# Non-Archimedean Combinatorics

María Angélica Cueto
(Columbia University, USA)

8th ANZMC - 2014

## §1. Setup:

Fix $(K, |\cdot|)$ alg. closed, <u>complete</u> non-Arch. field

$|\cdot| : K \longrightarrow \mathbb{R}_{\geq 0}$ (absolute value)

$|a| = 0 \iff a = 0.$

$|ab| = |a| \cdot |b|$

$|a+b| \leq \max\{|a|, |b|\}$ (non-Arch $\Delta$-ineq.)

$= \text{ if } |a| \neq |b|$

$\iff$ valuation $-\log(|\cdot|) : K \longrightarrow \mathbb{R} \cup \{+\infty\}$.

<u>E.g.</u> (1) $|\cdot|_K$ trivial $(|x| = 1 \ \forall x \neq 0)$.

(2) $K = \widehat{\overline{\mathbb{C}((t))}}$ t-norm = lowest exponent.

(3) $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ p-adic norm $\left[ \frac{a}{b} = p^r \frac{u}{v} \quad (u:p) = (v:p) = 1 \rightsquigarrow \left|\frac{a}{b}\right|_p = p^{-r} \right]$

- <u>ISSUE</u>: Topology on $K$ is <u>totally disconnected</u>.

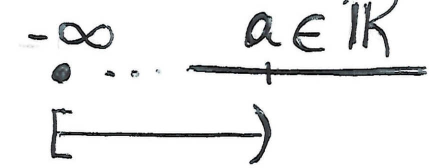
## §2. (Extended) Tropicalizations:

[$val_K$ non-trivial, $1 \in im(val_K)$]

Fix $X$ variety / $K$ **AND** $X \xhookrightarrow[cl.]{i} Y_\Delta$ = toric variety

(e.g. $Y_\Delta = \mathbb{G}_m^n = (K^*)^n,\ \mathbb{A}^n,\ \mathbb{P}^{n-1}$)

**Def:** $\mathrm{Trop}(X, i) = \text{closure}\{\text{image of } \underset{\in X(K)}{p} \longmapsto (-val(p_1), \dots, -val(p_n))\} \subseteq \mathrm{Trop}(Y_\Delta)$

- $\mathrm{Trop}(\mathbb{G}_m^n) = \mathbb{R}^n$ (Eucl. top.)
- $\mathrm{Trop}(\mathbb{A}^n) = \overline{\mathbb{R}}^n$ $\quad \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ 

(Prod. top.)
- $\mathrm{Trop}(\mathbb{P}^{n-1}) = \dfrac{\overline{\mathbb{R}}^n \smallsetminus \{(-\infty, \dots, -\infty)\}}{\mathbb{R} \cdot \underline{1}} := \mathbb{TP}^{n-1}$ (Quot. top.)

$\rightsquigarrow \mathrm{Trop}(Y_\Delta) := \bigsqcup_{\sigma \leq \Delta} \mathrm{Trop}(\mathcal{O}_\sigma)$

E.g. $\mathbb{A}^1 \hookrightarrow \mathbb{A}^2$ [line $y = x+1$]

$x \longmapsto (x, x+1) \rightsquigarrow (-\mathrm{val}(x), -\mathrm{val}(x+1))$

(A) $-\mathrm{val}(x) < 0 = -\mathrm{val}(1)$

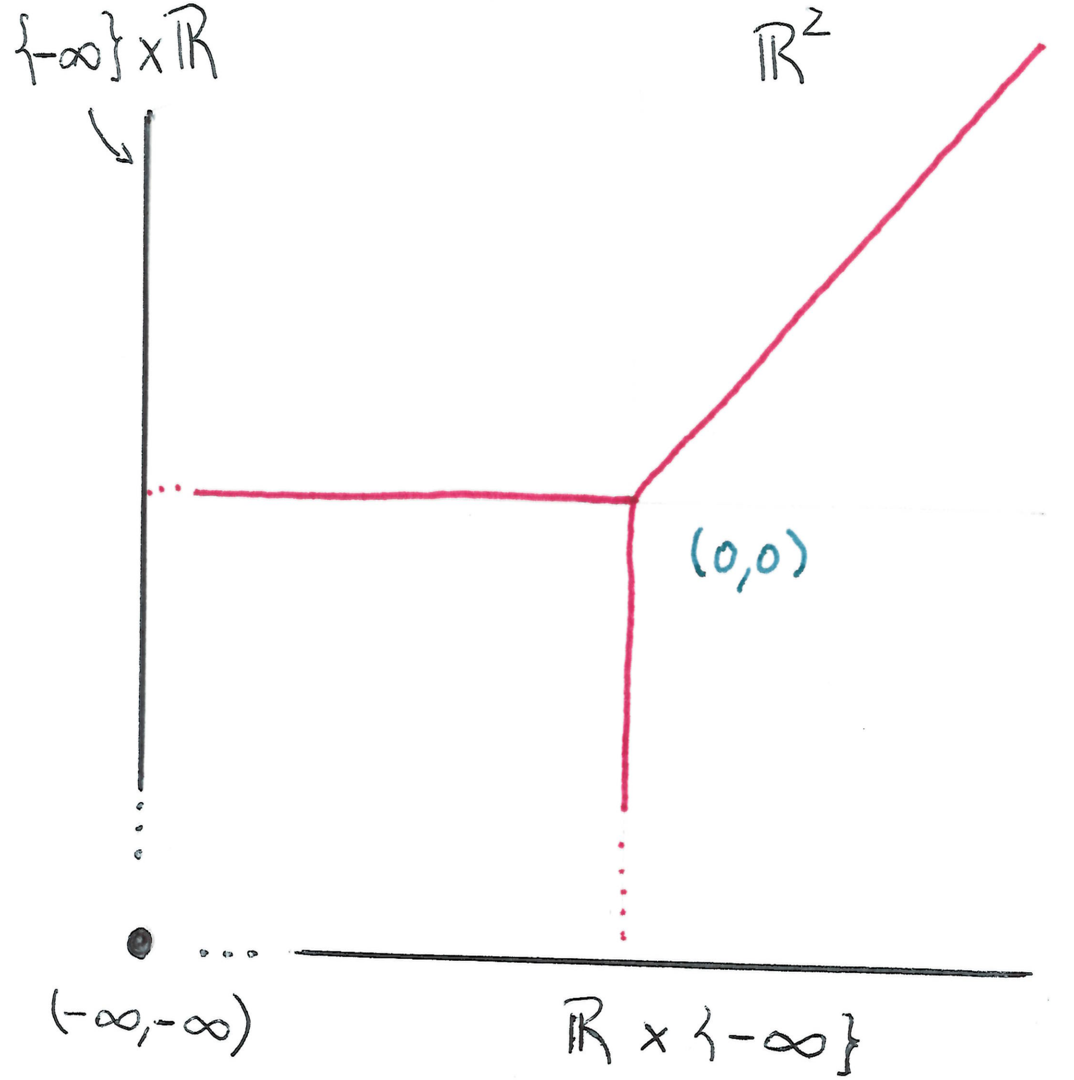
$\rightsquigarrow -\mathrm{val}(x+1) = 0$

(B) $-\mathrm{val}(x) > 0$

$\rightsquigarrow -\mathrm{val}(x+1) = -\mathrm{val}(x)$

(C) $-\mathrm{val}(x) = 0$

$\rightsquigarrow -\mathrm{val}(x+1) \leq 0$



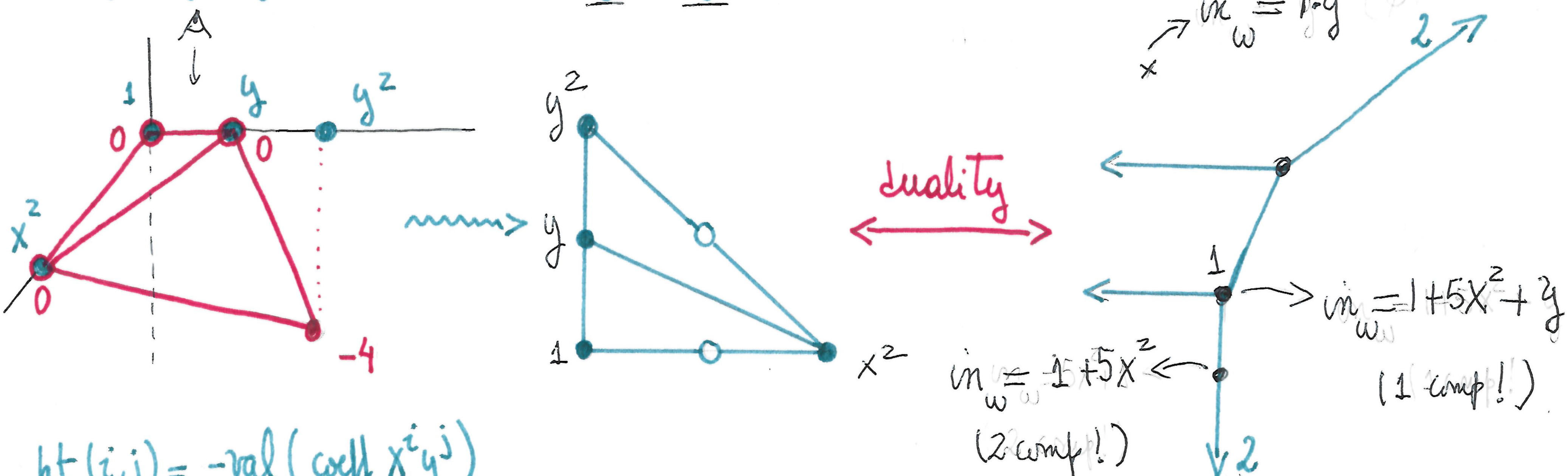
# Fund. Thm. of Trop. Geom.: (Kapranov, Speyer-Sturmfels, Draisma, Payne)

Let $I \subseteq K[y_1^{\pm}, \dots, y_n^{\pm}]$ be the defining ideal of $i(X) \cap \mathbb{G}_m^n$. Then,

$$\mathrm{Trop}(X \cap i^{-1}(\mathbb{G}_m^n), i) = \{ \underline{\omega} \in \mathbb{R}^n \mid \mathrm{in}_{\underline{\omega}}(I) = \langle \mathrm{in}_{\underline{\omega}}(g) : g \in I \rangle \neq 1 \}.$$

[ $\mathrm{in}_{\underline{\omega}}(I) \subseteq \tilde{K}[y_1^{\pm}, \dots, y_n^{\pm}]$ , $\tilde{K} = \{\mathrm{val} \geq 0\} / \{\mathrm{val} > 0\}$ ]

E.g. $g(x,y) = (5+t)\underline{x}^2 + t^4 \underline{y}^2 + \underline{y} + \underline{1}$ $\rightsquigarrow$ curve in $\mathbb{A}^2$.




$ht(i,j) = -\mathrm{val}(\mathrm{coeff}\ x^i y^j)$

$\rightsquigarrow m_\omega := \#\{\text{components of } \mathrm{in}_\omega(I)\}$ (counted w/ mult.)

- **Thm** (Bieri–Groves '94) (Sturmfels '02)

  $\mathrm{Trop}(X \cap i^{-1}(\mathbb{G}_m^n), i)$ is a polyhedral subcomplex of the Gröbner complex of $I$. It's pure of $\dim = \dim X$ & balanced at codim-1 cells.

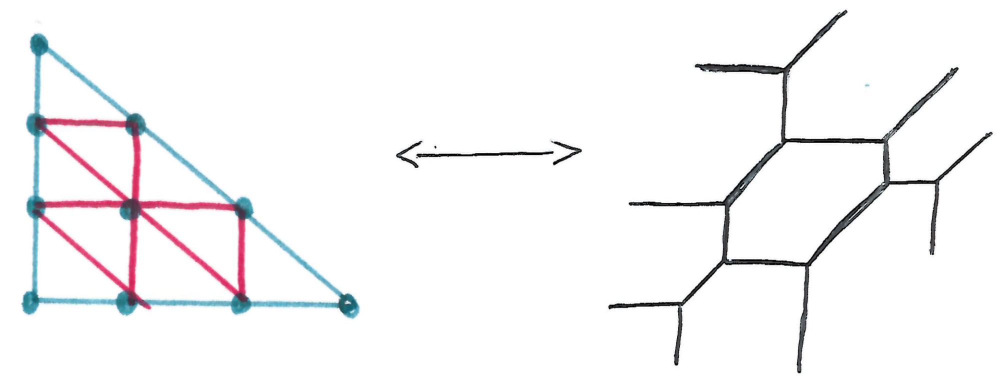
- **Thm** (Tevelev '07): Any $X \underset{cl.}{\subseteq} \mathbb{G}_m^n$ define over $K$ w/ trivial valuation has a (tropical) compactification in a smooth T.V. $Y_\Delta$ of $\mathbb{G}_m^n$ with "nice" divisorial boundary $\overline{X} \setminus X$. Moreover, $\Delta$ is supported on $\mathrm{Trop}(X, i)$

- Applications:
  - Enumerative geometry (Mikhalkin's Correspondence Thm ['05])
  - Implicitization problems
  - Mirror symmetry (Gross-Siebert program)
  - ...

Fix $E \subseteq \mathbb{P}^2$ smooth elliptic cubic $/K$ $\rightsquigarrow$ genus 1, $j(E) \in K$.

- **Thm** [Katz-Markwig$^2$ '07] $\mathrm{Trop}(E \cap \mathbb{G}_m^2)$ contains no cycle or it's cycle has length $\leq -\mathrm{val}(j(E))$. Equality holds if the trop. curve is trivalent.

- **Thm** [Chan-Sturmfels '13] $E$ admits a re-embedding in $\mathbb{P}^2$ into honeycomb form achieving $=$.

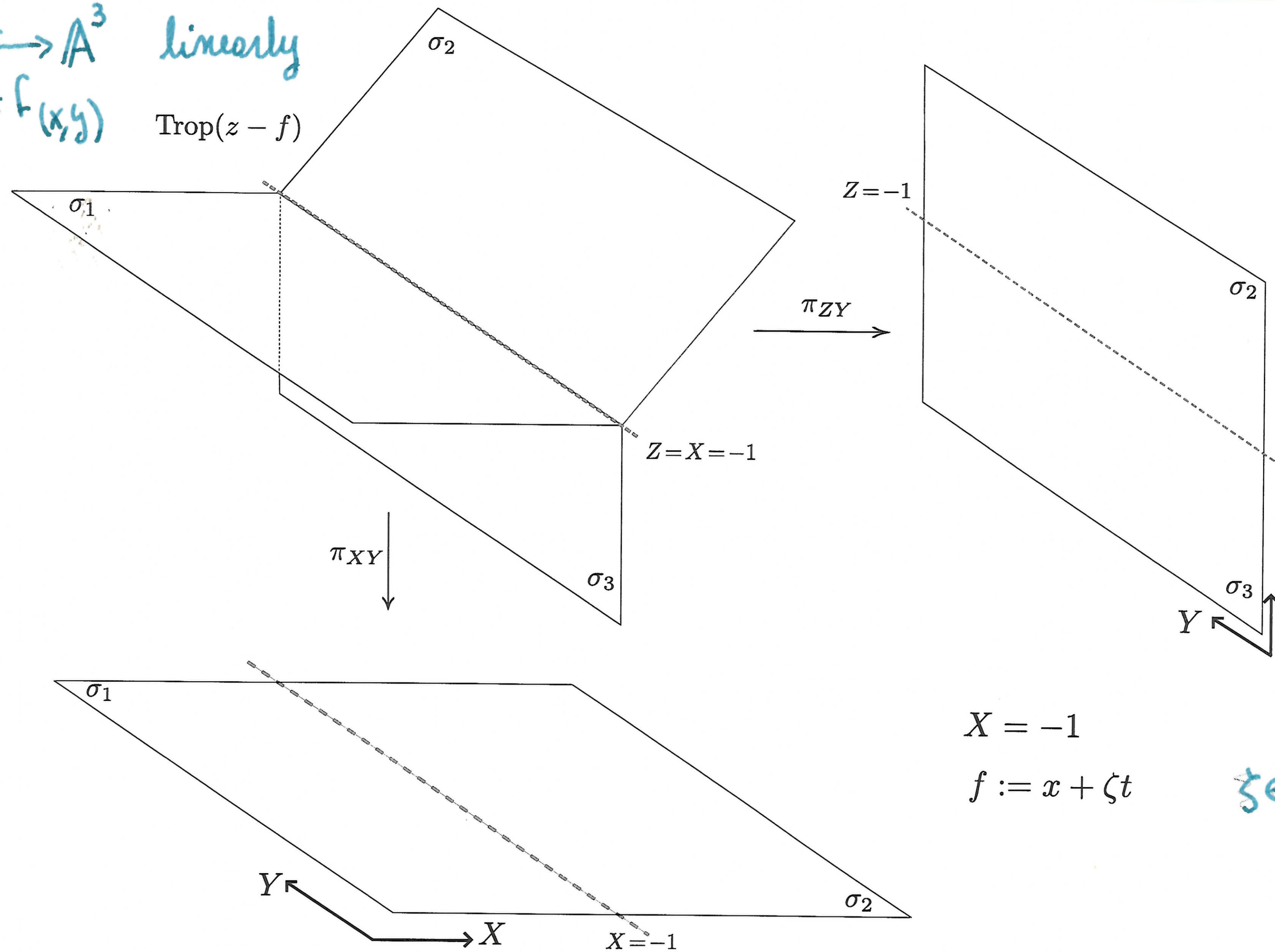


- **Thm** [C.-Markwig '14] Can effectively re-embed $E$ linearly in $\dim \leq 4$ and produce a tropical curve with the expected cycle length.

# Linear trop. modification of $\mathbb{R}^2$ along $\{X = l\}$

$\mathbb{A}^2 \hookrightarrow \mathbb{A}^3$ linearly

via $z = f_{(x,y)}$

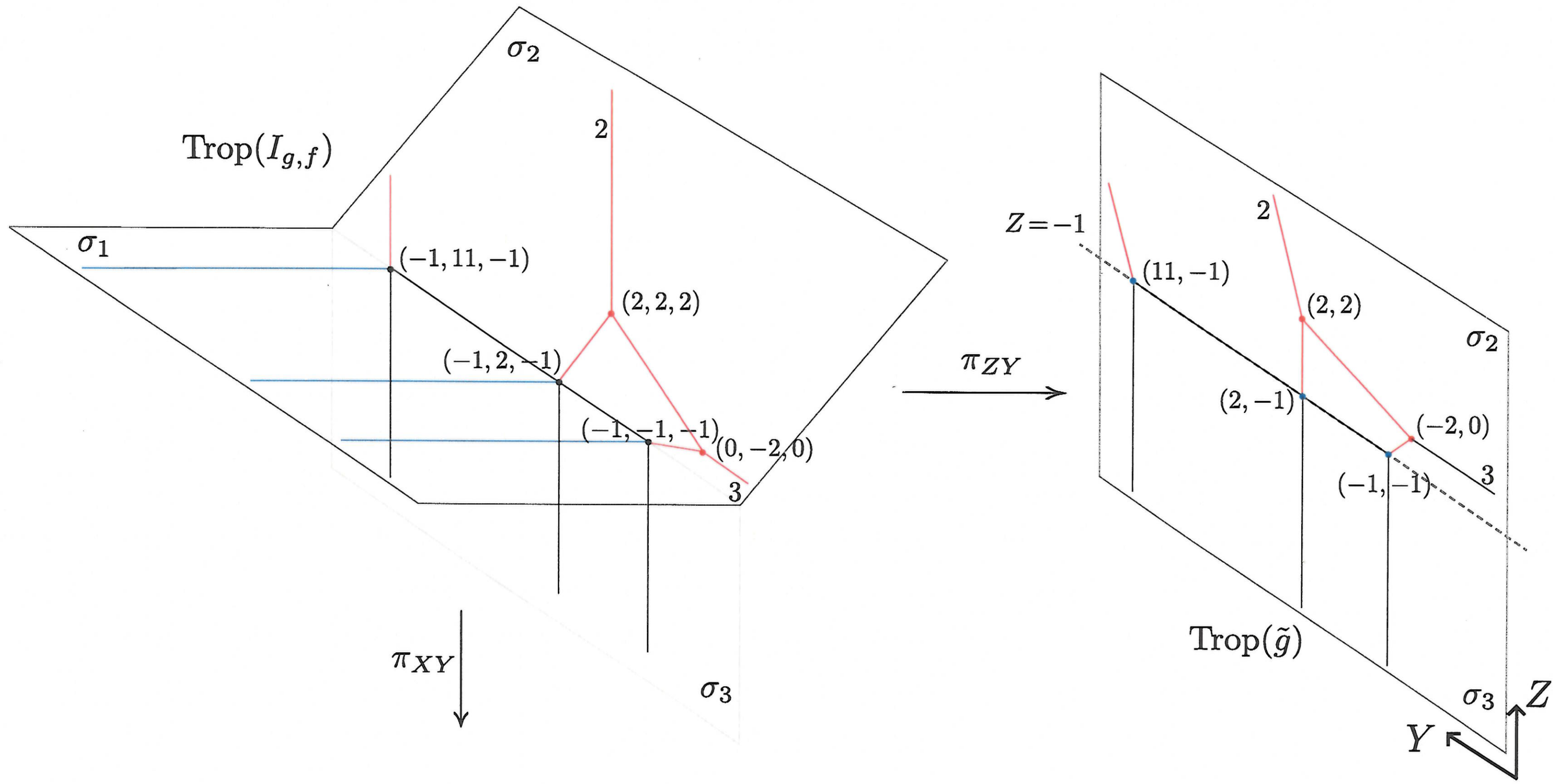


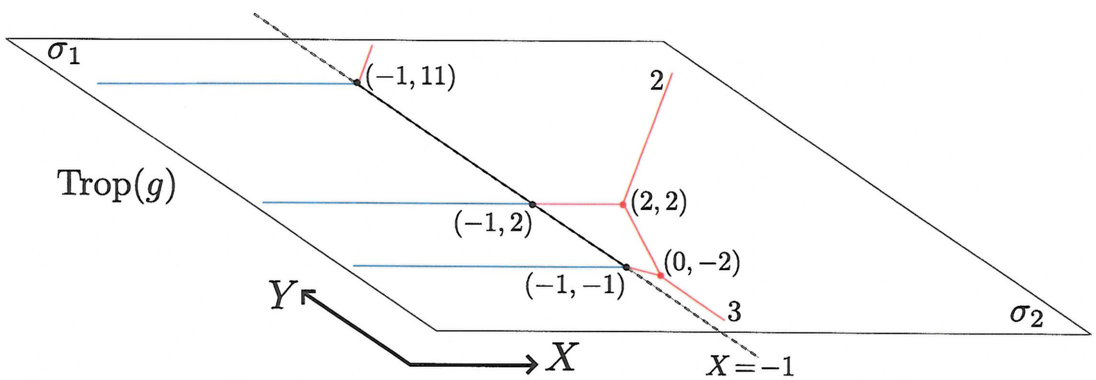

$X = -1$

$f := x + \zeta t$ $\quad \zeta \in \mathbb{C}^\times$

$$\sigma_1 = \{X \leqslant l, Z = l\},\ \sigma_2 = \{X \geqslant l, Z = X\},\ \sigma_3 = \{X = l, Z \leqslant l\}.$$

# Generic modification of a plane cubic along $\{X = I\}$



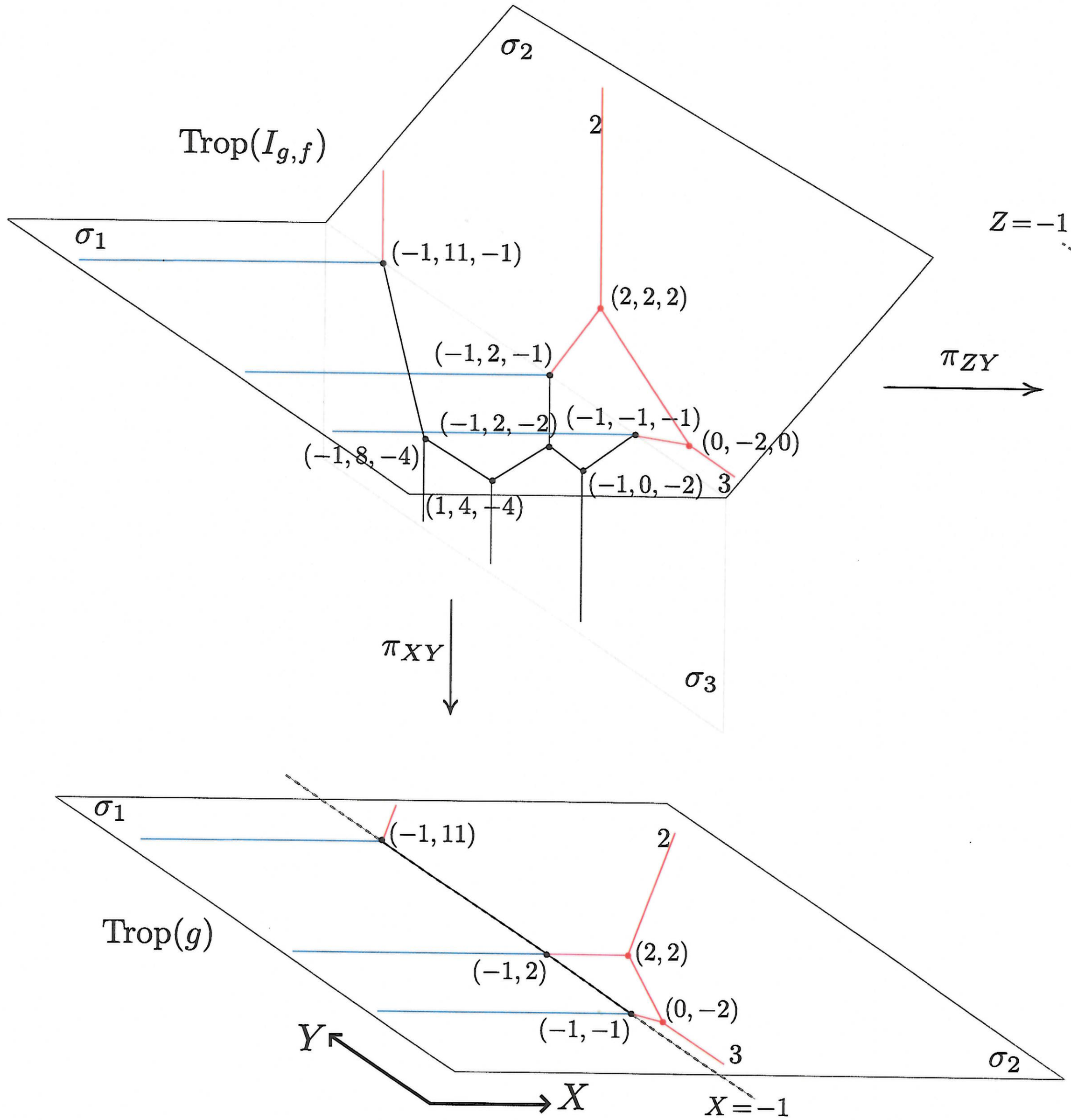

$X = -1$

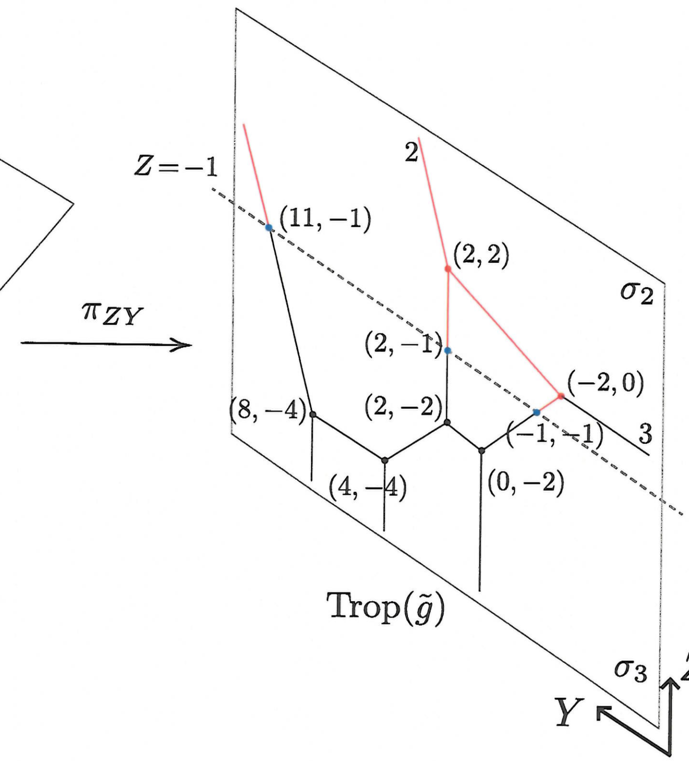
$f := x + \zeta t$

$\zeta$ generic

$$\sigma_1 = \{X \leqslant I, Z = I\},\ \sigma_2 = \{X \geqslant I, Z = X\},\ \sigma_3 = \{X = I, Z \leqslant I\}.$$

# Special modification of a plane cubic along $\{X = I\}$




$X = -1$

$f := x + 1t$ ⇝ linear

$I_{g,f} := \langle g, Z - f_{(x,y)} \rangle$

$$\sigma_1 = \{X \leqslant I, Z = I\},\ \sigma_2 = \{X \geqslant I, Z = X\},\ \sigma_3 = \{X = I, Z \leqslant I\}.$$

## §3. Berkovich's analytification: [ $K=\overline{K}$ complete, non-Arch. field ]

<u>Idea</u>: $X$ scheme of finite type$/K$ $\xrightarrow{\text{FUNCTORIAL}}$ $X^{an}$ = top. space + sheaf of analytic fncs

- Assume $X = \mathrm{Spec}(A)$ $\quad A = $ f.g. $K$-algebra (& glue!)

- <u>Def</u>: $X^{an} := \{ \|\cdot\| : A \longrightarrow \mathbb{R}_{\geq 0}$ mult. <u>seminorms</u> extending $|\cdot|_K \}$
  $(\|a\|=0 \not\Rightarrow a=0)$

  Topology: weakest s.t. $\{ ev_f : X^{an} \longrightarrow \mathbb{R}_{\geq 0} \quad \|\ \| \longmapsto \|f\| \}_{f\in A}$ are <u>all</u> cont.

- <u>Note</u>: $X(K) \hookrightarrow X^{an}$ $\quad p \longmapsto (\|\ \|_p : f \longmapsto |f(p)|_K)$

- <u>Thm</u> (Berkovich '90)
  - $X^{an}$ is loc. compact, loc. path connected
  - $X^{an}$ is Hausdorff $\Longleftrightarrow$ $X/K$ is separated
  - If $|\cdot|_K$ is non-trivial, then $X(K)$ is dense in $X^{an}$.

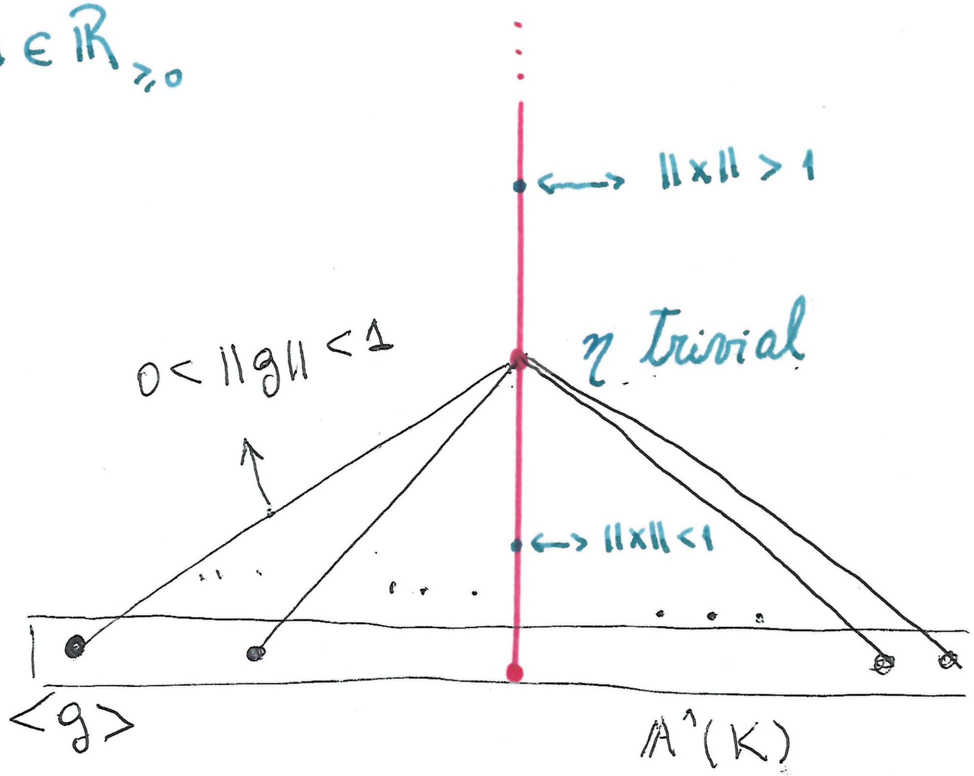
- <u>Fact</u>: $X^{an} = \bigsqcup_{\mathfrak{z}\in X} \{(K_{\mathfrak{z}}, \nu)$ extending $(K, val_K)\}$.

<u>Example</u>: (1) $\mathbb{A}^1 = \mathrm{Spec}(K[x])$ $\quad | \ |_K$ trivial

$f = \sum_{i=m}^{N} c_i x^i \quad (c_m, c_N \neq 0) \quad \rightsquigarrow \|x\| \in \mathbb{R}_{\geq 0}$

$$\|f\| = \begin{cases} \|x\|^N & \text{if } \|x\|>1, \\ \|x\|^m & \text{if } \|x\|<1, \\ ? & \text{if } \|x\|=1 \end{cases}$$

$\mathfrak{P} = \{ f : \|f\| < 1 \} = \langle g \rangle$
mxl ideal

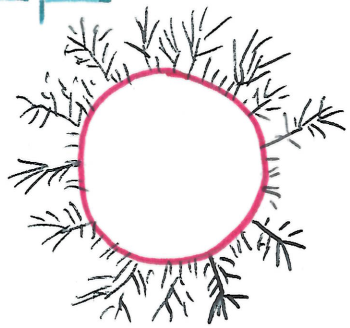



$f = g^s h \quad (g : h) = 1 \implies \|f\| = \|g\|^s \cdot 1 \quad \longleftrightarrow \quad \| \ \| = (\langle g \rangle, \|g\|)$

(2) $| \ |_K$ non-trivial $\rightsquigarrow (\mathbb{A}^1)^{an}$ is a (metric) $\mathbb{R}$-tree with <u>dense</u> set of branched pts.

(3) $X/K$ curve $\rightsquigarrow X^{an}$ locally homeo. to $(\mathbb{A}^1)^{an}$ with <u>global topology</u> captured by a <u>finite graph</u> (it's <u>skeleton</u>).

**Example:** $E$ elliptic curve $/K$ with mult. reduction, i.e. $-\mathrm{val}(j_{(E)})>0$)



Skeleton $= S^1$, with length $= -\mathrm{val}(j_{(E)})$.

**In general:** $X$ smooth $\Rightarrow X^{an}$ contains a PL structure (skeleton) & it deformation retracts to it.

- If $X \overset{i}{\hookrightarrow} Y_\Delta \supseteq \mathbb{G}_m^n$ with coords. $y_1, \dots, y_n$

$$\begin{tikzcd}
X^{an} \arrow[drr, Rightarrow, "(\mathrm{trop}, i)"] \arrow[dr, phantom, "\circlearrowleft"] & & \\
X(K) \arrow[u, hook] \arrow[rr] & & \mathrm{Trop}(X,i) \subseteq \mathrm{Trop}(Y_\Delta)
\end{tikzcd}$$

CONT. & SURJ.

$\| \ \| \longmapsto (\log\|y_1\|, \dots, \log\|y_n\|)$

$p \longmapsto (\log|p_j|)_{1\le j\le n}$

- **Thm.** (Payne '09) $X^{an} \underset{\text{homeo}}{\cong} \varprojlim_i \mathrm{Trop}(X,i)$

$$\begin{tikzcd}
X \arrow[r, hook, "i"] \arrow[dr, "i'"'] & Y_\Delta \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] & \supseteq \mathbb{G}_m^n \arrow[d] \\
& Y_{\Delta'} & \supseteq \mathbb{G}_m^r
\end{tikzcd}$$

equivariant.

**Q:** $\exists$ cont. section $\sigma$ to $(\mathrm{trop}, i) : X^{an} \twoheadrightarrow \mathrm{Trop}(X, i)$ ?

[Baker-Payne-Rabinoff ('11)] $m \equiv 1$ on $\mathrm{Trop}(X, i)$ is enough if $\dim X = 1$.

**Thms.** (C-Häbich-Werner '14)

① The Tropicalization of $Gr(2,n) \xhookrightarrow{\text{Plücker}} \mathbb{P}^{\binom{n}{2}-1}$ has a cont. section $\sigma$

② All trop. multiplicities are 1.

③ $\forall\, w \in \mathrm{Trop}\, Gr(2,n)$ : $\mathrm{trop}^{-1}(w) \subseteq Gr(2,n)^{an}$ has a $\underline{!}$ distinguished pt. $\rho$ $(= \sigma(w))$ satisfying $\|f\| \leq \rho(f)$ $\forall\, \|\cdot\| \in \mathrm{trop}^{-1}(w)$ $\forall f \in K[Gr(2,n)]$

**Thm** (Gubler-Rabinoff-Werner '14) If $X \overset{i}{\hookrightarrow} \mathbb{G}_m^n$ and <u>all</u> trop mult are 1, then $(\mathrm{trop}, i)$ admits a cont. section as in ③.

# §4. The tropical Grassmannian of planes:

$$Gr(2,n) := \mathbb{A}^{2\times n}_{rk=2} / GL_2 \xhookrightarrow{\varphi} \mathbb{P}^{\binom{n}{2}-1} \qquad \text{coords } (p_{ij})_{i<j}$$
$$X \longmapsto \Lambda^2 X = 2\times 2\text{-minors}$$

is defined by the Plücker relns. $\{ p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \;:\; i<j<k<l \}$

**Thm** (Speyer-Sturmfels '04) $\operatorname{Trop} \varphi^{-1}\left(\mathbb{G}_m^{\binom{n}{2}} / \mathbb{G}_m\right) \subseteq \mathbb{R}^{\binom{n}{2}} / \mathbb{R}\mathbb{1}$ is the space of phylogenetic trees on $n$ leaves of Billera-Holmes-Vogtmann.

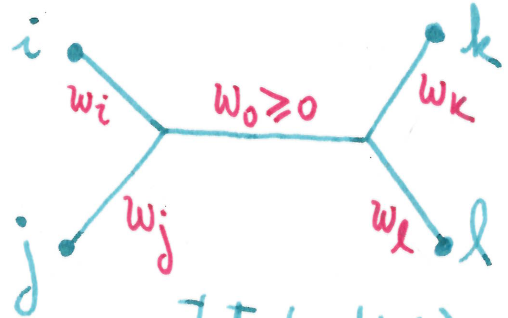
$$(T,w) = \begin{cases} T: \text{graph w/ no cycle \& no deg-2 vertices; leaves labeled 1 through } n. \\ w: E(T) \longrightarrow \mathbb{R} \;\; \& \;\; w(e) \geq 0 \text{ if } e \in E(T) \text{ not adj. to any leaf.} \end{cases}$$

$\rightsquigarrow$ fan structure: $\mathcal{C}$ cones $\longleftrightarrow$ $T$ tree

$\mathcal{C}_T \subseteq \mathcal{C}_{T'} \longleftrightarrow$ $T$ is a coarsening of $T'$.

( E.g. $\mathcal{C}_{{}^1_2 \times {}^3_4} \subseteq \mathcal{C}_{{}^1_2 \succ\!\!-\!\!\prec {}^3_4}$ )

**Why?** $(T,\omega) \rightsquigarrow \underline{x} \in \mathbb{R}^{\binom{n}{2}}$ as $x_{pq} = \sum_{e: p \to q} \omega(e) \quad (p<q)$




quartet $(ij|k\ell)$

$$\rightsquigarrow \begin{cases} x_{ij} = w_i + w_j \\ x_{ik} = w_i + w_0 + w_k \end{cases}, \text{ etc}$$

- $\underline{x}$ satisfies the 4-pt conditions = tropical Plücker relns :

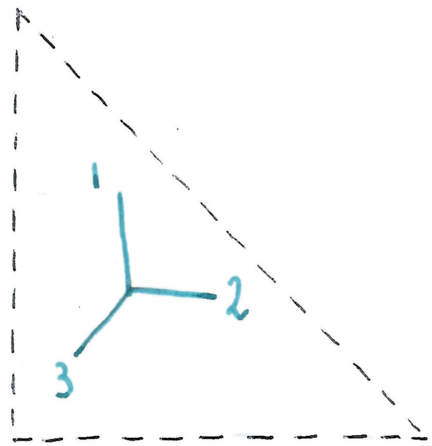
$$\max\{x_{ij} + x_{k\ell},\ x_{ik} + x_{j\ell},\ x_{i\ell} + x_{jk}\} \text{ is attained twice (at least)}$$

Claim : $(T,\omega) \xleftrightarrow{1-\text{to}-1} \underline{x}$ satisfying 4 pt. conditions

Pf./ (1) $\max\{x_{ij} + x_{k\ell},\ \underline{x_{ik} + x_{j\ell}},\ \underline{x_{i\ell} + x_{jk}}\} \iff$ quartet $(ij|k\ell)$

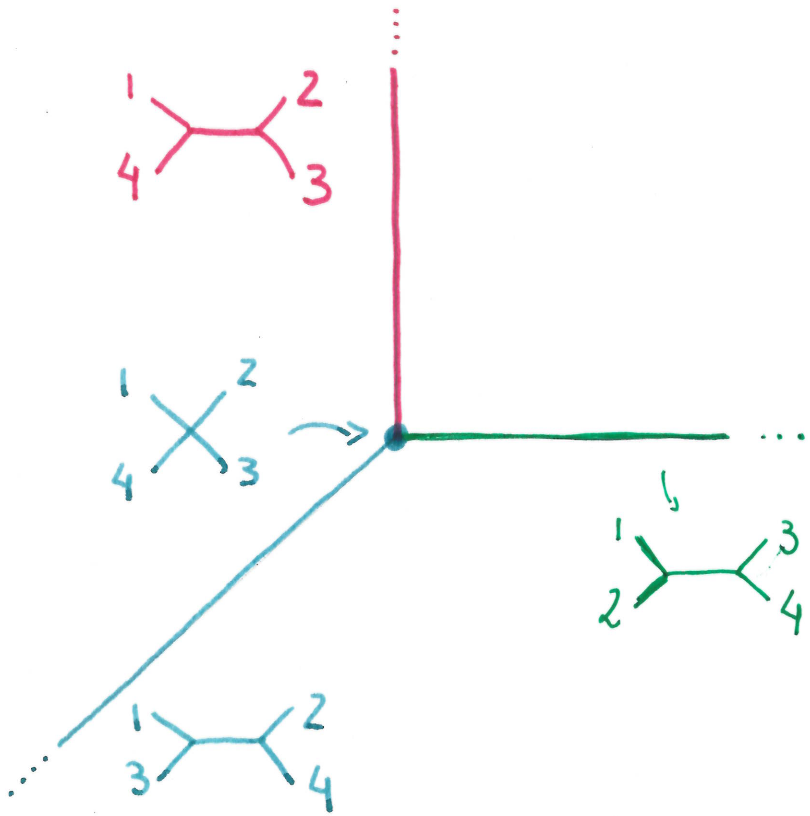
(2) Build T from quartets (so, from $\underline{x}$).

(3) From $x$ & T, can recover $\omega$ using linear algebra.




$= \mathbb{R}^3 / \mathbb{R}\cdot 1$

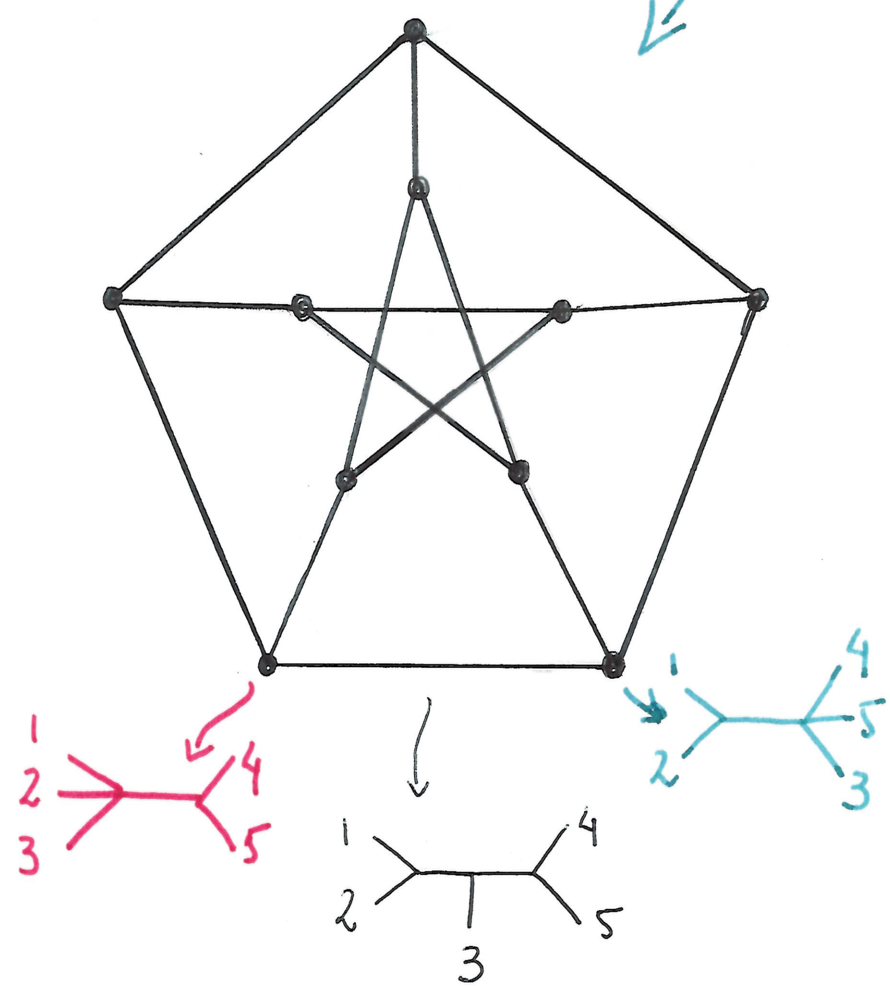
n=3




$\times \mathbb{R}^4 / \mathbb{R}\cdot 1$

n=4

n=5 $\frac{\mathbb{R}^5}{\mathbb{R}\cdot 1} \times$ (cone over the Petersen graph)



**Q**: What about non-real pts? $\rightsquigarrow$ $\operatorname{Trop} Gr(2,n) = \bigcup_{T \in \mathcal{T}_n} \overline{\mathcal{C}_T} \subseteq \mathbb{TP}^{\binom{n}{2}-1}$

Boundary strata = ? $\rightsquigarrow$ $Gr_J(2,n) := \varphi^{-1}\{ p_{ij} = 0 \iff ij \in J \}$, $J \subseteq \binom{[n]}{2}$.

(matroid stratif. of Gelfand-Goresky-MacPherson-Serganova ['87])

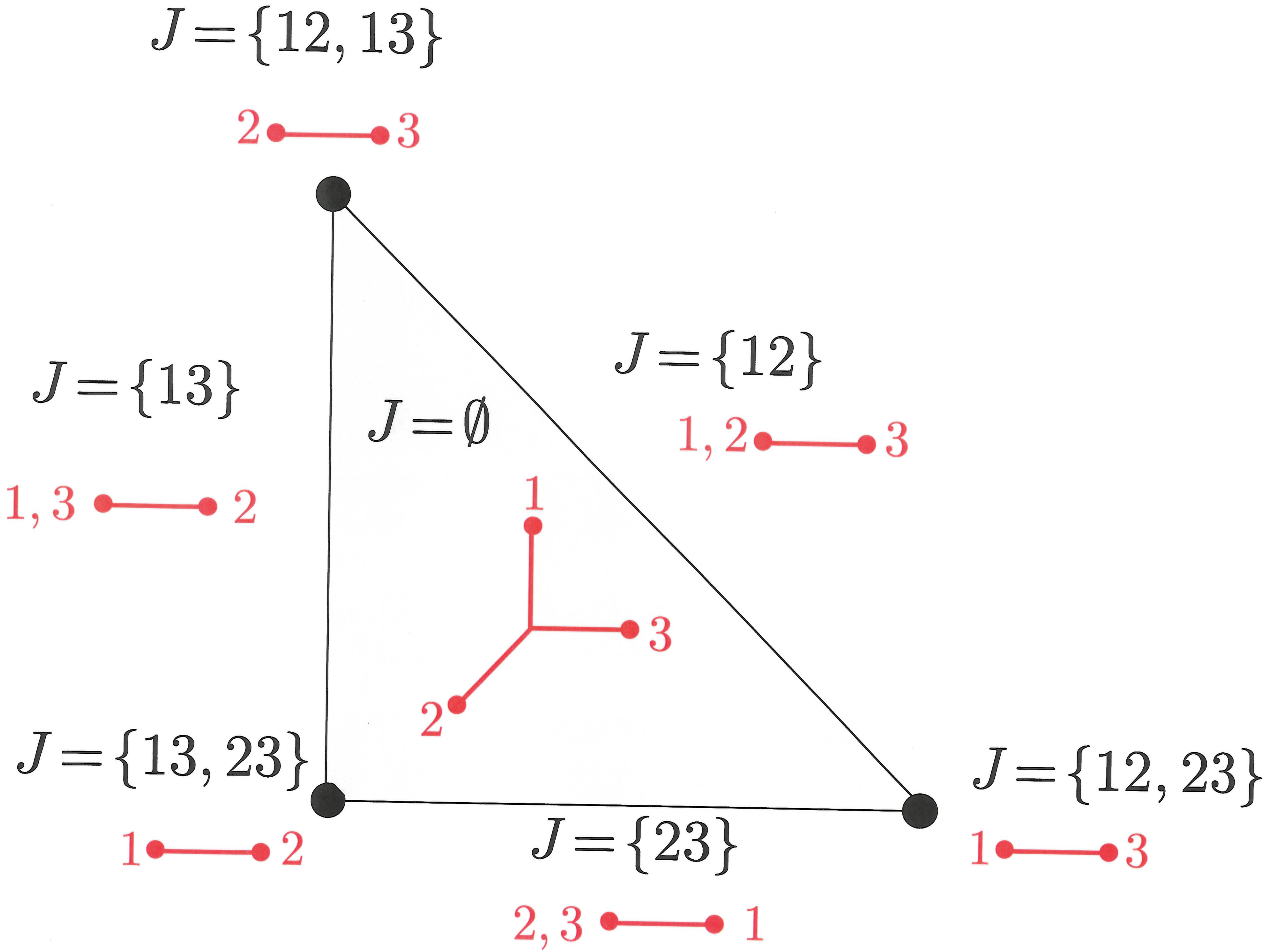
- $J = \emptyset$ gives $\varphi^{-1}\left(\mathbb{G}_m^{\binom{n}{2}} / \mathbb{G}_m\right) =: Gr_0(2,n)$

- $Gr_J(2,n) \neq \emptyset \iff J =$ pairs of l.d. cols of a $2 \times n$-matrix. $X$.

  Can write $X = \left( \underset{\text{l.d.}}{\boxed{B_1}} \overset{li}{\frown} \underset{\text{l.d.}}{\boxed{B_2}} \cdots \boxed{B_m} \; \boxed{0} \right) \rightsquigarrow$ pt in $Gr_0(2, m)$

**Thm** (C. '14). $\operatorname{Trop} Gr_J(2,n) = \mathcal{T}_m$ with labels $B_1, \ldots, B_m$ on the leaves.

- These cell-complex structures are compatible & glue to $\overline{\mathcal{T}_n}$.



$J = \{12, 13\}$
2
3
$J = \{13\}$
1, 3
2
$J = \emptyset$
1
3
2
$J = \{12\}$
1, 2
3
$J = \{13, 23\}$
1
2
$J = \{23\}$
2, 3
1
$J = \{12, 23\}$
1
3