}$, is the site whose underlying category is Sch/X and whose covers are exactly the* fpqc *covers.*

Remark 5.16

1. *Naïvely, one would expect to define the covers in the* fpqc *site to be jointy surjective families of faithfully flat and quasi-compact morphisms.*

   However, some open covers are not fpqc *in this sense. Because we want the* fpqc *site to be an enlargement of the Zariski site, we are led to this definition of* fpqc *morphism.*

2. *Condition* 1. *in Lemma 5.13 may be replaced with the following:*

   Any quasi-compact open in Y is the image of a quasi-compact open of X.

3. *We have the inclusion of sites*

   $$\text{Zariski} \subset \text{étale} \subset \text{fppf} \subset \text{fpqc}.$$

   All but the finally inclusion is immediate. For this last inclusion, use the fact that an fppf *morphism is open and surjective.*

4. *One may ask why we do not define an* fp *site where covers are taken to be jointly surjective families of morphisms of finite presentation. One may define such a site, but it is poorly behaved. One of the key features of the* fpqc *site is that representable functors are sheaves—this fails to be the case on an* fp *site.*

   For a good reference on this, see [Vis05]

For any site $\mathcal{C}$ we will write $Sh(\mathcal{C})$ for the category of sheaves of sets on $\mathcal{C}$ and $PSh(\mathcal{C})$ for the category of presheaves of sets. Similarly, $\mathrm{Ab}(\mathcal{C})$ will denote the category of sheaves of abelian groups on $\mathcal{C}$ and $\mathrm{PAb}(\mathcal{C})$ the category of presheaves of abelian groups.

6 Descent

There are two basic problems we must now attend to for these sheaves on a site:

1. For a site $\mathcal{C}$, what sort of properties does $\mathrm{Ab}(\mathcal{C})$ have? Is it an abelian category? Does it have enough injectives?

2. Can we write down any sheaves for the étale, fppf, or fpqc site? Are there analogues of Quasi-coherent sheaves? Can we describe representable functors as sheaves?

The answer to all of these questions is "yes," but some effort is needed to do so.

In order to verify that $\mathrm{Ab}(\mathcal{C})$ is an abelian category, we must show that it has kernels and cokernels. Kernels of a sheaf may be defined naïvely, but even in the classical definition of sheaves, the cokernel sheaf is defined through a process of *sheafification*. We must develop this for sheaves on more general sites.

For the second set of questions, given a quasi-coherent sheaf $\mathscr{F} \in QCoh(X)$ on the Zariski site we may define a functor

$$\begin{aligned}(\mathrm{Sch}/X)^{\mathrm{op}} &\longrightarrow \mathrm{Ab}\\ \left(T \xrightarrow{f} S\right) &\longmapsto \Gamma\left(T, f^*\mathscr{F}\right).\end{aligned}$$

This yields a natural definition for quasi-coherent sheaves on other sites, but we must verify it is a sheaf on the étale, fppf, and fpqc sites.

Similarly, if we have a representable functor

$$\begin{aligned}h_X\colon\ (\mathrm{Sch}/S)^{\mathrm{op}} &\longrightarrow \mathrm{Set}\\ \left(T \xrightarrow{f} S\right) &\longmapsto \mathrm{Hom}_S(T, X)\end{aligned}$$

this is a sheaf in the Zariski site, and a presheaf in the étale, fppf, and fpqc sites, but is it a sheaf?

One tool that is helpful in verifying whether a presheaf is a sheaf on the étale, fppf, or fpqc sites is the following.

Lemma 6.1 *A presheaf $\mathscr{F}$ is a sheaf on the fpqc site (resp. étale, fppf) if and only if*

1. *$\mathscr{F}$ is sheaf on the Zariski site.*

2. *For any cover of affine schemes $\left\{\mathrm{Spec}(B) \xrightarrow{f} \mathrm{Spec}(A)\right\}$ with f fpqc (resp. étale, fppf), the sheaf condition is satisfied.*

6.1 Lecture 8: 9/21

Question 6.2 *Given a cover $S' \to S$ when does a quasi-coherent sheaf on S' arise by pullback from S?*

Given any fpqc cover $\{S_i \xrightarrow{f_i} S\}_{i\in I}$ and $\mathscr{F} \in QCoh(S)$ we obtain $\mathscr{F}_i := f_i^*\mathscr{F} \in QCoh(S_i)$. When pulled back to $S_i \times_S S_j$ then there are isomorphisms between $\mathscr{F}_i$ that satisfy a cocycle condition. In what follows we will replace the cover $\{S_i \xrightarrow{f_i} S\}_{i\in I}$ with the 1-element cover $S' := \coprod S_i \to S$ and formulate the cocycle condition in this setting.

We work with $S' \xrightarrow{f} S$ and $\mathscr{F} \in QCoh(S)$. We have a composition of morphisms:

$$S'' := S' \times_S S' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S' \xrightarrow{f} S$$

and we obtain an isomorphism

$$p_1^*(f^*\mathscr{F}) \simeq (f \circ p_1)^*\mathscr{F} = (f \circ p_2)^*\mathscr{F} \simeq p_2^*(f^*\mathscr{F})$$

which we label with ϕ. We set $S''' := S' \times_S S' \times_S S'$. We have projections p_{12}, p_{13}, p_{23} to S' for any pair of the three factors. Then the cocycle condition can be writen as

$$p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi$$

More precisely we have the following commuting diagram:

$$\begin{array}{ccc} p_{13}^*(p_1^*f^*\mathscr{F}) & \xrightarrow{p_{13}^*\phi} & p_{13}^*(p_2^*f^*\mathscr{F}) \simeq p_{23}^*(p_2^*f^*\mathscr{F}) \\ \wr\Vert & & \uparrow p_{23}^*\phi \\ p_{12}^*(p_1^*f^*\mathscr{F}) & \xrightarrow{p_{12}^*\phi} & p_{12}^*(p_2^*f^*\mathscr{F}) \simeq p_{23}^*(p_1^*f^*\mathscr{F}) \end{array}$$

Definition 6.3 *Given $S' \xrightarrow{f} S$, a pair $(\mathscr{F}', \phi)$ consisting of $\mathscr{F}' \in QCoh(S')$ and an isomorphism $\phi : p_1^*\mathscr{F}' \xrightarrow{\sim} p_2^*\mathscr{F}'$ in $QCoh(S'')$ is called a* covering datum. *It is called a* descent datum *if the cocycle condition $p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi$ holds.*

A morphism $(\mathscr{F}', \phi) \to (\mathscr{G}', \psi)$ of covering data is a morphism $\alpha : \mathscr{F}' \to \mathscr{G}'$ in $QCoh(S')$ such that the diagram below commutes

$$\begin{array}{ccc} p_1^*\mathscr{F}' & \xrightarrow{\phi} & p_2^*\mathscr{F}' \\ p_1^*\alpha \downarrow & & \uparrow p_2^*\alpha \\ p_1^*\mathscr{G}' & \xrightarrow{\psi} & p_2^*\mathscr{G}' \end{array}$$

Thus, we can define a category of covering data for $S' \xrightarrow{f} S$.

Theorem 6.4 *Assume $S' \xrightarrow{f} S$ is fpqc. Then the pullback gives a functor*

$$QCoh(S) \longrightarrow \{\textit{quasi-coherent } S'\textit{-mods with covering data}\}$$

1. *that is fully faithful.*
2. *that has essential image given by the descent data.*

Remark 6.5 *Explicitly, the full faithfulness says that for $\mathscr{F}, \mathscr{G} \in QCoh(S)$ the diagram below is an equilizer*

$$\mathrm{Hom}_S(\mathscr{F}, \mathscr{S}) \xrightarrow{f^*} \mathrm{Hom}_{S'}(f^*\mathscr{F}, f^*\mathscr{G}) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Hom}_{S''}(q^*\mathscr{F}, q^*\mathscr{G})$$

Proof. We'll first show that f^* is faithful. It suffices to prove that for all $\mathscr{F} \xrightarrow{\gamma} \mathscr{G}$, $f^*\gamma = 0$ implies that $\gamma = 0$. We first reduce this to the affine case. For $\mathrm{Spec}(R) \subseteq S$, as f is fpqc there exists a quasi-compact $V \subseteq f^{-1}(\mathrm{Spec}(R))$ such that $V = \bigcup_{\text{finite}} \mathrm{Spec}(R)_i$ and $f(V) = \mathrm{Spec}(R)$. As the union is finite we end up with an fpqc cover $\mathrm{Spec}(R)' = \coprod \mathrm{Spec}(R)_i \to \mathrm{Spec}(R)$, where $R' := \prod R_i$ and $\gamma|_{\mathrm{Spec}(R)'} = 0$. Hence, we have reduced this to showing that if $R \to R'$ is faithfully flat and $\gamma : M \to N$ is an R-module morphism, then $\gamma = 0$, provided that $M \otimes_R R' \xrightarrow{\gamma \otimes id} N \otimes_R R'$ is 0. This follows by the following lemma:

Lemma 6.6 *Let $A \xrightarrow{f^\#} B$ be a flat ring homomorphism. The following are equivalent:*

1. $\mathrm{Spec}(B) \xrightarrow{f} \mathrm{Spec}(A)$ *is surjective.*
2. *For all A-modules M, if $M \otimes_A B = 0$, then $M = 0$.*
3. *For all maximal ideals $\mathfrak{m}$ of A, $\mathfrak{m}B \subsetneq B$.*

Proof. Assume that 2) holds. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then, as the residue field $\kappa(\mathfrak{p})$ is non-zero, $B \otimes_A \kappa(\mathfrak{p}) \neq 0$, which tells us that the fiber of $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ over $\mathfrak{p}$ is non-empty. Hence 1) holds.

Assume that 1) holds. Given $\mathfrak{m} \in MaxSpec\, A$, then there exists $\mathfrak{n} \in MaxSpec\, B$ such that $f(\mathfrak{n}) = \mathfrak{m}$. Thus, $\mathfrak{m}B \subseteq \mathfrak{n} \subsetneq B$.

Assume that 3) holds. Suppose that $M \neq 0$ is an A-module. Let $x \neq 0 \in M$ and $N = Ax \subseteq M$. It suffices to show that $N \otimes_A B \neq 0$. But, as N is cyclic, $N \simeq A/I$ for some ideal $I \subseteq A$. Thus, $N \otimes_A B \simeq B/IB$. As I is a proper ideal of A, there is a maximal ideal $\mathfrak{m}$ of A that contains it. Thus, $IB \subseteq \mathfrak{m}B \subsetneq B$, which implies that B/IB isn't the zero module. ■

Next, we show that that f^* is faithful. In particular, given $\alpha : f^*\mathscr{F} \to f^{\mathscr{G}}$ such that $p_1^*\alpha = p_2^*\alpha$, then $\alpha = f^*(\mathscr{F} \xrightarrow{\overline{\alpha}} \mathscr{G})$ for some $\overline{\alpha}$. Again, we can reduce to the case where both S and S' are affine by using Zariski gluing on S. In particular, we have reduced to the case $\mathrm{Spec}(R)' \xrightarrow{f} \mathrm{Spec}(R)$ being faithfully flat, M, N being R-modules and $\phi : M \otimes_R R' \to N \otimes_R R'$ a map of R'-modules such that $p_1^*\phi = p_2^*\phi$ and we want to show that $\phi = \overline{\phi} \otimes id$ for some $\overline{\phi} : M \to N$, a map of R-modules. We claim that it is enough to show that for all R-modules M, the diagram below is an equilizer:

$$M \xrightarrow{id \otimes 1} M \otimes_R R' \underset{id \otimes 1 \otimes id}{\overset{id \otimes id \otimes 1}{\rightrightarrows}} M \otimes_R R' \otimes_R R'$$

Indeed, via the identification $(M \otimes_R R') \otimes_{R'} R' \otimes_R R' \simeq M \otimes_R R' \otimes_R R'$ we get two maps $p_1^*\phi, p_2^*\phi : M \otimes_R R' \otimes_R R' \to N \otimes_R R' \otimes_R R'$. Explicitly $(p_1^*\phi)(m \otimes a \otimes b) = \phi(m \otimes a) \otimes b$ and $(p_2^*\phi)(m \otimes a \otimes b) = \phi(m \otimes b) \otimes a$. Then we get a commutative diagram

$$\begin{array}{ccccc}
M & \longrightarrow & M \otimes_R R & \underset{1 \otimes id}{\overset{id \otimes 1}{\rightrightarrows}} & M \otimes_R R' \otimes_R R' \\
 & & & & p_1^*\phi \Big\downarrow \Big\downarrow p_2^*\phi \\
N & \longrightarrow & N \otimes_R R & \underset{1 \otimes id}{\overset{id \otimes 1}{\rightrightarrows}} & N \otimes_R R' \otimes_R R'
\end{array}$$

with the horizontal rows being equilizers by the claim. Using the fact that $p_1^*\phi = p_2^*\phi$ and chasing the diagram we get that $\phi\big|_M$ maps to $eq(N \otimes_R R' \rightrightarrows N \otimes_R R' \otimes_R R')$, which is exactly N. Then, $\phi\big|_M$ is the desired morphism of R-modules $\overline{\phi}$.

Now, we claim that the equlizer property of the previous claim holds. By the properties of faithful flatness it is enough to show that for a faithfully flat $R \to A$, the base-changed diagram for $M \otimes_R A$ and $A' = R' \otimes_R A$ is an equilizer. This enables us to assume that $\mathrm{Spec}(R)' \to \mathrm{Spec}(R)$ has a section. Indeed, we can take $A = R'$ and then $\mathrm{Spec}(R)' \otimes_R R' \to \mathrm{Spec}(R) \otimes_R R' = \mathrm{Spec}(R)'$ has a section induced by $R' \xrightarrow{1 \otimes id} R' \otimes_R R' \xrightarrow{\text{mult.}} R'$.

Thus, given $R \xrightarrow{\pi} R' \xrightarrow{s} R$ such that $s \circ \pi = id$, if $\sum m_i \otimes r_i'$ satisfies $\sum m_i \otimes r_i' \otimes 1 = \sum m_i \otimes 1 \otimes r_i'$ by applying $id \otimes id \otimes s$ we get:

$$\begin{aligned}
\sum m_i \otimes r_i' &= (id \otimes id \otimes s)\left(\sum m_i \otimes r_i' \otimes 1\right) \\
&= (id \otimes id \otimes s)\left(\sum m_i \otimes 1 \otimes r_i'\right) \\
&= \sum m_i \otimes s(r_i') \\
&= \sum m_i s(r_i) \otimes 1
\end{aligned}$$

Therefore, $\sum m_i \otimes r_i'$ is in the image of $M \to M \otimes_R R'$, which proves that the diagram is an equilizer. This gives us the faithfulness of f^*.

We'll prove that the essential image is the descent data in the next lecture. ■.

Corollary 6.7 *For any $\mathscr{F} \in QCoh(S)$, the functor $(Sch/X)^{op} \longrightarrow$* ***Ab*** *given by $(T \xrightarrow{f} S) \to \Gamma(T, f^*\mathscr{F})$ is an fpqc sheaf.*

Proof.

$$\Gamma(T, f^*\mathscr{F}) \simeq \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{O}_T, f^*\mathscr{F}) = \mathrm{Hom}_{\mathscr{O}_T}(f^*\mathscr{O}_S, f^*\mathscr{F})$$

Then the equilizer diagram in Remark 5.21 is exactly the equilizer condition for an fpqc sheaf. ■

Let $f : S' \to S$ be a morphism. We get a functor $f^* : (Sch/S) \to (Sch/S)$ given by $(X \to S) \to (S' \times_S X \to S')$. As before, we have a commuting diagrams of fibered squares

$$\begin{array}{ccccc}
p_i^* f^* X & \longrightarrow & f^* X & \longrightarrow & X \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
S'' & \xrightarrow{p_i} & S' & \xrightarrow{f} & S
\end{array}$$

We get canonical isomorphism $\phi : p_1^*X \to p_2^*X$ in Sch/S' satisfying the cocycle condition $p_{23}^*\phi \circ p_{12}^*\phi = p_{13}^*\phi$.

Definition 6.8 *A* descent datum *on $X' \in Sch/S'$ is an S'-isomorphism $\phi : p_1^*X' \,\mathrm{tp}\, p_2^*X'$ satisfying the cocycle condition. A morphism of descent data $(X', \phi) \to (Y', \psi)$ is a S'-scheme morphism $\alpha : X' \to Y'$ such that the diagram below commutes.*

$$\begin{array}{ccc} p_1^*X' & \xrightarrow{\ \phi\ } & p_2^*X' \\ {\scriptstyle p_1^*\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle p_2^*\alpha} \\ p_1^*Y' & \xrightarrow{\ \psi\ } & p_2^*Y' \end{array}$$

Theorem 6.9 *Let $S' \xrightarrow{f} S$ be fpqc. Then:*

1. *$f^* : (Sch/S) \to \{$descent data on $Sch/S'\}$ is fully faithful.*
2. *Let $(X', \phi : p_1^*X' \to p_2^*X')$ be a descent datum for schemes over S' such that X' can be covered by opens U_i with $U_i \to S'$ an affine morphism such that ϕ restricts to $p_1^*U_i \simeq p_2^*U_i$. Then $(X' \to S', \phi)$ arises as the pullback of some $X \to S$.*

Remark 6.10 *The hypothesis in* 2) *is used via the equivalence of categories:*

$$\begin{array}{c} \{W \xrightarrow{\text{affine}} S'\} \longrightarrow \{\text{quasi-coherent } \mathscr{O}_{S'}\text{-algebras}\} \\ \pi \longrightarrow \pi_*\mathscr{O}_W \end{array}$$

Corollary 6.11 *For all $X \in Sch/S$, h_X is an fpqc sheaf.*

6.2 Lecture 9: 9/26

Proposition 6.12 *f^* is essentially surjective.*

Proof. As with full faithfulness, we reduce to the case S' and S are affine. Start there, and let $f : S' = \operatorname{Spec}(R)' \to S = \operatorname{Spec}(R)$. Let M' be an R'-module with a descent datum

$$\varphi : p_1^*M' \xrightarrow{\sim} p_2^*M'$$

as $R' \otimes R'$-modules, where $p_i : \operatorname{Spec}()(R' \otimes R') \to \operatorname{Spec}(R)'$ are the projection maps, and $p_2^*M' = M' \otimes_{R'} (R' \otimes_R R') \simeq M' \otimes_R R'$, where R' acts on $R' \otimes_R R'$ by $1 \otimes \mathrm{id}$. Then φ is an isomorphism $M' \otimes_R R' \to M' \otimes_R R'$ of R''-modules, where on the source R'' acts in the usual way, and on the target by

$$(a \otimes b)(m \otimes r) = bm \otimes ar,$$

i.e. φ gives an isomorphism $M' \otimes_R R' \simeq R' \otimes_R M'$ of R''-modules with the usual action. We obtain two maps $M' \to M' \otimes R'$, namely $\mathrm{id} \otimes 1$ and $\varphi^{-1}(1 \otimes \mathrm{id})$, so we can form

$$M := \mathrm{eq}\left(M' \rightrightarrows M' \otimes_R R'\right) \in R - \mathrm{Mod}$$

of these two maps, so we get a canonical map of R'-modules $M \otimes_R R' \to M$.

General calculations interpreting the cocycle condition:

- $M \otimes R' \to M'$ is injective:
  Suppose $\sum m_i \otimes a_i \mapsto 0$, i.e. $\sum a_i m_i = 0$ in M'. By definition it holds
  $$M = \left\{m' \in M' \colon m' \otimes 1 = \varphi^{-1}\left(1 \otimes m'\right)\right\}$$
  and thus
  $$0 = \varphi^{-1}\left(\sum 1 \otimes a_i m_i\right) = \sum m_i \otimes a_i.$$
  (We did not use the cocycle condition here.)

- Recall that for the projection maps $p_{ij} \colon \operatorname{Spec}()(R''' = R' \otimes_R R' \otimes_R R') \to \operatorname{Spec}()(R'')$ the cocycle condition on φ is
  $$p_{23}^* \circ p_{12}^* \varphi = p_{13}^* \varphi.$$
  We have isomorphisms of R'''-modules
  $$\begin{array}{ccc} p_{12}^*\varphi\colon\ (M' \otimes_R R') \otimes_{R'', p_{12}^*} R''' & \longrightarrow & (R' \otimes_R M') \otimes_{R'', p_{12}^*} R''' \\ \downarrow\simeq & & \downarrow\simeq \\ (M' \otimes_R R') \otimes_R R' & \xrightarrow{\varphi \otimes \mathrm{id}} & R' \otimes_R M' \otimes_R R', \end{array}$$
  where $(M' \otimes_R R') \otimes_{R'', p_{12}^*} R'''$ means that R'' acts as $\mathrm{id} \otimes \mathrm{id} \otimes 1$ on R'''. Similarly,
  $$\begin{array}{ccc} p_{23}^*\varphi\colon\ (M' \otimes_R R') \otimes_{R'', p_{23}^*} R''' & \longrightarrow & (R' \otimes_R M') \otimes_{R'', p_{23}^*} R''' \\ \downarrow\simeq & & \downarrow\simeq \\ (M' \otimes_R R') \otimes_R R' & \xrightarrow{\mathrm{id} \otimes \varphi} & R' \otimes_R M' \otimes_R R' \end{array}$$
  and
  $$\begin{array}{ccc} p_{13}^*\varphi\colon\ (M' \otimes_R R') \otimes_{R'', p_{13}^*} R''' & \longrightarrow & (R' \otimes_R M') \otimes_{R'', p_{13}^*} R''' \\ \downarrow\simeq & & \downarrow\simeq \\ (M' \otimes_R R') \otimes_R R' & \longrightarrow & R' \otimes_R M' \otimes_R R', \end{array}$$
  where the lower horizontal map is given by
  $$m \otimes 1 \otimes 1 \mapsto \sum a_i \otimes 1 \otimes x_i,$$
  if $\varphi(m \otimes 1) = \sum a_i \otimes x_i$. So $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$ becomes:
  If $\varphi(m \otimes 1) = \sum_i a_i \otimes x_i$ and $\varphi\left(x_i \otimes 1\right) = \sum_j b_{ij} \otimes y_{ij}$, then
  $$p_{23}^*\varphi \circ p_{12}^*\varphi(m \otimes 1 \otimes 1) = p_{23}^*\varphi\left(\sum_i a_i \otimes x_i\right) = \sum_{ij} a_i \otimes b_{ij} \otimes y_{ij},$$
  whereas
  $$p_{13}^*\varphi(m \otimes 1 \otimes 1) = \sum a_i \otimes 1 \otimes x_i.$$
  So we have
  $$\sum_{ij} a_i \otimes b_{ij} \otimes y_{ij} = \sum a_i \otimes 1 \otimes x_i \tag{3}$$

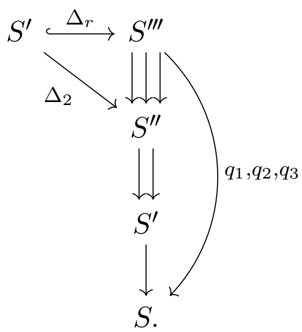
Now assume $\operatorname{Spec}(R)' \to \operatorname{Spec}(R)$ has a section $\sigma\colon \operatorname{Spec}(R) \to \operatorname{Spec}(R)'$. Applying $\sigma \otimes \mathrm{id} \otimes \mathrm{id}$ to (3) gives

$$\begin{aligned}\varphi\left(\sum_i \sigma(a_i)\, x_i \otimes 1\right) = \sum_{i,j} \sigma(a_i)\, b_{ij} \otimes y_{ij} \quad &= \quad \sum_i \sigma(a_i) \otimes x_i \\ &= \quad \sum_i 1 \otimes \sigma(a_i)\, x_i = 1 \otimes \left(\sum_i \sigma(a_i)\, x_i\right),\end{aligned}$$

i.e. $\sum_i \sigma(a_i)\, x_i \in M$ (the equalizer from above). Now apply $\mathrm{id} \otimes \sigma \otimes \mathrm{id}$ to (3) and compose with the action map $R' \otimes M' \to M'$ to get

$$\sum_i a_i \underbrace{\sum_j \sigma(b_{ij})\, y_{ij}}_{\in M} = \sum_{i,j} a_i \sigma(b_{ij})\, y_{ij} = \sum a_i x_i \in M'.$$

We want to show that the left-hand side lies in $\operatorname{im}(M \otimes R' \to M')$. This will be done by showing $\sum a_i x_i = m$. This last point is easier. Consider



We have a quasi-coherent sheaf $\mathscr{F}'$ on S' corresponding to M' and $\varphi\colon p_1^*\mathscr{F}' \xrightarrow{\sim} p_2^*\mathscr{F}'$. Also

$$p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi\colon q_1^*\mathscr{F}' \to q_3^*\mathscr{F}'$$

on S'''. Pull back along Δ_3:

$$\Delta_3^* \circ p_{ij}^* \cong (p_{ij} \circ \Delta_3)^* = \Delta_2^*$$

for all i, j and pullback of the cocycle condition is

$$\Delta_2^*\varphi \circ \Delta_2^*\varphi = \Delta_2^*\varphi,$$

and hence $\Delta_2^*\varphi = \mathrm{id}_{\mathscr{F}'}$, so $\Delta_2^*\varphi$ is an idempotent isomorphism. In our affine coordinates $\varphi(m \otimes 1) = \sum a_i \otimes x_i$ (after Δ_2^*) gives $m = \sum a_i x_i$. This finishes the affine case with a section.

Without a section, we still constructed M and $M \otimes R' \to M'$. To show this is an isomorphism, it suffices to do so after a faithfully flat base-change on R. As last time, use the fpqc $R \to R'$ to reduce to the case where we have a section.

Now we reduce to the affine case:

Knowing the result that there is a descent on a Zariski open cover of S, we win by Zariski glueing. So we may assume that S is affine (yet cocycle condition using full faithfulness in the general case, since the intersection of open affines in S might not be affine). Then we reduce to S' affine by applying the definition of fpqc morphisms to find a quasi-compact open $U' \subset S'$ with $f(U') = S$ open affine. Replace $U' = \bigcup_{\text{fin}} U_i'$ affine open with $\coprod U_i'$ is affine. Details are an exercise.

7 n'existe pas

8 Categories of (abelian) sheaves on a site

Let $\mathcal{C}, \mathrm{Cov}(\mathcal{C})$ be a site. We have the full subcategories $\mathrm{Sh}(\mathcal{C}) \subset \mathrm{PSh}(\mathcal{C})$ of sets and $\mathrm{Ab}(\mathcal{C}) \subset \mathrm{PAb}(\mathcal{C})$ of abelian groups. We want to find left adjoins: the functorial sheafifications.

Čech construction:

Let $\mathscr{F} \in \mathrm{PSh}(\mathscr{S})$, $\mathcal{U} = \{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.

Definition 8.1

$$\check{H}^0(\mathcal{U}, \mathscr{F}) := \left\{ (s_i) \in \prod_{i \in I} \mathscr{F}(U_i) : s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \text{ for all } i, j \in I \right\}$$

with its canonical map $\mathscr{F}(U) \to \check{H}^0(\mathcal{U}, \mathscr{F})$.

Definition 8.2 *A morphism* $\chi \colon \mathcal{U} \to \mathcal{V}$ *of coverings* $\mathcal{U} = \{U_i \to U\}_{i \in I}$, $V = \{V_j \to V\}_{j \in J}$ *is*

1. *a map* $\overline{\chi} \colon U \to V$, *and*
2. *a set map* $\alpha \colon I \to J$, *and*
3. *maps* $\chi_i \colon U_i \to V_{\alpha(i)}$ *for all* $i \in I$, *such that the diagram*

$$\begin{array}{ccc} U_i & \xrightarrow{\chi_i} & V_{\alpha(i)} \\ \downarrow & & \downarrow \\ U & \xrightarrow{\overline{\chi}} & V \end{array}$$

   commutes

When $\overline{\chi} = \mathrm{id}_U \colon U \to U (= V)$, *we call* $\chi \colon \mathcal{U} \to \mathcal{V}$ *a refinement. When we have* $\mathcal{U} \to \mathcal{V}$, *we get a map*

$$\check{H}^0(\mathcal{V}, \mathscr{F}) \to \check{H}^0(\mathcal{U}, \mathscr{F}), \ (s_j)_{j \in J} \mapsto \left(\chi_i^* \left(s_{\alpha(i)}\right)\right)_{i \in I},$$

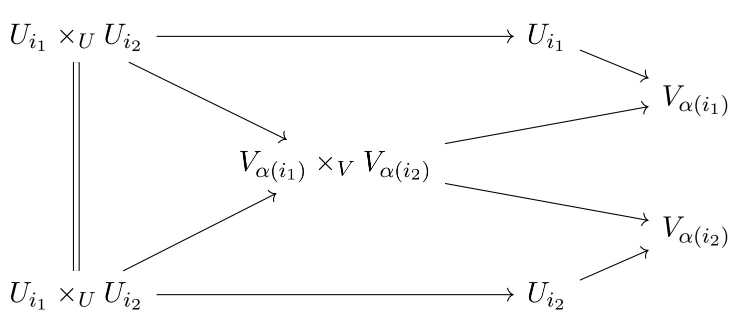
where $\chi_i^* \colon \mathscr{F}\left(V_{\alpha(i)}\right) \to \mathscr{F}(U_i)$ *is given by the presheaf structure.*

Lemma 8.3 *This map is well-defined, i.e. for all* $i_1, i_2 \in I$

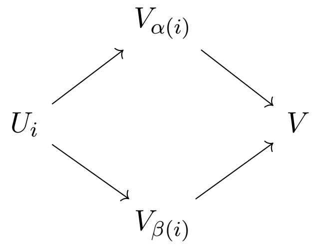
$$\chi_{i_1}^* \left(s_{\alpha(i_1)}\right) |_{U_{i_1} \times_U U_{i_2}} = \chi_{i_2}^* \left(s_{\alpha(i_2)}\right) |_{U_{i_1} \times_U U_{i_2}},$$

and for a fixed $\overline{\chi} \colon U \to V$, *it is independent of the choice of* α *and* $(\chi_i)_{i \in I}$.

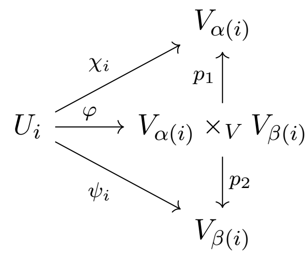
Proof. We have a commutative diagram



The first claim follows by pulling back along two paths. For the second claim suppose that we have $\psi \colon \mathcal{U} \to \mathcal{V} = \left(\overline{\chi}, \beta \colon I \to J, \psi_i \colon U_i \to V_{\beta(i)}\right)$. The diagram



commutes, so we get



and for $s = (s_j) \in \check{H}^0(\mathcal{V}, \mathscr{F})$ we get

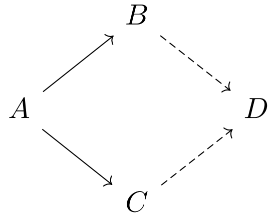
$$\psi^*(s) = \left(\psi_i^*\left(s_{\beta(i)}\right)\right)_{i \in I} = \left(\varphi^* p_2^*\left(s_{\beta(i)}\right)\right)_{i \in I} = \left(\varphi^* p_1^*\left(s_{\alpha(i)}\right)\right)_{i \in I} = \left(\chi_i^*\left(s_{\alpha(i)}\right)\right)_{i \in I} = \chi^*(s).$$

Definition 8.4 *Consider the category $\mathcal{I}_U$, whose objects are covers $\{U_i \to U\}$ of U and whose morphisms are refinements. Then we have a functor*

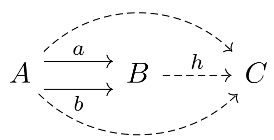
$$\check{H}^0(-, \mathscr{F}) : \mathcal{I}_U^{\mathrm{op}} \to \mathrm{Set}.$$

This diagram in sets is filtered, i.e.

1. it is not empty.
2. Given two maps $A \to B,\ A \to C$, we can find maps $B \to D,\ C \to D$ such that

   commutes.
3. For maps $a \colon A \to B,\ b \colon A \to B$ there exist a map $h \colon B \to C$ such that $ha = hb$, i.e.

commutes.

Proof. 1. $\{U \to U\} \in \mathcal{I}_U$, so it is not empty.

2. Two covers have a common refinement, because :
   Given $\mathcal{U}, \mathcal{V}$, form $\mathcal{U} \times \mathcal{V} = \{U_i \times V_j \to U\}$.

3. last lemma.

This makes colimits over $\mathcal{I}_U^{\text{op}}$ very explicit.

Definition 8.5
$$\check{H}^0(U, \mathscr{F}) := \mathscr{F}^+(U) := \operatorname{colim}_{\mathcal{I}_U^{\text{op}}} \check{H}^0(\mathcal{U}, \mathscr{F}),$$
i.e. any element of $\mathscr{F}^+(U)$ *is represented by some* $s_{\mathcal{U}} \in \check{H}^0(\mathcal{U}, \mathscr{F})$ *and two* $s_{\mathcal{U}}, s_{\mathcal{V}}$ *represent the same element of* $\mathscr{F}^+(U)$ *if and only if there is a common refinement*

$$\mathcal{T} \rightarrow \mathcal{U}, \quad \mathcal{T} \rightarrow \mathcal{V},$$

where $s_{\mathcal{U}}|_{\mathcal{T}} = s_{\mathcal{V}}|_{\mathcal{T}}$.

Theorem 8.6 *1.* $\mathscr{F}^+$ *is a separated presheaf.*

2. $\mathscr{F}^{\#} := (\mathscr{F}^+)^+$ *is a sheaf.*

3. $\mathscr{F} \to \mathscr{F}^{\#}$ *is the desired left-adjoint to* $\text{Sh}(\mathcal{C}) \subset \text{PSh}(\mathcal{C})$ *and* $\text{Ab}(\mathcal{C}) \subset \text{PAb}(\mathcal{C})$.

8.1 Lecture 10: 9/28

Last time we had a site induced by a pair $(\mathcal{C}, \text{Cov}(\mathcal{C}))$ and we had $\mathscr{F} \in \text{PSh}(\mathcal{C})$. We constructed $\mathscr{F}^+ \in \text{PSh}(\mathcal{C})$ given by

$$\mathscr{F}^+(U) = \operatorname*{colim}_{\underbrace{\mathcal{U}}_{\mathcal{I}_{\mathcal{U}}^{\text{op}}} \text{ of } U}^{\text{covers}} \check{H}^0(\mathcal{U}, \mathscr{F}).$$

Theorem 8.7 *The following hold.*

1. $\mathscr{F}^+$ *is a* <u>*separated*</u> *presheaf.*

2. Get canonical maps $\mathscr{F} \to \mathscr{F}^+$ *(use* $\mathcal{U} = \left\{U \xrightarrow{\text{id}} U\right\}$*).*

3. If $\mathscr{F}$ *is separated, then* $\mathscr{F}^+$ *is a* <u>*sheaf*</u>*, and* $\mathscr{F} \to \mathscr{F}^+$ *is injective.*

4. $\mathscr{F} \mapsto \mathscr{F}^\sharp := \mathscr{F}^{++}$ *is a left adjoint to* $\text{Sh}(\mathscr{F}) \hookrightarrow \text{PSh}(\mathcal{C})$*; i.e., we have functorial isos*
$$\text{Hom}_{\text{Sh}(\mathcal{C})}\left(\mathscr{F}^\sharp, \mathscr{G}\right) \cong \text{Hom}_{\text{PSh}(\mathcal{C})}(\mathscr{F}, \mathscr{G})$$
for all $\mathscr{F} \in \text{PSh}(\mathcal{C})$ *and* $\mathscr{G} \in \text{Sh}(\mathcal{C})$.

Proof. Left as exercise, but we will give one sample argument, showing

$$\mathscr{F} \text{ separated} \Rightarrow \mathscr{F}^+ \text{ is a sheaf.}$$

- $\mathscr{F} \to \mathscr{F}^+$ is injective (exercise).
- Next claim is that for all refinements $\mathcal{U} = \{U_i \to U\}_{i \in I} \xrightarrow{\xi} \mathcal{V} = \{V_j \to U\}_{j \in J}$ (that is, that we have a function $\alpha : I \to J$ such that for all $i \in I$ the diagram (4) commutes), the map $\check{H}^0(\mathcal{V}, \mathscr{F}) \to \check{H}^0(\mathcal{U}, \mathscr{F})$ is injective.

$$\begin{array}{ccc} U_i & \xrightarrow{\chi_i} & V_{\alpha(i)} \\ \downarrow & & \downarrow \\ U & \xrightarrow[\mathrm{id}_U]{} & U \end{array} \tag{4}$$

Indeed, assume $(s_j)_{j \in J}, (s'_j)_{j \in J} \in \check{H}^0(\mathcal{V}, \mathscr{F})$ are mapped to $(\chi_i^* s_{\alpha(i)})_{i \in I}$ and $\left(\chi_i^* s'_{\alpha(i)}\right)_{i \in I}$ respectively, and $(\chi_i^* s_{\alpha(i)})_{i \in I} = \left(\chi_i^* s'_{\alpha(i)}\right)_{i \in I}$.

Fix j. By site axioms, $\{U_i \times_U V_j \to V_j\}$ is a cover.

By definition of $\check{H}^0$, $s_{\alpha(i)}|_{V_{\alpha(i)} \times_U V_j} = s_{\alpha(j)}|_{V_{\alpha(i)} \times_U V_j}$, so

$$\chi_i^* s_j|_{U_i \times V_j} = \chi_i^* s_{\alpha(i)}|_{U_i \times V_j} = \chi_i^* s'_{\alpha(i)}|_{U_i \times V_j} = \chi_i^* s'_j|_{U_i \times V_j}.$$

Since $\left\{U_i \times V_j \xrightarrow{\chi_i \times \mathrm{id}}\right\} \in \mathrm{Cov}(\mathcal{C})$ and $\mathscr{F}$ is separated, $s_j = s'_j$ for all $j \in J$. Now let $\{U_i \to U\} \in \mathrm{Cov}(\mathcal{C})$, $s_i \in \mathscr{F}^+(U_i)$ are compatible on $U_{i_1} \times_U U_{i_2}$. Each s_i locally comes from sections of $\mathscr{F}$: i.e., there exist covers $\{U_{ij} \to U_i\}_{j \in J_i}$ and $s_{ij} \in \mathscr{F}(U_{ij})$ such that $s_i|_{U_{ij}}$ is the image of s_{ij} under $\mathscr{F} \to \mathscr{F}^+$.

Claim: the collection $(s_{ij})_{\substack{i \in I \\ j \in J_i}} \in \check{H}^0\left(\{U_{ij} \to U_i\}_{i,j}, \mathscr{F}\right) \to \mathscr{F}^+(U)$ is the desired glueing of the s_i's. Check that the s_{ij}'s are compatible on all $U_{i_1,j_1} \times_U U_{i_2,j_2}$: compatibility of s_i's means that for all i_1, i_2,

$$s_{i_1}|_{U_{i_1} \times_U U_{i_2}} = s_{i_2}|_{U_{i_1} \times_U U_{i_2}},$$

so for all j_1, j_2,

$$\underbrace{s_{i_1}|_{U_{i_1 j_1} \times_U U_{i_2 j_2}}}_{\substack{\text{(image of)} \\ s_{i_1 j_1}|_{U_{i_1 j_1} \times U_{i_2 j_2}}}} = \underbrace{s_{i_2}|_{U_{i_1 j_1} \times_U U_{i_2 j_2}}}_{\substack{\text{(image of)} \\ s_{i_2 j_2}|_{U_{i_1 j_1} \times U_{i_2 j_2}}}}.$$

Since $\mathscr{F} \to \mathscr{F}^+$ is injective, we actually get $s_{i_1 j_1}|_{U_{i_1 j_1} \times U_{i_2 j_2}} = s_{i_2 j_2}|_{U_{i_1 j_1} \times U_{i_2 j_2}}$. ■

Remark 8.8 *Similarly, the same construction* $\sharp$ *gives left adjoint of* $\mathrm{Ab}(\mathcal{C}) \subseteq \mathrm{PAb}(\mathcal{C})$.

Remark 8.9 *The previous construction required a colimit over* $\mathcal{I}_U^{\mathrm{op}}$*, which should have a set of objects for the colimit to make sense. In our strict sense of "site",* $\mathcal{C}$ *is a small category and* $\mathrm{Cov}(\mathcal{C})$ *is a set, so this is OK.*

This does not literally hold in our key examples ($\mathcal{C} = S_{\text{ét}}$), e.g. for any set I, $(\mathrm{Spec}(k) \to \mathrm{Spec}(k))_{i \in I} \in \mathrm{Cov}(S_{\text{ét}})$.

There are various workgrounds, the most direct being:

Lemma 8.10 *Fix a scheme U. There exists a* <u>*cofinal set*</u> *of covers in the (proper) class of all étale covers of U (apply the lemma to any object in $(\mathrm{Sch}/S)_{\acute{e}t}$ to replace $\operatorname{colim}_{\substack{\underline{all}\ \text{étale} \\ \text{covers}}}$ by $\operatorname{colim}_{\substack{\text{cofinal} \\ \text{set}}}$).*

Proof. First note that any cover $(U_i \to U)$ can be refined by a cover with all U_i's affine and mapping into affines in U.

U has a <u>set</u> of affine open subsets. Fix open subschemes W_i representing all of these. We can write down a set parametrizing up to isomorphism all finite presentation $\mathscr{O}(W_i)$-algebras (with cardinality a countable union of $\operatorname{card}(\mathscr{O}(W_i))$). ■

This lets us do Étale Cohomolohy in ZFC.

Same idea works for fppf covers, and we get fppf sheafification as well. **The analogue does not hold for fpqc covers**.

Lemma 8.11 *Let k be a field (same will hold for any nonzero ring). Then there is* <u>*NO*</u> *cofinal set in the proper class of all fpqc covers of $\operatorname{Spec}(k)$ (SO DO NOT FPQC SHEAFIFY).*

Proof. Consider for any set I the fpqc cover $\operatorname{Spec}(k(t_i)_{i\in I}) \to \operatorname{Spec}(k)$. Suppose there exists a <u>set</u> A of covers of $\operatorname{Spec}(k)$ cofinall in all of these. For all $\alpha \in A$, $\mathcal{U}_\alpha = \{U_{\alpha_j} \to \operatorname{Spec}(k)\}_{j\in J_\alpha}$. Then by assumption, for each I some $\mathcal{U}_\alpha$ refines $\operatorname{Spec}(k(t_i)_{i\in I}) \to \operatorname{Spec}(k)$, so there exists $j \in J_\alpha$ such that $U_{\alpha_j} \neq \varnothing$ and there exists $U_{\alpha_j} \to \operatorname{Spec}(k(t_i)_{i\in I})$, so any point $x \in U_{\alpha_j}$ has $\kappa(x) \hookleftarrow k(t_i)_{i\in I}$, so $|\kappa(x)| \geqslant |I|$. So get a contradiction by picking I such that $|I| > \sup_{\substack{\alpha\in A \\ j\in J_\alpha \\ x\in U_{\alpha_j}}} |\kappa(x)|$. ■

In any case, for $\mathcal{C} = S_{\text{ét}}$, $(\mathrm{Sch}/S)_{\text{ét}}$, $(\mathrm{Sch}/S)_{\text{fppf}}$, $\mathscr{F} \to \mathscr{F}^\sharp$ is OK.

8.1.1 Categories of abelian (pre)sheaves

Lemma 8.12 *$\mathcal{C}$-site. $\mathrm{PAb}(\mathcal{C})$ is an abelian category, with $\ker(\mathscr{F} \to \mathscr{G})(U) = \ker(\mathscr{F}(U) \to \mathscr{G}(U))$, $\operatorname{coker}(\mathscr{F} \to \mathscr{G})(U) = \operatorname{coker}(\mathscr{F}(U) \to \mathscr{G}(U))$. More generally, $\mathrm{PSh}(\mathcal{C})$ has all (small) limits and colimits, defined pointwise.*

Proof. Exercise. ■

Lemma 8.13 *$\mathrm{Sh}(\mathcal{C})$, $\mathrm{Ab}(\mathcal{C})$ have all limits: the presheaf limit is already a sheaf and it gives the limit in $\mathrm{Sh}(\mathcal{C})$.*

Proof. Exercise. ■

Lemma 8.14 *$\mathrm{Sh}(\mathcal{C})$, $\mathrm{Ab}(\mathcal{C})$ have all colimits: for any $F : I \to \mathrm{Sh}(\mathcal{C})$ write $\mathrm{Sh}(\mathcal{C}) \xrightarrow{\omega} \mathrm{PSh}(\mathcal{C})$ and set $\operatorname{colim} = (\operatorname{colim} \omega \circ F)^\sharp$.*

Proof. Let $\mathscr{G} \in \mathrm{Sh}(\mathcal{C})$.

$$\operatorname{Hom}_{\mathrm{Sh}(\mathcal{C})}\left((\operatorname{colim} \omega \circ F)^\sharp, \mathscr{G}\right) =$$

$$= \underbrace{\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}\left(\operatorname{colim}_I \omega \circ F, \omega(\mathscr{G})\right) = \lim_{i \in I} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(\omega \circ F(i), \omega(\mathscr{G}))}_{\text{two lemmas ago}} =$$

$$= \lim_{i \in I} \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(F(i), \mathscr{G}). \blacksquare$$

Lemma 8.15 *In* $\mathrm{Ab}(\mathcal{C})$*, a sequence* $\mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$ *with* $\beta \circ \alpha = 0$ *is exact*[4] *if and only if for all* $U \in \mathcal{C}$ *and all* $s \in \mathscr{G}(U)$ *such that* $\beta(s) = 0$ *(i.e.* $s \in (\ker(\beta))(U)$*) there exists* $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$*, and* $t_i \in \mathscr{F}(U_i)$ *such that* $\alpha(t_i) = s|_{U_i}$*.*

Proof. Assume sequence is exact. Let $s \in \mathscr{G}(U)$ be such that $\beta(s) = 0$.

$\mathrm{Im}(\alpha) := \ker\left(\mathscr{G} \xrightarrow{p} \mathrm{coker}(\alpha)\right)$, i.e., $t \in \mathscr{G}(U)$ such that $p(t) = 0$ in $\mathrm{coker}(\alpha)(U) = (\mathscr{G}(U)/\alpha\mathscr{F}(U))^\sharp$, which shows that $s \in \ker(\beta) \cong \mathrm{Im}(\alpha)$ arises locally from $\alpha(\mathscr{F}(U_i))$'s, by construction of $\sharp$. (Some details omitted).

The converse is an exercise. $\blacksquare$

Corollary 8.16 $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H}$ *in* $\mathrm{Ab}(\mathcal{C})$ *is exact if and only if* $0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U)$ *is exact in* Ab *for all* $U \in \mathcal{C}$*.*

Proof. Assume exactness. Then exactness at $\mathscr{F}(U)$ is immediate from Lemma 8.15. Remains to show $s \in \ker(\mathscr{G}(U) \to \mathscr{H}(U))$ comes from $\mathscr{F}(U)$. By Lemma 8.15, there exists a cover $(U_i \to U)$ and $t_i \in \mathscr{F}(U_i)$ such that $\alpha(t_i) = s|_{U_i}$. But $t_i|_{U_i \times U_j} = t_j|_{U_i \times U_j}$ because $\mathscr{F}(U_i \times_U U_j) \hookrightarrow \mathscr{G}(U_i \times_U U_j)$, so the t_i's glue (since $\mathscr{F}$ is a sheaf) to a $t \in \mathscr{F}(U)$, which must map to $s \in \mathscr{G}(U)$ ($\mathscr{G}$ is separated). Likewise converse. $\blacksquare$

Theorem 8.17 $\mathcal{C}$ *any site. Then* $\mathrm{Ab}(\mathcal{C})$ *is an abelian category with enough injectives.*

We will not prove this. The fact that it is abelian and we have an isomorphism between image and coimage is now an exercise using our lemmas. We will not prove that there are enough injectives in this generality, but (next class?) we will give a direct construction for $S_{\text{ét}}$ (using stalks and injectives in Ab).

Definition 8.18 *Let* $\mathcal{C}$ *be a site,* $\mathscr{F} \in \mathrm{Ab}(\mathcal{C})$*. Let* $X \in \mathrm{Obj}(\mathcal{C})$*. Define* $H^p_{\mathcal{C}}(X, \mathscr{F})$ *to be the pth right-derived functor of* $\Gamma(X, -) : \mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}$*, i.e.,* $\mathrm{Ab}(\mathcal{C})$ *is an abelian category with enough injectives, so for all* $\mathscr{F} \in \mathrm{Ab}(\mathcal{C})$*, there exists an injective resolution* $F \xrightarrow[\text{quasi-iso}]{\sim} I^\bullet$ *and we set* $H^p_{\mathcal{C}}(X, \mathscr{F}) = H^p(I^\bullet(X))$*.*

When $\mathcal{C} = X_{\text{ét}}$ *(which has final object* X*), we will write* $H^p(X_{\text{ét}}, \mathscr{F})$ *for* $H^p_{X_{\text{ét}}}(X, \mathscr{F})$*.*

9 Basic functorialities of the étale site

9.1 Lecture 11: 10/3

Suppose we have $j : U \to S$ étale. We get a restriction functor

$$j^{-1} : \mathrm{Ab}(S_{\text{ét}}) \to \mathrm{Ab}(U_{\text{ét}})$$

$$j^{-1}\mathscr{F}(U' \to U) := \mathscr{F}(U' \to S)$$

[4] We know how to define kernel and cokernel, so we can formulate what exactness means.

Clearly j^{-1} is exact, and we would want it to be compatible with cohomology, in the sense that

$$H^*_{X_{\text{ét}}}(U, \mathscr{F}) \cong H^*_{U_{\text{ét}}}(U, j^{-1}\mathscr{F}).$$

Another way of thinking about this is that given a sheaf $\mathscr{F}$ on X, and $j : U \to X$ étale, taking the cohomology of $\mathscr{F}$ on U in $X_{\text{ét}}$ and $U_{\text{ét}}$ gives the same result. We will see soon that this is true.

For any map of schemes $f : S' \to S$, we get a functors $f_* : \text{Ab}(S'_{\text{ét}}) \to \text{Ab}(S_{\text{ét}})$ and $f_* : \text{Sh}(S'_{\text{ét}}) \to \text{Sh}(S_{\text{ét}})$ defined by

$$f_*\mathscr{F}(U \to S) := \mathscr{F}(U \times_S S' \to S')$$

Proposition 9.1 *1. For any map of schemes $f : S' \to S$, the pushforward functor f_* has left a adjoint $f^{-1} : \text{Sh}(S_{\text{ét}}) \to \text{Sh}(S'_{\text{ét}})$.*

2. The functor f^{-1} is exact (i.e. it preserves finite limits and finite colimits) and the f_* is left exact.

3. The same holds replacing Sh *with* Ab*, and in that case, f_* preserves injective.*

Proof. We construct f^{-1} and the adjunction map, and leave the rest as an exercise.

Fix $\pi : U' \to S'$ étale. Let $\mathcal{D}$ be the category whose objects are diagrams

$$\begin{CD} U' @>>> U \\ @V{\pi}VV @VVV \\ S' @>>> S \end{CD} \quad ,$$

with $U \to S$ étale, and whose morphisms from a diagram with U_1 in the top right to a diagram with U_2 in the top right are S-morphisms $U_1 \to U_2$.

We claim this category $\mathcal{D}$ is cofiltered. Given

$$\begin{CD} U' @>>> U_i \\ @V{\pi}VV @VVV \\ S' @>>> S \end{CD} \quad ,$$

for $i = 1, 2$, we get

$$\begin{CD} U' @>>> U_1 \times_S U_2 \\ @V{\pi}VV @VVV \\ S' @>>> S \end{CD} \quad ,$$

and given

$$\begin{array}{ccccc} U' & \longrightarrow & U_1 & \overset{a}{\underset{b}{\rightrightarrows}} & U_2 \\ \downarrow{\scriptstyle \pi} & & \downarrow & \swarrow & \\ S' & \longrightarrow & S & & \end{array} \quad ,$$

we get

$$\begin{array}{ccc} U' & \longrightarrow & \widetilde{U} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ S' & \longrightarrow & S \end{array},$$

equalizing a, b, where $\widehat{U}$ is the fiber product

$$\begin{array}{ccc} \widetilde{U} & \longrightarrow & U_1 \\ \big\downarrow & & \big\downarrow{\scriptstyle (a,b)} \\ U_2 & \longrightarrow & U_2 \times_S U_2 \end{array},$$

Now, for $\mathscr{F} \in \mathrm{Sh}(S_{\text{ét}})$, set

$$f_p^{-1}\mathscr{F}(U' \to S') = \varinjlim_{\mathcal{D}} \mathscr{F}(U \to S)$$

where p stands for presheaf. As f_p^{-1} is defined by a filtered colimit, we have that $\mathscr{F} \mapsto f_p^{-1}\mathscr{F}$ commutes with all colimits, and finite limits. We define $f^{-1}\mathscr{F}$ as the sheafification of $f_p^{-1}\mathscr{F}$. As sheafification is exact, we have that $\mathscr{F} \mapsto f^{-1}\mathscr{F}$ commutes with finite limits and finite colimits.

The adjunction map

$$\beta : \mathrm{Hom}_{\mathrm{Sh}(S_{\text{ét}})}(\mathscr{F}, f_*\mathscr{G}) \to \mathrm{Hom}_{\mathrm{Sh}(S'_{\text{ét}})}(f^{-1}\mathscr{F}, \mathscr{G})$$

is given, for $U' \to S'$ étale, by the sheafification of

$$\beta_p(\varphi)(U') : f_p^{-1}\mathscr{F}(U') = \varinjlim_{U \in \mathcal{D}} \mathscr{F}(U) \xrightarrow{\varphi(U)} \varinjlim_{U \in \mathcal{D}} f_*\mathscr{G}(U) = \varinjlim_{U \in \mathcal{D}} \mathscr{G}(S' \times_S U) \to \mathscr{G}(U').$$

Example 9.2

1. *If $f : S' \to S$ is étale, then we have just defined two different f^{-1} functors, restriction and pull-back. There is little ambiguity, as the functors are naturally isomorphic. This can be checked by unravelling the construction of the latter in the case of an étale morphism.*

2. *Let S be a scheme, with a sheaf $\mathscr{F} \in \mathrm{Sh}\, S_{\text{ét}}$, and $s : \mathrm{Spec}(k) \to S$ me a morphism of schemes, k a field. Define $\mathscr{F}_s := s^{-1}\mathscr{F}$. This is the stalk of $\mathscr{F}$ at s.*

   When k is separably closed, $\mathrm{Sh}(\mathrm{Spec}(k)) \cong Set$ via $\mathscr{G} \mapsto \mathscr{G}(\mathrm{Spec}(k))$. When this is the case, we will conflate $\mathscr{F}_s$ and $\mathscr{F}_s(\mathrm{Spec}(k))$, as all of the information is contained in this set. We can compute this set

$$\mathscr{F}_s(\mathrm{Spec}(k)) = s^{-1}\mathscr{F}(\mathrm{Spec}(k)) = s_p^{-1}(\mathrm{Spec}(k)) = \varinjlim_{\mathcal{E}} \mathscr{F}(U)$$

   where $\mathcal{E}$ is the category of diagrams

$$\begin{array}{ccc} & & U \\ & \nearrow & \big\downarrow{\scriptstyle \text{ét}} \\ \mathrm{Spec}(k) & \xrightarrow{s} & S \end{array}$$

Definition 9.3 *When k is separably closed, a morphism $\bar{s} : \mathrm{Spec}(k) \to S$ is called a geometric point of S. An étale neighborhood of $\bar{s}$ is an étale map $f : U \to S$, and a geometric point $\bar{t} : \mathrm{Spec}(k) \to U$ of U so that $f \circ \bar{t} = \bar{s}$ (i.e. it is a commuting diagram like the one above).*

We have the following results on stalks at geometric points:

Proposition 9.4 *Let $\bar{s} : \mathrm{Spec}(k) \to S$ be a geometric point.*

1. *$\mathscr{F} \mapsto \mathscr{F}_{\bar{s}}$ is exact (on $\mathrm{Ab}(S_{\text{ét}})$ as well as $\mathrm{PAb}(S_{\text{ét}})$).*
2. *For $\mathscr{F} \in \mathrm{PAb}(S_{\text{ét}})$, the canonical map $\mathscr{F}_{\bar{s}} \to (\mathscr{F}^{\#})_{\bar{s}}$ is an isomorphism.*
3. *For all morphisms of schemes, $f : S \to T$ and $\mathscr{F} \in \mathrm{Ab}(T_{\text{ét}})$, the canonical map $f^{-1}\mathscr{F}_{\bar{s}} \to \mathscr{F}_{f \circ \bar{s}}$ is an isomorphism.*

Warning: Stalks of a pushforward along a morphism f_* are not as easy to describe.

For $f : U \to S$ we have a right adjoint to j^{-1}, namely j_*. But when f is étale, we also have a left adjoint:

Proposition 9.5 *Let $j : U \to S$ be étale.*

1. *j^{-1} has an exact left adjoint, $j_!$ (both for Sh and for Ab).*
2. *$H^p_{U_{\text{ét}}}(U, j^{-1}\mathscr{F}) \cong H^p_{X_{\text{ét}}}(U, \mathscr{F})$*

Proof.

- We should think about $j_!$ as extension by 0. When j is an immersion, this is literally what $j_!$ is.

  In general, we set

  $$j_!^p(\mathscr{F})(V \to S) = \coprod_{\mathcal{C}} \mathscr{F}(V \to U)$$

  where $\mathcal{C}$ is the category of commuting diagrams

  $$\begin{array}{ccc} V & & \\ \downarrow{\scriptstyle \text{ét}} & \searrow^{\text{ét}} & \\ U & \xrightarrow{\ j\ } & S \end{array}$$

  , and our definition of $j_!\mathscr{F}$ is the sheafification. The adjunction map can be described by taking $j_!^p\mathscr{F} \to \mathscr{G}$ to

  $$\mathscr{F}(V \to U) \to \coprod_{\mathcal{C}} \mathscr{F}(V' \to U) = j_!^p\mathscr{F}(V \to S) \to \mathscr{G}(V \to S) = j^{-1}\mathscr{G}(V \to U)$$

- We know j^{-1} is exact. It is a fact that exact functors with exact left adjoints preserve injectives. So given $\mathscr{F} \to I^\bullet$ an injective resolution, $j^{-1}\mathscr{F} \to j^{-1}I^\bullet$ is still an injective resolution, so

  $$H^p(U, j^{-1}\mathscr{F}) = H^p((j^{-1}I^\bullet)(U)) = H^p(I^\bullet(U)) = H^p(U, \mathscr{F})$$

  using our description of j^{-1} when j is étale at the very beginning of this lecture.

Remark 9.6 *Similar argument shows that cohomology is the same whether computed on the big étale (resp. Zariski) or small étale (resp. Zariski) site.*

Lemma 9.7 *Let $f : X \to Y$ be a morphism of schemes.*

- *For $\mathscr{G} \in \mathrm{Ab}(Y_{\text{ét}})$ we get $H^p(Y_{\text{ét}}, \mathscr{G}) \to H^p(X_{\text{ét}}, f^{-1}\mathscr{G})$.*
- *For $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$ we get $H^p(Y_{\text{ét}}, f_*\mathscr{F}) \to H^p(X_{\text{ét}}, \mathscr{F})$*

Proof.

- Let $\mathscr{G} \to I^\bullet$ be an injective resolution. Then $f^{-1}\mathscr{G} \to f^{-1}I^\bullet$ is a resolution, not necessarily injective. Let $f^{-1}\mathscr{G} \to J^\bullet$ be an injective resolution. We get an induced map of resolutions $f^{-1}I^\bullet \to J^\bullet$. Then we get

$$H^p(Y_{\text{ét}}, \mathscr{G}) = H^p(I^\bullet(Y)) \to H^p(f^{-1}I^\bullet(X)) \to H^p(J^\bullet(X)) = H^p(X_{\text{ét}}, f^{-1}\mathscr{G}).$$

- Using the previous bullet, we have $H^p(Y_{\text{ét}}, f_*\mathscr{F}) \to H^p(X_{\text{ét}}, f^{-1}f_*\mathscr{F}) \to H^p(X_{\text{ét}}, \mathscr{F})$, using the unit or counit map.

Example 9.8 *For any sheme S and group G, we get a constant sheaf, denoted by $G_S, \underline{G_S}, \underline{G}$, or G. We can describe this as $\underline{G_S}(U \to S) = \{$functions $U \to G$, Zariski-locally constant$\}$. This sheaf is represented by a group scheme $G_S := \coprod_G S$ with multiplication coming from that on G. For any $f : X \to Y$, $f^{-1}\underline{G_Y} \cong \underline{G_X}$, so we get $H^p(Y_{\text{ét}}, \underline{G_Y}) \to H^p(X_{\text{ét}}, \underline{G_Y})$. Typically we will have $G = \mathbb{Z}/n\mathbb{Z}$.*

Definition 9.9 *Let X be a scheme, ℓ a prime. Define*

$$H^p(X_{\text{ét}}, \mathbb{Z}_\ell) := \varprojlim_n H^p(X_{\text{ét}}, \underline{\mathbb{Z}/\ell^n\mathbb{Z}})$$

and

$$H^p(X_{\text{ét}}, \mathbb{Q}_\ell) := \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} H^0(X_{\text{ét}}, \mathbb{Z}_\ell)$$

For k separably closed, and $\ell \neq \mathrm{char}(k)$, this defined a Weil cohomology theory (which is what we used earlier to prove the Weil conjectures).

9.2 Lecture 12: 10/5

Let $f\colon X \to Y$ be a scheme morphism. Last time we constructed an adjoint pair $f^{-1}\colon \mathrm{Ab}\, Y_{\text{ét}} \to \mathrm{Ab}\, X_{\text{ét}}$, $f_*\colon \mathrm{Ab}\, X_{\text{ét}} \to \mathrm{Ab}\, Y_{\text{ét}}$ (f^{-1} the left adjoint, f^* the right). When f is étale, we also found a left adjoint $f_!$ to f^{-1}. The typical use of these functors is to construct functorial maps between the cohomology of different schemes, as we saw above.

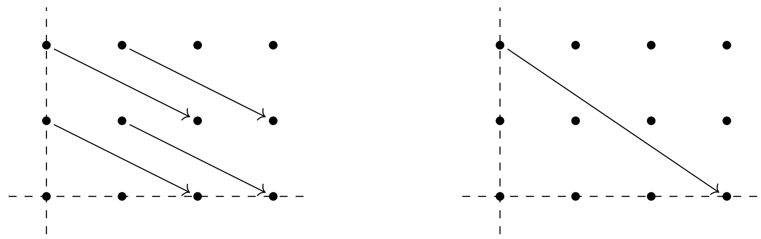
The functors f^{-1} and $f_!$ are exact, hence possess no interesting derived functors, but f_* is only left exact in general, so it admits (generally nontrivial) right derived functors R^pf_* (these are defined in the usual way, i.e. start with an injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$ and take cohomology: $R^pf_*\mathscr{F} := H^p(f_*\mathscr{I}^\bullet)$). The key computation tool for computing these is the Leray spectral sequence (actually, there are two of them). First recall this in the general setting:

Theorem 9.10 (Grothendieck(–Leray) spectral sequence) *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be Abelian categories, $F\colon \mathcal{A} \to \mathcal{B}$, $G\colon \mathcal{B} \to \mathcal{C}$ left-exact functors. Assume that $\mathcal{A}$, $\mathcal{B}$ have enough injectives and $F(I)$ is G-acyclic for $I \in \mathcal{A}$ injective (meaning $R^pG(F(I)) = 0$ for $p > 0$; e.g. $F(I)$ could be injective). Then there exists a first-quadrant spectral sequence $E_2^{p,q} = R^pG \circ R^qF(A) \Rightarrow R^{p+q}(G \circ F)(A)$.*

In order to explain this result, we'll give a brief summary, without any constructions or proofs, of spectral sequences. The Grothendieck spectral sequence for a given object $A \in \mathcal{A}$ consists of the following:

- Objects $E_r^{p,q} \in \mathcal{C}$ for each $r \geqslant 2$ and $p, q \geqslant 0$ such that $E_2^{p,q} = R^p G \circ R^q F(A)$. For (p, q) outside of the first quadrant, we set $E_r^{p,q} = 0$. We visualize the objects $E_r^{p,q}$, for fixed r, as occupying the integer lattice in $\mathbb{R}^2$ (i.e. at the coordinates (p, q)); they form the so-called *E_r-page* of the spectral sequence.

- Each E_r-page comes equipped with morphisms $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ for each p, q (the *differentials*). For example, the E_2- and E_3-pages and some of their differentials are pictured below:



- The data is required to satisfy that $E_{r+1}^{p,q}$ is formed by "taking the cohomology of E_r at (p, q)", i.e. we should have
$$E_{r+1}^{p,q} \cong \frac{\operatorname{Ker} d_r^{p,q}}{\operatorname{Im} d_r^{p-r,q+r-1}}.$$
Observe that for fixed p, q, since the nonzero objects of each page are concentrated in the first quadrant, the morphisms $d_r^{p,q}$ and $d_r^{p-r,q+r-1}$ will have trivial target and source, respectively, for sufficiently large r. Therefore each sequence $(E_r^{p,q})_r$ stabilizes, and we denote the stabilized object by $E_\infty^{p,q}$.

- Finally, the conclusion of the Grothendieck spectral sequence (the meaning of $R^p G \circ R^q F(A)$ "$\Rightarrow$" $R^{p+q}(G \circ F)(A)$) is that for all $n \geqslant 0$, $R^n(G \circ F)(A)$ admits a filtration (in $\mathcal{C}$)
$$R^n(G \circ F)(A) = \operatorname{Fil}^0 \supseteq \operatorname{Fil}^1 \supseteq \cdots \supseteq \operatorname{Fil}^n \supseteq \operatorname{Fil}^{n+1} = 0$$
whose jth graded piece (the successive quotient $\operatorname{Fil}^j / \operatorname{Fil}^{j+1}$) is isomorphic to $E_\infty^{j,n-j}$. (And eveything should be natural in A, but let us not expand on that.)

The point of the spectral sequence is to relate the objects $R^p G \circ R^q F(A)$ and $R^n(G \circ F)(A)$. This relationship is in a way only an "approximation" in that it expresses $R^n(G \circ F)(A)$ as filtered by certain subquotients of the $R^j G \circ R^{n-j} F(A)$, but sometimes this is enough to glean useful information about $R^n(G \circ F)(A)$.

Before describing the Leray spectral sequences, let's see the Grothendieck spectral sequence in action with some concrete examples and applications.

Example 9.11 *Suppose F is exact. (we leave the reader to consider the case when instead G is exact.) Then $R^p F = 0$ for $p > 0$, so the E_2-page has nonzero objects only along the line $p = 0$ (they are $R^q G \circ F(A)$). Immediately the spectral sequence degenerates—all the differentials of the E_2-page are forced to be trivial. So $E_\infty^{p,q} = E_2^{p,q}$ for all p, q and the corresponding filtration of $R^n(G \circ F)(A)$ has just one object, namely $R^n(G \circ F)(A) = R^n G \circ F(A)$.*

Example 9.12 (Hochschild–Serre spectral sequence) *Let G be a group. The "G-invariants" functor $(-)^G\colon \mathrm{Mod}_G \to \mathrm{Ab}$ is left exact, and its right-derived functors (the group cohomology of G) are denoted $H^p(G,-)$. Introducing a normal subgroup $H \subseteq G$, the cohomology groups $H^p(H,M)$ have a natural G/H-action, and there is the classical "inflation-restriction" exact sequence*

$$0 \longrightarrow H^1(G/H, M^H) \xrightarrow{\text{inf}} H^1(G,M) \xrightarrow{\text{res}} H^1(H,M)^{G/H}$$

which, if one is courageous, continues $\to H^2(G/H, M^H)$ (the so-called "transgression" map). This is in fact an instance of the Hochschild–Serre spectral sequence, which is the Grothendieck spectral sequence for the commutative diagram of functors

$$\begin{array}{ccc} \mathrm{Mod}_G & & \\ {\scriptstyle (-)^H}\downarrow & \searrow^{(-)^G} & \\ \mathrm{Mod}_{G/H} & \xrightarrow[(-)^{G/H}]{} & \mathrm{Ab}, \end{array}$$

giving $E_2^{p,q} = H^p(G/H, H^q(H,M)) \Rightarrow H^{p+q}(G,M)$. Indeed, we have a filtration $H^1(G,M) = \mathrm{Fil}^0 \supseteq \mathrm{Fil}^1 \supseteq 0$ with $\mathrm{Fil}^0 / \mathrm{Fil}^1 \cong E_\infty^{0,1}$ and $\mathrm{Fil}^1 \cong E_\infty^{1,0}$, i.e. a short exact sequence

$$0 \longrightarrow E_\infty^{1,0} \longrightarrow H^1(G,M) \longrightarrow E_\infty^{0,1} \longrightarrow 0.$$

Clearly $E_r^{1,0}$ has already stabilized at $r=2$, i.e. $E_\infty^{1,0} = H^1(G/H, M^H)$. The groups $E_r^{0,1}$ instead stabilize at $r=3$, giving $E_\infty^{0,1} = \mathrm{Ker}(H^1(H,M)^{G/H} \to H^2(G/H, M^H))$. Putting these pieces into the short exact sequence above yields the inflation-restriction sequence.

Caveat: Our argument does not show that the maps in the exact sequence coming from the Hochschild–Serre spectral sequence agree with those in the classical inflation-restriction sequence. This takes some amount of explicit computation.

Example 9.13 (Edge maps) *We return to the setting of the general Grothendieck spectral sequence. Observe that at each page, the differential $d_r^{p,0}$ is always trivial, so as we take cohomology to pass from one page to the next, $E_{r+1}^{p,0}$ is a quotient of $E_r^{p,0}$. Finally, $E_\infty^{p,0}$ appears as the smallest graded piece in the filtration of $R^p(G \circ F)(A)$, i.e. as a subobject. The composition*

$$R^pG \circ F(A) = E_2^{p,0} \twoheadrightarrow E_\infty^{p,0} \hookrightarrow R^p(G \circ F)(A)$$

is called an edge map of the spectral sequence. Dually, there is also an edge map

$$R^q(G \circ F)(A) \twoheadrightarrow E_\infty^{0,q} \hookrightarrow E_2^{0,q} = G \circ R^qF(A).$$

Now we state the two Leray spectral sequences, which for us are the main applications of the Grothendieck spectral sequence.

Corollary 9.14 (first Leray spectral sequence) *Let $f\colon X \to Y$, $g\colon Y \to S$ be maps of schemes. Then there exists a spectral sequence $R^pg_*(R^qf_*\mathscr{F}) \Rightarrow R^{p+q}(g \circ f)_*\mathscr{F}$ for $\mathscr{F} \in \mathrm{Ab}\, X_{\text{ét}}$.*

Of course, such a spectral sequence will also exist for fppf or Zariski sheaves, or for sheaves on topological spaces (the latter being what "Leray spectral sequence" classically refers to).

Proof. Indeed, f_*, g_* are left exact and f_*, having an exact left adjoint, takes injectives to injectives. So, the Grothendieck spectral sequence applies to $g_* \circ f_* = (g \circ f)_*$.

Corollary 9.15 (second Leray spectral sequence) *Consider $f\colon X \to Y$ a morphism of schemes. Then the Grothendieck spectral sequence associated to the diagram*

$$\begin{array}{ccc} \mathrm{Ab}\, X_{\text{ét}} & & \\ {\scriptstyle f_*}\downarrow & \searrow^{\Gamma(X,-)} & \\ \mathrm{Ab}\, Y_{\text{ét}} & \xrightarrow[\Gamma(Y,-)]{} & \mathrm{Ab} \end{array}$$

is $H^p(Y_{\text{ét}}, R^q f_* \mathscr{F}) \Rightarrow H^{p+q}(X_{\text{ét}}, \mathscr{F})$.

Example 9.16 *Let's consider the edge maps of the second Leray spectral sequence. The first ones take the form $H^p(Y_{\text{ét}}, f_*\mathscr{F}) \to H^p(X_{\text{ét}}, \mathscr{F})$, and in fact these are the "change-of-scheme" maps for étale cohomology defined previously! However, as with the Hochschild–Serre example, it is necessary to "look inside the machine" to check this. The other edge maps take the form $H^q(X_{\text{ét}}, \mathscr{F}) \to R^q f_* \mathscr{F}(Y)$. (We leave it as an exercise to construct these natural maps in a more concrete way.)*

Example 9.17 *Let $\iota\colon Z \hookrightarrow X$ be a closed immersion. Then ι_* is exact (exercise—check on stalks), so $R^q \iota_* = 0$ for all $q > 0$. Thus the second Leray spectral sequence collapses and gives $H^p(X_{\text{ét}}, \iota_* \mathscr{F}) \cong H^p(Z_{\text{ét}}, \mathscr{F})$ for $\mathscr{F} \in \mathrm{Ab}\, Z_{\text{ét}}$. The same is true (to be discussed later) for any finite morphism of schemes.*

Enough with the formalism (for now). So we next ask

10 Can we (ever) compute cohomology on the étale site?

Spoiler: amazingly, sometimes! We'll explain three examples and in the middle take a detour through the Čech cohomology $\Rightarrow$ derived-functor cohomology spectral sequence. For now we just give statements—proofs next time.

First, we've seen that quasicoherent Zariski sheaves on a scheme can be upgraded to a sheaf on the big fpqc site of that scheme, and hence on all the smaller sites we've considered. How does the classical sheaf cohomology compare to the cohomology on the fancier sites? In fact, it's the same:

Theorem 10.1 *Let $\mathscr{F}$ be a quasicoherent Zariski sheaf on a scheme S. Then the classical sheaf cohomology satisfies $H^p(S, \mathscr{F}) \cong H^p(S_\tau, \mathscr{F})$, where τ is any of the symbols* zar, ét, fppf, Zar, Ét, Fppf.

Example 10.2 *The sheaf $\mathbb{G}_{\mathrm{a}}\colon T \mapsto \Gamma(T, \mathscr{O}_T)$ gives $H^p(S_\tau, \mathbb{G}_{\mathrm{a}}) \cong H^p(S, \mathscr{O}_S)$.*

Example 10.3 *The sheaf $\mathbb{G}_{\mathrm{m}}\colon T \mapsto \Gamma(T, \mathscr{O}_T)^\times$ is not quasicoherent, so the Theorem does not apply. Nevertheless we have the following:*

Theorem 10.4 $H^1(S_\tau, \mathbb{G}_m) \cong H^1(S, \mathscr{O}_S^\times) \cong \mathrm{Pic}\, S$.

These results (and the next) give us some useful inroads toward computing étale and fppf cohomology, and in particular are a couple of the "seed" calculations needed for the general theoretical tools.

Let's now turn toward a more elementary example which gives a sense of what's different with our new étale cohomology theory. Let k be a field, $\Gamma_k := \mathrm{Gal}(k^{\mathrm{sep}}/k)$, $\mathscr{F} \in \mathrm{Sh}(\operatorname{Spec} k)_{\text{ét}}$. We construct a discrete left Γ_k-set:

$$M_{\mathscr{F}} := \operatorname*{colim}_{\substack{k'/k \text{ finite Galois} \\ k \subseteq k' \subseteq k^{\mathrm{sep}}}} \mathscr{F}(k').$$

Here, the right action of Γ_k on $\operatorname{Spec} k'$ induces compatible left actions of Γ_k on each $\mathscr{F}(k')$. That $M_{\mathscr{F}}$ is a "discrete" Γ_k-module means that all the stabilizers are open, which is clear since every element of $M_{\mathscr{F}}$ comes from some $\mathscr{F}(k')$.

Theorem 10.5 *The association $\mathscr{F} \mapsto M_{\mathscr{F}}$ induces equivalences of categories*

$$\mathrm{Sh}(\operatorname{Spec} k)_{\text{ét}} \xrightarrow{\sim} \{\text{discrete left } \Gamma_k\text{-sets}\},$$
$$\mathrm{Ab}(\operatorname{Spec} k)_{\text{ét}} \xrightarrow{\sim} \{\text{discrete left } \Gamma_k\text{-modules}\}.$$

Corollary 10.6 *Let $\mathscr{F} \in \mathrm{Ab}(\operatorname{Spec} k)_{\text{ét}}$. Then there exists a canonical isomorphism $H^p((\operatorname{Spec} k)_{\text{ét}}, \mathscr{F}) \cong H^p(\Gamma_k, M_{\mathscr{F}})$.*

(The latter cohomology group refers, of course, to the right derived functors of $(-)^{\Gamma_k}$ on the category of discrete Γ_k-modules.)

Proof. We have $H^0((\operatorname{Spec} k)_{\text{ét}}, \mathscr{F}) = \mathscr{F}(k)$, so by the Theorem it suffices to give a natural isomorphism $\mathscr{F}(k) \xrightarrow{\sim} M_{\mathscr{F}}^{\Gamma_k}$. We check this at the finite Galois levels, i.e. we fix k'/k finite Galois and want to show that $\mathscr{F}(k) \to \mathscr{F}(k')^{\mathrm{Gal}(k'/k)}$ is an isomorphism. For this, we write down the sheaf sequence corresponding to the étale cover $\{\operatorname{Spec} k' \to \operatorname{Spec} k\}$:

$$\mathscr{F}(k) \hookrightarrow \mathscr{F}(k') \underset{\mathrm{id}\otimes 1}{\overset{1\otimes \mathrm{id}}{\rightrightarrows}} \mathscr{F}(k' \otimes_k k').$$

Now recall that we also have an identification $\mathscr{F}(k' \otimes_k k') \cong \mathscr{F}(\prod_{\sigma \in \mathrm{Gal}(k'/k)} k')$, which is defined so that each of the following diagrams commutes:

$$\begin{array}{ccc} \mathscr{F}(k' \otimes_k k') & \xrightarrow{\sim} & \mathscr{F}(\prod_{\sigma\in\mathrm{Gal}(k'/k)} k') \\ & \searrow^{\mathrm{id}\cdot\sigma\colon a\otimes b \mapsto a\sigma(b)} & \downarrow{\scriptstyle \text{project onto the } \sigma\text{-factor}} \\ & & \mathscr{F}(k') \end{array}$$

Using this isomorphism, the sheaf sequence becomes

$$\begin{aligned} \mathscr{F}(k) &\cong \{t \in \mathscr{F}(k') \colon (\mathrm{id} \otimes 1)(t) = (1 \otimes \mathrm{id})(t)\} \\ &= \{t \in \mathscr{F}(k') \colon (\mathrm{id}\cdot\sigma) \circ (\mathrm{id} \otimes 1)(t) = (\mathrm{id}\cdot\sigma) \circ (1 \otimes \mathrm{id})(t) \text{ for all } \sigma\} \\ &= \mathscr{F}(k')^{\mathrm{Gal}(k'/k)}, \end{aligned}$$

as desired.

10.1 Lecture 13: 10/10

We start off with a few examples to the theorems of the last lecture.

Example 10.7 *Let k be a field and Γ_k be its absolute Galois group. Consider the constant sheaf $\mathbb{Z}/n\mathbb{Z} \in \mathrm{Ab}((\mathrm{Spec}\, k)_{\text{ét}})$, we have, from the last lecture, the following:*

$$H^1_{\text{ét}}(\mathrm{Spec}(k), \mathbb{Z}/n\mathbb{Z}) \cong H^1(\Gamma_k, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}_{cts}(\Gamma_k, \mathbb{Z}/n\mathbb{Z})$$

as we endow Γ_k with the Krull topology and consider it acting on $\mathbb{Z}/n\mathbb{Z}$ trivially. It's worth noting that these groups may well be infinite. For example, let $n = 2$ and $k = \mathbb{Q}$, we have $\mathrm{Hom}_{cts}(\Gamma_{\mathbb{Q}}, \mathbb{Z}/2\mathbb{Z})$ in correspondence to the trivial and quadratic extensions of $\mathbb{Q}$ of which there are infinitely many.

Example 10.8 *In the previous lecture we stated*

$$\mathrm{Pic}(S) \cong H^1_{Zar}(S, \mathscr{O}_S^\times) \cong H^1_{\text{ét}}(S, \mathbb{G}_m).$$

Take $S = \mathrm{Spec}(k)$, $\mathrm{Pic}(k) = 0$, hence

$$0 = \mathrm{Pic}(k) \cong H^1(\Gamma_k, (k^s)^\times)$$

which is exactly Hilbert's 90.

Example 10.9 *For scheme S and $n \geqslant 1$ we have $\mu_n \in Ab(S_{et})$ represented by group scheme $\mu_n = \mathrm{Spec}(\mathbb{Z}[X]/(X^n - 1))$, so $\mu_n(T) = \{x \in \Gamma(T, \mathscr{O}_T^\times) | x^n = 1\}$. If n is invertible in S, i.e. $n \in \mathscr{O}_S^\times$, then there is short exact sequence in $Ab(S_{\text{ét}})$*

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m \longrightarrow 1.$$

We also get Kummer sequence

$$\begin{array}{l} 1 \longrightarrow \mu_n(S) \longrightarrow \mathscr{O}_S^\times \longrightarrow \mathscr{O}_S^\times \\ \hookrightarrow H^1_{\text{ét}}(S, \mu_n) \longrightarrow H^1_{\text{ét}}(S, \mathbb{G}_m) \longrightarrow \cdots \end{array}$$

or

$$1 \longrightarrow \mathbb{G}_m(S)/(\mathbb{G}_m(S))^n \longrightarrow H^1_{\text{ét}}(S, \mu_n) \longrightarrow \mathrm{Pic}(S)[n] \longrightarrow 1.$$

Take $S = \mathrm{Spec}(k)$ we get

$$k^\times/(k^\times)^n \cong H^1(\Gamma_k, \mu_n)$$

which is Kummer's theory.

We will now give proofs of the three theorems from last times. First is the equivalence of categories one corresponding to Theorem 7.5. We will only prove the $Sh(\mathrm{Spec}(k))_{et})$ part, i.e. the equivalence of the following categories

$$Sh((\mathrm{Spec}(k))_{et}) \cong \{\text{discrete } \Gamma_k \text{ set}\} \cong (\mathrm{Spec}(k))_{\text{ét}}.$$

Proof. Let k be a field, k^s its separable closure, and Γ_k its absolute galois group. Given $X \longrightarrow \text{Spec}(k)$ étale, associating it to the discrete Γ_k set $\text{Hom}_{Sch/k}(\text{Spec}(k^s), X) = X(k^s)$ will give us the equivalence on the right $\{\text{discrete } \Gamma_k \text{ set}\} \cong (\text{Spec}(k))_{\text{ét}}$.
Now we want to show the essential surjectivity first. Given $\Gamma_k \circlearrowright S$ discretely, decompose S into orbits denoted by $\coprod S_i$ and pick $s_i \in S_i$ for each i. Now let $k_i := (k^s)^{\text{stab}(s_i)}$, which is a finite extension of k due to the discreteness of action of Γ_k. Letting $X := \coprod_i \text{Spec}(k)_i$ yields $X(k^s) \cong S$ as Γ_k sets.
Faithfulness however is left as an exercise.
For the fullness, given $X, Y \in (\text{Spec}(k))_{\text{ét}}$, since they are étale over $\text{Spec}(k)$ we can write

$$X \cong \coprod_{i \in I} \text{Spec}(k)_i \quad Y \cong \coprod_{j \in J} \text{Spec}(k)'_j.$$

We also let

$$S := \text{Hom}_{Sch/k}(\text{Spec}(k)^s, X) \cong \coprod_i \text{Hom}(k_i, k^s) = \coprod_i S_i,$$

$$T := \text{Hom}_{Sch/k}(\text{Spec}(k)^s, Y) \cong \coprod_j \text{Hom}(k'_j, k^s) = \coprod_j T_j.$$

and notice that $X(resp.Y)$ is associated to $S(resp.T)$. Given a Γ_k map $a : S \longrightarrow T$ (hence image of a single orbit of S lies in a single orbit of T, which gives $\alpha : I \longrightarrow J$ such that S_i is mapped to $T_{\alpha(i)}$). We claim that this induces a map $X \longrightarrow Y$ which would give the fullness. So the only thing left is the claim.
Fix base points $s_i \in S_i$ and $t_j \in T_j$ for each i, j such that $t_{\alpha(i)} = a(s_i)$. Identify by base points(i.e. identify k_i as $s_i(k_i) \subseteq k^s$) we can consider k_i and k'_j as subfields of k^s. For any $h \in \Gamma_k$ fix k_i, we have $a(h \cdot s_i) = a(s_i) = t_{\alpha(i)}$. On the other hand, since a is a Γ_k set map, we have $a(h \cdot s_i) = h \cdot a(s_i) = h \cdot t_{\alpha(i)}$, hence h also fix $k_\alpha(i)'$. By Galois theory we have $k'_{\alpha(i)} \subseteq k_i$ hence $X \longrightarrow Y$. Now we have the right equivalence. For the left equivalence, we will give a quasi-inverse to the functor $\mathscr{F} \mapsto M_{\mathscr{F}}$. Given an M, we constructed $X_M \in (\text{Spec}(k))_{\text{ét}}$ such that X_M is associated to M. Now we associate M to the sheaf $h_{X_M} \in Sh((\text{Spec}(k))_{\text{ét}})$ represented by X_M.

$$\begin{aligned} M_{h_{X_M}} &= \operatorname*{colim}_{\substack{k'/k \text{ finite Galois} \\ k \subseteq k' \subseteq k^s}} h_{X_M}(\text{Spec}(k)') \\ &= \operatorname*{colim}_{\substack{k'/k \text{ finite Galois} \\ k \subseteq k' \subseteq k^s}} \text{Hom}_{Sch/k}(\text{Spec}(k)', X_M) \\ &= \text{Hom}_{Sch/k}(\text{Spec}(k)^s, X_M) \\ &= M. \end{aligned}$$

This shows that $M \mapsto h_{X_M}$ is a quasi-inverse to $\mathscr{F} \mapsto M_{\mathscr{F}}$, hence the left equivalence, which concludes the proof.

Remark 10.10 *The isomorphism $H^p_{\text{ét}}(\text{Spec}(k), \mathscr{F}) \cong H^p(\Gamma_k, M_{\mathscr{F}})$ gives us a good reason to define our ℓ-adic cohomology as*

$$H^p_{\text{ét}}(\text{Spec}(k), \mathbb{Z}_\ell) := \varprojlim H^p_{\text{ét}}(\text{Spec}(k), \mathbb{Z}/\ell^n\mathbb{Z})$$

as opposed to the naïve definition

$$H^p_{\text{ét}}(\text{Spec}(k), \textit{constant sheaf } \mathbb{Z}_\ell)$$

because if we adopt the naïve definition, we will find that, given the isomorphism,

$$H^1_{\text{ét}}(\mathrm{Spec}(k), \textit{constant sheaf}\ \mathbb{Z}_\ell) \cong H^1(\Gamma_k, \mathbb{Z}_\ell) = \mathrm{Hom}_{cts}(\Gamma_k, \mathbb{Z}_\ell)$$

where Γ_k acts on $\mathbb{Z}_\ell$ trivially and $\mathbb{Z}_\ell$ is endowed with discrete topology. However we notice that continuous homomorphism from Γ_k, a compact group, to $\mathbb{Z}_\ell$ which has discrete topology has to have finite image, whereas $\mathbb{Z}_\ell$ has no finite subgroup; therefore the map has to be trivial, which in turn shows that the $H^1_{\text{ét}}(\mathrm{Spec}(k), \textit{constant sheaf}\ \mathbb{Z}_\ell)$ has to be trivial.

The other two theorems(Theorem 7.1 and 7.4) requires a detour to Čech cohomology.

11 Čech cohomology and first calculations of étale cohomology

Let $\mathcal{C}$ be a site, $\mathscr{F} \in \mathrm{Ab}(\mathcal{C})$ and $\mathcal{U} = \{U_i \longrightarrow U\}_{i\in I} \in Cov(\mathcal{C})$. These will give us a Čech complex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathscr{F})$

$$\textstyle\prod_{i_0\in I} \mathscr{F}(U_{i_0}) \longrightarrow \prod_{i_0,i_1\in I} \mathscr{F}(U_{i_0} \times_U U_{i_1}) \longrightarrow \cdots$$

with usual Čech differential given by

$$(ds)_{i_0,i_1,\cdots,i_{p+1}} := \sum_k (-1)^k s_{i_0,i_1,\cdots,\hat{i_k},\cdots,i_{p+1}}|_{U_{i_0}\times\cdots\times U_{i_{p+1}}}.$$

Definition 11.1

$$H^p(\mathcal{U}, \mathscr{F}) := H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathscr{F}))$$

In fact we used $\check{H}^0$ when defining sheafification.

Notice that $\mathscr{F} \mapsto \check{\mathcal{C}}(\mathcal{U}, \mathscr{F})$ is an exact functor from $\mathrm{PAb}(C)$ to $\mathrm{Ch(Ab)}$, and we know that short exact sequences of chain complexes give rise to long exact sequences, and we have the following lemma.

Lemma 11.2 *$\mathscr{F} \mapsto \check{\mathcal{C}}(\mathcal{U}, \mathscr{F})$ is effaceable hence a universal δ functor.*

Now we will introduce some preliminaries for the proof. The forget functor from $\mathrm{PAb}\,\mathcal{C}$ to $\mathrm{PSh}\,\mathcal{C}$ has a left adjoint given by

$$\mathscr{F} \mapsto \mathbb{Z}_{\mathscr{F}}(U \mapsto \bigoplus_{s\in\mathscr{F}(U)} \mathbb{Z}).$$

Take $\mathscr{F} = h_U$ represented by $U \in \mathcal{C}$ i.e. $\mathscr{F}(V) = \mathrm{Hom}_{\mathcal{C}}(V, U)$, then we have

$$\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbb{Z}_{h_U}, \mathscr{G}) = \mathrm{Hom}_{\mathrm{Sh}(\mathcal{C})}(h_U, \mathscr{G}) = \mathscr{G}(U).$$

Let's abbreviate $\mathbb{Z}_{h_U}$ to $\mathbb{Z}_U$. Notice that $\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbb{Z}_U, \mathscr{F}) = \mathscr{F}(U)$ and we rewrite the Čech complex into

$$\textstyle\prod_{i_0\in I} \mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbb{Z}_{U_{i_0}}, \mathscr{F}) \longrightarrow \prod_{i_0,i_1\in I} \mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(\mathbb{Z}_{U_{i_0}\times_U U_{i_1}}, \mathscr{F}) \longrightarrow \cdots$$

which equals to $\mathrm{Hom}_{PAb(\mathcal{C})}(\cdots \longrightarrow \bigoplus \mathbb{Z}_{U_{i_0}\times_U U_{i_1}} \longrightarrow \bigoplus \mathbb{Z}_{U_{i_0}}, \mathscr{F})$.

Proposition 11.3 *The following hold*

1. *Given cover* $\mathcal{U} = \{U_i \longrightarrow U\}_{i\in I}$*, the complex* $\bigoplus \mathbb{Z}_{U_{i_0}} \longleftarrow \bigoplus \mathbb{Z}_{U_{i_0}\times_U U_{i_1}} \longleftarrow \cdots$ *is exact in non-zero degree.*

2. *For any injective* I*,* $\check{H}^p(\mathcal{U}, I) = 0$ *for* $p > 0$.

3. $\check{H}^p(\mathcal{U}, -) : \mathrm{PAb}(\mathcal{C}) \longrightarrow \mathrm{Ab}$ *are the right derived functor of* $\check{H}^0(\mathcal{U}, -)$.

Proof. Assume 1., 2. follows because $\mathrm{Hom}_{\mathrm{PAb}(\mathcal{C})}(-, I)$ *is exact, which implies that* $\check{H}^p(\mathcal{U}, -)$ *is effaceable since* $\mathrm{PAb}(\mathcal{C})$ *has enough injectives. Given that* $\check{H}^p(\mathcal{U}, -)$ *is a delta functor, and we know effaceable* δ *functor is universal, so 3. follows. Now we turn to the proof of 1. Let* $\mathbb{Z}[S]$ *denote the free abelian group with basis* S*. To show 1., we must show, for any* $V \in \mathcal{C}$ *the exactness of the following at non-zero degree*

$$\bigoplus \mathbb{Z}[\mathrm{Hom}_{\mathcal{C}}(V, U_{i_0})] = \mathbb{Z}[\coprod \mathrm{Hom}_{\mathcal{C}}(V, U_{i_0})] \longleftarrow \mathbb{Z}[\coprod \mathrm{Hom}_{\mathcal{C}}(V, U_{i_0} \times_U U_{i_1})] \longleftarrow \cdots$$

To utilize the property of fiber product, i.e. $\mathrm{Hom}(A, B \times C) = \mathrm{Hom}(A, B) \times \mathrm{Hom}(A, C)$ *when the Hom set is appropriately restricted, we partition the basis of these free groups by the maps* $V \longrightarrow U$ *they induce: we write* $\mathrm{Hom}_{\mathcal{C}}(V, U_{i_0}) = \coprod_{\varphi\in\mathrm{Hom}_{\mathcal{C}}(V,U)} \mathrm{Hom}_{\mathcal{C}}(V, U_{i_0})_\varphi$*, we now can rewrite the complex into*

$$\bigoplus_{\varphi\in\mathrm{Hom}(V,U)} \left(\mathbb{Z}[\coprod \mathrm{Hom}(V, U_{i_0})_\varphi] \longleftarrow \mathbb{Z}[\coprod \mathrm{Hom}(V, U_{i_0})_\varphi \times \mathrm{Hom}(V, U_{i_1})_\varphi] \longleftarrow \cdots \right)$$

Each direct summand of this complex is in the form

$$\mathbb{Z}[S] \longleftarrow \mathbb{Z}[S \times S] \longleftarrow \mathbb{Z}[S \times S \times S] \longleftarrow \cdots$$

with the boundary map alternating sum of projection, which is exact at non-zero degree, hence so is their direct sum. This concludes the proof of 1.

11.1 Lecture 14: 10/12

We connect “presheaf cohomology" (Čech cohomology) to sheaf cohomology

Theorem 11.4 *Let* $\mathscr{F}$ *be a sheaf,* $\mathcal{U} = \{U_i \to U\}_{i\in I} \in Cov(\mathcal{C})$*,* $\mathscr{F} \in \mathrm{Ab}(\mathcal{C})$*. Let* $\underline{H}^q(\mathscr{F}) \in \mathrm{PAb}(\mathcal{C})$ *be the presheaf* $\underline{H}^q(\mathscr{F})(V) = H^q(V, \mathscr{F})$*. Then, there is a spectral sequence* $E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathscr{F})) \Rightarrow H^{p+q}(U, \mathscr{F})$.

Proof. The composite $\mathrm{Ab}(\mathcal{C}) \xrightarrow{\omega=\mathrm{Forget}} \mathrm{PAb}(\mathcal{C}) \xrightarrow{\check{H}^0(\mathcal{U},-)} \mathrm{Ab}$ is just $\mathscr{F} \to H^0(U, \mathscr{F})$ since $\mathscr{F}$ is a sheaf. ω preserves injectives since sheafification is an exact left adjoint; so we obtain a Grothendieck-Leray spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, R^q\omega(\mathscr{F})) \Rightarrow H^{p+q}(U, \mathscr{F})$$

We claim that $R^q\omega(\mathscr{F}) = \underline{H}^q(\mathscr{F})$. Indeed, they agree for $q = 0$, so it is enough to show that $\underline{H}^q(\mathscr{F})$ are the derived functors of $\underline{H}^0(\mathscr{F}) = \omega$. For $\mathscr{F} \to I^\bullet$, an injective resolution in $\mathrm{Ab}(\mathcal{C})$ we compute the p-th derived functors of ω as $\ker(I^p \to I^{p+1})/\operatorname{Im}(I^{p-1} \to I^p)$ in $PAb(\mathcal{C})$. This is the presheaf

$$V \longrightarrow \frac{\ker(I^p(V) \to I^{p+1}(V))}{\operatorname{Im}(I^{p-1}(V) \to I^p(V))} = H^p(V, \mathscr{F})$$

which is exactly $\underline{H}^p(\mathscr{F})$.■

Corollary 11.5 *The edge map $\check{H}^1(\mathcal{U}, \mathscr{F}) \longrightarrow H^1(U, \mathscr{F})$ is injective.*

When it makes sense (when $Cov(\mathcal{C})$ is a set, or when we make an adjustment as in the étale site example) we form $\check{H}^p(U, \mathscr{F}) := \operatorname{colim}_{\mathcal{U}} \check{H}^p(\mathcal{U}, \mathscr{F})$. We still get an injection $\check{H}^1(U, \mathscr{F}) \to H^1(U, \mathscr{F})$, which is now an isomorphism. Indeed, same reasoning as in the theorem shows that there is a spectral sequence $\check{H}^p(U, \underline{H}^q(\mathscr{F})) \Rightarrow H^{p+q}(U, \mathscr{F})$. Now, we use the fact that $\check{H}^0(U, \underline{H}^q(\mathscr{F})) = 0$ for $q > 0$. Indeed, any element of it arises from some $\check{H}^0(\mathcal{U}, \underline{H}^q(\mathscr{F}))$ for a cover $\mathcal{U} = \{U_i \to U\}_{i \in I}$. Then the claim follows from

Lemma 11.6 *Let $V \in \mathcal{C}, \xi \in H^q(V, \mathscr{F})$, $q > 0$. Then, there exists a $\mathscr{V} = \{V_j \to V\}_{j \in J}$ such that $\xi\big|_{V_j} = 0$ for all $j \in J$.*

Proof. We take an injective resolution $\mathscr{F} \to I^\bullet$. Then $H^q(V, \mathscr{F}) = H^q(I^\bullet(V))$. So, ξ is represented by $\tilde{\xi} \in \ker(I^q(V) \to I^{q+1}(V))$. Exactness of $I^\bullet$ in degree > 0 shows that there exists a cover $\mathscr{V} = \{V_j \to V\}_{j \in J}$ such that $\tilde{\xi}\big|_{V_j} \in \operatorname{Im}(I^{q-1}(V) \to I^q(V))$, i.e. $\xi\big|_{V_j} = 0$.■

This simplifies the spectral sequence and unpacking we get $\check{H}^1(U, \mathscr{F}) \simeq H^1(U, \mathscr{F})$.

Now, we return to our target results

Theorem 11.7 *For $\tau \in \{Zar, ét, fppf\}$, $\mathcal{C} = (Sch/S)_\tau$ (or the small sites) we write $\check{H}^\bullet_\tau$ and $H^\bullet_\tau$ for the Čech and the usual cohomology on $\mathcal{C}$. Then* $\operatorname{Pic}(S) \simeq \check{H}^1_\tau(S, \mathbb{G}_m) \simeq H^1_\tau(S, \mathbb{G}_m)$.

Proof. Let $\mathscr{L} \in \operatorname{Pic}(S)$. Then, there exists a Zariski cover (hence a τ-cover) $\mathcal{U} = \{U_i \to S\}_{i \in I}$ such that $\mathscr{L}\big|_{U_i}$ is trivial. Fix $\alpha_i : \mathscr{L}\big|_{U_i} \simeq \mathscr{O}_{U_i}$. The isomorphism $\mathscr{O}_{U_{ij}} \xrightarrow{\alpha_j^{-1}} \mathscr{L}\big|_{U_{ij}} \xrightarrow{\alpha_i} \mathscr{O}_{U_{ij}}$, where $U_{ij} = U_i \times_U U_j$ is multiplication by $g_{ij} \in \mathbb{G}_m(U_{ij})$ and from the cocycle condition (g_{ij}) defines a class in $\check{H}^1_\tau(\mathcal{U}, \mathbb{G}_m)$. So, we get a group homomorphism $\operatorname{Pic}(S) \longrightarrow \check{H}^1_\tau(\mathcal{U}, \mathbb{G}_m) \longrightarrow \check{H}^1_\tau(S, \mathbb{G}_m)$.

We claim that this composite is injective. Suppose, $\mathscr{L}$ is in its kernel. Then, there exist a refinement $\mathscr{V} \to \mathcal{U}$, with $\mathscr{V} = \{V_j \to S\}$, $\beta : J \to I$ together with maps $\chi_j : V_j \to U_{\beta(j)}$, such that if $\mathscr{L}$ is sent to c, then $c\big|_{\mathscr{V}}$ is trivial. This means that $c\big|_{\mathscr{V}} \in \prod_{j_1, j_2 \in J} \mathbb{G}_m(V_{j_1 j_2})$ given by $(c\big|_{\mathscr{V}})_{j_1 j_2} = (\chi_{j_1} \times \chi_{j_2})^*(g_{\beta(j_1)\beta(j_2)})$ is equal to $h_{j_2}^{-1}\big|_{V_{j_1 j_2}} h_{j_1}\big|_{V_{j_1 j_2}}$ for some $(h_j) \in \prod_{j \in J} \mathbb{G}_m(V_j)$.

We now modify the trivializations of $\mathscr{L}$. For all $j \in J$ we pullback $\alpha_{\beta(j)}$ to $\mathscr{L}\big|_{V_j} \xrightarrow{\chi_j(\alpha_{\beta(j)})} \mathscr{O}_{V_j}$. and alter to the new trivializations $\alpha'_j = h_j \cdot \chi_j^*(\alpha_{\beta(j)})$. The trivialization functions for the $(\alpha'_j)'s$ are now trivial. This means $\mathscr{O}_{V_{j_1 j_2}} \xrightarrow{\alpha'^{-1}_{j_2}} \mathscr{L}\big|_{V_{j_1 j_2}} \xrightarrow{\alpha'_{j_1}} \mathscr{O}_{V_{j_1 j_2}}$ is the identity map. Then, the full faithfulness statement in the flat descent theorem for QCoh shows that $\{\alpha_j\}_{j \in J}$ glue to a global isomorphism $\mathscr{O}_S \simeq \mathscr{L}$, which proves the injectivity.

For the surjectivity of $\mathrm{Pic}(S) \to \check{H}^1_\tau(S, \mathbb{G}_m)$ it suffices to prove that for any cover $\mathcal{U} = \{U_i \to S\}_{i\in I}$ any class $(g_{ij}) \in \check{H}^1_\tau(\mathcal{U}, \mathbb{G}_m)$ comes from a line bundle. We'll construct a line bundle by descent. Let $\mathscr{L}$ be the trivial bundle on $S' := \coprod_{i\in I} U_i$. We have isomorphisms $p_i^*\mathscr{L} \simeq p_j^*\mathscr{L}$ given by the $g'_{ij}s$ and the cocycle relation for g_{ij}'s implies that $p_{ik}^*\phi = p_{jk}^*\phi p_{ij}^*\phi$. Hence, we get a descent daturm. By the essential surjectivity in the flat descent theorem for QCoh we obtain $\overline{\mathscr{L}} \in QCoh(S)$, such that $p^*\overline{\mathscr{L}} \simeq \mathscr{L}$, where p is the projection $S' \to S$. It remains to check that $\overline{\mathscr{L}}$ is a line bundle. This reduction is possible because given an fpqc cover $S' \xrightarrow{p} S$ and $\mathscr{F} \in QCoh(S)$ to check

1. $\mathscr{F}$ is finitely generated
2. $\mathscr{F}$ is finitely presented
3. $\mathscr{F}$ is flat

can be done by checking it for $p_*\mathscr{F}$. Having this result, if we have $1) - 3)$ for $p^*\mathscr{F}$, as locally free $\iff$ flat of finite presentation we get that $\mathscr{F}$ is locally free. Then, an easy check yields that $rk\,\mathscr{F} = rk\,p^*\mathscr{F}$ if the latter has finite rank.

We explain how to prove 1) in the aforementioned result, and the rest of the claims follows similarly. In particular, we want to show that if $p^*\mathscr{F}$ is finitely generated, then so is $\mathscr{F}$. We reduce to the affine case $S' = \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and $\mathbb{F}$ is some A-module M. We know that $M \otimes_A B$ is finitely generated, i.e. there exists a surjection $B^r \twoheadrightarrow M \otimes_A B$. We choose generators $e_1, \dots, e_r$ of $M \otimes_A B$ over B. We write $e_i = \sum_{j=1}^{s_i} m_{ij} \otimes b_{ij}$. Let $N = \#\{m_{ij} \mid i = 1, \dots, r \text{ and } j = 1, \dots, s_i\}$. Then we define $\phi : A^N \to M$ by sending E_{ij} to m_{ij}. Certainly, $\phi \otimes_A B$ is surjective and as $A \to B$ is faithfully flat, so is ϕ.■

Finally, we have the other theorem

Theorem 11.8 *Let S be a scheme, $\mathscr{F} \in QCoh(S)$, $\tau \in \{$ Zar,ét,fppf $\}$. Then $H^p_{zar}(S, \mathscr{F}) \simeq H^p_\tau(S, \mathscr{F}_\tau)$, where $\mathscr{F}_\tau$ is the sheaf on $(Sch/S)_\tau$ associated to $\mathscr{F}$ by descent theory.*

Example 11.9 *Let $S = \mathrm{Spec}(k)$ and $\mathscr{F} = \mathbb{G}_a$. Then for $p > 0$ we get*

$$H^p_{zar}(\mathrm{Spec}(k), \mathbb{G}_a) = H^p_{\text{ét}}(\mathrm{Spec}(k), \mathbb{G}_a) = H^p(\Gamma_k, \overline{k}) = 0,$$

the last equality by the normal basis theorem.

Proof of Theorem: We'll show this only for S separated. We do that by induction on the degree p. The claim is clearly true for $p = 0$. We will first prove the result in the case when S is affine. We note that $H^p_{zar}(S, \mathscr{F}) = 0$ for all $p > 0$ for S affine and $\mathscr{F} \in QCoh(S)$ by Serre's Theorem. Now, for $p = 1$ we let $\xi \in H^1_\tau(S, \mathscr{F})$. We already showed that there exists τ-cover $\mathcal{U} = \{U_i \to U\}_{i\in I}$ such that $\xi\big|_{U_i} = 0$. We may assume that all U_i are affine and $|I| < \infty$. Then, the spectral sequence $\check{H}^p_\tau(\mathcal{U}, \underline{H}^q\mathscr{F}) \Rightarrow H^{p+q}_\tau(S, \mathscr{F})$ tells us that there exists $\tilde{\xi} \in \check{H}^1_\tau(\mathcal{U}, \mathscr{F})$ mapping to ξ.

Replacing $\mathcal{U}$ with $\mathscr{V} := \{\coprod_{i\in I} U_i \to S\}$ does not change the Č complex. Indeed, as $\mathscr{F}$ is a sheaf it takes $\coprod_{i\in I} U_i$ to $\prod_{i\in I} \mathscr{F}(U_i)$. So as U_i are affine and $|I| < \infty$ we can assume that $\mathscr{V}$ is of the form $\{\mathrm{Spec}(B) \to \mathrm{Spec}(A)\}$. Then $\check{H}^\bullet_\tau(\mathscr{V}, \mathscr{F})$ is the cohomology of the complex

$$M \otimes (B \longrightarrow B \times_A B \longrightarrow B \times_A B \times_A B \longrightarrow \cdots)$$

where M is the A-module corresponding to $\mathscr{F}$. By the descent lemma $A \to B \rightrightarrows B \otimes_A B$ is an equilizer for $A \to B$ faithfully flat. A minor variant of it shows that the complex is exact in degree > 0 and the first arrow has kernel M. Thus, $\check{H}^p(\mathscr{V}, \mathscr{F}) = 0$ for $p > 0$. Thus, $\tilde{\xi}$ and hence ξ vanishes.

Now, for S affine and $p > 1$, by induction we assume that for all affine u and $1 \leqslant s < p$, $H^s_\tau(U, \text{any QCoh}) = 0$. Then, as before for $\xi \in H^p_\tau(S, \mathscr{F})$ there exists a cover of affines $\{U_i \to U\}_{i \in I}$ with $|I| < \infty$ such that $\xi\big|_{U_i} = 0$. Then, from the spectral sequence $\check{H}^r_\tau(\mathcal{U}, \underline{H}^s(\mathscr{F})) \Rightarrow H^{r+s}_\tau(S, \mathscr{F})$, as $U_{i_o} \times_S \cdots \times_S U_{i_t}$ are affine schemes, $E_2^{r,s} = 0$ for $1 \leqslant s < r$ and ξ vanishes under the edge map to $E_2^{0,p} = \check{H}^0_\tau(\mathcal{U}, \underline{H}^p\mathscr{F})$. Hence there exists $\tilde{\xi} \in H^p_\tau(\mathcal{U}, \mathscr{F})$ mapping to ξ under the edge map. The same descent lemma argument as above shows that the groups vanish, hence $\xi = 0$.

In general, for S separated, another application of $\check{H}^p_\tau(\mathcal{U}, \underline{H}^p(\mathscr{F})) \Rightarrow H^{p+q}_\tau(S, \mathscr{F})$ with $\mathcal{U}$ an affine open cover of S allows us to reduce to the affine case as follows: By the result in the affine case the spectral sequence collapses and so

$$H^p_{zar}(S, \mathscr{F}) \simeq \check{H}^p_{zar}(\mathcal{U}, \mathscr{F}) = \check{H}^p_\tau(\mathcal{U}, \mathscr{F}) \simeq H^p_\tau(S, \mathscr{F})$$

where the middle equality follows since this is an affine open cover. ■

Lecture 15: 10/17

12 More on $R^q f_*$ and Stalks

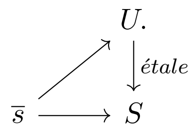
Lemma 12.1 *Let $f : X \to S$ be any scheme map and $\mathscr{F} \in \mathrm{Ab}(X_{ét})$. Then $R^q f_* \mathscr{F}$ is the sheafification of the presheaf given by $U \mapsto H^q_{ét}(X \times_S U, \mathscr{F}|_{X \times_S U})$.*

Proof. Let $\mathscr{F} \to I^\bullet$ be an injective resolution. By definition, $R^q f_* \mathscr{F} = H^q(f_* I^\bullet)$. So, $R^q f_* \mathscr{F}$ is:

$$\begin{aligned} &\left(U \mapsto \frac{\ker(f_* I^q(U) \to f_* I^{q+1}(U))}{\mathrm{Im}(f_* I^{q-1}(U) \to f_* I^q(U))}\right)^\# \\ =&\left(U \mapsto \frac{\ker(I^q(X \times_S U) \to I^{q+1}(X \times_S U))}{\mathrm{Im}(I^{q-1}(X \times_S U) \to I^q(X \times_S U))}\right)^\# \\ =&\,(U \mapsto H^q_{\text{ét}}(X \times_S U, \mathscr{F}|_{X \times_S U}))^\# \end{aligned}$$

which concludes the proof.

We next want to understand stalks $(R^p f_* \mathscr{F})_{\overline{s}}$ for $\overline{s} \to S$ a geometric point. Formally, this is $(R^p f_* \mathscr{F})_{\overline{s}} \cong \varinjlim H^p_{\text{ét}}(X \times_S U, \mathscr{F}|_{X \times_S U})$ in observance of the lemma, where the colimit is over



A special case would be suppose that $S = \mathrm{Spec}(k)$ with k^s a separable closure and that our geometric point is $\overline{s} = \mathrm{Spec}(k^s) \to S \xleftarrow{f} X$.

$(R^p f_*\mathscr{F})_{\overline{s}} \cong \varinjlim_{\substack{k'/k \text{ finite} \\ \text{seperable}}} H^p_{\text{ét}}(X_{k'}, \mathscr{F}_{k'})$. We hope that this is isomorphic to $H^p_{\text{ét}}(X_{k^s}, \mathscr{F}_{k^s})$. This is both true and convenient to formulate a vast generalization.

Let $(I, \leqslant)$ be a directed set. Let $(X_i, X_{i'} \xrightarrow{f_{i'i}} X_i \| \forall i, i' \geqslant i)$ be an inverse system of schemes over I. By a system of (abelian) sheaves on this inverse system, we mean

1. for all $i \in I$, $\mathscr{F}_i \in \text{Ab}(X_{i,ét})$

2. for all $i' \geqslant i$, $f_{i'i}\mathscr{F}_i \xrightarrow{\phi_{i'i}} \mathscr{F}_{i'}$ such that for all $i'' \geqslant i' \geqslant i$, $f^{-1}_{i''i} = f_{i'i} \circ f_{i''i'}$

$$\begin{array}{ccc} f^{-1}_{i''i}\mathscr{F}_i & \xrightarrow{\phi_{i''i}} & \mathscr{F}_{i''} \\ \cong\uparrow & & \phi_{ii'}\uparrow \\ f^{-1}_{i''i'}f^{-1}_{i'i}\mathscr{F}_i & \xrightarrow{f^{-1}_{i''i'}\phi_{i'i}} & f^{-1}_{i''i'}\mathscr{F}_{i'} \end{array}$$

commutes.

Example 12.2 *1. Take all $X_i = X$ with $f_{i'i} = \text{id}\, X$ and the data of the system of sheaves is a directed system ($\mathscr{F}_i \in \text{Ab}(X_{ét}), \mathscr{F}_i \to \mathscr{F}_{i'}$ for $i' \geqslant i$). So, we can form $\varinjlim \mathscr{F}_i \in \text{Ab}(X_{ét})$.*

2. Given a system of schemes $(S_i \to S, g_i : S_i \to S)$ over a fixed base scheme S, $\mathscr{F} \in \text{Ab}(S_{ét})$, we can set $\mathscr{F}_i = g_i^{-1}\mathscr{F}$ with $\phi_{i'i} : f^{-1}_{i'i}\mathscr{F}_i \cong f^{-1}_{i'i}g_i^{-1}\mathscr{F} \cong g_i^{-1}\mathscr{F} \to \mathscr{F}_i = g_i^{-1}\mathscr{F}$. This is our setup in the $\varinjlim H^p_{ét}(X_{k'}, \mathscr{F}_{k'})$ example.

In the full generality, for all $i' \geqslant i$, we get a map $H^p_{\text{ét}}(X_i, \mathscr{F}_i) \xrightarrow{f_i^{-1}} H^p_{\text{ét}}(X_{i'}, f^{-1}_{i'i}\mathscr{F}_i) \xrightarrow{\phi_{i'i}} H^p_{\text{ét}}(X_{i'}, \mathscr{F}_{i'})$(This was implicit in the Spec(k) example), so $\varinjlim_I H^p_{\text{ét}}(X_i, \mathscr{F}_i)$ makes sense. Is it isomorphic to $H^p_{\text{ét}}(\underbrace{\text{some scheme}}_{\text{“}\varinjlim X_i\text{”}}, \underbrace{\text{some sheaf}}_{\text{“}\varinjlim \mathscr{F}_i\text{”}})$?

Limits of schemes don't always exist, but inverse limits given $(I, \leqslant)$ do when the transition maps $X_{i'} \xrightarrow{f_{i'i}} X_i$ are <u>affine</u> morphisms (The idea is that if all $X_i = \text{Spec}(R_i)$ are affine, we set $X = \text{Spec}(\varinjlim R_i)$).

Then we set $X = \varinjlim_I X_i$.

In general, we fix some i_0 so that for all $i \geqslant i_0$, we have the affine map $X_i \xrightarrow{f_{ii_0}} X_{i_0}$ which yields a quasi-coherent sheaf of algebras $\mathscr{A}_i = (f_{ii_0})_*\mathscr{O}_{X_i}$ on X_{i_0}. Set $\mathscr{A} = \varinjlim_{i \geqslant i_0} \mathscr{A}_i \in$ QCohAlg(X_{i_0}), and it corresponds to some

$$\begin{array}{c} X \\ \big\downarrow \text{affine} \\ X_{i_0} \end{array}$$

Theorem 12.3 *Let $X = \lim_I X_i \xrightarrow{f_i} X_i$ be a limit of directed systems of qcqs (quasi-compact and quasi-separated) schemes X_i with affine transition maps ($X_{i'} \xrightarrow{f_{i'i}} X_i, i' \geqslant i$). Let $(\mathscr{F}_i, \varphi_{i'i} : f_{i'i}^{-1}\mathscr{F}_i \to \mathscr{F}_{i'})$ be a system of sheaves on (X_i). Set $\mathscr{F} = \varinjlim_i f_i^{-1}\mathscr{F}_i$, then $\varinjlim_{i\in I} H^p_{ét}(X_i, \mathscr{F}_i) \xrightarrow{\sim} H_{ét}(X, \mathscr{F})$.*

This includes our $\varinjlim_{k\subset k'\subset k^s} H^p_{\text{ét}}(X_{k'}, \mathscr{F}_{k'}) \xrightarrow{\sim} H^p_{\text{ét}}(X_{X_{k^s}, \mathscr{F}_{k^s}})$ examples; also for X fixed and $\mathscr{F}_i \to \mathscr{F}_{i'}$ a directed system), we have

$$\varinjlim_{i\in I} H^p_{\text{ét}}(X, \mathscr{F}_i) \xrightarrow{\sim} H^p_{\text{ét}}(X, \varinjlim_i \mathscr{F}).$$

We won't prove this theorem: see Stacks project.

Back to stalks of Rf_*, given

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle f} \\ \overline{s} & \xrightarrow[\text{geom pt}]{} & S \end{array}$$

$(R^p f_*\mathscr{F})_{\overline{s}} \cong \varinjlim H^p_{\text{ét}}(X \times_S U, \mathscr{F}\mid_{X\times_S U})$ where the colimit is over étale neighborhoods

$$\begin{array}{ccc} & \nearrow & U \\ & & \downarrow \\ \overline{s} & \longrightarrow & S \end{array}$$

The index category has a cofinal system obtained by assuming U affine. Therefore it equals $\varinjlim H^p(X \times_S \text{Spec}(A), \mathscr{F}\mid_{X\times_S \text{Spec}(A)})$. Should we assume X qcqs, then automatically so are all of $X \times_S \text{Spec}(A)$ since we know qcqs is stable under base change. Now we are in the setting of the theorem as the system of sheaves are given for free: we can simply take the sheaf on $X \times_S \text{Spec}(A)$ to be $\mathscr{F}\mid_{X\times_S \text{Spec}(A)}$. Hence we have $\varinjlim H^p(X \times_S \text{Spec}(A), \mathscr{F}\mid_{X\times_S \text{Spec}(A)}) \simeq H^p(\lim X \times_S \text{Spec}(A), \varinjlim \mathscr{F}\mid_{X\times \text{Spec}(A)}) \simeq H^p_{\text{ét}}(X \times \text{Spec}(\varinjlim A), p^{-1}\mathscr{F})$ where the limits/colimits are over all the diagrams

$$\begin{array}{ccc} & \nearrow & \text{Spec}(A) \\ & & \downarrow{\scriptstyle \text{étale}} \\ \overline{s} & \longrightarrow & S \end{array}$$

and p is the projection from the limit to X.

This $\text{Spec}(\varinjlim A)$ is the stalk at $\overline{s}$ of $\mathscr{O}_S^{\text{ét}}$ ($= \mathbb{G}_a$ on $S_{\text{ét}}$) which we write $\mathscr{O}^{\text{ét}}_{S,\overline{s}}$ or $\mathscr{O}_{S,\overline{s}}$. But what is it?

First, an example: If $S = \text{Spec}(k)$, k^s separable closure,

$$\overline{s} = \text{Spec}(k) \to \text{Spec}(k) = S$$

Then $\mathscr{O}_{S,\overline{s}} = k^s$.

Corollary 12.4 *We have the following:*

(1) $\mathscr{O}^{\acute{e}t}_{S,\bar{s}}$ *is the strict henselization* $\mathscr{O}^{sh}_{S,s}$ *of the Zariski local ring* $\mathscr{O}_{S,s}$ *(*s *= image of* $\bar{s}$ *in* S*).*

(2) For any $f : X \to S$ *qcqs,* $\bar{s} \to S$ *a geometric point, and* $\mathscr{F} \in \mathrm{Ab}(X_{\acute{e}t})$*, we have*

$$(R^p f_* \mathscr{F})_{\bar{s}} \cong H^p_{\acute{e}t}(X \times_S \mathrm{Spec}(\mathscr{O}^{sh}_{S,s}), \mathscr{F}|_{X \times_S \mathrm{Spec}(\mathscr{O}^{sh}_{S,s})} = p^{-1}\mathscr{F})$$

where p^{-1} *is the projection from* $X \times_S \mathrm{Spec}(\mathscr{O}^{sh}_{S,s})$ *to* X*.*

Granted (1), part (2) follows from last theorem. For (1), let's digress into some CA background.

Definition 12.5 *A local ring* $R, \mathfrak{m}$, $\kappa = R/\mathfrak{m}$ *is* ***henselian*** *if for every monic* $f \in R[T]$ *(*$\bar{f} \in \kappa[T]$*),* $a_0 \in \kappa$ *such that* $\bar{f}(a_0) = 0$ *and* $\bar{f}'(a_0) \neq 0$*, there exists (a necessarily unique) lift* $a \in R$ *of* a_0 *such that* $f(a) = 0$*.*

Example 12.6 (R, m) *a complete local ring.*

Some equivalent formulations:

1. For all $f \in R[t]$ and factorization $\bar{f} = g_0 h_0$ in $\kappa[t]$ with $(g_0, h_0) = 1$, there exist lifts $g \mapsto g_0$ and $h \mapsto h_0$ in $R[t]$ such that $f = gh$.

2. For all étale maps $\mathrm{Spec}(S) \xrightarrow{\mathrm{Spec}(\phi)} \mathrm{Spec}(R)$ and $q \in \mathrm{Spec}(S)$ such that $\phi^{-1}(q) = \mathfrak{m}$ and $\kappa(q) = \kappa$, there exists a section $\sigma : \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ of $\mathrm{Spec}(\phi)$ carrying $\mathfrak{m}$ to q.

Note 12.1 σ *must be étale, hence open, and a section of a separated morphism is a closed immersion. So* σ *is a iso onto a connected component of* $\mathrm{Spec}(S)$ *(*$S \cong R \times S'$*).*

Sketch of the implication "henselian $\implies$ 2": Localizing on $\mathrm{Spec}(S)$, we may assume the map is standard étale and $S = R[t]_g/f$ with f monic and f' invertible in $R[t]_g/f$. We've given $q \mapsto \mathfrak{m}$ with $\kappa(q) = \kappa$. Let $a_0 \in \kappa$ be the image of t under the natural map

$$S \longrightarrow S_q \longrightarrow \kappa(q) = \kappa.$$

Necessarily, $\bar{f}(a_0) = 0$ and $\bar{f}'(a_0) \neq 0$. Since R is henselian, there exists $a \in R$ with $a \mapsto a_0$ so that $f(a) = 0$. We then define $R[t]_g/f \to R$ to be given by $t \mapsto a$ for the section.

3. Any finite R-algebra is a finite product of local rings.

Sample argument: Let R be henselian, and let $R \to S$ be finite. Assume (and in fact the general case reduces to this one) that $S = R[t]/f$ with f monic. Let $\mathfrak{m} \subseteq R$ be any maximal ideal of R; any maximal ideal of S then lies over $\mathfrak{m}$ by the going-up theorem, so S is local if and only if $S/\mathfrak{m}S$ is local. If $\bar{f}$ is a power of an irreducible, then $S/\mathfrak{m}S \cong \kappa[t]/\bar{f}$ is clearly local, hence so is S, and we win. Otherwise, there is a nontrivial factorization $\bar{f} = g_0 h_0$ with $(g_0, h_0) = 1$. The henselian property 1 tells us that $f = gh$ for lifts g and

h, and then since $(g,h) = R[t]$, we get that $R[t]/f \cong R[t]/g \times R[t]/h$. The upshot of this special case is that we showed, given $S = R[t]/f$ for f and a non-trivial idempotent $x_0 \in S/\mathfrak{m}S$, you can find a non-trivial idempotent $x \in S$ that lifts x_0: you can easily verify this by writing $S/\mathfrak{m}S$ as a finite product of finite local algebra over κ (which are Artinian) and considering what a non-trivial idempotent look in it like (which then should lead to an easy lift). Now for the general case, suppose S not local, then from previous argument we have a non-trivial idempotent $\bar{x} \in S/\mathfrak{m}S$ with a lift $x \in S$. We also know that since $R \longrightarrow S$ is finite (a fortiori integral), x is a root of a monic polynomial $f \in R[t]$. Hence we obtain the map

$$R \to R[t]/f \xrightarrow{\varphi} S$$

by sending t to $\varphi(t) = x$, and we shall call this middle term A. Elementary arguments can show that

$$A/\mathfrak{m}A \cong \kappa[\bar{t}]/\bar{f} \cong \kappa[\bar{t}]/(\bar{t}^2 - \bar{t})^n \times \kappa[\bar{t}]/\bar{g}$$

for some $n > 0$ and $\bar{g}$ where $\bar{f} = (\bar{t}^2 - \bar{t})^n \bar{g}$ and $(\bar{g}, \bar{t}^2 - \bar{t}) = 1$; one can also verify that the induced map $\varphi' : A/\mathfrak{m}A \longrightarrow S/\mathfrak{m}S$ has kernel $(\bar{t}^2 - \bar{t}) \times \kappa[\bar{t}]/\bar{g}$. Notice that one of the non-trivial idempotent $(1,0) \in \kappa[\bar{t}]/(\bar{t}^2 - \bar{t})^n \times \kappa[\bar{t}]/\bar{g}$ corresponds to $h_0 \in A/\mathfrak{m}A$; from the special case h_0 has a non-trivial idempotent lift $h \in A$. Therefore $\varphi(h)$ is also an idempotent, which is a non-trivial one because

$$\overline{\varphi(h)} = \varphi'(h_0) = 1 - x.$$

Hence we can split S into a product of two finite $R-$algebra. This splitting process stops exactly when each factor is local. For converse, we assume 3, and given $f \in R[t]$ and $a_0 \in \kappa$ as in the assumption of henselian property, we consider $S = R[t]/f$ which is a finite $R-$algebra. We write $S \cong \prod A_i$ hence $S/\mathfrak{m}S \cong \prod A_i/\mathfrak{m}A_i$. $\bar{f}(a_0) = 0$ tell us that, WLOG, $A_1/\mathfrak{m}A_1 \cong \kappa[t]/(t - a_0)$. In particular, A_1, a direct summand of the free $R-$module S, is projective hence free $R-$module of rank 1: we can write $A_1 \cong R[t]/(t - a)$. Such a clearly is a lift a_0 and a root of f.

Definition 12.7 *A local ring $(R, \mathfrak{m})$ is* ***strictly henselian*** *if it is henselian and $R/\mathfrak{m}$ is separably closed.*

Lecture 16: 10/19

Corollary of our discussion of henselian rings: Let R be a henselian ring with residue field κ. Then $\text{FÉt}(\text{Spec}(R)) \to \text{FÉt}(\text{Spec}(\kappa))$ defined by

$$\begin{array}{c} X \\ \downarrow \\ \text{Spec}(R) \end{array} \mapsto X \times_{\text{Spec}(R)} \text{Spec}(\kappa)$$

is an equivalence of categories (Notation: For any scheme X, $\text{FÉt}(X)$ is the full subcategory of $X_{\text{ét}}$ with objects $Y \xrightarrow{f} X$ with f finite). As a result there is no non-trivial finite étale cover of $\text{Spec}(R)$.

Proof Sketch: I want to remark that any scheme finite over $\text{Spec}(R)$ is affine because finite morphisms are affine. For any finite étale $R \to S$, $S \cong A_1 \times A_2 \times \cdots \times A_r$ with local rings A_i. $S/\mathfrak{m}S \cong \prod A_i/\mathfrak{m}A_i$ and $R \to S$ étale implies each $A_i/\mathfrak{m}A_i$ is a finite separable extension of κ. Since objects on both sides of a functor canonically decompose as disjoint unions in a manner compatible with the functor, we can check essential surjectivity and

full faithfulness on connected objects.
Essentially surjective: Let L/κ be finite separable. Then $L \cong \kappa[t]/\overline{f}(t)$ for some monic irreducible $\overline{f} \in \kappa[t]$. Lift $\overline{f}$ to monic $f \in R[t]$ and looks at $S = R[t]/f(t)$. Then $R \to S$ is finite étale and reduces to L.
Full Faithfulness: Suppose $R \to S_1$ and $R \to S_2$ are finite étale with each S_i local. $S_1 \otimes_R S_2$ is finite étale over R and finite implies $S_1 \otimes_R S_2 \cong A_1 \times \cdots \times A_r$, a product of local finite étale R-algebras. Given a morphism $S_1 \xrightarrow{\varphi} S_2$, get

$$\begin{array}{ccc} S_1 \otimes_R S_2 & \xrightarrow{\phi \otimes \mathrm{id}} & S_2 \\ {\scriptstyle 1\otimes \mathrm{id}} \uparrow & \overset{\mathrm{id}}{\nearrow} & \\ S_2 & & \end{array}$$

and $S_1 \otimes_R S_2 \to S_2$ must be projections onto one of the A_i.
$\mathrm{Hom}_R(S_1, S_2) \xleftrightarrow{\text{bijection}} \{i \in 1, \ldots, r$ s.t. $S_2 \xrightarrow{1\otimes \mathrm{id}} S_1 \otimes S_2 \xrightarrow{\text{proj}} A_1$ is an isomorphism. The same description holds compatibly with $(\otimes_R \kappa)$ in $\mathrm{FÉt}(\kappa)$, so we get the full faithfulness. Q.E.D.
In particular, if R is strictly henselian (κ separable closed), R has no nontrivial (connected) finite étale covers.
Picture Think about $R = \mathbb{C}[[t]]$, $\kappa \in \mathbb{C}$. Spec(R) is a "formal unit disk". Spec$(\mathbb{C})$ is a point, is simply connected, as is the (formal) unit disk.

Theorem 12.8 *: Fix (R, m, κ) a local ring and κ^s a separable closure of κ. Then there exist flat local R-algebras $R_1 \to R^h \to R^{sh}$ (a "henselization" where R^{sh} is a strict henselization with respect to $\kappa \subset \kappa^s$) such that*

1. *R^h is henselian with maximal ideal $\mathfrak{m}R^h$ and residue field κ and*
2. *R^{sh} is strictly henselian with maximal ideal $\mathfrak{m}R^{sh}$ and residue field κ^s.*

Moreover they satisfy universal properties, e.g. $R \to R^h$ is initial among local homomophisms from R to any henselian ring.

Here is the construction
$$R^h = \varinjlim_{(S,q)} S$$
where $R \to S$ is étale, $q \in \mathrm{Spec}(S)$ lies over $\mathfrak{m}$ and $\kappa \to \kappa(q)$ is an isomorphism.
$$R^{sh} = \varinjlim_{(S,q,\alpha)} S$$
with S, q as before and $\alpha : \kappa(q) \hookrightarrow \kappa^s$. (Both of these colimits are filtered.) See Stack Project section 10.155 for the rest of the proof. Lemma S a scheme, $s \in S$, $\overline{s} \to S$ a generic point over s. Let κ^{sep} be the separable closure of $\kappa(s)$ in $\kappa(\overline{s})$. Then there exists a canonical iso $\mathscr{O}^{ét}_{S,\overline{s}} \cong \mathscr{O}^{sh}_{S,s}$ (with respect to $\kappa(s) \hookleftarrow \kappa^{\mathrm{sep}}$, where $\mathscr{O}^{ét}_{S,\overline{s}}$ is the stalk of the sheaf on $S_{ét}$ at $\overline{r}$).
See Stack Project 59.33 for proof.
Prop Let $f : X \to S$ be a finite morphism of schemes. Then for all $\mathscr{F} \in \mathrm{Ab}(X_{ét})$, $p \geqslant 1$, $R^p f_* \mathscr{F} = 0$.

Proof: Let $\overline{s} \to S$ be a generic point. Then (using last times discussion and the identification $\mathscr{O}^{ét}_{S,\overline{s}} \cong \mathscr{O}^{sh}_{S,s}$)

$$(R^p f_*\mathscr{F})_{\overline{s}} \cong H^p_{\text{ét}}(X \times_S \operatorname{Spec}(\mathscr{O}^{sh}_{S,s}), \mathscr{F}|_{X\times_S \operatorname{Spec}(\mathscr{O}^{sh}_{S,s})})$$

$X \times_S \operatorname{Spec}(\mathscr{O}^{sh}_{S,s})$ is finite over $\mathscr{O}^{sh}_{S,s}$ hence is $\operatorname{Spec}(A)$ where $A \cong \prod A_i$ is a product of finite local $\mathscr{O}^{sh}_{S,s}$-algebras. Each A_i is henselian since any finite A_i-algebra is finite over $\mathscr{O}^{sh}_{S,s}$. Hence this is isomorphic to a product of local rings. Since $\mathscr{O}^{sh}_{S,s}$ is strictly henselian, so is each A_i if the res field of A_i/κ^{sep} is separable. If not then the étale cohomology is no different: the étale site is invariant under universal homeomorphisms.

We are reduced to $S = \operatorname{Spec}(R)$.

Lemma 12.9 *Let R be a strictly henselian ring. Then $\Gamma(S, \cdot) : \operatorname{Ab}(S_{ét}) \to \operatorname{Ab}$ is exact (so for all $p \geqslant 1$ and $\mathscr{F} \in \operatorname{Ab}(S_{ét})$, $H^p_{ét}(S, \mathscr{F}) = 0$). Explicitly, $\Gamma(S, \mathscr{F}) \cong \mathscr{F}_{\overline{s}}$ where $\overline{s} \to S$ is the map $\operatorname{Spec}(R/\mathfrak{m}) \hookleftarrow \operatorname{Spec}(R)$.*

Proof. $\mathscr{F}_{\overline{s}}$ is the colimit of $\mathscr{F}(U)$ over the diagram

$$\begin{array}{ccc} & & U \\ & \overset{\overline{u}}{\nearrow} & \downarrow \text{étale} \\ \overline{s} & \longrightarrow & S \end{array}$$

but the pair $(S, \overline{s})$ is cofinal in the collection of étale neighborhoods $(U, \overline{u})$. To see this, we may assume $U = \operatorname{Spec}(A)$ and we have shown that $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$ has a section, and that section is an iso onto a connected component of A, so inducing A admits a splitting $A \cong R \times A'$). This gives $\Gamma(S, \mathscr{F}) \cong \mathscr{F}_{\overline{s}}$ and we know $\mathscr{F} \mapsto F_{\overline{s}}$ is exact.

Corollary 12.10 *$f : X \to S$ finite. For any geometric point $\operatorname{Spec}(k) \xrightarrow{\overline{s}} S$,*

$$(f_*\mathscr{F})_{\overline{s}} \cong \bigoplus \mathscr{F}_{\overline{x}}$$

Where the direct sum is over lifts $\overline{x}$ of the diagram

$$\begin{array}{ccc} & & X \\ & \overset{\textit{lifts } \overline{x}}{\nearrow} & \downarrow \\ \operatorname{Spec}(k) & \xrightarrow{\overline{s}} & S \end{array}.$$

For example if f is a closed immersion $f : Z \hookrightarrow X$, $(f_*\mathscr{F})_{\overline{x}} = \begin{cases} 0 & \textit{image}(\overline{x}) \notin Z \\ \mathscr{F}_{\overline{x}} & \textit{image}(\overline{x}) \in Z \end{cases}$

Proof.

$$\begin{aligned}(f_*\mathscr{F})_{\overline{s}} &= H^0_{\text{ét}}\left(X \times_S \operatorname{Spec}(\mathscr{O}^{sh}_{S,s}), \mathscr{F}|_{X\times_S \operatorname{Spec}(\mathscr{O}^{sh}_{S,s})}\right) \\ &= \bigoplus_{i=1}^{r} \Gamma\left(\operatorname{Spec}(A_i), \mathscr{F}|_{X\times_S \operatorname{Spec}(\mathscr{O}^{sh}_{S,s})}\right) \\ &= \bigoplus_{i=1}^{r} \mathscr{F}_{\overline{x_i}}\end{aligned}$$

the last equality follows from that inverse image preserves the stalks and the lifts $\{\overline{x_i}\}$ are in bijection with the maximal ideal of A_i.

13 Cohomology of curves and constructible sheaves, part 1

Two aspects:

1. Concrete calculation of $H^*_{\text{ét}}(X, \mu_n)$ where X is a smooth projective curves over a separably closed field k (or algebraically closed)

2. extension to softer but more general results: vanishing of higher coh. $H^{\geqslant 3}(X, \mathscr{F})$ for any curve X/k and torsion sheaf $\mathscr{F}$, and finiteness of $H^*(X, \mathscr{F})$ for "constructible“ sheaves with torsion prime to char k.

For Step 1, use results on Brauer graphs (and Picard varieties), so we will start with some background.

Definition 13.1 *Let K be a field, $Br(K)$ ={finite dimensional central simple algebras over K (CSA/K)}/∼.*

(CSA/K: associative but not necessarily commutative K-algebras with center K and no proper 2-sided ideals).

Example 13.2 *$M_d(K)$ the set of $d \times d$ matrices over K, is a division algebra over K with center K.*

The equivalence relation $\sim$ is $A \sim B$ if there exist $n, m \in \mathbb{Z}$ such that $M_n(A) \cong M_m(B)$. $Br(K)$ is a (commutative) group with identity and multiplication $- \otimes_K -$. (The inverse of A^K is A^{op}). This group turns out to be torsion, given A, $A \otimes_K \overline{K} \cong M_d(\overline{K})$ for some d and $A^{\otimes d}$ is trivial in $Br(K)$. In general $Br(K)$ is deep arithmetic invariant. Examples:

- K is separable and closed implies that $Br(K) = \{1\}$.
- $Br(\mathbb{R}) \cong \mathbb{Z}/2$
- $Br(\mathbb{F}_q) = \{1\}$
- $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$

$Br(\mathbb{Q})$ lies in a short exact sequence

$$0 \longrightarrow Br(\mathbb{Q}) \longrightarrow \bigoplus_{v \text{ all places of } \mathbb{Q}} Br(\mathbb{Q}_v) \xrightarrow{\Sigma \text{inv}_v} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

A consequence of non-abelian Hilbert 90: $Br(K) \cong H^2(G_K, (K^s)^\times) = H^2_{\text{ét}}(\text{Spec}(K), \mathbb{G}_m)$. Using

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \text{GL}_n \longrightarrow \text{PGL}_n \longrightarrow 1$$

and $H^1(\text{Spec}(K), \text{GL}_n) = 1$ (“Hilbert 90"). See Serre Local Fields or Galois Cohomology or Stacks Project.

13.1 Lecture 17: 10/24

Our key example of a Brauer group:

Theorem 13.3 *Let K be the function field of a curve (integral, 1-dimensional, finite type) over an algebraically closed field k. Then $Br(K) = 0$.*

Let $r \in \mathbb{Z}$. A field is C_r if $\forall d, n \in \mathbb{Z}_{\geqslant 0}$ such that $0 < d^r < n$, any homogeneous degree d polynomial $f \in K[t_1, \ldots, t_n]$ has a non-trivial solution, i.e., $f(\alpha) = 0$ with $\alpha \in K^n \backslash \{0\}$.

Lemma 13.4 *Let K be as in Theorem 13.3. Then K is C_1.*

Proof. We may assume $K = k(X)$ for X/k genus g, smooth, projective. Fix $f \in K[t_1, \ldots, t_n]$. There exists an ample line bundle $\mathscr{O}_X(D)$ with divisor $D \in \text{Div}(X)$ such that all coefficients of f lie in $H^0(X, \mathscr{O}_X(D)) = \{h \in K | \text{div} h \geqslant -D\}$. Consider tuples $\alpha \in K^n$ lying in $H^0(X, \mathscr{O}_X(D))$ for $e \geqslant 1$ allowed to vary. Riemann-Roch theorem implies that for $e \gg 0$ we have

$$\dim_k H^0(X, \mathscr{O}_X(eD)) = 1 - g + e \deg(D).$$

We can view f as a map $f : H^0(X, \mathscr{O}_X(eD))^{\oplus n} \to H^0(X, \mathscr{O}_X((ed+1)D)$. When $e \gg 0$, the domain and target have dimensions

$$\dim_k H^0(X, \mathscr{O}_X(eD))^{\oplus n} = n(1 - g + e \deg(D)),$$
$$\dim_k H^0(X, \mathscr{O}_X((ed+1)D) = 1 - g + (ed+1) \deg(D).$$

Assume $d < 0$, i.e. $d + 1 \leqslant n$. Then, we see that f has a non-trivial solution. So, K is C_1.

Then, the following lemma completes the proof of Theorem 13.3.

Lemma 13.5 *Let K be as in Theorem 13.3. If K is C_1, then $Br(K) = 0$.*

Proof. STP any division algebra D/K is trivial. Given D, $D \otimes K^s = M_d(K^s)$ for some $d \geqslant 1$. On D, we have the reduced norm

$$\text{Nm} : D \to K.$$

characterized by

$$D \otimes K^s \simeq M_d(K^s) \xrightarrow{\det} K^s.$$

Nm is a degree d homogeneous polynomial in d^2 variables. If $d > 1$, then $d < d^2$. So K being C_1 forces the existence of an element $x \in D \backslash 0$ such that $\text{Nm}(X) = 0$, which is impossible because D is a division algebra.

Galois cohomology consequence: Let $G_K = \text{Gal}(K^s/K)$.

Corollary 13.6 *(1) Let K be any field such that for all finite K'/K, $Br(K') = 0$. Then for all $p \geqslant 1$, $H^p(G_K, (K^s)^\times) = 0$ and for all torsion (and discrete) G_K-modules M, $H^p(G_K, M) = 0$ for all $p \geqslant 2$.*

(2) The hypothesis of part (1), hence the conclusion holds for any field K of transcendence degree 1 over an algebraically closed field k.

Proof. (Sketch) Vanishing $p \geqslant 2$, for torsion M, M is direct limit of its finite submodules, and $H^p(G_K, \text{filtered colim}) = \text{colim} H^p(G_K, \bullet)$. So we may assume $|M| < \infty$. Then,

$$M = \bigoplus_{\ell-\text{prime}} M_\ell.$$

So, we may assume $M = M_\ell$, since $H^*(G_K)$ commute with direct sums. There are two cases.

Case (a):The case we have $\ell \neq p = \text{char}(K)$. Then, we have Kummer sequnece

$$1 \to M_\ell \to (K^s)^\times \xrightarrow{z \mapsto z^\ell} (K^s)^\times \to 1,$$

and for any $L \subset K^s$, we get

$$H^{p-1}\left(G_L, (K^s)^\times\right) \to H^p\left(G_L, \mu_\ell\right) \to H^p\left(G_L, (K^s)^\times\right).$$

So, if $Br(L) = 0$, we deduce (using Hilbert 90) that $H^2(G_L, \mu_\ell) = 0$. Apply this observation with $L = (K^s)^{G_\ell}$ where G_ℓ is ℓ-Sylow subgroup. Then,

$$Br(L) = \underset{K \subset K' \subset L}{\text{colim}} Br(K') = 0 \quad \text{where } K \subset K' \text{ is finite.}$$

As a G_L-module, we have $\mu_\ell \simeq \mathbb{Z}/\ell$ ($L(\mu_\ell) = L$ because $\deg[L(\mu_\ell) : L]$ divides $\gcd(\ell - 1, \ell^\infty)$); therefore we have $H^2(G_L, \mathbb{Z}/\ell) = 0$. We deduce that for all ℓ-power torsion M, we have $H^2(G_L, M) = 0$. (The only simple $G_\ell = G_L$-module is the trivial module $\mathbb{Z}/\ell$. So, by looking at J-H series and LES we win using the case $M = \mathbb{Z}/\ell$.) From this and dimension shifting we deduce that for all ℓ_∞ torsion module M, we have $H^p(G_L, M) = 0$ for all $p \geqslant 2$. Since the restriction map is injective, we get $H^p(G_K, M) = 0$ for all $p \geqslant 2$.

Case (b): The case we have $\ell = p = \text{char}(K)$. Same argument reduces vanishing for all $p^\infty = \ell^\infty$-torsion module to the case of $M = \mathbb{Z}/p$. Instead of Kummer, use Artin-Schreir series. For any scheme S of characteristic p, we have a SES on $S_{\text{ét}}$:

$$0 \to \mathbb{Z}/p \to \mathscr{O}_S^{\text{ét}} \xrightarrow[a \mapsto a^p - a]{F-1} \mathscr{O}_S^{\text{ét}} \to 0.$$

We get

$$H^{n-1}_{\text{zar}}(S, \mathscr{O}_S) \simeq H^{n-1}_{\text{ét}}(S, \mathscr{O}_S^{\text{ét}}) \to H^n_{\text{ét}}(S, \mathbb{Z}/p) \to H^n_{\text{ét}}(S, \mathscr{O}_S^{\text{ét}}) \simeq H^n_{\text{zar}}(S, \mathscr{O}_S).$$

When S is affine and $n \geqslant 2$, we see that $H^n_{\text{ét}}(S, \mathbb{Z}/p) = 0$. In particular, $H^p_{\text{ét}}(\text{Spec}(K), \mathbb{Z}/p) = H^p \Gamma_K, \mathbb{Z}/p = 0$.
Now we turn to prove $H^p(G_K, (K^s)^\times) = 0$ for all $p \geqslant 3$ (we know it for for 1, 2). This follows from the torsion vanishing in degree greater than or equal to 2 and the exact sequence

$$0 \to \text{torsion} \to (K^s)^\times \to (K^s)^\times \otimes_{\mathbb{Z}} \mathbb{Q} \to \text{torsion} \to 0.$$

We know $H^p(\Gamma_K, \mathbb{Q}\text{-vector space}) = 0$ for all $p > 0$; In addition, $H^p(\Gamma_K, \text{torsion}) = 0$ for all $p \geqslant 2$. We obtain the result by taking LES.
(2) is a corollary of the previous theorem. Write

$$Br(K) = \underset{k \subset L \subset K}{\text{colim}} Br(L) \quad \text{where } k \subset L \text{ finitely generated of } \text{tr.deg}_k L = 1.$$

Since any such L is the function field of a curve over k we have $Br(L) = 0$. Hence, $Br(K) = 0$.

Now globalize. Until otherwise stated, let X/k be a (connected) smooth projective curve over an algebraically closed field k.

Goal: $H^q_{\text{ét}}(X, \mathbb{G}_m) = 0$ for all $q \geqslant 2$.

Relate this goal to the field theory results by

Lemma 13.7 *Let $j : \eta = \operatorname{Spec}(k(X)) \hookrightarrow X$ be the inclusion of the generic point and for all closed points $x \in X$ let $\iota_X : \{x\} \to X$. Then, there exists a short exact sequence on $X_{\acute{e}t}$:*

$$0 \to \mathbb{G}_{m,X} \to j_*\mathbb{G}_{m,\eta} \to \bigoplus_{x \in X_{cl}} (\iota_X)_*\mathbb{Z} \to 0.$$

(Lemma also holds if we replace X_{cl} by codimension 1 points on integral quasicompact scheme X.)

Proof. For $U \to X$ étale, $U = \coprod U_i$ with each U_i smooth connected curve. To construct the maps in the sequence, we may assume U is connected. ($\mathscr{F}(\coprod U_i) = \prod \mathscr{F}(U_i)$). Then, we have divisor sequence

$$(\star) \quad 0 \to \Gamma(U, \mathscr{O}_U^*) \to k(U)^\times \xrightarrow{\text{div}} \bigoplus_{u \in U_{cl}} \mathbb{Z}.$$

The sequence of the lemma evaluates this sequence $(\star)$ on U. This exactness for $(\star)$ shows

$$0 \to \mathbb{G}_{m,X} \to j_*\mathbb{G}_{m,\eta} \to \bigoplus_{x \in X_{cl}} \iota_{X*}\mathbb{Z}$$

is exact. The surjectivity on the right holds Zariski-locally because Weil divisors are Cartier in our setting. (Details of the proof is left as an exercise)

Using the LES associated to

$$0 \to \mathbb{G}_{m,X} \to j_*\mathbb{G}_{m,\eta} \to \bigoplus_{x \in X_{cl}} \iota_{X*}\mathbb{Z} \to 0$$

we get

$$H^{q-1}_{\text{ét}}(X, \bigoplus_{x \in X_{cl}} \iota_{X*}\mathbb{Z}) \to H^q_{\text{ét}}(X, \mathbb{G}_m) \to H^q_{\text{ét}}(X, j_*\mathbb{G}_{m,\eta})$$

where the leftmost element is

$$H^{q-1}_{\text{ét}}(X, \bigoplus_{x \in X_{cl}} \iota_{X*}\mathbb{Z}) \simeq \bigoplus_{x \in X_{cl}} H^{q-1}_{\text{ét}}(X, \iota_{X*}\mathbb{Z}) \simeq \bigoplus_{x \in X_{cl}} H^{q-1}_{\text{ét}}(\{x\}, \mathbb{Z}) = 0.$$

We have

$$H^p_{\text{ét}}(X, R^q_{j_*}\mathbb{G}_{m,\eta}) \implies \underbrace{H^{p+q}_{\text{ét}}(\eta, \mathbb{G}_{m,\eta})}_{\text{Galois Cohomology}}.$$

Lemma 13.8 *For all $q \geqslant 1$, we have $R^q j_*\mathbb{G}_{m,\eta} = 0$.*

Granting this, we get the following corollary. We will prove corollary first and will get back to proof of lemma.

Corollary 13.9 *For all $p \geqslant 1$, $H^p_{\acute{e}t}(X, j_*\mathbb{G}_{m,\eta}) = 0$.*

Proof. If $R^q j_* \mathbb{G}_{m,\eta} = 0$, then for all $q \geqslant 1$ E_2 page of Leray is zero and for all p we have

$$0 \simeq H^p_{\text{ét}}(\eta, \mathbb{G}_{m,\eta}) \simeq H^p_{\text{ét}}(X, j_* \mathbb{G}_{m,\eta}).$$

since $Br(k(X)) = 0$.

The corollary gives us the following.

Theorem 13.10 *For all $q \geqslant 2$, we have $H^q_{ét}(X, \mathbb{G}_m) = 0$.*

It remains to prove lemma.

Proof. (Proof of Lemma) For any geometric point $\bar{x}$ of X, we show $(R^q j_* \mathbb{G}_{m,n})_{\bar{x}} = 0$. We have two cases:

1) $\bar{x}$ over a closed point,

2) $\bar{x}$ over the generic point η.

Case 1: The point x lies in some affine open $\operatorname{Spec}(A)$. So, we have

$$(R^q j_* \mathbb{G}_{m,\eta})_{\bar{x}} \simeq H^q(\eta \times \operatorname{Spec}(\mathscr{O}^{sh}_{X,\bar{x}}), \mathbb{G}_m)$$

$$\eta \times_X \operatorname{Spec}(\mathscr{O}^{sh}_{X,\bar{x}}) \simeq \operatorname{Spec}(\operatorname{Frac} A \otimes_A \mathscr{O}^{sh}_{X,\bar{x}}) \simeq \underbrace{\operatorname{Spec}(\operatorname{Frac}(\mathscr{O}^{sh}_{X,\bar{x}}))}_{\text{Call this } K^{sh}_{\bar{x}}}.$$

$K^{sh}_{\bar{x}}$ is an algebraic (but not finite) extension of K. So, again our Brauer group result show $Br(K^{sh}_{\bar{x}}) = 0$. Hence, we have $H^q_{\text{ét}}(\operatorname{Spec}(K^{sh}_{\bar{x}}), \mathbb{G}_m) = 0$ for all $q \geqslant 1$.

Case 2: Left as an exercise.

13.2 Lecture 18: 10/26

Applying the last time's vanishing result, and following last lecture's notation, we have

Theorem 13.11 *Let X/k be a connected smooth projective curve of genus g over a separably closed field, and let $n \in \mathbb{Z}_{\geqslant 1} \cap k^*$. Then we have*

$$\begin{aligned} H^0_{ét}(X, \mu_n) &= \mu_n(k) \cong \mathbb{Z}/n \\ H^1_{ét}(X, \mu_n) &= \operatorname{Pic}^0(X)[n] \cong (\mathbb{Z}/n)^{2g} \\ H^2_{ét}(X, \mu_n) &= \mathbb{Z}/n \\ H^p_{ét}(X, \mu_n) &= 0, p > 2 \end{aligned}$$

Proof. From the LES coming from the Kummer sequence (note that $H^0(X, \mathbb{G}_m) = k^\times$, so H^0 part is easy), we have

$$0 \to H^1_{ét}(X, \mu_n) \to H^1_{ét}(X, \mathbb{G}_m) \to H^1_{ét}(X, \mathbb{G}_m) \to H^2_{ét}(X, \mu_n) \to H^2_{ét}(X, \mathbb{G}_m) = 0$$

(The surjectivity of $H^0_{ét}(X, \mathbb{G}_m) \to H^0_{ét}(X, \mathbb{G}_m)$ comes from the fact that $n \in k^*$.) The fact that $H^p_{ét}(X, \mu_n) = 0$ is because $H^{p-1}_{ét}(X, \mathbb{G}_m)$ and $H^p_{ét}(X, \mathbb{G}_m)$ are both 0 for $p \geqslant 3$. Since $H^1_{ét}(X, \mathbb{G}_m) \cong \operatorname{Pic}(X)$, we find $H^1_{ét}(X, \mu_n) \cong \operatorname{Pic}(X)[n]$ and the cokernel of multiplication-by-n map on $\operatorname{Pic}(X)$ is isomorphic to $H^2_{ét}(X, \mu_n)$.
So we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 0\\ & & \downarrow n & & \downarrow n & & \downarrow n & & \\ 0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}(X) & \xrightarrow{\deg} & \mathbb{Z} & \longrightarrow & 0\end{array}$$

, then we use the fact that there is an abelian variety Pic^0_X defined over k such that its k-points are $\mathrm{Pic}^0(X)$, and that $[n]$ is surjective on Pic^0_X/k, together with snake lemma, we conclude that $H^1_{ét}(X,\mu_n) = \mathrm{Pic}(X)[n] \cong \mathrm{Pic}^0(X)[n] \cong (\mathbb{Z}/n)^{2g}$ and $H^2_{ét}(X,\mu_n) = \mathbb{Z}/n$; this follows from general theory of Abelian Variety (for abelian variety AV/k, $AV(k)[n] \cong (\mathbb{Z}/n)^{2\dim AV}$) and general theory of Picard Scheme of Curves ($\dim \mathrm{Pic}^0_X = g$): see Stack Project 03RP and 0BA0.

Remark 13.12 *(1) The use of Kummer sequence is analogous to the use of exponential sequence for X a compact Riemann surface:*

$$0 \longrightarrow 2\pi i\mathbb{Z} \longrightarrow \mathscr{O}_X \xrightarrow{f\to e^f} \mathscr{O}_X^* \longrightarrow 0$$

(2) In the theorem, if X is defined over a general field k, then the above calculation applies to $H^*_{ét}(X_{k^s},\mu_n)$. The isomorphisms in the theorem are $\mathrm{Gal}(k^s/k)$*-equivariant, i.e., they are isomorphic as Galois modules.*

More generally, for X/k an affine scheme defined over a characteristic $p>0$ field, we use Artin-Schreier sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathscr{O}_X \xrightarrow{Fr-id} \mathscr{O}_X \longrightarrow 0\ .$$

Then since $H^1_{ét}(X,\mathscr{O}_X) = H^1_{zar}(X,\mathscr{O}_X) = 0$ (X is affine), $H^p_{ét}(X,\mathscr{O}_X) = 0$ as well. So $H^1_{ét}(X,\mathbb{Z}/p)$ is the cokernel of $Fr - id$ on $\mathscr{O}_X(X)$. However, this may not be a finite dimensional $\mathbb{Z}/p$ vector space. For example, consider the cokernel of $Fr - id : \overline{\mathbb{F}_p}[t] \to \overline{\mathbb{F}_p}[t]$. But if X/k is proper, then this H^1 will not be infinite dimensional.

Theorem 13.13 *Let X/k be a separated scheme of finite type defined over a separably closed field k, with* $\dim(X) \leqslant 1$*. Then:*

(1) For any torsion sheaf $\mathscr{F} \in Ab(X_{ét})$ and $q \geqslant 3$, $H^q_{ét}(X,\mathscr{F}) = 0$. And if moreover X is affine, $H^q_{ét}(X,\mathscr{F}) = 0$ for all $q \geqslant 2$.

(2) For any q any constructible sheaf $\mathscr{F}$ with torsion coprime to $\mathrm{char}(k)$*, $H^q_{ét}(X,\mathscr{F})$ is finite. And if moreover X/k is proper, $H^1_{ét}(X,\mathscr{F})$ is finite for all constructible sheaves $\mathscr{F}$.*

(3) For any torsion sheaf $\mathscr{F}$ of order prime to $\mathrm{char}(k)$*, and for any $k \subseteq k'$, where k' is separably closed, the natural map:*

$$H^q_{ét}(X,\mathscr{F}) \to H^q_{ét}(X_{k'},\mathscr{F}_{k'})$$

is an isomorphism. And if X/k is moreover proper, this holds for all torsion $\mathscr{F}$.

Definition 13.14 *Let X be a scheme, a sheaf $\mathscr{F}$ of sets or abelian groups on $X_{ét}$ is locally constant if there is an étale covering $\{U_i \to X\}_{i\in I}$ such that $\mathscr{F}|_{U_i}$ is a constant sheaf.(i.e., $\mathscr{F}$ is the sheafification of $(V_{ij} \to \Sigma_i)$, where V_{ij} are étale coverings of U_i.*

Definition 13.15 *Notation as above, $\mathscr{F}$ is said to be finite locally constant (locally constant constructible, l.c.c or lcc) if Σ_i's are all finite.*

Lemma 13.16 *(1) Let $f : X \to Y$ be a morphism of schemes, Let $\mathscr{F}$ be a locally constant sheaf, then $f^{-1}\mathscr{F}$ is locally constant.*

(2) If $f : X \to Y$ is finite étale and $\mathscr{F}$ is locally constant(resp. finite locally constant), then $f_*(\mathscr{F})$ is locally constant(resp. finite locally constant).

Proof. (1) there is an étale covering $\{U_i \to X\}_{i\in I}$ such that $\mathscr{F}|_{U_i} \cong \underline{\Sigma_i}$. Then $(f^{-1}\mathscr{F})|_{X\times_Y U_i} \cong (\mathscr{F}|_{U_i})|_{X\times_Y U_i}$ is constant, since pullback of a constant sheaf is constant.

(2) To prove (2), first prove a theorem:

Theorem 13.17 *Let S be any scheme. The functor $S_{ét} \to Sh(S_{ét}), X \to h_X$, restricts to an equivalence of categories $F\acute{E}t_S \to \{$l.c.c. sheaves of sets on $S_{ét}\}$.*

Proof of theorem: To see this functor is well defined, first note that a finite *é*tale morphism is étale locally isomorphic to a disjoint unions of isomorphisms. To prove this, we let $\{U_n \to S\}$ be a Zariski cover such that $f|_{U_n}$ is finite *é*tale of constant degree n(e.g., finite unramified extension of Dedekind domains). So we can assume $deg(f) = n$.

Induct on n, and we have such a diagram

$$\begin{array}{ccc} X\times_S X & \longrightarrow & X \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle f} \\ X & \longrightarrow & S \end{array}$$

Since X is proper and thus separated, $\triangle_X$ is a closed immersion. Since pr_2 is a pullback of an étale morphism, pr_2 is étale. Being a section of pr_2, $\triangle_X$ is étale. Thus, $\triangle_X$ is a closed and open immersion. So $X\times_S X = \triangle_X(X) \amalg W$, and restrict pr_2 to W, the degree of pr_2 is $n-1$(since the cardinality of fiber is $n-1$), so by induction, we are done.

Thus, there is a covering $\{U_i \to S\}$ such that $X\times_S U_i$ is a finite disjoint union of U_i, say $\amalg_{\Sigma_i} U_i$. Then $h_X|_{U_i}$ is the constant sheaf $\underline{\Sigma_i}|_{U_i}$. So this functor is well defined. And the full faithfulness is guaranteed by Yoneda's lemma.

To prove the essential surjectivity, we first note that it is enough to represent $\mathscr{F}$ on a Zariski cover.(For if $S = \cap S_i$ is a Zariski cover of S, and suppose we have $\widetilde{S_i}$ representing $\mathscr{F}|_{S_i}$, then $h_{\widetilde{S_i}\times_{S_i}(S_i\cap S_j)} \cong \mathscr{F}|_{S_i\cap S_j} \cong h_{\widetilde{S_j}\times_{S_j}(S_i\cap S_j)}$ gives, by the full faithfulness of Hom functor, gives an isomorphism $\widetilde{S_i}\times_{S_i}(S_i\cap S_j)$ with $\widetilde{S_j}\times_{S_j}(S_i\cap S_j)$, so we can glue $\widetilde{S_i}$'s together.

Thus, we can assume that there is a single covering space $f : U \to S$, such that $\mathscr{F}|_U$ is a constant sheaf, say $\underline{\Sigma}$. And we want to use the descent theory to conclude(Milne

Etale cohomology, chap I, Thm 2.23, Rmk 2.24). First, there is an $\alpha : \mathscr{F}|_U \cong h_{\Sigma \times U}$. So, with notations in the following diagram,

$$U' = U \times_S U \overset{p_1}{\underset{p_2}{\rightrightarrows}} U \xrightarrow{p} S$$

we have an isomorphism $\phi : p_1^*(\Sigma \times U) \cong p_2^*(\Sigma \times U)$, and this isomorphism satisfies cocycle condition. Thus there is a Cartesian square

$$\begin{array}{ccc} \Sigma \times_S U & \xrightarrow{p_1} & \widetilde{S} \\ \downarrow & & \downarrow \\ U & \longrightarrow & S \end{array},$$

and $\widetilde{S}$ is finite and *é*tale over S and represents $\mathscr{F}$.

proof of (2) Now to prove (2), note that if $\mathscr{F}$ is represented by $U \to X$, then $f_*(\mathscr{F})$ is represented by $U \to Y$, by definition of f_*. Since f is finite *é*tale, the result is clear.

Definition 13.18 ((Constructible sheaf)) *Let X be a Noetherian scheme. Let $\mathscr{F} \in Ab(X_{ét})$. $\mathscr{F}$ is constructible if there is a finite partition $X = \amalg X_i$ as sets, and X_i's are locally closed subspace of X, such that for all i, $\mathscr{F}|_{X_i}$ is l.c.c.*

Though there is a potential ambiguity in this definition that $\overline{X_i}$ can be given different structure, we have the following theorem:

Theorem 13.19 *Let $f : X \to S$ be a universal homeomorphism (equivalently, integral, universally injective and surjective). Then*

$$\{\text{étale schemes}/S\} \to \{\text{étale schemes}/X\} \qquad (Y \to S) \to (X \times_S Y \to S)$$

is an equivalence of categories, and so $Sh(S_{ét}) \cong Sh(X_{ét})$.

14 Statement of the proper base change theorem and some applications

14.1 Lecture 19 (cont)

Proposition 14.1 *Let $f : X \to S$ be a finite morphism such that for any $s \in S$, the preimage $f^{-1}(\{s\})$ of s under f is a singleton, denoted by $\{x\}$, and the field extension $\kappa(s) \to \kappa(x)$ is purely inseparable (or equivalently, for any geometric point $\bar{s} : \mathrm{Spec}(k) \to S$ with k algebraically closed, there exists a geometric point $\bar{x} : \mathrm{Spec}(k) \to X$ which is a lifting of $\bar{s}$). Then $\mathrm{Sh}\, X_{\text{ét}} \overset{f_*}{\underset{f^{-1}}{\rightleftarrows}} \mathrm{Sh}\, S_{\text{ét}}$ are quasi-inverse equivalences of categories. The same conclusion holds for abelian sheaves.*

We give two examples that satisfy the conditions in the proposition.

Example 14.2 *(1) The scheme morphism $X_{red} \to X$ for any scheme X.*

(2) Let X be a scheme over the field k and $k \to k'$ a purely inseparable extension. We consider the scheme morphism $X_{k'} \to X$.

Proof. Let $\bar{s} : \mathrm{Spec}(k) \to S$ be a geometric point with k algebraically closed. For $\mathscr{F} \in \mathrm{Sh}\, X_{\text{ét}}$, $(f_*\mathscr{F})_{\bar{s}} \cong \mathscr{F}_{\bar{x}}$, where $\bar{x}$ is the unique lift of $\bar{x}$, by our formula for $f_*\mathscr{F}$ when f is finite. We also know for any $\mathscr{G} \in \mathrm{Sh}\, S_{\text{ét}}$, $f^{-1}(\mathscr{G})_{\bar{x}} \cong \mathscr{G}_{\bar{s}}$. Then we can check that the maps $\mathscr{G} \to f_*f^{-1}\mathscr{G}$ and $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ are compatible with the stalk isomorphisms we've described. Thus the two maps are isomorphisms.

Consequences of the above proposition:

(i) Our definition of constructible is unambiguous.

(ii) In proving theorems like the big curve theorem, we may assume X is reduced.

(iii) We have a tool for replacing X/k^{sep} with $X/\bar{k}$.

15 Statement of the proper base change theorem and some applications.

In topology, for proper map $X \xrightarrow{f} S$ of locally compact spaces, for any $\mathscr{F} \in \mathrm{Ab}(X)$ and any $s \in S$, we have

$$(R^p f_*\mathscr{F})_s \cong H^p(X_s, \mathscr{F}|_{X_s}),$$

where $X_s = f^{-1}(s)$. The analogue in étale cohomology is one of the key fundamental theorems. We give two equivalent formulations of the proper base change theorem.

Theorem 15.1 (Proper base change theorem) *1. Let $f : X \to S$ be a proper morphism, $\bar{s} \to S$ a geometric point, and let $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})_{tor}$ be a torsion sheaf (this means that for any $U \subset X_{\text{ét}}$ and $t \in \mathscr{F}(U)$, locally on U t is killed by some integer). Then we have the following:*

$$(R^p f_*\mathscr{F})_{\bar{s}} \cong H^p_{\acute{e}t}(X_{\bar{s}}, \mathscr{F}|_{X_{\bar{s}}}).$$

2. Let $f : X \to S$ be proper and $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})_{tor}$. Let $T \to S$ be a morphism of schemes. We lable the canonical morphism as following:

$$\begin{array}{ccc} X \times_S T = X_T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

Then the canonical base change morphism

$$g^{-1}R^p f_*\mathscr{F} \to R^p f'_*(g'^{-1}\mathscr{F})$$

is an isomorphism.

Remark 15.2 *When $T = \mathrm{Spec}(\kappa(\bar{s}))$, (2) implies (1). (1) implies (2) by looking at the stalks.*

Proof. For any commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

we give the construction of base change map.
First we consider the case when $p = 0$. Appyting f_* to the adjunction map $\mathscr{F} \to g'_* g'^{-1}\mathscr{F}$, we get the map

$$f_*\mathscr{F} \to f_* g'_* g'^{-1}\mathscr{F} = g_* f'_* g'^{-1}\mathscr{F}.$$

We then take the adjoint map and get

$$g^{-1} f_*\mathscr{F} \to f'_* g'^{-1}\mathscr{F}.$$

Now we consider the case when $p > 0$. Choose an injective resolutions $\mathscr{F} \xrightarrow{\sim} \mathscr{I}^\bullet$ and $g'^{-1}\mathscr{I} \xrightarrow{\sim} \mathscr{J}^\bullet$, where the notation " $\sim$ " denote quasi-isomorphism. Then we have maps

$$\begin{array}{ccc} g^{-1} f_* \mathscr{I}^\bullet & \overset{\alpha}{\dashrightarrow} & f'_* \mathscr{J}^\bullet \\ & {\scriptstyle p=0 \text{ case}} \searrow \quad \nearrow & \\ & f'_* g'^{-1} \mathscr{I}^\bullet & \end{array}.$$

By definition, $H^p(f'_*\mathscr{J}^\bullet) = R^p f'_*(g'^{-1}\mathscr{F})$. And since g^{-1} is exact, we have $H^p(g^{-1} f_*\mathscr{I}^\bullet) \simeq g^{-1} R^p f_*\mathscr{F}$, so $H^p(\alpha)$ gives the base change map.

Next we give a few consequences of proper base change.

Theorem 15.3 *Let $f : X \to S$ be proper and S quasi-compact and quasi-separable. Let $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$ be constructible. (For our purpose, we should restrict to the case when X and S are Noetherian.) Then for all $p \geqslant 0$, $R^p f_*\mathscr{F}$ are constructible.*

Example 15.4 *Let X be a proper scheme over k, where k is a separably closed field. Let $\mathscr{F}$ be a constructible funtor in $\mathrm{Ab}(X_{\text{ét}})$. Then for any $p \geqslant 0$, $H^p_{\text{ét}}(X, \mathscr{F})$ is finite: $H^p_{\text{ét}}(X, \mathscr{F})$ is isomorphic to $R^p f_*\mathscr{F}$, where $f : X \to \mathrm{Spec}(k)$ is the structure map.*

We have following theorems as corollaries.

Theorem 15.5 *Let $f : X \to S$ be proper. Suppose all fibers have dimension no greater than d. Then for any $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})_{tor}$, $R^p f_*\mathscr{F} = 0$ for $d > 2d$.*

Theorem 15.6 *Let X be an affine scheme over a separably closed field k with $\dim X = d$. Then for any $p > d$ and $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})_{tor}$, $H^p_{\text{ét}}(X, \mathscr{F}) = 0$.*

Analogously, for any complex manifold $X \hookrightarrow \mathbb{C}^n$, $H^p(X, \mathbb{Q}) = 0$ for any $p > n$.
One would like some "proper base change", finiteness, and vanishing results for non-proper maps as well. This is done by using compatification and cohomology of compact support. In topology, let $f : X \to S$ be a map of nice spaces and $\mathscr{F} \in \mathrm{Ab}(X)$. We define sections with compact support of $f_*\mathscr{F}$ by

$$(f_!\mathscr{F})(U) = \{s \in \mathscr{F}(f^{-1}(U)) | \mathrm{supp}(s) \to S \text{ is proper}\}.$$

As an example, if we take S to be a singleton, then

$$(f_!\mathscr{F})(S) = \{s \in \mathscr{F}(X) | \mathrm{supp}(s) \text{ is compact}\}.$$

Now we define for a space X,

$$\Gamma_c\mathscr{F} := \{s \in \mathscr{F} | \mathrm{supp}(s) \text{ is compact}\},$$

and the cohomology of sompact support

$$H^p_c(X, \mathscr{F}) := R^p\Gamma_c\mathscr{F}.$$

Example 15.7 *We have* $H^i_c(\mathbb{R}^n, \mathbb{Z}) = \begin{cases} 0 & i \neq n \\ \mathbb{Z} & i = n \end{cases}$.

We note that mimicking this definition for étale cohomology works out badly. For example, we take $X = \mathbb{A}^1_{\mathbb{C}} \to \mathrm{Spec}(\mathbb{C})$. Topologically, we would get $H^i_c(\mathbb{C}, \mathbb{Z}) = \begin{cases} 0 & i \neq 2 \\ \mathbb{Z} & i = 1. \end{cases}$. In étale cohomology, we have

$$\begin{aligned} \Gamma_c(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Z}/l) &= \{\text{sections with proper support}\} \\ &= \{s \in \mathbb{Z}/l(\mathbb{A}^1_c) = \mathbb{Z}/l | \mathrm{supp}(s) \to \mathrm{Spec}(\mathbb{C}) \text{ is proper}\} \\ &= (0) \end{aligned}$$

We consider the derived functors. Rewrite

$$\Gamma_c(\mathbb{A}^1_{\mathbb{C}}) = \operatorname*{colim}_{\substack{Z \xhookrightarrow{\text{closed}} \mathbb{A}^1_c \\ \text{proper} \searrow \quad \downarrow \\ \mathrm{Spec}(\mathbb{C})}} \Gamma_Z(\mathbb{A}^1_{\mathbb{C}}, \mathscr{F}),$$

where, for any closed immersion $Z \hookrightarrow X$ of schemes and $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$, we set $\Gamma_Z(X, \mathscr{F}) = \ker(\mathscr{F}(X) \to \mathscr{F}(X \backslash Z))$. Then for an injective resolution $\mathscr{F} \to \mathscr{I}^\bullet$, we have

$$R^p\Gamma_c(\mathbb{A}^1_{\mathbb{C}}, \mathscr{F}) = H^p(\operatorname*{colim}_{\substack{Z \xhookrightarrow{\text{closed}} \mathbb{A}^1_c \\ \text{proper} \searrow \quad \downarrow \\ \mathrm{Spec}(\mathbb{C})}} \Gamma_Z(\mathbb{A}^1_{\mathbb{C}}, \mathscr{I}^\bullet)) = \mathrm{colim}_Z H^p_Z(\mathbb{A}^1_{\mathbb{C}}, \mathscr{F}) = \bigoplus_{\substack{\text{closed point} \\ z \in \mathbb{A}^1_{\mathbb{C}}}} H^p_{\{z\}}(\mathbb{A}^1_{\mathbb{C}}, \mathscr{F}).$$

To compute these, we do a general calculation for H^p_Z. Let $Z \overset{i}{\hookrightarrow} X$ be a closed immersion. Denote by j the map $U = X \backslash Z \hookrightarrow X$. Then we have a short exact sequence of sheaves on $X_{\text{ét}}$:

$$0 \to j_!j^{-1}\mathbb{Z} \to \mathbb{Z} \to i_*i^{-1}\mathbb{Z} \to 0.$$

Taking $\mathrm{Hom}(\cdot, \mathscr{F})$, we obtain

$$0 \longrightarrow \mathrm{Hom}(j_!j^{-1}\mathbb{Z}, \mathscr{F}) \simeq \Gamma_Z(X, \mathscr{F}) \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathscr{F}) \longrightarrow \mathrm{Hom}(i_*i^{-1}Z, \mathscr{F}) \simeq \Gamma_Z(X, \mathscr{F})$$

$$\longrightarrow \mathrm{Ext}^1(i_*i^{-1}\mathbb{Z}, \mathscr{F}) \longrightarrow \mathrm{Ext}^1(\mathbb{Z}, \mathscr{F}) \longrightarrow \mathrm{Ext}^1(j_!j^{-1}\mathbb{Z}, \mathscr{F}) \dashrightarrow$$

$$\dashrightarrow \mathrm{Ext}^r(i_*i^{-1}\mathbb{Z}, \mathscr{F}) \simeq H^r_Z(X, \mathscr{F}) \longrightarrow \mathrm{Ext}^r(\mathbb{Z}, \mathscr{F}) \simeq H^r(X, \mathscr{F}) \longrightarrow \mathrm{Ext}^r(j_!j^{-1}\mathbb{Z}, \mathscr{F}) \simeq H^r(U, \mathscr{F})$$

$$\longrightarrow \cdots$$

then we obtain

$$\cdots \to H^r_{\{z\}}(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Z}/l) \to H^r_{\text{ét}}(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Z}/l) \to H^r_{\text{ét}}(\mathbb{A}^1_{\mathbb{C}} \backslash \{z\}, \mathbb{Z}/l) \to \cdots,$$

and therefore

$$H^r_{\{z\}}(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Z}/l) = \begin{cases} 0 & r = 0 \\ 0 & r = 1 \\ \mathbb{Z}/l & r = 2 \end{cases},$$

which implies that $R^2\Gamma_c(\mathbb{A}^1_{\mathbb{C}}, \mathbb{Z}/l)$ is infinite. This is in conflict with the topological intuition.

15.1 Lecture 20: 11/2

In topology, there is an equivalent definition of $H^*_c(X, \mathscr{F})$ or of $R^n f_! \mathscr{F}$ when we have $f = \overline{f} \circ j : X \to S$ where $j : X \hookrightarrow \overline{X}$ is an open immersion and $\overline{f}$ is proper. Namely, $f_!\mathscr{F} = \overline{f}_* j_! \mathscr{F}$. There is an identification between

$$f_!\mathscr{F}(S) = \{t \in \mathscr{F}(X) \mid \mathrm{supp}(t) \to S \text{ is proper}\}$$

with

$$(\overline{f}_* j_! \mathscr{F})(S) = (j_!\mathscr{F})(\overline{X}) = \{t \in \mathscr{F}(X) \mid \mathrm{supp}(t) \to \overline{X} \text{ is proper}\}.$$

Under mild hypotheses on X and S, this identification passes to an identification

$$R^n f_! \mathscr{F} \simeq R^n \overline{f}_*(j_!\mathscr{F})$$

(check that $j_!$ takes injectives to acyclics).

This motivates the correct definition in étale cohomology.

Definition 15.8 *Let $f : X \to S$ be separated of finite type. Let $j : X \hookrightarrow \overline{X}$ be an open immersion into a proper S-scheme $\overline{f} : \overline{X} \to S$ (j is a "compactification"). The higher direct images with proper support of f are by definition, for $\mathscr{F} \in \mathrm{Ab}(X_{\acute{e}t})_{\mathrm{torsion}}$,*

$$R^n f_! \mathscr{F} = R^n \overline{f}_*(j_!\mathscr{F})$$

In particular we use $f_!$ to denote $R^0 f_!$ (note that this is not the R^n of $f_!$). When $S = \mathrm{Spec}(k)$ for k a separably closed field, set $H^*_c(X, \mathscr{F}) = H^*(\overline{X}, j_!\mathscr{F})$ ($\simeq \Gamma(S, R^* f_! \mathscr{F})$), the cohomology with compact support.

Remark 15.9 • *Given f as in the definition, assuming S is qcqs, such a compactification exists. When S is Noetherian, this is the Nagata compactification (hard), and the qcqs case can be reduced to this.*

- *The construction of $Rf_!\mathscr{F}$ is canonically independent of the choice of compactification, but his depends on proper base change.*
- *We have that $j_!$ is exact, so $R^\bullet f_!$ is still a δ-functor (even though it is not $R^n(\overline{f}_* \circ j_!)$).*
- *When f is étale (where we previously defined $f_!$), this agrees with the previous definition.*

Sketch the relation: Since f is quasi-finite, Zariski's Main Theorem implies that there exists a factorization $f = \overline{f} \circ j$ where $j : X \hookrightarrow \overline{X}$ is open and $\overline{f}$ is finite. To build a map $f_!\mathscr{F} \to \overline{f}_* \circ j_!\mathscr{F}$, it suffices to give a map $\mathscr{F} \to f^{-1}\overline{f}_* j_!\mathscr{F}$. Given $\phi : U \to X$ étale, we want

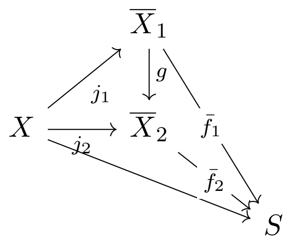
$$\mathscr{F}(U) \to (f^{-1}\overline{f}_* j_!\mathscr{F})(U) = (\overline{f}_* j_!\mathscr{F})(U) = (j_!\mathscr{F})(U \times_S \overline{X}).$$

The map $(\mathrm{id}, j \circ \phi) : U \to U \times_S \overline{X}$ is an isomorphism onto an open and closed subscheme (étale by 2/3, proper since $U \times_S X \to U$ is separated, and monomorphism). So $U \times_S \overline{X} = U \coprod W$ (W complementary open and closed). So,

$$(j_!\mathscr{F})(U \times_S \overline{X}) \simeq (j_!\mathscr{F})(U) \times (j_!\mathscr{F})(W).$$

The map we wanted above is then given by $s \in \mathscr{F}(U) \mapsto (s, 0)$ under the above identification.

Returning to the remark. Why is $R^n f_!\mathscr{F}$ independent of compactification? For any two $j_i :\hookrightarrow \overline{X}_i$ with $\overline{f}_i : \overline{X}_i \to S$ both are dominated by $(j_1, j_2) : X \hookrightarrow \overline{X}_1 \times_S \overline{X}_2$; we upgrade (j_1, j_2) to a compactification by restricting its target to the closure of its image, which makes it an open immersion. So, by comparing both to (j_1, j_2), it suffices to consider the case where $\overline{X}_1 \to \overline{X}_2$ (over S). We obtain the following diagram



We now compute:

$$\begin{aligned}
R^n f_{2!}\mathscr{F} &= R^n\overline{f}_{2*}(j_{2!}\mathscr{F}) \\
&\cong R^n\overline{f}_{2*}(g_* j_{1!}\mathscr{F}) \\
&\cong R^n(\overline{f}_{2*} \circ g_*)(j_{1!}\mathscr{F}) \\
&= (R^n\overline{f}_{1*})(j_{1!}\mathscr{F}) = R^n f_{1!}\mathscr{F}.
\end{aligned}$$

For the intermediate isomorphisms, use $R^p g_* j_{1!}\mathscr{F} = \begin{cases} j_{2!}\mathscr{F} & p = 0 \\ 0 & p > 0 \end{cases}$ and Leray spectral sequence. To see this we introduce the following

Lemma 15.10 *For all cartesian diagrams of schemes $X \times_Y U$ for $f : X \to Y$ and $j : U \to Y$ (and $f' : X \times_Y U \to U$, $j' : X \times_Y U \to X$) where j is étale, and for all $\mathscr{F} \in \mathrm{Ab}(U_{\acute{e}t})$, the canonical map $j'_! f'^{-1}\mathscr{F} \to f^{-1} j_! \mathscr{F}$ is an isomorphism.*

Proof. Exercise: use the explicit stalk description.

Stalk calculation. Let $\overline{y} \to \overline{X}_2$ be a geometric point, we consider the following two Cartesian diagrams.

$$\begin{array}{ccc} \overline{X}_{1,\overline{y}} & \longrightarrow & \overline{X}_1 \\ \downarrow & & \downarrow \\ \overline{y} & \longrightarrow & \overline{X}_2 \end{array} \qquad\qquad \begin{array}{ccc} X_{\overline{y}} & \xrightarrow[j'_1]{} & \overline{X}_{1,\overline{y}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{j_1} & \overline{X}_1 \end{array}$$

By proper base change and the above lemma,

$$(R^p g_* j_{1!}\mathscr{F})_{\overline{y}} \cong H^p(\overline{X}_{1,\overline{y}}, (j_{1!}\mathscr{F})|_{\overline{X}_{1,\overline{y}}}) \simeq H^p(\overline{X}_{1,\overline{y}}, j'_{1!}(\mathscr{F}|_{X\overline{y}})).$$

This is 0 if $\overline{y} \notin j_2(X)$, and if $\overline{y} \in j_2(X)$, then $X_{\overline{y}}$ is a single point and H^p vanishes for $p > 0$ and for $p = 0$ get $\mathscr{F}_{\overline{y}}$. We should further check that the isomorphism $R^n f_{2!}\mathscr{F} \simeq R^n f_{1!}\mathscr{F}$ is independent of the choice of the third compactification dominating the first two; we omit this check.

We now list some corollaries of proper base change.

Corollary 15.11 *Let $f : X \to S$ be separated of finite type, with S qcqs. Let $g : T \to S$ be any map, and let $\mathscr{F} \in \mathrm{Ab}(X_{\acute{e}t})_{\mathrm{tor}}$. Then, there exist a canonical base change isomorphism $g^{-1}R^n f_! \mathscr{F} \simeq R^n f'_!(g'^{-1}\mathscr{F})$ where $g' : X_T \to X$ and $f' : X_T \to T$ are the projections from the fiber product $X_T = X \times_S T$.*

(Taking $T = \overline{s} \to S$ a geometric point yields $(R^n f_! \mathscr{F})_{\overline{s}} \simeq H^p_c(X_{\overline{s}}, \mathscr{F}|_{X_{\overline{s}}})$.)

Corollary 15.12 *For $X \xrightarrow{g} Y \xrightarrow{f} S$ both separated of finite type maps of qcqs schemes, and for $\mathscr{F} \in \mathrm{Ab}(X_{\acute{e}t})_{\mathrm{tor}}$, there is a spectral sequence*

$$E_2^{p,q} = R^p f_!(R^q g_! \mathscr{F}) \Rightarrow R^{p+q}(f \circ g)_! \mathscr{F}.$$

Corollary 15.13 *Let $f : X \to S$ be separated of finite type with S qcqs. For $\iota : Z \hookrightarrow X$ closed and $j : U = X \backslash Z \hookrightarrow X$ open, set $f_U = f \circ j$ and $f_Z = f \circ \iota$. Then, for $\mathscr{F} \in \mathrm{Ab}(X_{\acute{e}t})_{\mathrm{tor}}$, there is a long exact sequence in $\mathrm{Ab}(S_{\acute{e}t})$*

$$\cdots \to R^n f_{U!}(\mathscr{F}|_U) \to R^n f_! \mathscr{F} \to R^n f_{Z!}(\mathscr{F}|_Z) \to R^{n-1} f_{U!}(\mathscr{F}|_U) \to \cdots$$

(Taking $S = \mathrm{Spec}(k^s)$, we get $H^n_c(U, \mathscr{F}|_U) \to H^n_c(X, \mathscr{F}) \to \cdots$)

Idea: We have a short exact sequence

$$0 \to j_! \mathscr{F}|_U \to \mathscr{F} \to \iota_* \mathscr{F}_Z \to 0$$

and $R^n f_!(j_! \mathscr{F}|_U) \simeq R^n (f \circ j)_! \mathscr{F}|_U$ because proper base change shows $R^k j_! \mathscr{F}|_U = 0$ for $k > 0$. Likewise for $\iota_* = \iota_!$. Then, apply the δ-functor $R^n f_!$ to the short exact sequence above.

Theorem 15.14 (Generalizing the vanishing statement from before) *Let $f : X \to S$ be separated of finite type with S qcqs such that all fibers of f have dimension $\leqslant d$. Let $\mathscr{F} \in \mathrm{Ab}(X_{ét})_{\mathrm{tor}}$. Then, for all $q > 2d$, $R^q f_! \mathscr{F} = 0$.*

Proof. It suffices to check on stalks. By proper base change (for $f_!$), it suffices to check that $H^q_c(X, \mathscr{F}) = 0$ for $q > 2d$ when $X \to \mathrm{Spec}(k)$ is separated of finite type and dimension d where k is a separably closed field.

There is an affine open $j : U \hookrightarrow X$ with $\iota : X \backslash U \hookrightarrow X$ and $\dim(X\backslash U) < d$. From the short exact sequence

$$0 \to j_! j^{-1}\mathscr{F} \to \mathscr{F} \to \iota_* \iota^{-1}\mathscr{F} \to 0$$

by induction on d, it suffices to show $H^q_c(U, j^{-1}\mathscr{F}) = H^q_c(U, j_! j^{-1}\mathscr{F}) = 0$ for all $q > 2d$. Thus, by induction, we may assume that X is affine, i.e. $X = \mathrm{Spec}(A)$ where $A = k[x_1, \dots, x_n]/I$, and the algebra inclusions $k \subseteq k[x_1] \subseteq \cdots \subseteq k[x_1, \dots, x_n]$ induce

$$X = X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} X_{n-2} \to \cdots \to X_1 \xrightarrow{f_1} X_0 = \mathrm{Spec}(k)$$

where $X_i = \mathrm{Spec}(\text{subalgebra of } A \text{ generated by } x_1, \dots, x_i)$. Each $f_i : X_i \to X_{i-1}$ has fibers with dimension $\leqslant 1$. Repeated use of "Leray" spectral sequence shows that it suffices to prove that $R^p f_{i!} \mathscr{G}_i = 0$ for all $p \geqslant 3$ and $\mathscr{G}_i \in \mathrm{Ab}(X_{i,\text{ét}})_{\mathrm{tor}}$. (inductively,

$$R^p(f_1 f_2 \cdots f_{i-1})_! R^q f_{i!} \mathscr{G} \Rightarrow R^{p+q}(f_1 \cdots f_i)_! \mathscr{G}$$

check $R^n(f_1 \cdots f_i)_! \mathscr{G} = 0$ for $n > 2i$). But this follows from proper base change which reduces you to showing $H^p_c(Y, \mathscr{G}) = 0$ for all Y affine finite type over k of dimension $\leqslant 1$ and $\mathscr{G} \in \mathrm{Ab}(Y_{\text{ét}})_{\mathrm{tor}}$ and $p \geqslant 3$. Such a Y embeds in a projective $\overline{Y}$ over k of dimension $\leqslant 1$ ($\alpha : Y \hookrightarrow \overline{Y}$) and $H^p_c(Y, \mathscr{G}) = H^p(\overline{Y}, \alpha_! \mathscr{G})$, which is zero by the curve theorem we have previously stated (but not proved).

Now, generalizing the constructibility statement from last time, we have:

Theorem 15.15 *Let $f : X \to S$ be separated of finite type with X and S Noetherian, and let $\mathscr{F} \in \mathrm{Ab}(X_{ét})$ constructible. Then, $R^q f_! \mathscr{F}$ is constructible for all q.*

15.2 Lecture 21: 11/7

There is one more consequence of proper base change we want to mention, before turning back the cohomology of curves: the comparison between étale cohomology and singular cohomology.

Let X be a separated scheme of finite type over $\mathbb{C}$. To X we associate a complex-analytic space X^{an} whose underlying set is $X(\mathbb{C})$ (e.g. in the smooth projective case, X^{an} is a submanifold of complex projective space). The analytification X^{an} has an étale site, whose maps are analytic maps that are local homeomorphisms, and there is a functor $X_{\text{ét}} \to X^{\mathrm{an}}_{\text{ét}}$ which does the obvious thing: $(X \to Y) \mapsto (X^{\mathrm{an}} \to Y^{\mathrm{an}})$. There is a corresponding "direct image" functor of sheaves $\mathrm{Sh}\, X^{\mathrm{an}}_{\text{ét}} \to \mathrm{Sh}\, X_{\text{ét}}$ given by $\mathscr{F} \mapsto (U \mapsto \mathscr{F}(U^{\mathrm{an}}))$. This functor has a left adjoint $\mathscr{F} \mapsto \mathscr{F}^{\mathrm{an}} \colon \mathrm{Sh}\, X_{\text{ét}} \to \mathrm{Sh}\, X^{\mathrm{an}}_{\text{ét}}$. Here are two ways to construct the analytification of a sheaf:

- Mimic the construction of the inverse image of a sheaf.

- It's clear how a representable sheaf should be analytified: $h_U^{\mathrm{an}} := h_{U^{\mathrm{an}}}$. Now a general fact about sheaves on a site is that any sheaf is a colimit of representable sheaves, so we can define in general $(\operatorname{colim}_U h_U)^{\mathrm{an}} := \operatorname{colim}_U h_{U^{\mathrm{an}}}$.

Theorem 15.16 *Let X be a separated scheme of finite type over $\mathbb{C}$ and $\mathscr{F} \in \mathrm{Ab}\, X_{\text{ét}}$ torsion.*

1. *Then $H_c^*(X_{\text{ét}}, \mathscr{F}) \cong H_c^*(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$ (the latter refers to cohomology on the usual site of open subsets of the topological space X^{an}).*

2. *Assume $\mathscr{F}$ is constructible. Then $H^*(X_{\text{ét}}, \mathscr{F}) \cong H^*(X^{\mathrm{an}}, \mathscr{F}^{\mathrm{an}})$.*

Remarks on the proof One first constructs isomorphisms between cohomology on the étale sites $X_{\text{ét}}$ and $X_{\text{ét}}^{\mathrm{an}}$. Next one uses that the site $X_{\text{ét}}^{\mathrm{an}}$ is sandwiched between the étale and usual sites of the topological space X^{an}, and that cohomology on the étale site of a topological space computes the usual cohomology.

(1). This result is a rather formal consequence of proper base change: follow the proof of vanishing in high degrees. Notice that when X is proper, this becomes a statement about the usual cohomology and strengthens (2).

(2). This is a difficult result; the usual proof uses resolution of singularities.

This is nice because it allows us to directly import results from the classical theory to the étale setting.

16 Proof of the theorem on the cohomology of curves

Recall the statement:

Theorem 16.1 *Let k be a separably closed field, X a separated scheme of finite type over k of dimension $\leqslant 1$, and let $\mathscr{F} \in \mathrm{Ab}\, X_{\text{ét}}$ be torsion.*

1. *One has $H_{\text{ét}}^q(X, \mathscr{F}) = 0$ for $q \geqslant 3$.*

(1′) *If X is affine, (1) holds for $q \geqslant 2$.*

2. *If $\mathscr{F}$ is constructible of torsion coprime to $\operatorname{char} k$, the groups $H_{\text{ét}}^q(X, \mathscr{F})$ are finite for all q.*

(2′) *If X is proper over k, (2) holds without the assumption on $\operatorname{char} k$.*

3. *If the torsion of $\mathscr{F}$ is coprime to $\operatorname{char} k$ and k'/k is an extension of separably closed fields, then $H_{\text{ét}}^q(X, \mathscr{F}) \to H_{\text{ét}}^q(X_{k'}, \mathscr{F}_{k'})$ is an isomorphism.*

(3′) *If X is proper over k, (3) holds without the assumption on $\operatorname{char} k$.*

Below, continue to let k, X, $\mathscr{F}$ be as in the Theorem (with additional assumptions stated as needed).

We separate the proof into several parts. First of all, we can and do assume k is algebraically closed, since performing a purely inseparable field extension leaves the étale site, and hence étale cohomology, unchanged. It's easy to further reduce to the case when X is connected of dimension exactly 1. Recall also that we've already treated the case of X smooth projective over k.

Lemma 16.2 *Assume X is smooth and affine. Then the Theorem holds for $\mathscr{F} = \mathbb{Z}/n$ (the case of n prime suffices for our later use).*

Proof. We can write $X = \overline{X} \backslash S$, where $\overline{X}$ is smooth projective over k and S is a finite nonempty set of closed points. The case when $\operatorname{char} k \mid n$ (items ($2'$, $3'$)) is left as an exercise in applying the Artin–Schreier sequence. So we assume $\operatorname{char} k \nmid n$, which implies $\mathbb{Z}/n \cong \mu_n$ on X since k is separably closed. Consider the Kummer sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_{\mathrm{m}} \longrightarrow \mathbb{G}_{\mathrm{m}} \xrightarrow{z \mapsto z^n} 1.$$

Now take the long exact sequence. First, since $H^q(X, \mathbb{G}_{\mathrm{m}}) = 0$ for $q \geqslant 2$, we get that $H^q(X, \mu_n) = 0$ for $q \geqslant 3$. Thus (1) is verified.

Next, using $H^1(X, \mathbb{G}_{\mathrm{m}}) = \operatorname{Pic} X$, the long exact sequence gives an isomorphism $\operatorname{Pic} X / n \operatorname{Pic} X \cong H^2(X, \mu_n)$. But given $D \in \operatorname{Div} X$, we can extend D to some $\overline{D} \in \operatorname{Div}^0 \overline{X}$; using that multiplication by n is surjective on $\operatorname{Pic}^0 \overline{X}$, it follows that $\overline{D} = nE + \operatorname{div}(f)$, therefore $D = nE_{|X} + \operatorname{div}(f)_{|X}$. It's clear now that $[n]$ is surjective on $\operatorname{Pic}(X)$, thus $H^2(X, \mu_n) = 0$, verifying ($1'$).

Now look a bit earlier in the long exact sequence to separate out the short exact sequence

$$1 \longrightarrow \mathscr{O}_X(X)^\times / \mathscr{O}_X(X)^{\times n} \longrightarrow H^1(X, \mu_n) \longrightarrow (\operatorname{Pic} X)[n] \longrightarrow 0.$$

We want to show that both $\mathscr{O}_X(X)^\times / \mathscr{O}_X(X)^{\times n}$ and $(\operatorname{Pic} X)[n]$ (the latter meaning n-torsion) are finite, which will prove (2) in this case.

We first consider the divisor map on $\overline{X}$ restricted to $\mathscr{O}_X(X)^\times$, which evidently takes the form $\mathscr{O}_X(X)^\times \to \mathbb{Z}^S$, and whose kernel is $\mathscr{O}_{\overline{X}}(\overline{X})^\times \cong k^\times$. Then we can consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k^\times & \longrightarrow & \mathscr{O}_X(X)^\times & \longrightarrow & \operatorname{div}_{\overline{X}} \mathscr{O}_X(X)^\times & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle n} & & \\
1 & \longrightarrow & k^\times & \longrightarrow & \mathscr{O}_X(X)^\times & \longrightarrow & \operatorname{div}_{\overline{X}} \mathscr{O}_X(X)^\times & \longrightarrow & 0,
\end{array}$$

(all the vertical maps are multiplication by n). Since the nth-power map is surjective on $k^\times$, the (easy part of the) snake lemma furnishes an isomorphism $\mathscr{O}_X(X)^\times / \mathscr{O}_X(X)^{\times n} \cong \operatorname{div}_{\overline{X}} \mathscr{O}_X(X)^\times / n \operatorname{div}_{\overline{X}} \mathscr{O}_X(X)^\times$. Since $\operatorname{div}_{\overline{X}} \mathscr{O}_X(X)^\times$ embeds into $\mathbb{Z}^S$, the finiteness of $\mathscr{O}_X(X)^\times / \mathscr{O}_X(X)^{\times n}$ follows.

Next, we show that $(\operatorname{Pic} X)[n]$ is finite. Begin with the morphism of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{div}_{\overline{X}} \mathscr{O}_X(X)^\times & \longrightarrow & \operatorname{PDiv} \overline{X} & \longrightarrow & \operatorname{PDiv} X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Div} S & \longrightarrow & \operatorname{Div} \overline{X} & \longrightarrow & \operatorname{Div} X & \longrightarrow & 0,
\end{array}$$

where PDiv is the principal divisor and apply the snake lemma to get an exact sequence

$$0 \longrightarrow \operatorname{Div} S / \operatorname{div}_{\overline{X}} \mathscr{O}_X(X)^\times \longrightarrow \operatorname{Pic} \overline{X} \longrightarrow \operatorname{Pic} X \longrightarrow 0.$$

Now, like in the previous paragraph, take multiplication by n on this sequence to get a morphism of exact sequences and apply again the snake lemma. Since $\operatorname{Div} S \cong \mathbb{Z}^S$, the relevant piece of the snake-lemma exact sequence takes the form

$$(\operatorname{Pic} \overline{X})[n] \longrightarrow (\operatorname{Pic} X)[n] \longrightarrow \text{finite}.$$

By the theory of smooth projective curves over an algebraically closed field we have $(\operatorname{Pic} \overline{X})[n] \cong (\mathbb{Z}/n)^{2g}$, where g is the genus of $\overline{X}$. This at last finishes the proof of the present special case of (2).

What remains is (3). This follows from a sufficiently close inspection of the argument given above.

Remark 16.3 *One can refine the calculations of the preceding proof to obtain that* $H^1(X, \mathbb{Z}/n) \cong (\mathbb{Z}/n)^{2g+\#S-1}$ *when* $\operatorname{char} k \nmid n$.

Next, we make the main reduction for general X.

Lemma 16.4 *It suffices to prove the Theorem for* $\mathscr{F}$ *of the form* $j_!\mathscr{G}$*, where* $\mathscr{G}$ *is an lcc sheaf of* $\mathbb{Z}/\ell$*-vector spaces on* U *and* $j\colon U \hookrightarrow X$ *is the inclusion of an irreducible open subscheme.*

Proof. Any arbitrary torsion sheaf is a direct colimit of constructible sheaves (proof omitted), and since cohomology commutes with colimits, we reduce to the case of $\mathscr{F}$ constructible and moreover ℓ^∞-torsion. Repeatedly using the exact sequence

$$0 \longrightarrow \mathscr{F}[\ell] \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\mathscr{F}[\ell] \longrightarrow 0,$$

we inductively reduce to the ℓ-torsion case (here use that subquotients of constructible sheaves are constructible). Thus there exists an open inclusion $j\colon U \hookrightarrow X$ such that $j^{-1}\mathscr{F}$ (i.e. $\mathscr{F}|_U$) is an lcc sheaf of $\mathbb{Z}/\ell$-vector spaces. Shrinking U, we may assume $U = \bigsqcup_i U_i$ where each U_i is an open irreducible subscheme of U (throw out the intersections of the irreducible components of U). For $Z := X \backslash U$, with inclusion $i\colon Z \hookrightarrow X$, consider the adjunction short exact sequence

$$0 \longrightarrow j_! j^{-1}\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow i_* i^{-1}\mathscr{F} \longrightarrow 0.$$

Since Z is finite, we have

$$i_* i^{-1}\mathscr{F} \cong \bigoplus_{z \in Z} i_{z,*}\mathscr{F}_{\overline{z}},$$

where i_z is the inclusion $\{z\} \hookrightarrow X$ and $\overline{z}$ is a geometric point above z; thus

$$H^q(X, i_* i^{-1}\mathscr{F}) = \begin{cases} \text{finite}, & q = 0 \\ 0, & q \geqslant 1. \end{cases}$$

The reduction now follows upon taking the long exact sequence associated to the adjunction short exact sequence.

We now begin to attack the special case of the Theorem described in the preceding lemma. We first consider the yet more special case of constant $\mathscr{G}$.

Lemma 16.5 *The Theorem holds for $\mathscr{F} = j_!(\mathbb{Z}/\ell)_U$, where $j\colon U \hookrightarrow X$ is the inclusion of an irreducible open subscheme.*

Proof. We may of course assume X is reduced, since the étale site is not affected by passing to the reduction.

First assume X is smooth. Use the short exact sequence

$$0 \longrightarrow j_!(\mathbb{Z}/\ell)_U \longrightarrow (\mathbb{Z}/\ell)_X \longrightarrow i_*i^{-1}(\mathbb{Z}/\ell)_X \longrightarrow 0$$

considered in the previous lemma; we know the cohomology of $i_*i^{-1}(\mathbb{Z}/\ell)_X$ as stated above, and we also know the Theorem for $(\mathbb{Z}/\ell)_X$ by Lemma 16.2. So, we get the Theorem for $j_!(\mathbb{Z}/\ell)_U$ by chasing the long exact sequence.

Now for general X, we will look at the normalization $f\colon \widetilde{X} \to X$ (a finite map) and try to relate the X-cohomology to $\widetilde{X}$-cohomology. We'll treat the case $\operatorname{char} k \nmid n$ (the other case again being an application of Artin–Schreier). Consider the diagram

$$\begin{array}{ccc} \widetilde{U} & \xhookrightarrow{\tilde{j}} & \widetilde{X} \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ U & \xhookrightarrow{j} & X, \end{array}$$

where $\widetilde{U} := f^{-1}U$. We know the theorem for $\tilde{j}_!$ in place of $j_!$. There is a canonical map $j_!(\mathbb{Z}/\ell) \to f_*\tilde{j}_!(\mathbb{Z}/\ell)$: write it down explicitly at the level of sections, or construct the adjoint map $f^{-1}j_!(\mathbb{Z}/\ell) \to \tilde{j}_!\tilde{f}^{-1}(\mathbb{Z}/\ell)$ using that $j_!$ commutes with arbitrary base change. This map is injective and its cokernel is a (constructible) skyscraper sheaf supported on some finite set $S \subseteq X$. Using the long exact sequence, we see that the theorem holds for $j_!(\mathbb{Z}/\ell)$ if and only if it holds for $f_*\tilde{j}_!(\mathbb{Z}/\ell)$. But f is finite, so f_* is exact, so the Leray spectral sequence degenerates, so $H^*(X, f_*\tilde{j}_!(\mathbb{Z}/\ell)) \cong H^*(\widetilde{X}, \tilde{j}_!(\mathbb{Z}/\ell))$. This reduces to the smooth case already proven.

Finally we need to deduce the whole Theorem from the case just established. For this we need a new construction, which is maybe the most interesting part of the entire proof, namely of the "trace map".

Let $f\colon U \to X$ be étale, X any scheme. Given an Abelian étale sheaf $\mathscr{F}$ on X, we have adjunction maps $f_!f^{-1}\mathscr{F} \to \mathscr{F}$ (using that f is étale) and $\mathscr{F} \to f_*f^{-1}\mathscr{F}$. Assume that f is moreover finite: then $f_! = f_*$ (an exercise in checking on stalks). Thus we get $\operatorname{tr}\colon f_*f^{-1}\mathscr{F} \to \mathscr{F}$ and a composition

$$\mathscr{F} \longrightarrow f_*f^{-1}\mathscr{F} \xrightarrow{\operatorname{tr}} \mathscr{F}$$

Example 16.6 *In the special case $U = \bigsqcup_{i=1}^{d} X \to X$, we have $f_*f^{-1}\mathscr{F} \cong \mathscr{F}^{\oplus d}$, and the trace is the sum map. Thus the composition $\mathscr{F} \to \mathscr{F}$ is multiplication by d.*

16.1 Lecture 22 11/9

Lemma 16.7 (Proper Base Change for Finite Morphisms) *Given a cartesian diagram*

$$\begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f \text{ finite}} \\ T & \xrightarrow[g]{} & S \end{array}$$

and $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$*, the canonical morphism*

$$g^{-1} f_* \mathscr{F} \to f'_* g'^{-1} \mathscr{F}$$

is an isomorphism.

Proof. Compute the stalks.

Given an étale morphism $f : U \to X$, we have a **restriction map** and a **trace map**.

$$\mathscr{F} \xrightarrow{res_f} f_* f^{-1} \mathscr{F} \text{ and } f_! f^{-1} \mathscr{F} \xrightarrow{tr_f} \mathscr{F}$$

coming from the adjunctions $f^{-1} \dashv f_*$ $f_! \dashv f^{-1}$. If f is additionally finite, using the fact that $f_! = f_*$ if f is finite we have the composition

$$\mathscr{F} \xrightarrow{res_f} f_* f^{-1} \mathscr{F} = f_! f^{-1} \mathscr{F} \xrightarrow{tr_f} \mathscr{F}.$$

In the special case where $U = \coprod_{i=1}^d X$ $f_* f^{-1} \mathscr{F} = \mathscr{F}^{\oplus d}$, the composition is just multiplication by d.

Moreover, traces behave well with localization: Given a cartesian diagram

$$\begin{array}{ccc} U \times_X Y & \xrightarrow{g'} & U \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f \text{ finite étale}} \\ Y & \xrightarrow[g \text{ étale}]{} & S \end{array}$$

the pullback

$$g^{-1} tr_f : g^{-1} f_* f^{-1} \mathscr{F} \to g^{-1} \mathscr{F}$$

agrees with

$$tr_{f'} : f'_* f'^{-1} (g^{-1} \mathscr{F}) \to g^{-1} \mathscr{F}$$

via the identification

$$g^{-1} f_* f^{-1} \mathscr{F} \cong f'_* g'^{-1} f^{-1} \mathscr{F} \cong f'_* f'^{-1} g^{-1} \mathscr{F}$$

where the first isomorphism is the base change morphism and the second one is the canonical isomorphism.

Therefore, putting this result with the previous special case, in addition to the fact that every finite etale U over X étale-locally looks like $\coprod X$ (as we have seen before), we deduce the following: For a finite étale morphism $f : U \to X$ of constant degree d, the the composite

$$\mathscr{F} \to f_* f^{-1} \mathscr{F} \xrightarrow{tr_f} \mathscr{F}$$

is the multiplication by d map, denoted $[d]$.

In particular, if $[d]$ is an isomorphism for a sheaf $\mathscr{F}$, then the composite

$$H^p(X, \mathscr{F}) \to H^p(U, f^{-1}\mathscr{F}) \cong H^p(X, f_* f^{-1}\mathscr{F}) \xrightarrow{tr_f} H^p(X, \mathscr{F})$$

is the multiplication by d map, hence an isomorphism. (Note that the isomorphism in the middle comes from the fact that f is finite so the Leray spectral sequence degenerates). So, the pull-back map $H^p(X, \mathscr{F}) \to H^p(U, f^{-1}\mathscr{F})$ is injective.

Proposition 16.8 *Let X be a separated finite type scheme of dimension $\leqslant 1$ over an algebraically closed field k. Assume that X is reduced. Let $j : U \to X$ be an open immersion, $\mathscr{G}$ a lcc sheaf of $\mathbb{Z}/\ell$-module on U. Then the theorem for curves holds for $j_! \mathscr{G}$.*

Proof. We will prove later the following fact: There exists a finite étale morphism $f : V \to U$ of degree coprime to ℓ such that $f^{-1}\mathscr{G}$ has a filtration with all graded pieces isomorphic to $\mathbb{Z}/\ell$, i.e.,

$$f^{-1}\mathscr{G} = F^0 \supseteq F^1 \supseteq F^2 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

such that $F^i/F^{i+1} \cong \mathbb{Z}/\ell$ for all $i = 0, \ldots, r$.

Assuming the fact, the composite

$$\mathscr{G} \to f_* f^{-1}\mathscr{G} \xrightarrow{tr_f} \mathscr{G}$$

is an isomorphism. So, if we prove the theorem for $j_! f_* f^{-1}\mathscr{G}$, then we will be done because $j_!\mathscr{G}$ is a direct summand of $j_! f_* f^{-1}\mathscr{G}$.

By Zariski's Main theorem, we can factor $j \circ f$ as

$$\begin{array}{ccc} V & \longrightarrow & Y \\ & \underset{j\circ f}{\searrow} & \downarrow{\scriptstyle \overline{f} \text{ finite}} \\ & & X \end{array}$$

where $V \to Y$ is an open immersion. Observe that the diagram

$$\begin{array}{ccc} V & \xrightarrow{j'} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f} \text{ finite}} \\ U & \xrightarrow[j]{} & X \end{array}$$

is cartesian, i.e. the morphism $V \xrightarrow{(f,j')} U \times_X Y$ is an isomorphism. Hence, finite base change theorem applied to $j'_! f^{-1}\mathscr{G}$ yields an isomorphism

$$j^{-1}\overline{f}_* j'_! f^{-1}\mathscr{G} \xrightarrow{\sim} f_* j'^{-1} j'_! f^{-1}\mathscr{G} = f_* f^{-1}\mathscr{G}.$$

Applying $j_!$ and using adjunction, we get a morphism

$$\alpha : j_! f_* f^{-1}\mathscr{G} \to \overline{f}_* j_! f^{-1}\mathscr{G}.$$

The restriction of α to U is clearly an isomorphism. On the other hand, for $\overline{x} \in X - U$, the stalk of LHS at $\overline{x}$ is zero because $j_!$ is extension by zero and the stalk of RHS is also zero because

$$(\overline{f}_* j'_! f^{-1}\mathscr{G})_{\overline{x}} = \bigoplus_{\overline{y} \in Y} (j'_! f^{-1}\mathscr{G})_{\overline{y}}$$

where the direct sum is over the geometric points $\overline{y}$ of Y lying over $\overline{x}$. But, for each geometric point $\overline{y}$ of Y lying over $\overline{x}$, $(j'_! f^{-1}\mathscr{G})_{\overline{y}} = 0$ because $\overline{x} \in X - U$. Therefore, α is an isomorphism.

Hence, it suffices to prove the theorem for

$$H^*(X, \overline{f}_* j'_! f^{-1}\mathscr{G}) = H^*(Y, j'_! f^{-1}\mathscr{G}).$$

But, $j'_! f^*\mathscr{G}$ has a filtration with graded pieced isomorphic to $j'_!\mathbb{Z}/\ell$ and we know the theorem for this case.

rmk 16.9 *The curve theorem holds for non-separated curves. In order to prove the theorem for non-separated curves, cover it by affine open subschemes and use Mayor-Vietoris sequences.*

Next, we discuss the fact we use in the proof. Namely, if $\mathscr{G}$ is a lcc sheaf of $\mathbb{Z}/\ell$-modules on U, then there is a finite étale morphism $f : V \to U$ of degree coprime to ℓ such that $f^{-1}\mathscr{G}$ has a filtration with graded pieces isomorphic to $\mathbb{Z}/\ell$.

For $U = \mathrm{Spec}(k)$ where k is a field, this can be seen as follows: a lcc sheaf of $\mathbb{Z}/\ell$-module corresponds to a finite $\mathbb{Z}/\ell[G_K]$-module where G_k is the absolute Galois group of k. Such a finite module gives a rational representation $G_k \to \mathrm{GL}_n(\mathbb{Z}/\ell)$ for some n. Since the latter is finite, this representation factors through some $G = \mathrm{Gal}(L/k)$ where L/k is a finite Galois extension. Now let G_ℓ be an Sylow-ℓ subgroup of G, then up to conjugacy, G_ℓ is mapped into $SU_n(\mathbb{F}_\ell)$. Let $L_\ell = L^{G_\ell}$ be the Galois subextension of L/k fixed by G_ℓ. Then, G_{L_ℓ}-module $M|_{G_{L_\ell}}$ has filtration with graded pieces isomorphic to $\mathbb{Z}/\ell$ by construction.

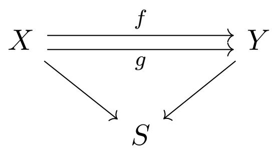
This generalizes as long as we have a formalism of the absolute Galois group for a general scheme, the étale fundamental group of a scheme.

17 $\pi_1^{\text{ét}}$

17.1 Lecture 23: 11/14

We start with this rigidity lemma:

Lemma 17.1 *If we have a diagram*



where X is connected and $Y \to S$ is separated étale, and $\overline{x}$: $\mathrm{Spec}(k) \to X$ *is a geometric point. Then $f(\overline{x}) = g(\overline{x})$ implies $f = g$.*

Proof. *Consider $\triangle : Y \longrightarrow Y \times_S Y$. It's closed immersion by separatedness, and etale (2/3 over S) hence open, so $\triangle(Y)$ is close and open therefore $Y \times_S Y = \triangle(Y) \coprod W$. For $f \times g : X \longrightarrow Y \times Y$, we know $f(\overline{x}) = g(\overline{x})$, we have $f \times g(X)$ meets $\triangle(Y)$, but X is connected, hence $f \times g$ factors through $\triangle(Y)$ which is saying $f = g$.*

Let $f : X \to S$ finite étale, X, S connected. Suppose $\deg(f) = n$. Then $\#\mathrm{Aut}(X/S) \leqslant n$ by the lemma. (Take $Y = X$ in the above diagram.)

Lemma 17.2 (Galois closures) *In the previous setup, there exists a connected $X' \to X$ finite étale with* $\#\mathrm{Aut}\,(X'/S) = [X' : S]$.

Proof. Let $n = [X : S]$ and fix a geometric point $\overline{x}\colon \mathrm{Spec}(k) \to S$. Let $X^n := X \times_S \cdots \times_S X = \coprod_{t \in T} Z_t$ with each Z_t connected. Choose t_0 such that $Z_{t_0,\overline{s}} \subset X^n_{\overline{x}}$ contains a point $(x_1, \dots, x_n)$ with $x_i \neq x_j$ for all $i \neq j$. Then the same holds for all $(x'_1, \dots, x'_n) \in Z_{t_0,\overline{s}}$: Suppose not, so $x'_i = x'_j$ for some $i \neq j$. Then there is a diagonal map $\Delta_{ij}\colon X^{n-1} \to X^n$ (repeat the same coordinate in i and j position) whose images contains $(x'_1, \dots, x'_n)$. Δ_{ij} is an étale closed immersion and hence must surject onto $Z_{t_0,\overline{s}}$ (the latter being connected). This contradicts $x_i \neq x_j$.

What is $\mathrm{Aut}\,(Z_{t_0}/S)$? Let $G_{t_0} \subset S_n$ be the stabilizer of Z_{t_0} under the action of the symmetric group S_n on X^n. Then $G_{t_0} = \{\sigma \in S_n \colon \sigma\,(x_1, \dots, x_n) \in Z_{t_0}\}$. Thus G_{t_0} acts transitively on $Z_{t_0,\overline{s}}$, since any element $(x'_1, \dots, x'_n)$ of $Z_{t_0,\overline{s}}$ satisfies $x'_i \neq x'_j$ for all $i \neq j$. Hence there is a $\sigma \in S_n$ such that $\sigma\,(x_1, \dots, x_n) = (x'_1, \dots, x'_n)$. It also acts freely, so $\#G_{t_0} = \#Z_{t_0,\overline{s}}$. Thus taking

$$\begin{array}{ccc} X' = Z_{t_0} & \longrightarrow & X \\ & \searrow & \downarrow \\ & & S, \end{array}$$

where for the horizontal map we choose any one of the projections, we have $\#\mathrm{Aut}\,(X'/S) = [X' : S]$.

Definition 17.3 *A finite étale map $X \to S$ between connected schemes is* <u>*Galois*</u>*, if* $\#\mathrm{Aut}\,(X/S) = [X : S]$*. Set* $\mathrm{Gal}\,(X/S) := \mathrm{Aut}\,(X/S)^{\mathrm{op}}$ *(for consistency with* $\mathrm{Gal}(L/K)$*).* $\mathrm{Aut}(X/S)$ *acts on the left on X, so* $\mathrm{Gal}(X/S)$ *acts on the right on X.*

<u>Another interpretation of the Galois condition:</u>

$X \times \mathrm{Gal}(X/S) \to X \times_S X, (x, g) \mapsto (x, xg)$ is an isomorphism (it is a finite étale degree one map. ($X \times \mathrm{Gal}(X/S)$ is just disjoint copies of X. Clearly this map is injective and local isomorphism; If X/S is galois it's surjective for degree reason, for both of the are degree (degree of $X/S)^2$ over S.)

<u>Construction of the étale fundamental group:</u>

Fix a connected scheme S and fix a geometric point $\overline{s}\colon \mathrm{Spec}(k) \to S$. Consider two connected pointed Galois covers

$$\begin{array}{ccc} (X, \overline{x}) & & \\ & \searrow & \\ & & (S, \overline{s}) \\ & \nearrow & \\ (Y, \overline{y}) & & \end{array}$$

such that

$$\begin{array}{ccc} \overline{x}\colon \mathrm{Spec}(k) & \longrightarrow & X \\ & \overset{\overline{s}}{\searrow} & \downarrow \\ & & S \end{array} \qquad \begin{array}{ccc} \overline{y}\colon \mathrm{Spec}(k) & \longrightarrow & Y \\ & \overset{\overline{s}}{\searrow} & \downarrow \\ & & S \end{array}$$

We know there is at most one $g\colon X \to Y$ such that $g(\overline{x}) = \overline{y}$. Suppose there is such a map g. Then we can construct a surjective group homomorphism $\mathrm{Aut}(X/S) \twoheadrightarrow \mathrm{Aut}(Y/S)$: Given $f \in \mathrm{Aut}(X/S)$, look at $g(f(\overline{x}))$. Since Y is Galois, there is exactly one $f_Y \in \mathrm{Aut}(Y/S)$ such that $f_Y(g(\overline{x})) = g(f(\overline{x}))$. In a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f_Y} & Y. \end{array}$$

By the rigidity lemma, $f_y \circ g = g \circ f$. One can check that this is a surjective homomorphism, and induces $\mathrm{Gal}(X/S) \twoheadrightarrow \mathrm{Gal}(Y/S)$, $f \mapsto f_Y$.

Definition 17.4 $\pi_1^{ét}(S, \overline{s}) := \varprojlim \mathrm{Aut}(X/S)^{\mathrm{op}}$, *where the limit is over all diagrams*

$$\begin{array}{ccc} & & X \\ & \overset{\overline{x}}{\nearrow} & \downarrow{\scriptstyle \text{fin. et. Gal.}} \\ \mathrm{Spec}(k) & \xrightarrow{\overline{s}} & S. \end{array}$$

where X is connected. The transition maps $(X, \overline{x}) \to (Y, \overline{y})$ are as we constructed above.

Example 17.5 $\overline{s} = \mathrm{Spec}(k^{\mathrm{sep}}) \to S = \mathrm{Spec}(k)$. *Then* $\pi_1^{ét}(S, \overline{s}) \simeq \mathrm{Gal}(k^{\mathrm{sep}}/k)$.

Lemma 17.6 (Functoriality lemma) $\pi_1^{ét}$ *is a functor on pointed schemes: Given $f\colon (T, \overline{t}) \to (S, \overline{s})$, there is a group homomorphism $f_*\colon \pi_1^{ét}(T, \overline{t}) \to \pi_1^{ét}(S, \overline{s})$ making $\pi_1^{ét}$ into a functor.*

Proof. Let $f\colon (X, \overline{x}) \to (S, \overline{s})$ be a connected finite étale Galois cover with geometric point $\overline{x}$ over $\overline{s}$. The pullback $X_T = X \times_S T$ is finite étale with degree $[X_T : T] = [X : S]$ and it comes with a basepoint $\overline{z} = \overline{x} \times_{\overline{s}} \overline{t}$, which hits a unique component $Z \subset X_T$ (the latter may not be connected). In fact, $Z \to T$ is a Galois cover. More precisely, $\mathrm{Gal}(X/S)$ acts simply transitively on $X_{\overline{s}} \cong (X_T)_{\overline{t}}$. For the subgroup $H < \mathrm{Gal}(X/S)$ given by $\mathrm{Stab}Z$, $\#H = [Z : T]$. We define $\pi_1(T, \overline{t}) \to \mathrm{Gal}(Z/T) = H < \mathrm{Gal}(X/S)$. Passing to the limit, we get a map $\pi_1(T, \overline{t}) \to \pi_1(S, \overline{s})$.

Theorem 17.7 (Fundamental theorem of Galois theory) *Let S be a connected scheme, $\overline{s}\colon \mathrm{Spec}(k) \to S$ a geometric point. There is an equivalence of categories*

$$\textit{FÉt}_S \to \left\{ \textit{finite left discrete } \pi_1^{ét}(S, \overline{s}) - \textit{sets} \right\}$$

given by

$$(X \to S) \mapsto X_{\overline{s}}.$$

Connected objects on the left correspond to transitive π_1-sets.

Remark 17.8 *(a) $\pi_1^{ét}$ is defined in SGA I as: define the "fiber functor" $F_{\overline{s}}\colon \textit{FÉt}_S \to$ Set, $(X \to S) \mapsto X_{\overline{s}}$. Then $\pi_1^{ét}(S, \overline{s}) = \mathrm{Aut}(F_{\overline{s}})$*

(b) This equivalence is functorial in $(S, \overline{s})$: Given a map $(T, \overline{t}) \to (S, \overline{s})$ and $(X \to S) \in \textit{FÉt}_S$, then

$$\begin{array}{ccc}(X \times_S T)_{\overline{t}} & \simeq & X_{\overline{s}} \\ \circlearrowleft & & \circlearrowleft \\ \pi_1^{\acute{e}t}(T, \overline{t}) & \xrightarrow{f_*} & \pi_1^{\acute{e}t}(S, \overline{s})\end{array}$$

are compatible.

(c) *The theorem suggests (and would allow you to prove) that for a second geometric point* $\overline{s}'$: $\mathrm{Spec}(k) \to S$, $\pi_1(S, \overline{s})$ *and* $\pi_1(S, \overline{s}')$ *are isomorphic. This is true, and the isomorphism is unique up to inner automorphism (compare to the topological* π_1*, or classical Galois theory with different choices of separable closures and a choice of isomorphism between them).*

Corollary 17.9 *Let* S *be a connected scheme,* $\overline{s}$ *a geometric point. Then there is an equivalence (functorial in* $(S, \overline{s})$*)*

$$\{\textit{lcc sheaves of sets on } S_{\acute{e}t}\} \to \{\textit{finite left discrete } \pi_1(S, \overline{s}) - \textit{sets}\}$$

given by

$$\mathscr{F} \mapsto \mathscr{F}_{\overline{s}}.$$

Proof. Combine the theorem with the earlier result that $\mathrm{FÉt}_S \simeq$ {lcc sheaves} by $X \mapsto h_X$.

Remark 17.10 *For any finite ring* Λ*, this extends to an equivalence*

$$\{\textit{lcc sheaves of } \Lambda - \textit{modules}\} \simeq \{\textit{discrete left } \Lambda[\pi_1] - \textit{modules}\}.$$

We'll see later a breakdown to non-lcc sheaves/ infinite Λ.

Proof of the fundamental theorem of Galois theory To construct the action on $X_{\overline{s}}$, we may assume X is connected. There is a scheme $X' \xrightarrow{g} X \xrightarrow{f} S$ with X'/S is Galois (X'/X is also Galois). Fix a base point $\overline{x}'$: $\mathrm{Spec}(k^s) \to X'$ over $\overline{s}$. Then $\mathrm{Hom}_S(X', X) \to X_{\overline{s}}$, $h \mapsto h \circ \overline{x}'$ is injective by the rigidity lemma and surjective because X'/S is Galois. $\mathrm{Aut}(X'/S)$ acts on X' on the left, so by precomposition acts on $\mathrm{Hom}_S(X', X)$ on the right by $h \cdot \sigma = h \circ \sigma$. This bijection gives $X_{\overline{s}}$ the structure of a right $\mathrm{Aut}(X'/S)$: for $\overline{y} \in X_{\overline{s}}$ write $\overline{y}$ as $g \circ \tau \circ \overline{x}'$ for some $\tau \in \mathrm{Aut}(X'/S)$ and then set $\overline{y} \cdot \sigma = g \circ \tau \circ \sigma(\overline{x}')$. Thus $X_{\overline{s}}$ is a left $\pi_1(S, \overline{s}) \twoheadrightarrow \mathrm{Aut}(X'/S)^{\mathrm{op}}$-set. The stabilizer of $g(\overline{x}')$ is $\mathrm{Aut}(X'/X)^{\mathrm{op}}$, identifying $\mathrm{Aut}(X'/X)^{\mathrm{op}} / \mathrm{Aut}(X'/S)^{\mathrm{op}} \simeq X_{\overline{s}}$ as left π_1-sets.

For Full faithfulness, use the rigidity lemma.

Essential surjectivity: any discrete left π_1-set A is a disjoint union of orbits so we may assume A is a transitive π_1-set with the action factoring through some $\mathrm{Aut}(X/S)^{\mathrm{op}}$ with $(X \to S)$ Galois. Fix $a \in A$ and let $H < \mathrm{Aut}(X/S)^{\mathrm{op}}$ be the stabilizer of a. Then we can form $X \to X/H \to S$, where one can check that $X/H \to S$ is still finite étale, and $(X/H)_{\overline{s}} \simeq A$.

In this case scenario, let $f : X \to S$ finite étale and $g \in H \subset \mathrm{Gal}(X/S)$, since g is over S and for affine $U = \mathrm{Spec}(A)$, $f^{-1}(U) = \mathrm{Spec}(B)$, g induces automorphism on $\mathrm{Spec}(B)$ hence on B. We consider $\mathrm{Spec}(B^H)$ (B_i^H means elements fixed by H) where $\mathrm{Spec}(B_i) = f^{-1}(\mathrm{Spec}(A_i))$ and $\{\mathrm{Spec}(A_i)\}$ covers S, and we are able to glue them to get X/H.

17.2 Lecture 24: 11/16

Last time we defined $\pi_1^{\text{ét}}(S, \bar{s})$ and showed the equivalence between categories FÉt_S and $\{\text{left discrete } \pi_1(S, \bar{s}) - \text{sets}\}$, this lecture we are going to see its relation to cohomology.

Let $S = \text{Spec}(k)$, we showed earlier that

$$H^*_{\text{ét}}(S, \mathscr{F}) \cong H^*(\text{Gal}(k^s/k) = \pi_1^{\text{ét}}(S, \bar{s}), M_{\mathscr{F}})$$

For general S this is however too much to hope for. For one, not all topological spaces satisfy this. Indeed, we have

$$H^*(\mathbb{CP}^1, \mathbb{Z}) \neq H^*(\pi_1^{\text{top}}(\mathbb{CP}^1), \mathbb{Z})$$

Furthermore, the étale subtlety is related to lcc condition. To investigate the relation between the fundamental group and cohomology, we begin with a definition.

Definition 17.11 *$\mathscr{F} \in \text{Sh}(S_{\text{ét}})$ is a left $\mathscr{G} \in \text{Ab}(S_{\text{ét}})$ torsor if it comes with an action*

$$\mathscr{G} \times \mathscr{F} \longrightarrow \mathscr{F}$$

meaning

1. *When $\mathscr{F}(U)$ is not empty, this action makes $\mathscr{F}(U)$ a principal homogeneous space under $\mathscr{G}(U)$, i.e. for fixed $\alpha \in \mathscr{F}(U)$, $\mathscr{G}(U) \longrightarrow \mathscr{F}(U)$ given by $g \mapsto g \cdot \alpha$ is a bijection.*

2. *There exists an étale cover $\{U_i \longrightarrow S\}$ such that $\mathscr{F}(U_i)$ is not empty. So by the previous property we have*
$$\mathscr{F}_{|U_i} \cong \mathscr{G}_{|U_i}$$
as $\mathscr{G}_{|U_i}$ torsors.

Proposition 17.12 *Let $(S, \bar{s})$ be as before, $\mathscr{G} \in \text{Ab}(S_{\text{ét}})$, we have the following two results.*

(1) $H^1_{\text{ét}}(S, \mathscr{G}) \cong \{\text{left } \mathscr{G} \text{ torsor on } S_{\text{ét}}\}$

(2) If $\mathscr{G}$ is lcc, then we have

$$H^1_{\text{ét}}(S, \mathscr{G}) \cong H^1(\pi_1^{\text{ét}}(S, \bar{s}), \mathscr{G}_{\bar{s}})$$

Proof. *(1) holds for any site if you identify $H^1_{\text{ét}}(S, \mathscr{G})$ with $\check{H}^1_{\text{ét}}(S, \mathscr{G})$ and regard Čech 1-cocycle on some cover $\{U_i \longrightarrow S\}$ as glueing data for the trivial $\mathscr{G}-$torsor on each U_i. For (2), let $\mathscr{F}$ be a $\mathscr{G}-$torsor, by definition there exists $\{U_i \longrightarrow S\}$ such that $\mathscr{F}_{|U_i} \cong \mathscr{G}_{|U_i}$, then $\mathscr{F}$ is also lcc. By classification theorem of lcc sheaves, we obtain that $\mathscr{F}_{\bar{s}}$ is a $\mathscr{G}_{\bar{s}}-$torsor in the category of $\pi_1^{\text{ét}}(S, \bar{s})$ sets. $\mathscr{G}_{\bar{s}} \times \mathscr{F}_{\bar{s}} \longrightarrow \mathscr{F}_{\bar{s}}$ is a $\pi_1^{\text{ét}}(S, \bar{s})$ equivariance in the sense that for $\sigma \in \pi_1^{\text{ét}}(S, \bar{s})$ we have*
$$\sigma(g \cdot \alpha) = \sigma(g) \cdot \sigma(\alpha)$$
This torsor gives a class in $H^1(\pi_1^{\text{ét}}(S, \bar{s}), \mathscr{G}_{\bar{s}})$ by (1).

Concretely we get a cocycle representing a class $H^1(\pi_1^{\text{ét}}(S, \bar{s}), \mathscr{G}_{\bar{s}})$ as follow: Choose $x \in \mathscr{F}_{\bar{s}}$, we define $\pi_1^{\text{ét}}(S, \bar{s}) \longrightarrow \mathscr{G}_{\bar{s}}$ by sending $\sigma \in \pi_1^{\text{ét}}(S, \bar{s})$ to the unique g such that $\sigma(x) = g_\sigma \cdot x$, then we have

$$g_{\sigma\tau} \cdot x = \sigma(\tau(x)) = \sigma(g_\tau) \cdot \sigma(x) = \sigma(g_\tau) g_\sigma \cdot x$$

hence $g_{\sigma\tau} = \sigma(g_\tau) g_\sigma$ therefore it is a cross homomorphism. And we can check that this is indeed a isomorphism that gives (2).

We will use these result for proper BC.

Remark 17.13 *In the previous proposition, (1) is a rather general fact as it holds for all sites, while (2) is more special to the lcc condition, and it is not true in general for non-lcc sheaves, as shown in the following example.*

Example 17.14 *Let S be the nodal cubic curve cut by $zy^2 = x^3 + zx^2$ in $\mathbb{P}^2$ has normalization $\mathbb{P}^1 \longrightarrow S$, we can show that $H^1_{\text{ét}}(S, \underline{\mathbb{Z}}) \cong \mathbb{Z}$ whereas $\pi_1^{\text{ét}}(S, \bar{s})$ is profinite so $H^1(\pi_1^{\text{ét}}(S, \bar{s}), \mathbb{Z}) \cong 0$ (note that $\underline{\mathbb{Z}}_{\bar{s}} = \mathbb{Z}$). Indeed, let $\tilde{S}$ be the a chain of $\mathbb{P}^1$ indexed by $\mathbb{Z}$ (each copy of $\mathbb{P}^1$ has a unique intersection with another copy if any only if they are indexed by i, j where $|i - j| = 1$), so we have a morphism $\tilde{S} \longrightarrow S$ which the normalization morphism when restricted on any copy of $\mathbb{P}^1$, and $\tilde{S}$ represents a non-trivial $\underline{\mathbb{Z}}-$torsor, hence $H^1_{\text{ét}}(S, \underline{\mathbb{Z}}) \cong \mathbb{Z}$ is non-trivial.*

Alternatively, we can also look at the Leray spectral sequence. Let $f : \mathbb{P}^1 \longrightarrow S$ we have ses

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow f_*\underline{\mathbb{Z}} \longrightarrow f_*\underline{\mathbb{Z}}/\underline{\mathbb{Z}} \longrightarrow 0$$

we obtained les

$$0 \longrightarrow H^0_{\text{ét}}(S, f_*\underline{\mathbb{Z}}/\underline{\mathbb{Z}}) \longrightarrow H^1_{\text{ét}}(S, \underline{\mathbb{Z}}) \longrightarrow H^1_{\text{ét}}(S, f_*\underline{\mathbb{Z}})$$

We first notice that $H^0_{\text{ét}}(S, f_*\underline{\mathbb{Z}}/\underline{\mathbb{Z}}) \cong \mathbb{Z}$ and $H^1_{\text{ét}}(S, f_*\underline{\mathbb{Z}}) \cong H^1_{\text{ét}}(\mathbb{P}^1, \underline{\mathbb{Z}})$, to show $H^1_{\text{ét}}(S, \underline{\mathbb{Z}}) \cong \mathbb{Z}$, it will be sufficient to show $H^1_{\text{ét}}(\mathbb{P}^1, \underline{\mathbb{Z}}) \cong 0$. This is true because when S (in this case $S = \mathbb{P}^1$) is normal, the equivalence of categories between $\{$lcc sheaves on $S_{\text{ét}}\}$ and $\{$finite discrete $\pi_1^{\text{ét}}(S, \bar{s}) -$ sets$\}$ can be extended to equivalence between

$$\{\textit{lcc sheaves of } \Lambda-\textit{module of finite type on } S_{\text{ét}}\}$$

and

$$\{\textit{finite type } \Lambda-\textit{module with } \pi_1^{\text{ét}}(S, \bar{s}) \textit{ continuously acting on it}\}$$

This gives us

$$H^1_{\text{ét}}(\mathbb{P}^1, \underline{\mathbb{Z}}) \cong H^1(\pi^1_{\text{ét}}(\mathbb{P}^1, \bar{x}), \mathbb{Z}) \cong 0$$

Proposition 17.15 *Let $f : X \longrightarrow Y$ be a morphism of connected schemes, $\bar{x} : \mathrm{Spec}(k) \longrightarrow X$ be a geometric point, and $\bar{y} = f(\bar{x})$, then $f_* : \pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(Y, \bar{y})$ is surjective if and only if for all connected finite étale cover $Y' \longrightarrow Y$, $X \times_Y Y'$ is connected.*

Proof. *Surjectivity is equivalent to that for all finite etale Galois $Y' \longrightarrow Y$, $\bar{y'}$ over $\bar{y}$, the composition*

$$\pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow \pi_1^{\text{ét}}(Y, \bar{y}) \longrightarrow \mathrm{Gal}(Y'/Y)$$

is surjective. Given such cover, $\bar{x} \times_{\bar{y}} \bar{y'}$ is a geometric point of $X \times_Y Y'$ lying in a unique connected component X' which is Galois over X. By construction of f_* (see functoriality lemma) we have the above composition is the following one

$$\pi_1^{\text{ét}}(X, \bar{x}) \longrightarrow \mathrm{Gal}(X'/X) \hookrightarrow \mathrm{Gal}(Y'/Y)$$

This composition is surjective if and only if $[X' : X] = [Y' : Y]$. We also have $[X' : X] \leqslant [X \times_Y Y' : X] = [Y' : Y]$, and the equality holds if and only if $X' = X \times_Y Y'$. Eventually, we have that it's surjective if and only if $X' = X \times_Y Y'$ i.e. $X \times_Y Y'$ is connected.

Corollary 17.16 *Let X be an irreducible normal scheme, η be the generic point, $K = \mathscr{O}_{X,\eta}$, K^s be its separable closure. $\bar{y} : \mathrm{Spec}(K^s) \longrightarrow X$, then the map $\pi_1^{\text{ét}}(\mathrm{Spec}(K), \bar{y}) \longrightarrow \pi_1^{\text{ét}}(X, \bar{y})$ factor through the isomorphism*

$$\mathrm{Gal}(L/K) \longrightarrow \pi_1^{\text{ét}}(X, \bar{y})$$

where

$$L := \bigcup_{\substack{K \subseteq M \subseteq K^s \\ M \text{ finite extension} \\ X_M \longrightarrow X \text{ is étale} \\ X_M \text{ is the normalization of } X \text{ in } M}} M$$

Example 17.17 *Let $X = \mathrm{Spec}(\mathbb{Z}[1/n])$, then*

$$\pi_1^{\text{ét}}(X) \cong \mathrm{Gal}(L/\mathbb{Q})$$

where L is the maximal extension unramified away from n. In particular, $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Z})) = \{1\}$ since there is no unramified extension over $\mathbb{Q}$.

18 Proof of proper BC

Theorem 18.1 *Given $f : X \longrightarrow S$ proper, $g : T \longrightarrow S$ as indicated,*

$$\begin{array}{ccc} X \times_S T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

then $g^{-1}R^n f_* \mathscr{F} \cong R^n f'_* g'^{-1} \mathscr{F}$ for all n for $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})_{tor}$.

Remark 18.2 *We also have "non-abelian" variants for $n = 0, 1$.*

1. *If $\mathscr{F} \in \mathrm{Sh}(X_{\text{ét}})$, then the BC map*

$$g^{-1} f_* \mathscr{F} \longrightarrow f'_* g'^{-1} \mathscr{F}$$

is an isomorphism.

2. *Let $\mathscr{F}$ be a sheaf of groups (not necessarily abelian). If $\mathscr{F}$ is ind-finite, meaning all the stalks are ind-finite (finite subsets generate finite subgroup), then the BC map*

$$g^{-1} R^1 f_* \mathscr{F} \longrightarrow R^1 f'_* g'^{-1} \mathscr{F}$$

is an isomorphism. Here $R^1 f_* \mathscr{F}$ is the sheaf associated to the presheaf of pointed set

$$U \mapsto H^1_{\text{ét}}(X \times U, \mathscr{F}_{|X \times U})$$

The key special case to prove the theorem is when S is the spectrum of a (strictly) henselian local ring, and T is the closed point of S, call it s, then the theorem becomes that the restriction map $H^*_{\text{ét}}(X, \mathscr{F}) \longrightarrow H^*_{\text{ét}}(X_s, \mathscr{F}_{|X_s})$ is isomorphism. For $* = 0$, we have the map on global sections $\mathscr{F}(X) \longrightarrow \mathscr{F}_{|X_s}(X_s)$ is an isomorphism. When $\mathscr{F} = \underline{\mathbb{Z}/n}$, $\mathscr{F}(X)$

is all the Zariski locally constant maps $X \longrightarrow \mathbb{Z}/n$. Likewise for X_s, the theorem in this case amounts to a bijection between closed and open sets in X and those in X_s given by $U \mapsto U \cap X_s$.

When each connected component of X_s is closed and open (for example when X_s is locally noetherian), the bijection above also gives a bijection between $\pi_0(X)$ and $\pi_0(X_s)$.

Similarly for $* = 1$, and $\mathscr{F} = \underline{G}$, G finite abelian, then the theorem says X connected implies X_s connected, and the restriction map

$$Hom(\pi_1^{\text{ét}}(X, \bar{x}), G) \longrightarrow Hom(\pi_1^{\text{ét}}(X_s, \bar{x}), G)$$

is a bijection of pointed sets. Varying G, this amounts to showing

$$\pi_1^{\text{ét}}(X_s, \bar{x}) \cong \pi_1^{\text{ét}}(X, \bar{x})$$

To prove proper BC, we will directly address $* = 0, 1$ and reduce the general case to the case where $X \longrightarrow S$ has fibre of dimension $\leqslant 1$. We will then use the curve theorems.

18.1 Lecture 25: 11/21

In this lecture and the next one we will prove the proper base change theorem 18.1.

Key Cases: S=Henselian local, T=closed point $s \in S$

$$\pi_0(X_s) \xrightarrow{\text{bijection}} \pi_0(X)$$

and

$$\pi_1(X_s) \xrightarrow{\text{iso}} \pi_1(X) \quad \text{when } X \text{ is connected.}$$

Part I: General Reductions

Proposition 18.3 *(a) It suffices to prove proper base change replacing S by $\overline{S} = \operatorname{Spec}(\mathscr{O}^{sh}_{S',s'})$ where $S' \to S$ is globally finite type and $s' \in S'$ that is closed in its fiber, and when $g : T \to \overline{S}$ is the map*

$$\operatorname{Spec}(k(\mathscr{O}^{sh}_{S',s'})) \hookrightarrow \operatorname{Spec}(\mathscr{O}^{sh}_{S',s'}).$$

(b) It then further suffices to prove proper base cahnge for $f : X \to S$ projective and S Noetherian. We can even take f to be $\mathbb{P}^n_S \to S$.

Firstly, we will provide a sketch proof of part (a). Since we only have to check the base change map is an isomorphism at all geometric points E of T, we may assume $S = \operatorname{Spec}(A)$ and $T = \operatorname{Spec}(B)$ are affine.

Write $B = \operatorname{colim} B_i$ of finite-type A-subalgebras $B_i \subset B$. The square diagram

$$\begin{array}{ccc} X_T & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S \end{array}$$

is a limit of square diagrams (on U_i^*)

$$\begin{array}{ccc} X_{\mathrm{Spec}(B_i)} & \xrightarrow{g_i} & X \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f} \\ \mathrm{Spec}(B_i) & \longrightarrow & \mathrm{Spec}(A) \end{array}$$

and the old theorem on commuting H^* with limits implies

$$\mathrm{colim} U_i^{-1} R^n f_{i*} g_i^{-1} \mathscr{F} \xrightarrow{\sim} R^n f_* g^{-1} \mathscr{F}$$

where $U_i : \mathrm{Spec}(B_i) \longrightarrow T$. So, we may further assume $T \to S$ is finite type. To show the BC map is an isomorphism, it suffices to prove it is an isomorphism on stalks at all (geometric points over) $t \in T$ that are closed in their fiber over S (exercise).

Now let $t \in T$ be closed in its fiber over $s \in S$. Let

$$\mathrm{Spec}(k(\bar{t})) \xrightarrow{\bar{t}} T$$

be a geometric point over T and let $g \circ \bar{t} = \bar{s}$ be the corresponding geometric poin

$$\mathrm{Spec}(k(\bar{s})) \to S$$

over S. Consider

$$\begin{array}{ccc} \mathrm{Spec}(k(\bar{t})) & = & \mathrm{Spec}(k(\bar{s})) \\ \downarrow{\scriptstyle \bar{t}} & & \downarrow{\scriptstyle \bar{s}} \\ T & \xrightarrow{g} & S \end{array}.$$

The $\bar{t}$-stalk of the base change map is

$$(g^{-1} R^n f_* \mathscr{F})_{\bar{t}} \longrightarrow (R^n f'_* g'^{-1} \mathscr{F})_{\bar{t}}.$$

Note that we have

$$(g^{-1} R^n f_* \mathscr{F})_{\bar{t}} \simeq (R^n f_* \mathscr{F})_{g \circ \bar{t} = \bar{s}} \simeq H^n(X_S \times \underbrace{\mathrm{Spec}(\mathscr{O}_{S,s}^{sh})}_{= \overline{S}}, \mathscr{F}|_{X \times_S \overline{S}})$$

and

$$(R^n f'_* g'^{-1} \mathscr{F})_{\bar{t}} \simeq H^n(X_T \times_T \overline{T}, \mathscr{F}|_{X_T \times_T \overline{T}}) \simeq H^n(X \times_S \overline{T}, \mathscr{F}|_{X \times_S \overline{T}}).$$

Then, the above map becomes

$$H^n(X \times_S \overline{S}, \mathscr{F}|_{X \times_S \overline{S}}) \xrightarrow{\text{pull back}} H^n(X \times_S \overline{T}, \mathscr{F}|_{X \times_S \overline{T}})$$

Then the commutative diagram above gives the following diagram (at this point we stop explicitly writing restriction/pullback of sheaf):

$$\begin{array}{ccc} H^n(X_{\bar{s}}, \mathscr{F}) & \xrightarrow{\simeq} & H^n(X_{\bar{t}}, \mathscr{F}) \\ \uparrow & & \uparrow \\ H^n(X \times_S \overline{S}, \mathscr{F}) & \longrightarrow & H^n(X \times_S \overline{T}, \mathscr{F}) \end{array}$$

So, it suffices to show the two vertical maps are isomorphisms. Part(a): Assume that $\kappa(t)^{\mathrm{sep}} =: k(\bar{t}) = k(\bar{s}) \supset \kappa(s)^{\mathrm{sep}}$ the right hand map is an isomorphism. To see this, consider this Cartesian diagram

$$\begin{array}{ccc} X_{\bar{t}} & \xhookrightarrow{\;g'\;} & X \times_S \overline{T} \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \bar{t} & \xhookrightarrow{\;\;g\;\;} & \overline{T} \end{array}$$

, and we assumed that PBC holds on this diagram, so we do the following calculation:

$$\begin{aligned} H^n(X \times_S \overline{T}, \mathscr{F}) &\simeq R^n f_*(\mathscr{F})(\overline{T}) \text{ because } \Gamma(\overline{T}, -) \text{ is exact from 12.9} \\ &= R^n f_*(\mathscr{F})_{\bar{t}} \text{ again, from 12.9} \\ &= g^{-1} R^n f_*(\mathscr{F}) \\ &\simeq R^n f'_*(g'^{-1}\mathscr{F}) \text{ because PBC holds for this diagram} \\ &\simeq H^n(X_{\bar{t}}, \mathscr{F}) \end{aligned}$$

(It in fact shows that in the case of such diagram, PBC holds is equivalent to $H^n(X \times_S \overline{T}, \mathscr{F}) \simeq H^n(X_{\bar{t}}, \mathscr{F})$.) The same argument shows

$$H^n(X \times_S \overline{S}, \mathscr{F}) \simeq H^n(X_{\kappa(s)^{\mathrm{sep}}}, \mathscr{F}).$$

Since t is closed in its (finite type) fiber over s, $\kappa(s) \subset \kappa(t)$ is finite, so $\kappa(s)^{\mathrm{sep}} \subset k(\bar{s}) = k(\bar{t}) = \kappa(t)^{\mathrm{sep}}$ is a finite purely inseparable extension. Thus,

$$H^n(X_{\bar{s}}) \simeq H^n(X_{\kappa(s)^{\mathrm{sep}}}, \mathscr{F})$$

by invariance of étale cohomology under purely inseparable field extensions. This completes the reduction of Part(a).

Part(b):(Sketch, mostly omitted) To reduce the proper case to the projective case, use Chow's lemma. Given that

$$\begin{array}{ccc} X & \xhookrightarrow{\;i\;} & \mathbb{P}^n_S \\ & \searrow^{f} & \big\downarrow{\scriptstyle \overline{f}} \\ & & S \end{array}$$

with f is projection, i is closed immersion, we show proper base change for $\overline{f}$ and we know proper base change for i (We have shown proper base change holds for all finite maps) together imply proper base change $f = \overline{f} \circ i$. Indeed ,using a spectral sequence argument (see Prop 4.4(ii) of SGA4 XII), consider the following diagram

$$\begin{array}{ccc} X & \xleftarrow{\;\;g''\;\;} & X_T \\ \big\downarrow{\scriptstyle f_2} & & \big\downarrow{\scriptstyle f'_2} \\ Y & \xleftarrow{\;\;g'\;\;} & Y_T \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f'_1} \\ S & \xleftarrow{\;\;g\;\;} & T \end{array}$$

more generally, if you know proper base change for f_1 and f_2 you can deduce it for $f_1 \circ f_2$ by

$$\begin{array}{ccc} g^{-1}R^p f_{1*}R^q f_{2*}\mathscr{F} & \Longrightarrow & g^{-1}R^{p+q}f_*\mathscr{F} \\ \downarrow{\scriptstyle \text{BC map}} & & \downarrow{\scriptstyle \text{BC map}} \\ R^p f_{1*}\circ R^q f_{2*}(g''^{-1}\mathscr{F}) & \Longrightarrow & R^{p+q}(f_1'\circ f_2')g''^{-1}\mathscr{F} \end{array}$$

gives a morphism of (Leray) spectral sequences. Base change isomorphism for $f = f_1 \circ f_2$ follows formally. Thus it suffices to establish proper base change for S strictly henselian and the projection

$$\mathbb{P}^n_S \longrightarrow S.$$

Writing

$$\mathscr{O}_S(S) = \text{colim(strict henselizations of finite type } \mathbb{Z}\text{-sub algebras)}$$

you can further assume S is Noetherian.

Reference for details: SGA4 chapter XII.

For the general theorem we will want to reduce from general $\mathscr{F} \in \text{Ab}(X_{\text{ét}})_{\text{tors}}$ to the case $\mathscr{F} = \mathbb{Z}/n$ where we prefer concrete calculations.

Fact: Let X be a Noetherian scheme

(1) Any $\mathscr{F} \in \text{Ab}(X_{\text{ét}})_{\text{tors}}$ is filtered colimit of constructible sheaves

(2) $\mathscr{F} \in \text{Ab}(X_{\text{ét}})$ is constructible if an only if it can be written as a subsheaf

$$\mathscr{F} \subset \bigoplus_{i=1}^{d} \text{P}\,(\mathbb{Z}/n_i)$$

for some integers $d, n_1, \ldots, n_d$, where $p_i : X_i \to X$ is a finite morphism.

For (2), see SGA4 IX 2.14
Part II: The case $n = 0$

Proposition 18.4 *Proper base change holds in degree* 0*.*

We have seen that we may assume $f : X \to S$ where S is Noetherian, strictly henselian and $T = s \hookrightarrow S$ is the closed points, and we must show

$$H^0(X, \mathscr{F}) \longrightarrow H^0(X_s, \mathscr{F}|_{X_s})$$

is an ismorphism for all $\mathscr{F} \in \text{Ab}(X_{\text{ét}})_{\text{tors}}$.

Claim: It suffices to prove this assertion (for all such $X \longrightarrow S$) when $\mathscr{F} = \mathbb{Z}/n$ for all n.

Proof of claim: Assume we know the $\mathbb{Z}/n$ case. Write general $\mathscr{F} = \text{colim}\mathscr{F}_i$ with $\mathscr{F}_i$ constructible. Since

$$H^*(-, \text{colim}) = \text{colim} H^*(-, -)$$

we may assume $\mathscr{F}$ is constructible. Our constructible $\mathscr{F}$ is a subsheaf

$$\mathscr{F} \subset \bigoplus_{i=1}^{d} p_{i*}\mathbb{Z}/n_i$$

for some finite maps

$$\begin{array}{ccc} p_i : X_i & \longrightarrow & X \\ & \searrow & \downarrow \\ & & S \end{array} .$$

Thus, we know by assumption proper base change for the $\mathbb{Z}/n_i$ on X_i, and consequently for

$$\mathscr{G} := \bigoplus_{i=1}^{d} p_{i*}\mathbb{Z}/n_i$$

on X (because the p_i are finite we have $H^\bullet(X, p_{i*}\mathbb{Z}/n_i) = H^\bullet(X_i, \mathbb{Z}/n_i)$). Proper base change in degree 0 follows formally for $\mathscr{F}$:

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{G}/\mathscr{F} \to 0$$

LES:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(X,\mathscr{F}) & \longrightarrow & H^0(X,\mathscr{G}) & \longrightarrow & H^0(X,\mathscr{G}/\mathscr{F}) \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ 0 & \longrightarrow & H^0(X_s,\mathscr{F}_s) & \longrightarrow & H^0(X_s,\mathscr{G}_s) & \longrightarrow & H^0(X,\mathscr{G}_s/\mathscr{F}_s) \end{array}$$

Since β is an isomorphism, the map α is injective, and base change map in degree 0 is injective for all constructible sheaves, including $\mathscr{G}/\mathscr{F}$. Thus γ is also injective, and a chase shows α is then surjective as well. This completes the proof of claim.

Part II of the proof: The case $n = 0$.

Proposition 18.5 *Proper base change in degree* 0*.*

We have seen we may assume everything is Noetherian where the $\mathscr{F} = \mathbb{Z}/n$ case will follow from

Claim: $\pi_0(X_s) \to \pi_0(X)$ is a bijection.
Setup

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \downarrow & & \downarrow f \text{ proper} \\ \operatorname{Spec}(\kappa(s)) & \xrightarrow{g} & \operatorname{Spec}(R) = S \end{array}$$

with R strictly henselian and local Noetherian. $H^0(X, \mathscr{O}_X) = H^0(S, f_*\mathscr{O}_X)$ is a finite $R-$algebra because f is proper. Since R is henselian, $H^0(X, \mathscr{O}_X)$ is a finite product of finite local $R-$ algebras.

For the bijection, STP if X is connected and non-empty, then X_s is connected and non-empty. If X is non-empty and f is proper then $f(X)$ is non-empty and closed, hence contains $s \in S$, hence X_s is non-empty. So we thus have to show X is connected imples X_s connected.

For X connected, $A = H^0(X, \mathscr{O}_X)$ is a local finite $R-$algebra (If it were a product, we would get orthogonal idempotents disconnecting X). Consider the float base change $R \to \hat{R}$:

$$\begin{array}{ccc} X_{\hat{R}} & \longrightarrow & X \\ \downarrow & & \downarrow {\scriptstyle f \text{ proper}} \\ \operatorname{Spec}(\hat{R}) & \xrightarrow{g} & \operatorname{Spec}(R) \end{array}$$

X and $X_{\hat{R}}$ have the same special fiber.

Now, $H^0(X_{\hat{R}}, \mathscr{O}_{\hat{R}}) = H^0(X, \mathscr{O}_X) \otimes_R \hat{R} = A \otimes_R \hat{R} = \hat{A}$. We may and do assume R is complete. Suppose X_s is not connected. Then, we obtain compatible orthogonal idempotents on $X_n = X \times_S \operatorname{Spec}(\mathscr{O}_{S,s}/\mathfrak{m}^{n+1})$ for all $n \geqslant 0$. So, we obtain orthogonal idempotents in $\varprojlim H^0(X_n, \mathscr{O}_{X_n})$.

Theorem of formal functions says: $f : X \to Y$ proper map of (locally) Noetherian schemes, $y \in Y$, $\mathscr{F}$ a coherent sheaf on X.

$$(R^q f_* \mathscr{F})^{\wedge}_y \xrightarrow{\cong} \varprojlim H^q(X \times_Y \operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1}), \mathscr{F}).$$

In our case,

$$(f_* \mathscr{O}_X)_s \xrightarrow{\simeq} \varprojlim H^0(X_n, \mathscr{O}_n)$$

where $(f_* \mathscr{O}_X)_s = H^0(X, \mathscr{O}_X)$ since we saw global functions on X are complete.

Thus, we obtain orthogonal idempotents in $H^0(X, \mathscr{O}_X)$ contradicting the assumption that X is connected.

18.2 Lecture 26: 11/28

We have already reduced the theorem to the case:

$$\begin{array}{ccc} \mathbb{P}^n_{\kappa(s)} & \longrightarrow & \mathbb{P}^n_S = X \\ \downarrow & & \downarrow {\scriptstyle f} \\ T = \operatorname{Spec} \kappa(s) & \longrightarrow & S = \operatorname{Spec} R \end{array}$$

, where R is Noetherian, strictly henselian, and s is a closed point. We continue our reduction.

Part III: Reduction to the case of relative curves. Suppose the theorem is know for any $f : X \to S$ proper with all fibers of dimension less than or equal to 1. Then the theorem holds.

Proof. From the assumption, we deduce that PBC(proper base change) holds for a product of projective lines over S , since we can express the product as a sequence of relative curves ($(\mathbb{P}^1_S)^n \to (\mathbb{P}^1_S)^{n-1} \to \cdots \to S$) and that PBC map being an isomorphism is stable under composition.

Next we observe that there is a finite map $p : (\mathbb{P}^1_S)^n \to \mathbb{P}^n_S$(quotient by symmetric group). Let $\mathscr{F} \in \operatorname{Ab}(S_{\text{ét}})_{tor}$, the canonical adjunction map $\mathscr{F} \to p_* p^{-1}(\mathscr{F})$ is injective by stalk considerations. We have PBC for the sheaf $p_* p^{-1}(\mathscr{F})$ since there is a commutative diagram:

$$\begin{array}{ccc} H^q(\mathbb{P}^n_S, p_*p^{-1}(\mathscr{F})) & \xrightarrow{\simeq} & H^q((\mathbb{P}^1_S)^n, p^{-1}(\mathscr{F})) \\ \downarrow & & \downarrow \simeq \\ H^q(\mathbb{P}^n_{\overline{s}}, p_*p^{-1}(\mathscr{F})) & \xrightarrow{\simeq} & H^q((\mathbb{P}^1_{\overline{s}})^n, p^{-1}(\mathscr{F})) \end{array}$$

, so the left vertical arrow is also an isomorphism.

To deduce the PBC for $\mathscr{F}$, we use that the short exact sequence

$$0 \to \mathscr{F} \to p_*p^{-1}\mathscr{F} \to \mathscr{G} := p_*p^{-1}\mathscr{F}/\mathscr{F} \to 0$$

gives a long exact sequence:

$$\begin{array}{ccccccccc} H^{q-1}(\mathbb{P}^n_S, p_*p^{-1}(\mathscr{F})) & \longrightarrow & H^{q-1}(\mathbb{P}^n_S, \mathscr{G}) & \longrightarrow & H^q(\mathbb{P}^n_S, \mathscr{F}) & \longrightarrow & H^q(\mathbb{P}^n_S, p_*p^{-1}(\mathscr{F})) & \longrightarrow & H^q(\mathbb{P}^n_S, \mathscr{G} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{q-1}(\mathbb{P}^n_{\overline{s}}, p_*p^{-1}(\mathscr{F})_{\overline{s}})) & \longrightarrow & H^{q-1}(\mathbb{P}^n_{\overline{s}}, \mathscr{G}_{\overline{s}})) & \longrightarrow & H^q(\mathbb{P}^n_{\overline{s}}, \mathscr{F}_{\overline{s}})) & \longrightarrow & H^q(\mathbb{P}^n_{\overline{s}}, p_*p^{-1}(\mathscr{F}_{\overline{s}})) & \longrightarrow & H^q(\mathbb{P}^n_{\overline{s}}, \mathscr{G} \end{array}$$

We induct on q for both injectivity and surjectivity of the middle vertical arrow.

First for injectivity, by some version of snake lemma, the middle vertical arrow is injective.

Next the surjectivity is established by another version of snake lemma, since the injectivity part also applies to the sheaf $\mathscr{G}$.

<u>Part IV:</u> Application of the curve thm.

Proposition 18.6 *Assume that we can show, for all Noetherian strictly henselian S and projective $f : X \to S$ with fiber dimension $\leqslant 1$, that*

1. $H^1(X, \mathbb{Z}/n) \to H^1(X_s, \mathbb{Z}/n)$ *is surjective;*
2. $Pic(X) \to Pic(X_s)$ *is surjective.*

Then the full PBC holds.

Proof. Similar as the reduction we did in Part I, we can assume that $T \to S$ is $\mathrm{Spec}\kappa(\overline{s}) \to S = \mathrm{Spec}R$, where R is Noetherian and strictly henselian. And we want to show that the map $H^q(X, \mathscr{F}) \to H^q(X_{\overline{s}}, \mathscr{F}_{\overline{s}})$ is an isomorphism for any $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})_{\text{tor}}$.

As before, we can assume $\mathscr{F}$ is a constructible sheaf of $\mathbb{Z}/\ell^a$-modules for l prime and $a \geqslant 1$. Let $n = l^a$. We saw any such $\mathscr{F}$ is a subsheaf of $\bigoplus_{i=1}^r p_{i*}(\mathbb{Z}/n)$ for some $p_i : X_i \to X$ finite. And we have a commutative diagram:

$$\begin{array}{ccc} H^q(X, \bigoplus_{i=1}^r p_{i*}(\mathbb{Z}/n)) & \xrightarrow{\simeq} & \oplus H^q(X_i, \mathbb{Z}/n) \\ \downarrow & & \downarrow \\ H^q(X_{\overline{s}}, \bigoplus_{i=1}^r p_{i*}(\mathbb{Z}/n)) & \xrightarrow{\simeq} & \oplus H^q(X_{i,\overline{s}}, \mathbb{Z}/n) \end{array}$$

, where $X_i \to S$ is still a relative curve, and so the map on H^1 of X_i is surjective.

When $q = 2$, we have two cases:

The first case is when $l = \mathrm{char}(\kappa(s))$. Then the vertical arrows are surjective by cohomological dimension considerations and Artin-schreier type argument.

The second case is where $l \neq \mathrm{char}(\kappa(s))$, then by kummer type argument, we have a commutative diagram

$$\begin{array}{ccc} H^2(X_i, \mu_n) & \longrightarrow & H^2(X_i, \mathbb{Z}/n) \\ \downarrow & & \downarrow \\ Pic(X_i) & \longrightarrow & Pic(X_{i,\bar{s}}) \end{array}$$

(The verical arrows comes from the LES induced by Kummer sequence). Then since $H^2(X_{i,\bar{s}}, \mathbb{G}_m) = 0$($X_{i,\bar{s}}$ is a curve over separably closed field), the surjectivity is established.

Now, for $q \geqslant 3$, $H^q(X_i, \mathbb{Z}/n) = 0$, so we deduce that $\mathscr{F}$ is a subsheaf of a sheaf $\mathscr{G}$ of $\mathbb{Z}/n$-mods such that for any $q > 0$, $H^q(X, \mathscr{G}) \to H^q(X_{\bar{s}}, \mathscr{G}_{\bar{s}})$ is surjective and is an isomorphism for $q = 0$.(Proof to be completed by the proof of the following claim).

Claim 18.7 *This conclusion implies that $H^q(X, \mathscr{F}) \to H^q(X_{\bar{s}}, \mathscr{F}_{\bar{s}})$ is an isomorphism for $q \geqslant 0$.*

Proof. Let $\mathscr{F} \subset \mathscr{G}$ as above. The abelian category of sheaves of $\mathbb{Z}/n$-mods on $X_{étale}$ has enough injectives, so let $\mathscr{G} \subset \mathscr{J}$ be an inclusion of $\mathscr{G}$ into an injective sheaf of $\mathbb{Z}/n$-mods.

Then $\mathscr{J}$ is acyclic for the global section functor, i.e., $H^q(X, \mathscr{J}) = 0$ for all $q > 0$. This can be seen by checking the Godement resolution and realizing $\mathscr{J}$ as a direct summand of its Godement resolution. Taking LES induced by

$$0 \to \mathscr{G} \to \mathscr{J} \to \mathscr{J}/\mathscr{G} \to 0$$

, and use a dimension shifting argument as before, we can conclude that $H^q(X, \mathscr{G}) \to H^q(X_{\bar{s}}, \mathscr{G}_{\bar{s}})$ is injective for $q > 0$.

Now, we can again do a similar trick, but this time with the SES

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{G}/\mathscr{F} \to 0$$

and its associated long exact sequence, and then conclude that $H^q(X, \mathscr{F}) \to H^q(X_{\bar{s}}, \mathscr{F}_{\bar{s}})$ is injective for $q > 0$.

Part V: $q = 1$. We prove something stronger.

Theorem 18.8 *Let $S = \mathrm{Spec}(R)$, where R is only assumed to be henselian (local), $s \in S$ the closed point, $f : X \to S$ proper. Then pullback induces an equivalence $F\acute{\mathrm{E}}\mathrm{t}(X) \to F\acute{\mathrm{E}}\mathrm{t}(X_s)$, and this proves that $\pi_1(X, *) \cong \pi_1(X_s, *)$ under the assumption that X_s is connected.*

Proof. 3 main ingredients.

1. Let $X_n = X \times_S \mathrm{Spec}(\mathscr{O}_{S,s}/\mathfrak{m}_S^{n+1})$ (we will conflate X_0 with X_s). Then for any $n \geqslant 0$, $F\acute{\mathrm{E}}\mathrm{t}(X_{n+1}) \to F\acute{\mathrm{E}}\mathrm{t}(X_n)$ is an equivalence. (We stated that for any nilpotent thickening which is an universal homeomorphism $Z \to Z'$, $\acute{\mathrm{E}}\mathrm{t}(Z') \to \acute{\mathrm{E}}\mathrm{t}(Z)$ is an equivalence, and we proved the equivalence of sheaves. Then restricting to lcc sheaves we get the equivalence $F\acute{\mathrm{E}}\mathrm{t}(Z') \to F\acute{\mathrm{E}}\mathrm{t}(Z)$.)

2. First we reduce to the case where $S = \mathrm{Spec}(R)$ with R Noetherian and henselian(since we can write R as a colimit of finite type $\mathbb{Z}$-subalgebra of R, replace the R_i by their henselization at m_{R_i})

   Then we can check that

   $$F\acute{\mathrm{E}}\mathrm{t}(X) \to \lim F\acute{\mathrm{E}}\mathrm{t}(X_{R_i})$$

   is an equivalence (direct limit formalism).

   Then given a finite étale $X' \to X$, we get a compactible collection of finite étale $X'_n \to X_n (\forall n \geqslant 0)$.

   Then we replace $X \to \mathrm{Spec}(R)$ by $X_{\hat{R}} \to \mathrm{Spec}(\hat{R})$, assume R is complete. We'll finish the proof of the theorem assuming R is complete. (Later on we prove that this case in fact implies the general (henselian but not necessarily complete) case in ingredient 3.) We then consider the formal scheme $\hat{X} := \mathrm{colim}\, X_n$ which is a locally ringed space with underlying space $X_0 = X_s$ and structure sheaf $\lim \mathscr{O}_{X_n}$.

   Use the Grothendieck existence theorem in formal geometry or formal GAGA.

Theorem 18.9 *Let R be a Noetherian ring complete with respect to an ideal I. Let $X \to \mathrm{Spec}(R)$ be proper. The the functor between categories of coherent sheaves*

$$\mathrm{Coh}(X) \to \mathrm{Coh}(\hat{X})$$
$$\mathscr{F} \to \hat{\mathscr{F}} := \lim i_n^*(\mathscr{F})$$

is an equivalence.

We first prove the essential surjectiveness of the functor in the theorem.

For each $X'_0 \to X_0$ finite étale, we obtain $X'_n \to X_n$ for all n because $F\acute{\mathrm{E}}\mathrm{t}(X_n) \simeq F\acute{\mathrm{E}}\mathrm{t}(X_0)$. From each of these we have coherent $\mathscr{O}_{X_n}-$algebra $\mathscr{F}_n$. They form a compatible system on $\hat{X}$,denoted by $\hat{\mathscr{A}}$. By formal GAGA this corresponds to a coherent $\mathscr{O}_X-$module, denoted by $\mathscr{A} \in coh(X)$. Each $\mathscr{F}_n$ is an $\mathscr{O}_{X_n}-$algebra, therefore we can make $\hat{\mathscr{A}}$ an $\mathscr{O}_{\hat{X}}$ algebra. The morphisms $\hat{\mathscr{A}} \otimes \hat{\mathscr{A}} \to \hat{\mathscr{A}}$ and $\mathscr{O}_{\hat{X}} \to \hat{\mathscr{A}}$ correspond to $\mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ and $\mathscr{O}_X \to \mathscr{A}$, hence we upgrade $\mathscr{A}$ to a coherent $\mathscr{O}_X-$algebra, which gives a finite morphism $X' \to X$. Clearly this morphism base change to $X'_0 \to X_0$, so we are left to prove that it is étale. Since $X' \to S$ is proper hence closed, any closed point has to be sent to the only closed point $s \in S$, so every point in X' specializes to some point in $X'_0 = X'_s$. Moreover we know the étale locus in X' over X is open. If we can prove the étale locus contains all of points in X'_0 (because $X'_0 \longrightarrow X_0$ is étale) then it has to be the entire X'. First it's unramified at every point in X'_0 because $\Omega_{X'/X}$ vanishes at X'_0. To show it's flat at X'_0, we use the following

Lemma 18.10 *Let A be a ring, B an $A-$algebra, $\mathfrak{a}$ an ideal of A such that $\mathfrak{a}B$ is in the jacobson ideal of B, M a $B-$module. Then M is a flat $A-$module if and only if $M/\mathfrak{a}^n$ is a flat $A/\mathfrak{a}^n-$module.*

For a point $p \in X'_0$ with image $q \in X_0$, We take A to be $\mathscr{O}_{X,q}$, B to be $\mathscr{O}_{X',p}$ and $\mathfrak{a}$ to be m_s. Because $X'_n \to X_n$ is étale, $B/\mathfrak{a}^n$ is flat $A/\mathfrak{a}^n-$module, and we apply the lemma.

18.3 Lecture 27: 11/30

Next step: for R complete, we show the full faithfulness. First we show faithfulness. Let $X' \to X$ and $X'' \to X$ be finite étale and

$$h_1, h_2 : X' \to X''$$

such that the following diagram is commutative:

$$\begin{array}{ccc} X' & \overset{h_1}{\underset{h_2}{\rightrightarrows}} & X'' \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

We want to show that if $h_1|_{X'_s} = h_2|_{X'_s}$ holds, then $h_1 = h_2$. We consider the following diagram

$$\begin{array}{ccc} \mathrm{eq}(h_1, h_2) & \longrightarrow & X' \\ \downarrow & & \downarrow \Gamma_{h_2} \\ X' & \xrightarrow{\Gamma_{h_1}} & X' \times_X X'' \end{array}$$

We observe that $\mathrm{eq}(h_1, h_2) \to X'$ is finite étale, so it is an open and closed immersion by the properties of equalizers. Thus the locus where $h_1 = h_2$ is open. If the locus contains X'_s, since $X' \to S$ is proper, the locus must be equal to X' (the same argument as earlier). Now we show the fullness. Let $X' \to X$ and $X'' \to X$ be given as before. Let $\overline{h} : X'_s \to X''_s$ be a given morphism such that the diagram

$$\begin{array}{ccc} X'_s & \xrightarrow{\overline{h}} & X''_s \\ & \searrow \quad \swarrow & \\ & X_s & \end{array}$$

is commutative. We view $\overline{h}$ as the embedding $X'_s \overset{\Gamma_{\overline{h}}}{\hookrightarrow} X'_s \underset{X_s}{\times} X''_s$. Applying the essential surjectiveness to the base $X' \underset{X}{\times} X''$, we deduce that there exists a finite étale morphism $W \to X' \underset{X}{\times} X''$, pulling back $\Gamma_{\overline{h}}$ to a special fiber. Since $W \to X' \underset{X}{\times} X'' \xrightarrow{\mathrm{pr}_1} X'$ is finite étale and pulls back to an isomorphism on the special fiber, $W \to X'$, denoted by φ, is in fact an isomorphism: the locus where a finite étale map is an isomorphism is open on the base and contains the special fiber over s (see Stack Project 04DH), so it has to be the entire X'. Then

$$\begin{array}{ccccc} X' & \xrightarrow{\varphi^{-1}} & W & \longrightarrow & X'' \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

lifts $\overline{h}$.

Step ③ (the third ingredient mentioned in the last lecture): reduction from complete to henselian case. Again we assume R is Noetherian and henselian. Let $\widehat{R}$ be the completion of R, then $\widehat{R}$ is also Noetherian. Let $F : X \to \mathrm{Spec}(R)$ be a proper morphism. First, we use a limiting argument to reduce to a setting where Artin approximation applies. Write $R = \operatorname{colim}_{i \in I} R_i$, where R_i's are finite-type $\mathbb{Z}$-algebras. Then $R = \operatorname{colim}_{i \in I} R^h_{i,(m_i)}$, where $m_i = R \cap m_R$. In the following, we denote by R_i the ring $R^h_{i,(m_i)}$. Since R is Noetherian, $X \to \operatorname{Spec}(R)$ is finitely presented, so a limiting argument shows that there exists $i_1 \in I$ and a proper morphism $X(i_1) \to \operatorname{Spec}(R_{i_1})$ such that for all the diagram

$$\begin{array}{ccc} X(i_1) & \longrightarrow & \operatorname{Spec}(R_{i_1}) \\ \uparrow & & \uparrow \\ X & \longrightarrow & \operatorname{Spec}(R) \end{array}$$

is Cartesian.
Let $X'_0 \to X_0 = \lim_{i \geqslant i_1} X(i)_0$ be a finite étale map where for every $i \geqslant i_1$ we write $X(i) = X(i_1) \underset{\operatorname{Spec}(R_{1_1})}{\times} \operatorname{Spec}(R_i)$. Similarly, there exists $i_2 \geqslant i_1$ such that $X'_0 \to X_0$ arises by base change from a finite étale map $X'(i_2)_0 \to X(i_2)_0$ over $\operatorname{Spec}(R_{i_2})$. Hence it suffices to prove the essential surjectivity when R is the henselization at a prime ideal of a finite-type $\mathbb{Z}$-algebra. (We use the same argument used earlier for full-faithfulness.)
The core of the proof uses Artin aprroximation theorm.

Theorem 18.11 (Artin approximation) *Let $R = R^h_{0,\mathfrak{p}}$, where R_0 is finite type over $\mathbb{Z}$ and $\mathfrak{p}$ a prime ideal. Given a system $\{f_i(Y_1, Y_2, \ldots, Y_n) = 0\}$, where f_i's are elements in the polynomial ring $R[Y_1, Y_2, \ldots, Y_n]$, a solution $(\widehat{y}_1, \widehat{y}_2, \ldots, \widehat{y}_n) \in \widehat{R}$, and $c \geqslant 1$, there exists a solution $(y_1, y_2, \ldots, y_n) \in R^n$ such that for any i, $y_i \equiv \widehat{y}_i \bmod m_{\widehat{R}}$.*

Now, for a given finite étale cover $X'_s \to X_s$, from ② we know there exists a finite étale map $\widetilde{X} \to X_{\widehat{R}}$, pulling back to $X'_s \to X_s$. Write $\widehat{R}$ as the colimit of finite-type R-subalgebras. Another limiting argument shows that there exists a finite-type R-subalgebra $A \subset \widehat{R}$ and a finite étale map $\widehat{X}_A \to X_A$ such that the diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{X}_A \\ \downarrow & & \downarrow \\ \operatorname{Spec}(\widehat{R}) & \longrightarrow & \operatorname{Spec}(A) \end{array}$$

is Cartesian. Concretely, $A \cong R[Y_1, \ldots, Y_n]/(f_1, \ldots, f_g)$ and $A \subset \widehat{R}$ corresponds to a solution $(\widehat{y}_1, \ldots, \widehat{y}_n) \in \widehat{R}^n$ to $f_i = 0$ for $1 \leqslant i \leqslant g$. By Artin approximation theorem, there exists an R-algebra homomorphism $\psi : A \to R$ such that the diagram

$$\begin{array}{ccccc} A & \hookrightarrow & \widehat{R} & \longrightarrow & \widehat{R}/m_{\widehat{R}} \\ {\scriptstyle \mathrm{id}}\uparrow & & & & \uparrow{\scriptstyle \cong} \\ A & \xrightarrow{\psi} & R & \longrightarrow & R/m_R \end{array}$$

commutes. Now form $\widetilde{X}_A \underset{\operatorname{Spec}(A),\psi}{\times} \operatorname{Spec}(R)$, which agrees on special fiber with $\widetilde{X} \underset{\operatorname{Spec}(A)}{\times} \operatorname{Spec}(R/m_R) = X'_s$ by transitivity of pullback and commutativity of the diagram. This

establishes the essential surjecvitity of the map $\text{FÉT}(X) \to \text{FÉT}(X_s)$, and the full faithfulness is as in step ②. This completes the proof of π_1 theorem.
Part VI. End of the proof. By discussion last time, the proof of proper base change is reduced to the following proposition.

Proposition 18.12 *Let S be Noetherian and strictly henselian and s a closed point of S. Let $f : X \to S$ be projective with all fibers of dimension no greater than 1. Then $\mathrm{Pic}(X) \to \mathrm{Pic}(X_s)$ is surjective.*

Proof. The proof is similar to degree 1 case. Let $\mathscr{L}_0 \in \mathrm{Pic}(X_s)$

(a) Extend $\mathscr{L}_0$ to each $X_n = X \underset{S}{\times} \mathrm{Spec}(\mathscr{O}_{S,s}/m_S^{n+1})$, and we want to show for any n, $\mathrm{Pic}(X_{n+1}) \to \mathrm{Pic}(X_n)$ is surjective. We identify this map with $H^1(X_0, \mathscr{O}_{X_{n+1}}^\times) \to H^1(X_0, \mathscr{O}_{X_n}^\times)$. Since X_n's all have same underlying topologcial space, we can work on Zariski site. We have a short exact sequence of sheaves:

$$0 \longrightarrow \mathscr{I} := \ker(\mathscr{O}_{X_{n+1}} \to \mathscr{O}_{X_n}) \longrightarrow \mathscr{O}_{X_{n+1}}^\times \longrightarrow \mathscr{O}_{X_n}^\times \longrightarrow 0$$

$$\alpha \longmapsto 1 + \alpha$$

Then $\mathscr{I}$ is coherent, and this together with $\dim(X_0) \leqslant 1$ implies that $H^2(X_0, \mathscr{I}) = 0$. The surjectivity then follows.

(b) Since $\mathscr{L}_0$ extends to compatible collections $(\mathscr{L}_n)_{n \geqslant 0}$, where $\mathscr{L}_n \in \mathrm{Pic}(X_n)$ by Grothendieck's existence theorem, there exists a coherent sheaf $\widetilde{\mathscr{L}}$ on $X_{\hat{R}}$, where R is the ring such that $S = \mathrm{Spec}(R)$. We need to check that $\widetilde{L} \in \mathrm{Pic}(X_{\hat{R}})$.

(c) Another Artin approximation argument shows that there exists $\mathscr{L} \in \mathrm{Pic}(X)$ such that $\mathscr{L}|_{X_0} \cong \widetilde{\mathscr{L}}|_{X_0} \cong \mathscr{L}_0$. It remains to check that $\mathscr{L}$ descends to X_A for some finite-type R-subalgebra $A \subset \hat{R}$. The details are left as an exercise.

Now we are done with proper base change theorem. We've already discussed variants and corollaries of proper base change theorem. In the following, we reemphasize a variant of proper base change theorem.

Theorem 18.13 *Let $f; X \to S$ be proper of finite presentation. Let $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$ be constructible (and for us S is Noetherian). Then for any q, $R^q f_* \mathscr{F} \in \mathrm{Ab}(S_{et})$ is constructible.*

Example 18.14 *Let X/k be proper, where k is separably closed. Then $|H^q(X, \mathbb{Z}/n)| < \infty$ for any n.*

19 Statement and discussion of smooth base change theorem

Motivation: recall Ehresmann's theorem.

Theorem 19.1 *If $f : X \to S$ is a proper submersion of smooth manifolds, then locally on S, f is trivial: for any $t \in S$, there exists a neighborhood N of t such that $f^{-1}(N) \simeq N \times X_t$,and we have the following commutative diagram*

$$\begin{array}{ccc} f^{-1}(N) & \xrightarrow{\simeq} & N \times X_t \\ \downarrow & \swarrow_{\text{pr}} & \\ N & & \end{array}.$$

Therefore, for any $p \in X$ and any constant sheaf Λ on X, $R^p f_* \Lambda$ is the sheaf that is associated to $U \to H^p(f^{-1}(U), \Lambda)$. For U small enough and contractible, $H^p(f^{-1}(U), \Lambda) \simeq H^p(U \times X_t, \Lambda) \simeq H^p(X_t, \Lambda)$, hence it's locally constant. This induces an action of $\pi_1(S, t)$ on $H^p(X_t, \Lambda)$.
Next we state an analogue of Ehresmann's theorem in étale cohomology.

Theorem 19.2 (Smooth and proper base change) *Let $f : X \to S$ be a smooth proper morphism of schemes, and let $\mathscr{F} \in \mathrm{Ab}(X_{\text{ét}})$ such that*

1. *$\mathscr{F}$ is locally constant constructible;*
2. *all torsion in $\mathscr{F}$ is invertible on S.*

Then for any p, $R^p f_* \mathscr{F}$ is locally constant constructible on $S_{\text{ét}}$.

Example 19.3 *Let $X_{\mathbb{Q}} \to \mathrm{Spec}(\mathbb{Q})$ be a smooth projective variety. For some $N \geqslant 1$, $X_{\mathbb{Q}}$ spreads out the following :*

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Spec}(\mathbb{Z}\left[\frac{1}{N}\right]) \\ \uparrow & & \uparrow \\ X_{\mathbb{Q}} & \longrightarrow & \mathrm{Spec}(\mathbb{Q}) \end{array}.$$

Fix a prime ℓ. For sheaves $\mathscr{F} := \mathbb{Z}/\ell^n$, we can apply the theorem to $f : X\left[\frac{1}{l}\right] \to \mathbb{Z}\left[\frac{1}{Nl}\right]$, so $R^p f_* \mathscr{F}$ is locally constant constructible on $\mathrm{Spec}(\mathbb{Z}\left[\frac{1}{Nl}\right])$. Picking a geometric point $\overline{s} : \mathrm{Spec}(\overline{\mathbb{Q}}) \to \mathrm{Spec}(\mathbb{Z}\left[\frac{1}{Nl}\right])$, it corresponds to a $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Z}\left[\frac{1}{Nl}\right]), \overline{s})$ action on $R^p f_* \mathscr{F}_{\overline{s}}$. $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Z}\left[\frac{1}{Nl}\right]), \overline{s}) \simeq G_{\mathbb{Q},Nl}$ and $R^p f_* \mathscr{F}_{\overline{s}} \simeq H^p(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^n)$, give $G_{\mathbb{Q},Nl}$ action on $H^p(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^n)$. The lcc condition tells us that stalks at different geometric points are non-canonically isomorphic (more on this later). For p does not divide Nl, condiering the geometric point $\overline{s}' : \mathrm{Spec}(\overline{\mathbb{F}_p}) \to \mathrm{Spec}(\mathbb{Z}\left[\frac{1}{Nl}\right])$ this gives

$$H^p(X_{\overline{\mathbb{F}_p}}, \mathbb{Z}/\ell^n) \simeq H^p(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/l^n).$$

In fact, these two are isomorphic as $G_{\mathbb{Q},Nl}$-module if we also identify

$$G_{\mathbb{Q},Nl} \simeq \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Z}\left[\frac{1}{Nl}\right]), \overline{s}') \simeq \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Z}\left[\frac{1}{Nl}\right]), \overline{s}).$$

Remark 19.4 *For any finite type $X/\mathbb{Q}$, it still holds that the action of $G_{\mathbb{Q}}$ on $H^*(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^n)$ (each automorphism of $\mathbb{Q}$ gives an autormophism of $X_{\overline{\mathbb{Q}}}$, giving action on Cohomology) is almost everywhere unramified.*

The same holds for $H^p(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell) := \varprojlim_n H^p(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/\ell^n)$. *The representations* $G_{\mathbb{Q},Nl}$ *on* $H^p(X_{\overline{\mathbb{Q}}}, \mathbb{Z}_\ell) \bigotimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$, *which are equal to* $H^p(X_{\overline{\mathbb{Q}_\ell}}, \overline{\mathbb{Q}_\ell})$, *satisfy another deep general property: the restriction to* $\mathrm{Gal}(\overline{\mathbb{Q}_\ell}/\mathbb{Q}_\ell)$ *(via* $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$*) is de Rham (or potentially semistable) in sense of Fontaine.*

Conjecture 19.5 (Fontaine-Mazur) *Let* $\rho : G_{\mathbb{Q}} \to GL_n(\overline{\mathbb{Q}}_\ell)$ *be a continuous map. Assume the following conditions hold:*

1. ρ *is irreducible;*

2. ρ *is almost everywhere unramified;*

3. $\rho|_{G_{\mathbb{Q}_\ell}}$ *is de Rham.*

Then there exists smooth projective $X/\mathbb{Q}$ *and integers* r *and* s *such that* ρ *is isomorphic to a subquotient of* $H^r(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)(s)$.

19.1 Lecture 28: 12/5

Recall from lecture the theorem on smooth proper base change:

Theorem 19.6 *Let* $f\colon X \to S$ *be a smooth and proper morphisms and* $\mathscr{F}$ *an l.c.c. sheaf on* $X_{\acute{e}t}$ *with all torsion orders invertible on* X *(and hence* S*). Then* $R^p f_* \mathscr{F}$ *remains l.c.c.*

The l.c.c. condition implies that the stalks of $R^p f_* \mathscr{F}$ at different geometric points $\bar{s} \in S$ are isomorphic. More precisely, we may state the following:

Definition 19.7 *Let* S *be a scheme,* $s, \eta \in X$ *with* $s \in \overline{\{\eta\}}$. *Take* $\bar{s}$ *and* $\bar{\eta}$ *to be geometric points over* s *and* η *respectively. We define a* specialization *of* $\bar{\eta}$ *to* $\bar{s}$ *to be a choice*

$$\iota\colon\ \mathrm{Spec}(\kappa(\bar{\eta})) \to \mathrm{Spec}(\mathscr{O}_{S,s}^{\mathrm{sh}})$$

making the diagram commute:

$$\begin{array}{ccccc} \mathrm{Spec}(\kappa(\bar{\eta})) & \xrightarrow{\iota} & \mathrm{Spec}(\mathscr{O}_{S,s}^{\mathrm{sh}}) & \xleftarrow{\mathrm{sh w.r.t.}\kappa(s)} & \mathrm{Spec}(\kappa)(\bar{s}) \\ & \searrow^{\bar{\eta}} & \downarrow & \swarrow_{\bar{s}} & \\ & & S & & \end{array}$$

For $\mathscr{F} \in \mathrm{Ab}(S_{\acute{e}t})$ *we get a (co)specialization map*

$$\mathscr{F}_{\bar{s}} = \varinjlim_{U \text{ an et. nbhd of } \bar{s}} \mathscr{F}(U) \xrightarrow{\iota^*} \mathscr{F}_{\bar{\eta}}$$

via the composition

$$\mathrm{Spec}(\kappa(\bar{\eta})) \to \mathrm{Spec}(\mathscr{O}_{S,s}^{\mathrm{sh}}) \to U.$$

Proposition 19.8 *Let* X *be a Noetherian scheme,* $\mathscr{F} \in \mathrm{Ab}(X_{\acute{e}t})$ *constructible. Then* $\mathscr{F}$ *is l.c.c. if and only if all specialization maps are isomorphisms.*

As a contrasting example, let $E/\mathbb{Q}_p$ be an elliptic curve with multiplicative reduction. There is a model $\mathscr{E} \to \mathrm{Spec}(\mathbb{Z}_p)$ such that $\mathscr{E}_{\mathbb{F}_p}$ is a nodal curve. (Take for example E with Weierstrass equation $y^2 = x^3 - x^2 + p$.) Then we have

$$\overline{\mathbb{Q}_p} \supset \mathbb{Z}_p^{\mathrm{ur}} \twoheadrightarrow \overline{F}_p$$

and a (co)specialization map

$$H^1\left(\mathscr{E}_{\overline{\mathbb{F}_p}}, \mathbb{Z}/\ell^r\right) \to H^1\left(\mathscr{E}_{\overline{\mathbb{Q}_p}}, \mathbb{Z}/\ell^r\right)$$

whenever $\ell \neq p$. The above map is $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-equivarient, and in particular we have an injective map to the intertia invariants

$$H^1\left(\mathscr{E}_{\overline{\mathbb{F}_p}}, \mathbb{Z}/\ell^r\right) \hookrightarrow H^1\left(\mathscr{E}_{\overline{\mathbb{Q}_p}}, \mathbb{Z}/\ell^r\right)^{I_{\mathbb{Q}_p}}.$$

Note 19.1 *This may not be a surjective map, but it will be upon taking inverse limits over r.*

One may think of this example by using Tate's p-adic analytic parametrization of E.

The main technical ingredient in smooth and proper base change is the Smooth Base Change Theorem:

Theorem 19.9 (Smooth Base Change) *Let $f\colon X \to S$ be any scheme map. Consider the cartesian diagram*

$$\begin{array}{ccc} X_T & \longrightarrow & X \\ \downarrow & & \downarrow f \\ T & \xrightarrow{g} & S \end{array}$$

Let $g\colon T \to S$ be smooth or $T = \varprojlim S_i$ where $S_i \to S$ is smooth and (S_i) form an inverse system with affine transition maps. Then for $\mathscr{F} \in \mathrm{Ab}\left(X_{\acute{e}t}\right)_{\mathrm{tors}}$ with all torsion orders invertible in S, the base change map

$$g^{-1}R^p f_*\mathscr{F} \to R^p f'_* g'^{-1}\mathscr{F}$$

is an isomorphism.

Example 19.10 *Let $T = \mathrm{Spec}(L) \to S = \mathrm{Spec}(K)$ where $K \subset L$ is any extension of separably closed fields (both L and K are separably closed) and $(n, \mathrm{K}) = 1$. Then $T = \varprojlim S_i$ is nice in the sense of the theorem above. Therefore, for X/K,*

$$H^*\left(X, \mathbb{Z}/n\right) \to H^*\left(X_L, \mathbb{Z}/n\right)$$

is an isomorphism. As we already know such an isomorphism exists under a purely inseparable extension, we see such an isomorphism exists whenever $K \subset L$ is an inclusion of separably closed fields and $n \in K^*$.

An important use of this is the following: let $X/\mathbb{Q}$ be a variety. Then

$$H^*\left(X_{\overline{\mathbb{Q}}}, \mathbb{Z}/n\right) = H^*\left(X_{\mathbb{C}}, \mathbb{Z}/n\right) = H^*_{\mathrm{sing}}\left(X_{\mathbb{C}}^{an}, \mathbb{Z}/n\right).$$

20 Statement of Cohomological Purity

Recall that for a scheme X and $Z \hookrightarrow X$ a closed subscheme and $U = X \backslash Z$, we defined

$$\Gamma_Z(X, \mathscr{F}) = \ker(\mathscr{F}(X) \to \mathscr{F}(U)).$$

Then taking derived functors, we get $H^*_Z(X, \mathscr{F})$—the cohomology supported on Z. This lives in a long exact sequence

$$\cdots \to H^*_Z(X, \mathscr{F}) \to H^*(X, \mathscr{F}) \to H^*(U, \mathscr{F}) \to \cdots$$

Cohomological Purity will (in certain cases) help us compute H^*_Z.

Theorem 20.1 (Cohomological Purity) *Let K be a field, X/K smooth, $Z \xrightarrow{i} X$ with Z/K smooth. Take $\mathscr{F} \in Ab(X_{\acute{e}t})$ be an l.c.c. sheaf of $\mathbb{Z}/n$-modules with $n \in K^*$. Then there is a canonical isomorphism $H^{r-2c}(Z, \mathscr{F}(-c)) \cong H^r_Z(X, \mathscr{F})$ where $Z \hookrightarrow X$ has pure codimension c.*

Here $\mathscr{F}(-c)$ is the *Tate twist* $\mathscr{F} \otimes_{\mathbb{Z}/n} \mu_n^{\otimes -c}$ where $\mu_n = \mathrm{Hom}(\mu_n, \mathbb{Z}/n)$.

If $K = K^{\mathrm{sep}}$, then $\mu_n^{\otimes r} \cong \mathbb{Z}/n$ non-canonically (the isomorphism depends on a choice of primitive generator). This twist is important because it may modify a $\mathrm{Gal}(K^{sep}/K)$-action if Z is defined over K.

Example 20.2 *1. If $2c > r+1$, then $H^r(X, \mathscr{F}) \cong H^r(U, \mathscr{F})$.*

2. Take $Z = \mathrm{Spec}(k) \hookrightarrow \mathbb{A}^1_k$ (with k algebraically closed and n invertible in k), then $c = 1$ and the theorem states

$$H^{r-2}(\mathrm{Spec}(k), \mu_n) \cong H^r_Z(\mathbb{A}^1_k, \mu_n)$$

and this is canonically isomorphic to $\mathbb{Z}/n$ for $r = 2$ and 0 otherwise.

On the otherhand

$$\begin{array}{ccccc} H^r_Z(\mathbb{A}^1_k) & \longrightarrow & H^r(\mathbb{A}^1_k, \mu_n) & \longrightarrow & H^r(\mathbb{G}_{m,k}, \mu_n) \\ & & \| & & \| \\ & & \begin{cases} \mu_n(k) & r = 0 \\ 0 & else \end{cases} & & \begin{cases} \mu_n(k) & r = 0 \\ \mathbb{Z}/n & r = 1 \\ 0 & else \end{cases} \end{array}$$

This implies that

$$H^r_Z(\mathbb{A}^1_k) = \begin{cases} 0 & r \neq 2 \\ \mathbb{Z}/n & r = 2. \end{cases}$$

Corollary 20.3 *Let $Z \subset X/k$ be as in the theorem, $n \in k^*$, and c the pure codimension of Z. Take $\mathscr{F} = \mathbb{Z}/n(c)$, $r = 2c$, then*

$$H^0(Z, \mathbb{Z}/n) \to H^{2c}_Z(X, \mathbb{Z}/n(c)) \to H^{2c}(X, \mathbb{Z}/n(c))$$

In identifying $H^0(Z, \mathbb{Z}/n) \cong \mathbb{Z}/n$, this map is given by $1 \mapsto \mathrm{cl}_X(Z)$.

This coincides with the cycle class map for smooth subvarieties. We may also extend this to the cycle class map

$$Z^c(X) \to H^{2c}(X, \mathbb{Z}/n(c)).$$

Now fix $(\ell, \text{char } k) = 1$ and tensoring with $\mathbb{Q}_\ell$ (by taking an inverse limits over $\mathbb{Z}/\ell^r$) we get

$$Z^c(X) \otimes \mathbb{Q}_\ell \to H^{2c}(X, \mathbb{Z}_\ell(c)) \otimes \mathbb{Q}_\ell.$$

In particular, if k^s/k is a separable closure, we have

$$Z^c(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^{2c}(X_{k^s}, \mathbb{Q}_\ell(c))^{\mathrm{Gal}(k^s/k)}. \tag{5}$$

Conjecture 20.4 (Tate) *If k is finitely generated over the prime field, then* (5) *is an isomorphism.*

Note 20.1 *To prove the purity theorem, it is convenient to have a sheaf version: take $i\colon Z \hookrightarrow X \hookleftarrow U : j$, n, and c be as before. Then there is a functor*

$$i^!\mathscr{F} = i^*\left(\ker(\mathscr{F} \to j_* j^{-1}\mathscr{F})\right) \in \mathrm{Ab}(Z_{\acute{e}t}).$$

This is a right adjoint to $i_*\colon \mathrm{Ab}(Z_{\acute{e}t}) \to \mathrm{Ab}(X_{\acute{e}t})$. In particular it is left exact and preserves injectives so we may take the derived functors $R^q i^!\mathscr{F}$.

Theorem 20.5

$$R^r i^! \mathscr{F} = \begin{cases} 0 & r \neq 2c \\ (i^{-1}\mathscr{F})(c) & r = 2c. \end{cases}$$

This implies the previous theorem by the spectral sequence

$$E_2^{r,s} = H^r(Z, R^s i^! \mathscr{F}) \Rightarrow H_Z^{r+s}(X, \mathscr{F}).$$

(Note $\Gamma(Z, i^!\mathscr{F}) = \Gamma_Z(X, \mathscr{F})$.)

Then one uses this sheafy version to affines inside of affines.

20.1 A Thought Experiment of Deligne

Suppose X is smooth projective over $\mathbb{F}_q$ and $D = \sum_{i\in I} D_i$ a simple normal crossings divisor on X. (So D_i is smooth, irreducible and they have transverse smooth intersection.) For $J \subseteq I$, we write $D_J = \bigcap_{j\in J} D_j$. Let $U = X \backslash \bigcup D_i$.

By the Leray spectral sequence,

$$E_2^{r,s} = H^r(X_{\overline{\mathbb{F}_q}}, R^s j_* \mathbb{Q}_\ell) \Rightarrow H^{r+s}(U_{\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell).$$

(The ℓ-adic sheaf $\mathbb{Q}_\ell$ will be defined later.) As a consequence of purity, we may compute

$$R^s j_* \mathbb{Q}_\ell \cong \bigoplus_{|J|=s} \mathbb{Q}_\ell(-s)_{D_J}$$

where $\mathbb{Q}_\ell(s)_{D_J} = i_* \mathbb{Q}_\ell(s)$ for $i : D_J \hookrightarrow X$. Then

$$E_2^{r,s} = \bigoplus_{|J|=s} H^r(D_{J,\overline{\mathbb{F}_q}}, \mathbb{Q}_\ell(-s))$$

and the $D_{J,\overline{\mathbb{F}_q}}$ carry the action of Frobenius.

The Riemann Hypothesis implies the Eigenvalues of Frobenius on $E_2^{r,s}$ are algebraic integers with complex absolute value $q^{\frac{r+2s}{2}}$. We say that $E_2^{r,s}$ is *pure of weight* $r + 2s$.

But

$$d_2\colon E_2^{r,s} \to E_2^{r+2,s-1}$$

has pure weight $r + 2s$ and

$$d_3\colon E_3^{r,s} \to E_3^{5+3,s-1}.$$

But the source of d_3 is pure of weight $r + 2s$ and the target is pure of weight $r + 2s - 1$. As the d are equivarient under the Frobenius, d_3 must be trivial and so the spectral sequence degenerates at the E_3-page.

Moreover, we may describe d_2 in terms of the purity theorem and the Gysin map

$$D_J \hookrightarrow D_{J\setminus\{i\}}.$$

At E_∞, we get an increasing filtration on $H^r(U_{\overline{\mathbb{F}_q}}, \mathbb{Q}_p)$ with graded pieces $\mathrm{Gr}_k^W \cong E_3^{2n-k,k}$ pure of weight k.

Deligne imported this heuristic over $\mathbb{C}$ and discovered mixed Hodge theory for smooth, non-projective varieties.

20.2 Lecture 29: 12/7

21 ℓ-adic Sheaves and Poincare Duality

Our principle motivation for defining étale cohomology was to prove the Weil conjectures. We saw that to do this, we needed a cohomology theory with coefficients in characteristic 0. So far, all of our cohomology groups we have been able to prove good things about have been finite, and thus not characteristic 0. We now see how to stitch these groups together to give somethign of characteristic 0.

Definition 21.1 *Let X be a Noetherian scheme. Am ℓ-adic sheaf on X is an inverse system*

$$\mathscr{F} = (\mathscr{F}_n, \mathscr{F}_{n+1} \to \mathscr{F}_n)_{n\geqslant 1}$$

where $\mathscr{F}_n$ is a sheaf of $\mathbb{Z}/\ell^n$-modules, and $\mathscr{F}_{n+1} \otimes_{\mathbb{Z}/\ell^{n+1}} \mathbb{Z}/\ell^n \to \mathscr{F}_n$ are isomorphisms of $\mathbb{Z}/\ell^n$ modules. Morphisms between ℓ-adic sheaves are morphisms of inverse systems. We say an ℓ-adic sheaf is lisse if each $\mathscr{F}_n$ is locally constant constructible.

Example 21.2 *Here are some examples of ℓ-adic sheaves. In each instance, the morphisms $\mathscr{F}_{n+1} \to \mathscr{F}_n$ are obvious, so we omit them.*

- $\mathscr{F} = (\mathbb{Z}/\ell^n)_{n\geqslant 1}$ *(We sometimes write this simply as $\mathbb{Z}_\ell$.)*
- $\mathscr{F} = (\mu_{\ell^n})_{n\geqslant 1}$, $\ell \in \mathscr{O}_X^\times$
- *For $\mathscr{F}$ a sheaf of $\mathbb{Z}/\ell^n$ modules, we obtain an ℓ-adic sheaf $(\mathscr{F}/\ell^n)_{n\geqslant 1}$.*

We have the usual sheaf operations on ℓ-adic sheaves, such as f^{-1}, f_*, etc.

It is not actually true that for $\mathscr{F} = (\mathscr{F}_n)$ is an ℓ-adic sheaf on X and $f : X \to Y$, we get that $R^q f_!(\mathscr{F}_n)$ is an ℓ-adic sheaf. This can be fixed, as it is isomorphic to an ℓ-adic sheaf in the following category:

Definition 21.3 *Let X be a Notherian scheme. The Artin-Rees category is the category whose objects are projective systems of sheaves of abelian groups on X, and whose morphisms are given by*

$$\mathrm{Hom}((\mathscr{F}_n), (\mathscr{G}_n)) := \varinjlim \mathrm{Hom}(\mathscr{F}[d], \mathscr{G})$$

where d is the shift operator.

Definition 21.4 *Given an ℓ adic sheaf $\mathscr{F}$, the stalk at a geometric point $\bar{x}$ of X is defined to be $\mathscr{F}_{\bar{x}} = \varprojlim (\mathscr{F}_n)_{\bar{x}}$. It is a $\mathbb{Z}_\ell$-module.*

As a consequence of what we did on our connection between sheaves and $\pi_1(X, \bar{x})$, we have the following:

Corollary 21.5 *Suppose X is a connected Noetherian scheme, and $\bar{x}$ is a geometric point of X. Then*

$$\{\textit{lisse } \ell\textit{-adic sheaves}\} \to \{\textit{continuous representations of } \pi_1(X, \bar{x}) \textit{ on finitely generated } \mathbb{Z}_\ell\textit{-modules}\}$$

$$\mathscr{F} \mapsto \mathscr{F}_{\bar{x}}$$

is an equivalence of categories.

Definition 21.6 *The category of $\mathbb{Q}_\ell$-sheaves on a Notherian scheme X has objects ℓ-adic sheaves, with the hom sets being equal to the morphisms of ℓ-adic sheaves, tensored over $\mathbb{Z}_\ell$ with $\mathbb{Q}_\ell$.*

When we view an ℓ-adic sheaf, $\mathscr{F}$, in this category, we write it as $\mathscr{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, or just $\mathscr{F} \otimes \mathbb{Q}_\ell$. We also write $\mathbb{Q}_\ell$ for the $\mathbb{Q}_\ell$ sheaf associated to the ℓ-adic sheaf $\mathbb{Z}_\ell$. The stalk of an ℓ-adic sheaf $\mathscr{F} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ at a geometric point $\bar{x}$ is $\mathscr{F}_{\bar{x}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. We can modify the above corollary to work with $\mathbb{Q}_\ell$-sheaves and $\mathbb{Q}_\ell$-representations.

Definition 21.7 *Let X be a Notherian scheme, $\mathscr{F}$ an ℓ-adic sheaf on X. We define*

$$H^*(X, \mathscr{F}) := \varinjlim H^*(X, \mathscr{F}_n)$$

and

$$H^*(X, \mathscr{F} \otimes \mathbb{Q}_\ell) := H^*(X, \mathscr{F}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

By our previous work on cohomology of constructible sheaves, these are finitely generated $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$ modules, respectively.

With these definitions in hand, we state a version of Poincare Duality (which can be further generalized):

Theorem 21.8 *Let k be a separably closed field, X/k a smooth, geometrically connected variety of dimension d. Then*

- *There is a canonical isomorphism $\mathrm{Tr} : H_c^{2d}(X, \mathbb{Q}_\ell(d)) \to \mathbb{Q}_\ell$ called the trace map.*
- *For any lisee $\mathbb{Q}_\ell$ sheaf $\mathscr{F}$ on X, cup product and the trace map combine to give a perfect pairing*

$$H^r(X, \mathscr{F}) \times H_c^{2d-r}(X, \mathscr{F}^\vee(d)) \xrightarrow{\cup} H_c^{2d}(X, \mathbb{Q}_\ell(d)) \xrightarrow{\mathrm{Tr}} \mathbb{Q}_\ell$$

(where $\mathscr{F}^\vee := \mathscr{H}om(\mathscr{F}, \mathbb{Q}_\ell)$ which is the sheaf associated to $(U \mapsto \mathrm{Hom}(\mathscr{F}|_U, \mathbb{Q}_\ell|_U))$).

This theory of $X \mapsto H^*(X, \mathbb{Q}_\ell)$ does give a Weil Cohomology Theory for any separably closed field of characteristic not equal to ℓ, though there are details we have not verified. So, up to these details, we have proven the rationality of the Zeta function, the functional equation, and the factorization into characteristic polynomials.

As mentioned earlier, the Riemann Hypothesis part of the Weil Conjectures does not follow just from the existence of a Weil Cohomology Theory. We turn our attention now to how Deligne proved this.

22 Generalization of Lefschetz Fixed Point Formula

For this section, let k be a finite field, X/k separated and of finite type, $\mathscr{F}$ a $\mathbb{Q}_\ell$-sheaf on X, ℓ a prime not equal to the characteristic of k.

Recall the zeta function $Z(X/k, t) := \prod_{x \in X_{\mathrm{cl}}} (1 - t^{\deg(x)})^{-1}$, where $\deg(x) := [\kappa(x) : k]$. A priori, this lives in $\mathbb{Z}[\![t]\!]$. We now generalize this to any ℓ-adic sheaf:

Definition 22.1 *Let $\mathscr{F}$ be an ℓ-adic sheaf on X. Set*

$$L(X/k, \mathscr{F}, t) := \prod_{x \in X_{cl}} \det(1 - t^{\deg(x)} F_{\kappa(x)} |_{\mathscr{F}_{\overline{x}}})^{-1}$$

where $F_{k'} \in \mathrm{Gal}(\bar{k}/k')$ is defined for finite extensions k'/k to be the inverse of $a \mapsto a^{|k'|}$.

The above $F_{k'}$ is called the geometric Frobenius. It must act on cohomology for the above definition to make sense. We explain how this works now. For $x \in X$ a closed point, we pick geometric point $\overline{x} : \mathrm{Spec}(\overline{k}) \to X$ over x. Each element in $\mathscr{F}_{n\overline{x}}$ is represented by $(U, \overline{u}, t \in \mathscr{F}_n(U))$ where U is étale over X and $\overline{u}$ is a geometric point lying over $\overline{x}$. Now we send this to $(U, u \circ \mathrm{Spec}(F_{\kappa(x)}), t \in \mathscr{F}_n(U))$, so we get an $F_{\kappa(x)}$ action on $\mathscr{F}_{n\overline{x}}$. Taking the limit we get $F_{\kappa(x)}$ action on $\mathscr{F}_{\overline{x}}$.

Theorem 22.2 *Let $\mathscr{F}$ be an ℓ-adic sheaf on X. We have that*

$$L(X/k, \mathscr{F}, t) = \prod_i \det(1 - tF|_{H^r_c(X_{\overline{k}}, \mathscr{F}|_{X_{\overline{k}}})})^{(-1)^{r+1}}$$

We now explain the action F on $H^r_c(X_{\bar{k}}, \mathscr{F}|_{X_{\overline{k}}})$. We define $F_X := id_X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(F_k)$ called Frobenius automorphism, and there is a canonical identification from functoriality of inverse image $F_X^{-1}(\mathscr{F}|_{X_{\overline{k}}}) \to \mathscr{F}|_{X_{\overline{k}}}$. Therefore we have the map we call F

$$F : H^r_c(X_{\overline{k}}, \mathscr{F}|_{X_{\overline{k}}}) \to H^r_c(X_{\overline{k}}, F_X^{-1}(\mathscr{F}|_{X_{\overline{k}}})) \to H^r_c(X_{\overline{k}}, \mathscr{F}|_{X_{\overline{k}}}).$$

This looks a lot like our result for the Zeta function, but before we had the Frobenius map $\varphi_{X_{\bar{k}},k}$ (defined by $a \to a^{|k|}$ on affine open of X and product with $id_{\mathrm{Spec}(\overline{k})}$) instead of our new F_X. It turns out that they have the same action on cohomology. (The action of $\varphi_{X_{\overline{k}},k}$ comes from a canonical isomorphism $\varphi^{-1}_{X_{\overline{k}},k}(\mathscr{F}|_{X_{\overline{k}}}) \to \mathscr{F}|_{X_{\overline{k}}}$.) Note that

$$\varphi_{X_{\bar{k}},k} \circ F_X^{-1}$$

is the absolute Frobenius morphism (the morphism that is set theoretically the identity on $X_{\bar{k}}$, and on open affines sends a to $a^{|k|}$). Denote this by $F_{X,k}$.

The surprising fact is that $F_{X,k}$ acts trivially on cohomology. Note that this then gives that $\varphi_{X_{\bar{k}},k}$ and F_X act the same on cohomology, as stated above. To get an action on cohomology from the absolute Frobenius, note that by pull back, we have a morphism $H^i(X, \mathscr{F}) \to H^i(X, F_{X,k}^{-1}\mathscr{F})$, and so all we need is a morphism back the other way.

Lemma 22.3

- *There are canonical isomorphisms $F_{X,k}^{-1}\mathscr{F} \to \mathscr{F}$, and $\mathscr{F} \to (F_{X,k})_*\mathscr{F}$.*
- *Composing the induced map from the first iso with the map described above yields the identity on $H^i(X, \mathscr{F})$.*

For a proof, see SGA 5, around page 456.

23 Some ideas in the proof of the Riemann Hypothesis

[Note: we ran out of time and did not complete this discussion. More may be added later.]

For this section, let k be a finite field of size q, and let X/k be smooth projective, geometrically connected. The Riemann Hypothesis says that

$$\det(t - F|_{H^i(X_{\bar{k}}, \mathbb{Q}_\ell)})$$

is in $\mathbb{Z}[t]$, independent of ℓ, and its roots have complex absolute value $q^{i/2}$ for any embedding into the complex numbers. We say that $H^i(X_{\bar{k}}, \mathbb{Q}_\ell)$ is pure of weight i.

Lemma 23.1 *To prove the Riemann Hypothesis, it suffices to show that for all even dimensional varieties Y of dimension d, the eigenvalues of F_k on $H^e(Y, \mathbb{Q}_\ell)$ are algebraic, and all complex embeddings have absolute value $\leqslant q^{\frac{d}{2}+\frac{1}{2}}$.*

Proof. Suppose we have proven this statement. Poinacre duality gives that we also have a lower bound on the absolute values by $q^{\frac{d}{2}-\frac{1}{2}}$. For an arbitrary variety X of dimension d, and any even number r, we have

$$H^d(X, \mathbb{Q}_\ell)^{\otimes r} \subseteq H^{dr}(X^{2r}, \mathbb{Q}_\ell)$$

via Kunneth, and so if α is an eigenvalue of $H^d(X, \mathbb{Q}^\ell)$, we have that

$$q^{\frac{rd}{2}-\frac{1}{2}} \leqslant |\iota\alpha|^r \leqslant q^{\frac{rd}{2}+\frac{1}{2}}$$

Taking r-th roots and letting r go to infinity gives $|\iota\alpha| = q^{\frac{d}{2}}$. Thus, for any variety of dimension d, we have that the eigenvalues are algebraic with the correct absolute values in middle degree.

To prove this for $H^i(X, \mathbb{Q}_\ell)$ for $i < d$, we induct on dimension and use a weak Lefschetz Theorem, which says that for a smooth projective variety X, there is a smooth hyperplane section $Y \subseteq X$ with $H^i(X, \mathbb{Q}_\ell) \to H^i(Y, \mathbb{Q}_\ell)$ injective for $i < \dim(X)$. We can find a smooth hyperplane section by Bertini's Theorem. The statement on cohomology follows from our long exact sequence on cohomology with support in Y, using that the complement is affine. As $\dim(Y) < \dim(X)$ by induction its eigenvalues satisfy the correct condition, then so do the eigenvalues on $H^i(X, \mathbb{Q}_\ell)$.

To get the eigenvalues to have the correct absolute value for $i > \dim(X)$, we can use Poincare Duality, and the fact that it is true for $i < \dim(X)$.

So now we have that all eigenvalues are algebraic, and of the correct absolute values. This then implies that all of these characteristic polynomials are in $\mathbb{Z}[t]$ and independent of ℓ, as their alternating product is a rational function in $\mathbb{Q}(t)$, and there can be no cancellation due to the distinct eigenvalue absolute values.

The hard part of the Riemann Hypothesis is to show what we assumed in the last lemma. We briefly discuss ideas involved here:

Definition 23.2 *Let Z be a variety over a field k'. A Lefschetz pencil over X is a diagram*

$$\begin{array}{ccc} X & \xleftarrow[g]{} & \widetilde{X} \\ & & \downarrow f \\ & & \mathbb{P}^1_{k'} \end{array}$$

where g is birational, f is flat, projective, smooth away from a finite set, and the fibers over that finite set have ordinary double points as singularities.

From the definition, we have that the fibers of f give hyperplane sections of X. It is a fact that the induced map $H^d(X, \mathbb{Q}_\ell) \to H^d(\widetilde{X}, \mathbb{Q}_\ell)$ is injective, so it is enough to prove the statement on eigenvalues in middle degree for $\widetilde{X}$.

We can use the Grothendieck-Leray spectral sequence to get that $H^r(\mathbb{P}^1_{\bar{k}}, R^s f_* \mathbb{Q}_\ell) \implies H^{r+s}(\widetilde{X}_{\bar{k}}, \mathbb{Q}_\ell)$. We only need to check the eigenvalues on $E_2^{0,d}, E_2^{1,d-1}, E_2^{2,d-2}$. Standard theory of Lefshetz pencils covers $E_2^{2,d-2}, E_2^{0,d}$. But , $E_2^{1,d-1}$ is hard. The following theorem proves a large piece of what we want:

Theorem 23.3 *Let U/k be a smooth affine gemetrically connected curve, $\mathscr{G}$ a lisee $\mathbb{Q}_\ell$ sheaf on U. For any geometric point $\bar{x}$ of U, we get a representation $\pi_1(U, \bar{x}) \to \mathscr{G}_{\bar{x}}$. Suppose*

1. $\det(t - F_x|_{\mathscr{G}_{\bar{x}}}) \in \mathbb{Q}[t]$ *for all* $x \in U_{\mathrm{cl}}$
2. *there is a nondegenerate alternating form $\psi : \mathscr{G} \times \mathscr{G} \to \mathbb{Q}_\ell(-a)$, for some $a \in \mathbb{Z}$ that is π_1 equivariant*
3. *The image of the image of $\pi_1(U)$ in $Sp(\mathscr{G}, \psi)$ is Zariski dense.*

Then we have

- *For all closed points $x \in U_{\mathrm{cl}}$, the eigenvalues of $F_{\kappa(x)}$ acting on $\mathscr{G}_{\bar{x}}$ are algebraic numbers with absolute value $q^{\frac{a}{2}\deg(x)}$ and*
- *the eigenvalues of F_k acting on $H^1(U, \mathscr{G})$ have absolute value $\leqslant q^{\frac{a}{2}+1}$*

Here, to show that the first conclusion implies the second, you use the Grothendieck trace formula for $L(U, \mathscr{G}, t)$. The proof of the first conclusion is the master-stroke involving "Rankin's trick" that rests on the positivity of the coefficients of $L(U, \mathscr{G}^{\otimes 2k}, t)$ for all $k \in \mathbb{Z}_{>0}$.

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