### Weil II and Perverse Sheaves Seminar, Fall 2023

References are to [KW01] unless otherwise specified. The plan below takes us through (more or less) the main theorem of [Del80]. Time permitting, we'll use remaining weeks this term to start discussing abstract t-structures on triangulated categories and the perverse t-structure on $D_c^b(X, \overline{\mathbb{Q}}_\ell)$, for $X/\mathbb{F}_q$ a variety.

We'll meet in BE136 (Baker Systems) on Fridays from 3:30-5:00.

1. 8/25 (SP). Preliminary talk. Recollections on some basic principles of étale cohomology.
2. 9/1 (Yifei) §I.1 of [KW01]. Explain carefully the various Frobenii and their relations to each other (as morphisms, as endomorphisms of cohomology, as endomorphisms of stalks). State the Grothendieck trace formula. Define Weil sheaves on a variety $X/\mathbb{F}_q$, describe the interpretation in terms of representations of the Weil group, and prove Theorem I.1.4, which implies the trace formula also holds for Weil sheaves.
3. 9/8 §I.2 (Stefan N.) Define weights (Definition I.2.1), and state the main theorem of the chapter, Theorem I.9.3. Discuss pages 13-20: weights and the relation to zeroes and poles of the L-function, semicontinuity of weights.
4. 9/15 §I.2. (Mehmet) Introduce the sheaf-function correspondence and its relation to the weight of a mixed sheaf (pages 20-25).
5. 9/22 §I.3 (Min) Algebraic monodromy. §1.3. Describe rank 1 Weil sheaves. Prove that geometrically semisimple lisse sheaves have semisimple geometric monodromy group. Consequences for determinantal weights.
6. 9/29 §I.4 (Jake) The Rankin-Selberg method. Real sheaves are mixed, real lisse sheaves are pure.
7. 10/6 (Luke) Interlude: introduce the derived category of an abelian category and derived functors (at the level of the derived category); specialize the discussion to our sheaf-theoretic cases of interest as in [FK88, Appendix II]. See also [KW01, §II.1] and for more details [Har66, Chapter I].
8. 10/13: Fall Break, no talk (if there's no NT seminar on Monday, we might have a make-up talk)
9. 10/20 (Will) §I.5 $\ell$-adic Fourier transform. Define Artin-Schreier sheaves and the Fourier transform. Prove the Fourier inversion formula. (Note we have not rigorously constructed $D_c^b(X, \overline{\mathbb{Q}}_\ell)$ and the sheaf-theoretic operations. This is a technical problem best avoided for now: just assume this situation behaves like the derived category at torsion level.)
10. 10/27 (Yifei) §I.6. Prove the main theorem in the case of curves (Theorem I.6.1).
11. 11/3 §I.7. Prove the first version of the main theorem (without local monodromy refinements), Theorem 7.1. With the remaining time, sketch the argument that direct images of mixed complexes are mixed (Theorem I.9.4; assume Theorem I.9.3(2), since so far we have only proven the analogous statement for $\tau$-mixed sheaves).
12. 11/10: Veterans Day, no talk (if there's no NT seminar on Monday, we might have a make-up talk)
13. 11/17 (Kyle) §I.8-I.9. Prove Theorem I.9.3, the main theorem with the local monodromy refinement.
14. 12/1 (Jake) §II.1-II.4 t-structures on triangulated categories, heart of a t-structure, cohomology sequence in the heart associated to an exact triangle, etc.
15. 12/8 (Min) §III.1-III.3 Definition of the perverse t-structure. The smooth case. Abstract gluing.

### Spring 2023

References: KW: [KW01]. dCM: [dCM09]

1. 1/19 (Yifei) (III.3-III.4) The perverse t-structure is a t-structure (application of the gluing construction from 12/8). Exactness properties with respect to open and closed immersions.

2. 1/26 (Will) (III.5) Intermediate extension: construction, application to classifying simple perverse sheaves. Define intersection complexes and intersection cohomology and prove Poincaré duality for intersection cohomology (apply III.5.3).Deligne's successive truncation formula (BBD 2.1.11, which includes the case of other perversities; just do the case of the middle perversity on a variety. Another reference is [Dim04, 5.2.10]) and its application to compute the intersection complex and intersection cohomology for a variety with an isolated singularity (see also [Dim04, 5.4.4]); use this to illustrate (for eg a nodal curve, or more general projective varieties with isolated singularities) how intersection cohomology is pure in spite of the ordinary cohomology being mixed.

3. 2/1 (Min) Do a couple more examples continuing the example of last time: ordinary (can do ordinary with compact supports too) and intersection cohomology of: the affine cone on, say a nonsingular genus g projective curve (dCM 2.2.2), the projective cone on a quadric surface (dCM 2.2.4). Emphasize what's going on with weights in these examples. Give the geometric explanation of the shape of the weight filtration on $H^k(U)$ for a smooth (non-projective) variety $U$, as in [Del71, §6] (for $U$ over a finite field, assume there is a SNCD smooth compactification, as would be known in characteristic zero–you can imagine $U$ comes by spreading out and specializing from something over $\mathbb{Q}$). Define pure and mixed Hodge structures, describe the case of smooth projective complex varieties (due to Hodge) and explain ([Del71, §7]) how Deligne discovered the mixed Hodge structure on smooth complex varieties using finite field weight heuristics. (If you have time, you can sketch the construction of the Hodge filtration in this case: see eg [Voi07, Chapter 8].)

4. 2/8 (Min) Continuation of the last talk.

5. 2/15 ...

6. 2/22 (Yifei) Hodge theory and point counts: Katz's appendix to [HRV08].

7. 2/29 (Jake) Hodge theory and point counts and PORC problems: the example from [dSVL12], Higman's PORC conjecture and the (limited) positive result in [Hig60], du Sautoy's algebro-geometric structuring of the PORC conjecture in [dS00].

8. 3/7, 3/14: no talks (spring break)

9. 3/21 () Tame ramification and Abhyankar's lemma. Recall the simplest version (tame extensions of local fields). Then discuss SGA 1 Exposeée XIII.5 (Proposition 5.1, Proposition 5.2, Corollaire 5.3, and the relative Abhyankar lemma, Prop.5.4- Prop. 5.5). Do careful proofs at least of 5.1-5.3.

10. 3/28 () Recall the specialization isomorphism for the fundamental group of a proper scheme over a henselian local ring (done in class last year: see [Art69, Theorem 3.1] or [Sta24, Tag 0BQC]). Then describe (show where the maps come from, without a full proof) the prime-to-$p$ and tame specialization maps (isomorphisms and surjections respectively) for the fundamental group of non-proper schemes: see Appendix A, especially A.12 of [LO10]. Give an example of a Galois extension where all inertia groups have prime to $p$ order but the global cover does not. Give an application: geometrically irreducible $\overline{\mathbb{Q}}_p$-local systems on a smooth scheme over a number field are automatically unramified almost everywhere (see [Pet23, Prop. 6.1]).

11. 4/4 () In the case of the complement $X \setminus D$ of a NCD in a (nice enough) scheme $X$, explain that tameness (along the boundary divisor) of an étale cover of $X \setminus D$ is equivalent to tameness after pull-back to curves: [KS10, Proposition 4.1-4.2], building on §2 of *loc. cit.* (and Abhyankar's lemma).

12. 4/11 () Discuss Deligne's proof ([Del73, Théorème 9.8]) that on curves over a finite field, tameness is preserved under taking $\ell$-adic companions.

13. 4/11 () State L. Lafforgue's theorem on the global Langlands correspondence over function fields and its implications for the curve case of Deligne's conjecture ([Laf02]). State Deligne's result ([Del12]) on the "field of coefficients" in the higher-dimensional case. Begin discussing Drinfeld's proof of Deligne's conjecture ([Dri12].

14. 4/18 () Drinfeld's proof.

## References


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