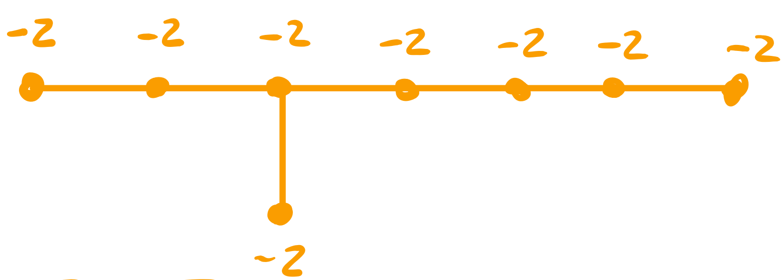
Local Tropicalizations of splice type surface singularities

M. Angelica Cueto

(The Ohio State University)

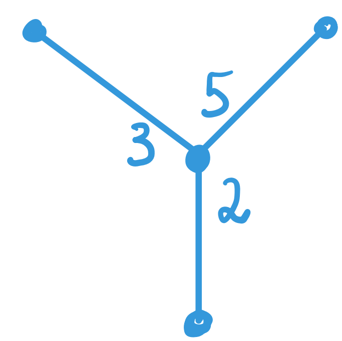
joint with Patrick Popescu-Pampu & Dmitry Stepanov (arXiv: 2108.05912)

G:



Resolution Graph

Γ:



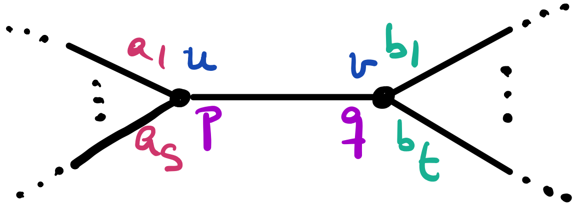
Splice Diagram

$$z_1^2 + z_2^3 + z_3^5 = 0$$ (E_8-singularity)

SPLICE-TYPE SURFACE SINGULARITIES

- Generalize Brieskorn complete intersections $\Sigma(p_1,\ldots,p_k)$ (including A_n, E_6 & E_8 du Val sing)
- INPUT: A splice type diagram $\boxed{\Gamma}$ with 2 properties
  - Γ = tree with no valency 2 vertices (nodes = vertices with valency > 1, leaves = vertices with valency = 1)
  - Γ has positive integer weights on the edges surrounding each node.

Properties: ① "Edge Determinant condition" for each pair of adjacent nodes



$$\det([u,v]) := pq - a_1\cdots a_s\, b_1\cdots b_t > 0$$

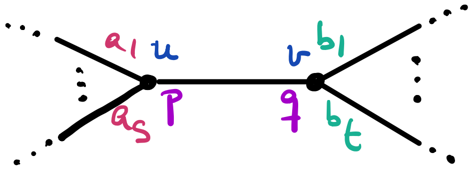
② "Semigroup Condition" for each pair (v,e): v node, e edge, $v\in e$

$\boxed{\ell_{v,w}}$ = "linking number of vertices v & w" = product of all weights adjacent to but NOT ON the path joining v & w in Γ.

$\boxed{d_v}$ = $\ell_{v,v}$ = product of all weights around v

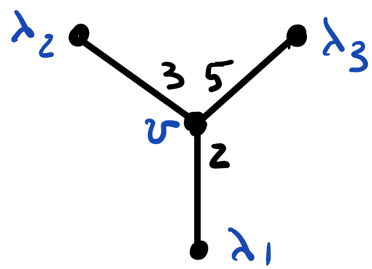
→ Semigroup Cond. (v,e): $d_v \in \mathbb{N}_0 \langle \ell_{v,\lambda} : \lambda \text{ leaf of } \Gamma \text{ \& } e \text{ in path } [v,\lambda] \subseteq \Gamma \rangle$

Props

① 

$\det([u,v]) := pq - a_1 \cdots a_s b_1 \cdots b_t > 0$

② $d_v \in \mathbb{N}_0 \langle \ell_{v,\lambda} : \lambda$ leaf of Γ & e in path $[v,\lambda] \subseteq \Gamma \rangle$

EXAMPLES: ① 

Brieskorn C.I. $\rightsquigarrow \Sigma(2,3,5)$

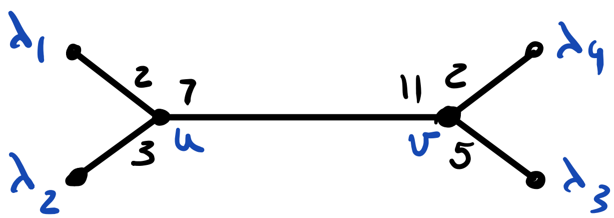
$d_v = 30 = 2\cdot 3\cdot 5$

$30 \in \mathbb{N}_0\langle 15\rangle$ ✓ $(v, [v,\lambda_1])$

$30 \in \mathbb{N}_0\langle 10\rangle$ ✓ $(v, [v,\lambda_2])$

$30 \in \mathbb{N}_0\langle 6\rangle$ ✓ $(v, [v,\lambda_3])$

Obs: Semigroup cond is automatically true for pairs $(v,(v,\lambda])$ with λ leaf of Γ.

② 

• $\det([u,v]) = 7\cdot 11 - 2\cdot 3\cdot 2\cdot 5 = 77-60 = 17 > 0$ ✓

• Non-trivial Semigp. Conditions to check : $(u,[u,v])$ & $(v,[u,v])$

• $(u,[u,v])$ $d_u = 42 \in \mathbb{N}_0\langle \ell_{u,\lambda_3}; \ell_{u,\lambda_4}\rangle = \mathbb{N}_0\langle 2\cdot 3\cdot 2, 2\cdot 3\cdot 5\rangle = \mathbb{N}_0\langle 12,30\rangle$

$\iff 7 \in \mathbb{N}_0\langle 2,5\rangle$ ✓ since $7 = 1\cdot 2 + 1\cdot 5$

• $(v,[u,v])$ $d_v = 110 \in \mathbb{N}_0\langle \ell_{v,\lambda_1}; \ell_{v,\lambda_2}\rangle = \mathbb{N}_0\langle 2\cdot 3\cdot 5, 2\cdot 5\cdot 2\rangle = \mathbb{N}_0\langle 30,20\rangle$

$\iff 11 \in \mathbb{N}_0\langle 3,2\rangle$ ✓ in 2 ways

$11 = 1\cdot 3 + 4\cdot 2$

$11 = 3\cdot 3 + 1\cdot 2$

Q: Why these 2 conditions?

$(X,0) \hookrightarrow (\mathbb{C}^n,0)$ isol. normal surf sing
$\Sigma = X \cap S^{2n-1}_{\varepsilon}(0)$ link (cpt conn. oriented 3-fold)

- Siebenmann (1980) Coprime splice diagrams (pairwise coprime weights near each node) encode $\mathbb{Z}$-homology sphere links ($H^1(\Sigma,\mathbb{Z})=0$) that are graph manifolds (by "plumbing")
- Eisenbud-Neumann (1985) Isolated surface singularities over $\mathbb{C}$ with $\mathbb{Z}$-HS^3 links arise from coprime splice diagrams with edge determinant condition.
- Neumann (1983): Quasi-homogeneous normal complex surface singularities with $\mathbb{Q}$-homology sphere links ($H^1(\Sigma,\mathbb{Q})=0$) have Brieskorn complete intersections as universal abelian covers. (Pham-Brieskorn-Hamm systems)
- Neumann-Wahl (2005) Coprime splice diagrams give isolated complete intersection surface singularities with $\mathbb{Z}$-HS^3 links if Γ satisfies the semigp condition.
  $\rightsquigarrow$ splice type systems
- Neumann-Wahl (2002-2005) Fix a splice diagram Γ with edge det & semigroup conditions from a $\mathbb{Q}$-HS^3 link. Then, its universal abelian cover is of splice type.
- Okuma (2006) Universal ab covers of rational or $\mathbb{Q}$-HS^3 minimally elliptic & non cuspidal surface singularities are of splice type.

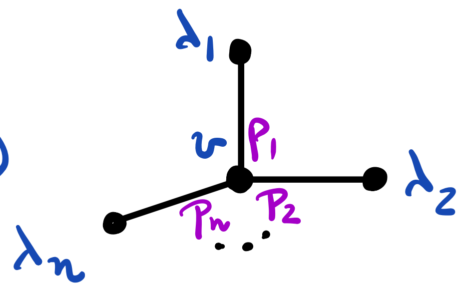
Q: How to build splice type systems from Γ?

A: Use the semigroup condition to build $\mathrm{val}(v)-2$ equations for each node v of Γ with the same support (Pham-Brieskorn-Hamm systems)

variables = leaves of Γ (n in total), $\#\text{eqns} = \sum_{v \text{ node}} (\mathrm{val}(v)-2) = \#\text{leaves of } \Gamma - 2$ ($\rightsquigarrow$ expect dim 2)

- **Brieskorn**

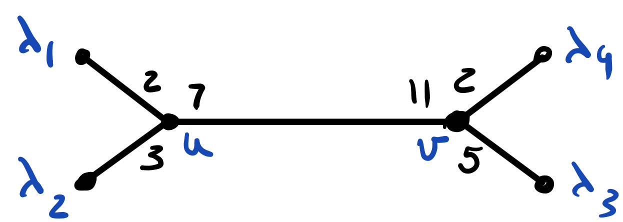
$(X,0) \hookrightarrow (\mathbb{C}^n,0)$



$$\begin{cases} a_{11} z_1^{p_1} + a_{12} z_2^{p_2} + \cdots + a_{1n} z_n^{p_n} = 0 \\ \quad \vdots \\ a_{n-2,1} z_1^{p_1} + a_{n-2,2} z_2^{p_2} + \cdots + a_{n-2,n} z_n^{p_n} = 0 \end{cases}$$

Pham-Brieskorn-Hamm: All maximal minors of (a_{ij}) are nonzero $\rightsquigarrow$ (isolated sing)

- **Splice system by example:**



$$7 = \overset{\lambda_3}{1\cdot 2} + \overset{\lambda_4}{1\cdot 5} \qquad (u,[u,v])$$

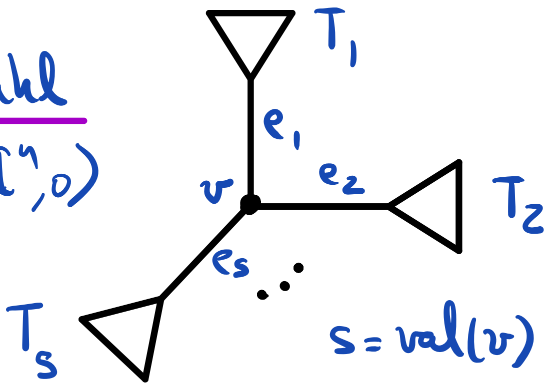
$$\begin{cases} 11 = 1\cdot 3 + 4\cdot 2 \\ 11 = \underset{\lambda_1}{3\cdot 3} + \underset{\lambda_2}{1\cdot 2} \end{cases} \qquad (v,[u,v])$$

$\rightsquigarrow$ 2 different splice systems:

$$(\mathrm{I}) \begin{cases} z_1^2 + z_2^3 + \underline{z_3^1 z_4^1} = 0 \\ \underline{z_1^1 z_2^4} + z_3^5 + z_4^2 = 0 \end{cases} \quad \& \quad (\mathrm{II}) \begin{cases} z_1^2 + z_2^3 + \underline{z_3 z_4} = 0 \\ \underline{z_1^3 z_2^1} + z_3^5 + z_4^2 = 0 \end{cases}$$

- **Neumann–Wahl**

$(X,0) \hookrightarrow (\mathbb{C}^n,0)$



Each T_i gives one monomial $z^{m_{v,e_i}}$ via the semigroup condition for (v, e_i) (admissible monomials)

- $(s-2)$ Equations at v $\left((f_{v,i})_{1 \le i \le s-2} \right)$

$$\begin{cases} a_{1,1}^{(v)} z^{m_{v,e_1}} + a_{1,2}^{(v)} z^{m_{v,e_2}} + \cdots + a_{1,s}^{(v)} z^{m_{v,e_s}} = 0 \\ \quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ a_{s-2,1}^{(v)} z^{m_{v,e_1}} + a_{s-2,2}^{(v)} z^{m_{v,e_2}} + \cdots + a_{s-2,s}^{(v)} z^{m_{v,e_s}} = 0 \end{cases}$$

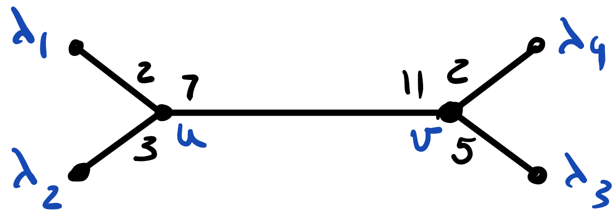
STRICT SPLICE SYST.

- Coefficients: $(a_{i,j}^{(v)})_{i,j}$ has all maximal minors non zero. (Pham-Brieskorn-Hamm determinant condition)

- $\boxed{S(\Gamma)}$: Splice Type systems: Allow Tails of convergent power series on strict splice systems.

- Tails for v: each monomial must have weight $> d_v$ with respect to the weight vector $\boxed{W_v} := (\ell_{v,\lambda_1}, \ldots, \ell_{v,\lambda_n}) \in \mathbb{Z}^n_{\geq 0}$.

EXAMPLE:



$W_u = (21, 14, 12, 30)$

$W_v = (30, 20, 22, 55)$

	u-node eqn.			
	z_1^2	$+ \; z_2^3$	$+ \; z_3 z_4$	$=0$
W_u-wt	42	42	42	d_u
W_v-wt	60	60	77	$l_{u,v}$

	v-node eqn.			
	$z_1 z_2^4$	$+ \; z_3^5$	$+ \; z_4^2$	$=0$
W_u-wt	77	60	60	$l_{u,v}$
W_v-wt	110	110	110	d_v

NW: Splice systems are quasi-homogeneous with respect to $\{W_v : v \text{ node of } \Gamma\}$

Theorem (NW 2005) If Γ satisfies the edge determinant & semigp. conditions, then $S(\Gamma)$ (with tails) determines an ICIS.

Proof • $V(S(\Gamma), z_\lambda = z_\mu = 0) = \{\underline{0}\}$ in $\mathbb{C}^n$ for each pair of leaves $\lambda \neq \mu$

$\Rightarrow \dim V(S(\Gamma)) = 2$ & we get CI.

• Inductive explicit resolutions of $V(S(\Gamma)) \subseteq (\mathbb{C}^n, 0)$ by weights blow-ups (with analytic patches) confirm $\underline{0}$ is an isolated singularity. $\square$

NATURALNESS of $S(\Gamma)$

Q: Why do we need tails & how does the construction depend on the choice of admissible monomials?

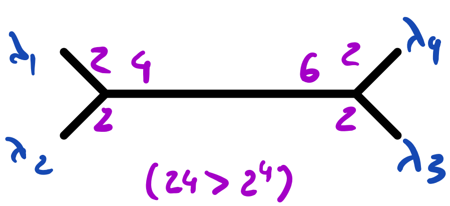
- $\mathcal{M} = \{ z^{m_{v,e}} : v, e \}$ a set of admissible monomials for Γ.
- $S(\Gamma)_{\mathcal{M}} := \{$ splice type systems using monomials from $\mathcal{M} \}$
- $X_{\mathcal{M}} := \{$ germs $(X,0) \hookrightarrow \mathbb{C}^n$ defined by systems from $S(\Gamma)_{\mathcal{M}} \}$

THM1 (NW, C-P-P-S): If Γ is coprime, $X_{\mathcal{M}}$ is independent of $\mathcal{M}$.
(can swap admissible monomials by adding tails)

THM2 (NW): If Γ is general, we are OK if we only use equivariant tails w.r.t. the discriminant group of the starting plumbing graph of the singularity.

⚠ For general Γ, THM1 is false

EXAMPLE: [C, P-P, S]



$$\begin{cases} z_1^2 + z_2^2 + z_3 z_4 = 0 \\ z_1^3 + z_3^2 + z_4^2 = 0 \end{cases} \quad \text{vs} \quad \begin{cases} z_1^2 + z_2^2 + z_3 z_4 = 0 \\ b_1 z_1^2 z_2 + b_2 z_3^2 + b_3 z_4^2 + (hot) = 0 \end{cases}$$

$4 = 2+2 \;,\; 6 = 3\cdot 2 + 0 \cdot 4 \qquad\qquad 4 = 2+2 \;,\; 6 = 2\cdot 2 + 2$

$\boxed{\text{MAIN RESULTS}}$ [C, Popescu-Pampu, Stepanov]

THM 1: Elementary combinatorial proof of NW Theorem. (via Tropical geometry)

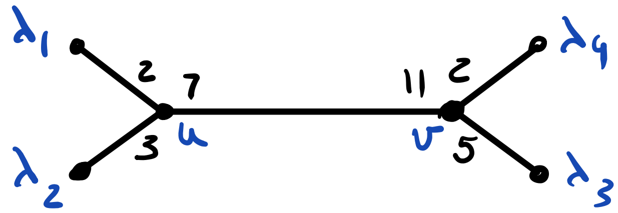
THM 2: $S(\Gamma)$ is a Newton non-degenerate CI system in the Khovanskii sense.

- $\{f_{v,i}\}_{v,i}$ is a regular sequence in $\mathbb{C}\{z_1, \ldots, z_n\}$;
- for each $w \in (\mathbb{R}_{>0})^n$ $\{in_w f_{v,i} = 0\}_{v,i}$ defines a normal crossings divisor in a neighborhood of $\bigcap_{v,i} V(in_w f_{v,i}) \subseteq (\mathbb{C}^*)^n$ if nonempty. [equiv. $\{\nabla in_w f_{v,i}\}_{v,i}$ are lin indep at every pt of this intersection]

THM 3: We view $\Gamma \subseteq \Delta_{n-1} = (n-1)$ standard simplex in $\mathbb{R}^n$ & consider the fan $\mathcal{F}_\Gamma = \mathbb{R}_{\geq 0}\Gamma$. Then, the birational map $\pi_{\mathcal{F}_\Gamma} : \underbrace{\mathcal{X}_{\mathcal{F}_\Gamma}}_{\text{Toric variety}} \longrightarrow \mathbb{C}^n$ induces a modification $\pi: \tilde{X} = \pi^{-1}(X) \longrightarrow X = V(S(\Gamma))$ such that $(\tilde{X}, \partial \mathcal{X}_{\mathcal{F}_\Gamma} \cap \tilde{X})$ is a Toroidal pair ($\rightsquigarrow$ resolve $\tilde{X}$ by Toric modifications!)

- Thm 3 extends result of [de Felipe, González-Pérez, Mourtada (2021)] = "Any germ of a plane curve can be resolved by one Toric modification after reembedding its ambient sm. germ of surface in some $(\mathbb{C}^n, 0)$"

EXAMPLE:



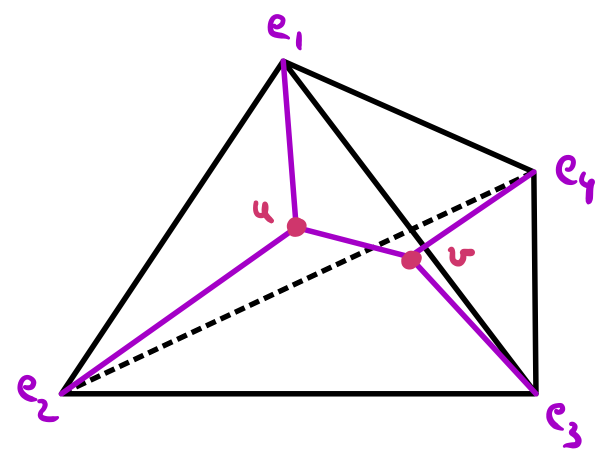
$$S(\Gamma) = \begin{cases} z_1^2 + z_2^3 + z_3 z_4 = 0 \\ z_1 z_2^4 + z_3^5 + z_4^2 = 0 \end{cases}$$

$\Gamma \hookrightarrow \Delta_3 \subseteq \mathbb{R}^4$

$v \longmapsto \dfrac{W_v}{|W_v|}$ $\quad v = \text{node}$

$\lambda_i \longmapsto e_i$ $\quad \lambda_i = \text{leaf}$

+ extend linearly along edges



$W_u = (21, 14, 12, 30)$

$W_v = (30, 20, 22, 55)$

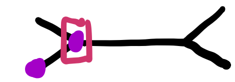
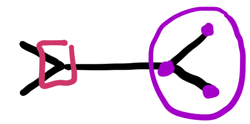
$|W_u| = 77$, $|W_v| = 127$

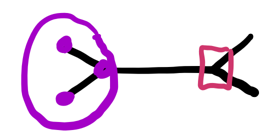
- Strong convexity properties ensure this map is injective (inductive procedure = simplices from "star-full subtrees of Γ")
- Check initial forms of 2 equations for w in $\Gamma \cap \Delta_3^\circ$ or in $\Delta_3^\circ \setminus \Gamma$

(*) $\begin{cases} \text{- 5 edges} \\ \text{- 2 nodes} \end{cases}$

get a monomial in $\mathbb{C}$-span of some $\{in_w f_{v,i}\}_i$ (so $\emptyset$ intersection in $(\mathbb{C}^*)^4$)

(*) In all 7 cases $\{in_w(f_u), in_w(f_v)\}$ is a regular sequence in $\mathbb{C}\{z_1, \ldots, z_4\}$

$f_u =$	z_1^2	$+\ z_2^3$	$+\ z_3 z_4$		$in_w(f_u)$ (terms with MIN w-weight)	
w_u-wt	42	42	42	(d_u)	f_u	
$[\lambda_1,u]$-wt	$2\alpha+(1-\alpha)\frac{42}{77}$	$(1-\alpha)\frac{42}{77}$	$(1-\alpha)\frac{42}{77}$		$z_2^3 + z_3^2 z_4$	
$[\lambda_2,u]$-wt	$(1-\alpha)\frac{42}{77}$	$>$	$(1-\alpha)\frac{42}{77}$		$z_1^2 + z_3^2 z_4$	
w_v-wt	60	60	77	($l_{u,v}$)		
$[\lambda_3,v]$-wt	$\not<$	$\not<$	$>$		$z_1^2 + z_2^3$	
$[\lambda_4,v]$-wt	$\not<$	$\not<$	$>$			
$[u,v]$-wt	$\not<$	$\not<$	$>$			

$f_v =$	$z_1 z_2^4$	$+\ z_3^5$	$+\ z_4^2$		$in_w(f_v)$	
w_v-wt	110	110	110	(d_v)	f_v	
w_u-wt	77	60	60	($l_{u,v}$)		
$[\lambda_1,u]$	$>$	$\not<$	$\not<$		$z_3^5 + z_4^2$	
$[\lambda_2,u]$	$>$	$\not<$	$\not<$			
$[u,v]$	$>$	$\not<$	$\not<$			
$[\lambda_3,v]$	$\not<$	$>$	$\not<$		$z_1 z_2^4 + z_4^2$	
$[\lambda_4,v]$	$\not<$	$\not<$	$>$		$z_1 z_2^4 + z_3^5$	

Main Technique: Local Tropicalization (PP-S (2013))

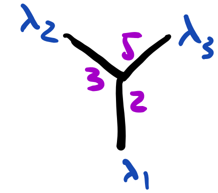
- **Slogan**: Version of tropical geometry in the local setting ($\mathcal{O} = \mathbb{C}\{z_1, \ldots z_n\}$ = ring)
- $(X,0) \subseteq (\mathbb{C}^n, 0)$ defined by an ideal $I \subseteq \mathcal{O}$
- $\mathrm{Trop}(X,0) = \{ w \in (\mathbb{R}_{\geq 0})^n : \mathrm{in}_w(I) = \langle \mathrm{in}_w f : f \in I \rangle$ has no monomial$\}$
- $\mathrm{Trop}_>(X,0) = \{ w \in (\mathbb{R}_{>0})^n : \rule{6cm}{0.4pt} \}$

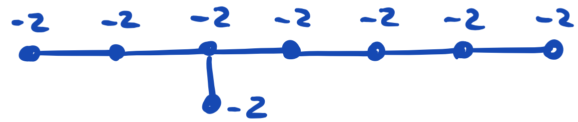
- **Key example**: If I is generated by polynomials $\{f_1, \ldots f_s\}$, set $Y = V(f_1, \ldots f_s)$

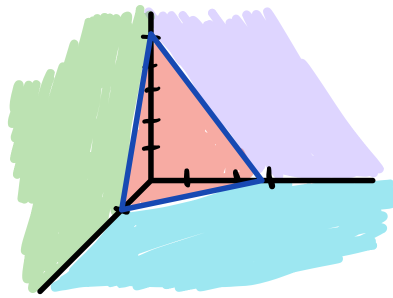
$$\mathrm{Trop}(X,0) = \mathrm{Trop}(Y \cap (\mathbb{C}^*)^n) \cap \mathbb{R}^n_{\geq 0}$$ (restrict global trop to $\mathbb{R}^n_{\geq 0}$)

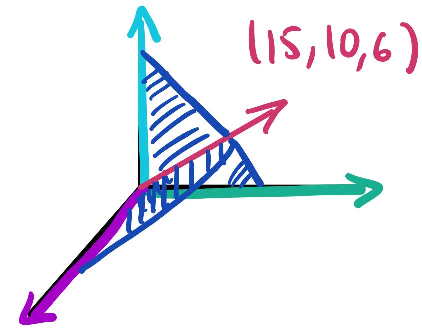
Theorem: [P-P, S]
- $\mathrm{Trop}(X,0)$ is a fan in $\mathbb{R}^n_{\geq 0}$ of dimension $= \dim X$ (local Bieri-Groves)
- If $(X,0)$ is pure dim'l, then $\mathrm{Trop}_>(X,0)$ is a balanced fan along codim-1 faces (if top-faces are weighted appropriately as in the global case)

Ex: $X = V(f)$ $\quad \mathrm{Trop}(X,0)$ = codim 1 skeleton of Newton fan of f.

Ex: (E_8 - singularity) $z_1^2 + z_2^3 + z_3^5 = 0$ $\quad \Gamma =$ 

Resolution dual graph 

Newton polyhedron: 

2-skeleton of Newton fan 

= local tropicaliz. $= \mathcal{F}_\Gamma$

Q: What does $\mathrm{Trop}(X,0)$ know about $(X,0)$?

Theorem [C, P-P, S] (Local Version of Tevelev's Lemma)

Assume $(X,0) \hookrightarrow (\mathbb{C}^n,0)$ is the closure of $X \cap (\mathbb{C}^*)^n$ & X is equidim. Fix a fan $\mathcal{F}$ with $|\mathcal{F}| \subseteq \mathbb{R}^n_{\geq 0}$ & let $\pi_{\mathcal{F}} : \mathcal{X}_{\mathcal{F}} \longrightarrow \mathbb{C}^n$ be the toric morphism defined by $\mathcal{F}$.

Consider $\tilde{X}$ the strict transform of X by $\pi_{\mathcal{F}}$ & the restriction $\pi : \tilde{X} \to X$

Then: (1) π is proper $\iff |\mathcal{F}| \supseteq \mathrm{Trop}(X,0)$

(2) If $|\mathcal{F}| \supseteq \mathrm{Trop}(X,0)$, then:

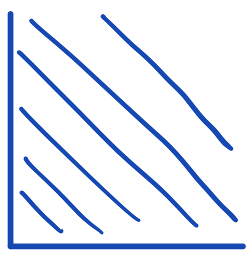
$|\mathcal{F}| = |\mathrm{Trop}(X,0)| \iff \mathrm{codim}_{\tilde{X}}(\mathcal{O}_\sigma \cap \tilde{X})$ is the expected one.
$= \dim \sigma$

Def: A <u>standard tropicalizing fan</u> for $(X,0) \hookrightarrow (\mathbb{C}^n,0)$ is a fan $\mathcal{F}$ with $|\mathcal{F}| = |\mathrm{Trop}(X,0)|$ & with $\mathrm{in}_w I$ constant along $w \in \tau^\circ$ for each cone τ of $\mathcal{F}$

Theorem [P-P,S] Standard tropicalizing fans always exist. (construct tropical bases $\sim$ enhanced standard bases)

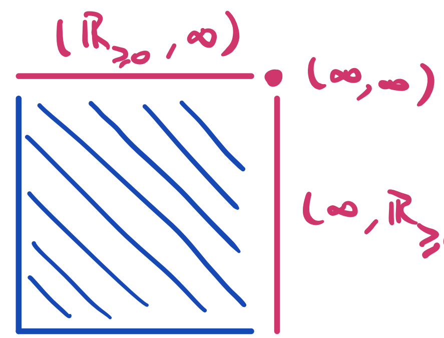
- Extended Version is needed to study $X \cap V(z_{i_1},\ldots,z_{i_k})$ ($\sim$ Kajiwara-Payne global extended Trop)

Ex



$\mathrm{Trop}(\mathbb{C}^2,0) = (\mathbb{R}_{\geq 0})^2$

vs



Extended $\mathrm{Trop}(\mathbb{C}^2,0)$

3 boundary strata

Theorem [C, P-P, S] The cone $\mathcal{F}_\Gamma$ over the embedded splice diagram $\Gamma \subseteq \Delta_{n-1}$ is a standard tropicalizing fan for $S(\Gamma)$.

Theorem [C, P-P, S] The cone $\mathcal{F}_\Gamma$ over the embedded splice diagram $\Gamma \subseteq \Delta_{n-1}$ is a standard tropicalizing fan for $S(\Gamma)$.

Proof outline:

1. $|\mathrm{Trop}(X,0)| \hookrightarrow |\mathcal{F}_\Gamma|$: by stellar subdivisions of Δ_{n-1} using $\frac{w_v}{|w_v|}$ exploiting convexity properties of subtrees of Γ, removing simplices until only $\Gamma \cap \Delta^\circ_{n-1}$ remains.

   Then: use extended tropicalization to conclude $|\mathrm{Trop}(X,0) \cap \partial\Delta_{n-1}| =$ standard rays. & $\mathrm{Trop}(X,0)$ is pure of dim $= 2$.

2. $|\mathcal{F}_\Gamma| \hookrightarrow |\mathrm{Trop}(X,0)|$: by balancing condition (after assigning weights to top-cones of $\tilde{\mathcal{F}}_\Gamma$.)

3. $\mathcal{F}_\Gamma$ is a standard tropicalizing fan (check $\{in_w f_{v,i}\}_{v,i}$ is a reg. seq. in $\mathbb{C}\{\underline{z}\}$ & use a Lemma of NW that implies $\{in_w f_{v,i}\}_{v,i}$ generate the initial ideal $in_w(\langle S(\Gamma)\rangle)$.

Proof of NND: Explicit computation & analysis of $\{in_w f_{v,i}\}$ for w in each cell of $\Gamma \cap \mathbb{R}^n_{>0}$.