Faithful tropicalization for the Grassmannian of planes

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Non-Archimedean Berkovich spaces

- Fix $(K, |\cdot|)$ complete non-Archimedean field, $|\cdot|\colon K \to \mathbb{R}_{\geqslant 0}$

(1) $|a| = 0 \iff a = 0$
(2) $|ab| = |a||b|$ (multiplicative)
(3) $|a + b| \leqslant \max\{|a|, |b|\}$ (with $=$ if $|a| \neq |b|$) (non-Arch. triangle ineq.)

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- **Topology:** coarsest s.t. all $\mathrm{ev}_f : \|\cdot\| \mapsto \|f\|$ $(f \in A)$ are continuous.
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Example: Skeleton (semi) norm on $(\mathbb{A}^n)^{an}$ for each $\rho \in \overline{\mathbb{R}}^n$.

$$\delta(\rho)\colon K[x_1, \ldots, x_n] \to \mathbb{R}_{\geqslant 0} \quad \sum_\alpha c_\alpha x^\alpha \longmapsto \max_\alpha\{|c_\alpha| \exp(\sum_{i=1}^n \alpha_i\, \rho_i)\}.$$

$\delta(\rho)(x_i) = \exp(\rho_i)$ and it is **maximal** with this property.
Note: If $\rho_i \neq -\infty$, we can extend $\delta(\rho)$ to $K[x_1, \ldots, x_i^{\pm}, \ldots, x_n]$.

Analytification is the limit of all tropicalizations [Payne]

Fix X=K-scheme of fin. type and $X \xhookrightarrow[\text{cl.}]{i} Y_\Sigma$ (TV with dense torus $\mathbb{G}_m^n$).
Assume $i(X)$ meets $\mathbb{G}_m^n$ and write $\{y_1, \ldots, y_n\}$ basis of characters.

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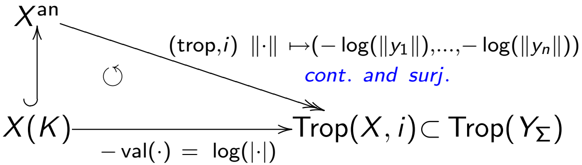

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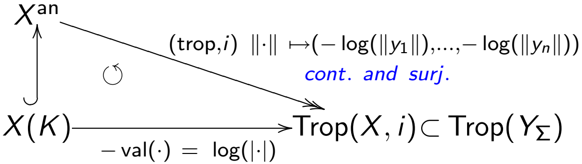

Question (after [Payne]): Does there exist a continuous section $\sigma\colon \text{Trop}(X,i) \to X^{\text{an}}$ to (trop, i)? If so, i induces a faithful tropicalization.

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X(K) & \xrightarrow[-\text{val}(\cdot)\ =\ \log(|\cdot|)]{} & \text{Trop}(X,i) \subset \text{Trop}(Y_\Sigma)
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cont. and surj.

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Theorem (C.-Häbich-Werner)

The Grassmannian $\text{Gr}(2, n)$ *of 2-planes in* $\mathbb{A}^n$ *is tropicalized faithfully by the Plücker map. The cont. section* $\sigma\colon\ \text{Trop}\,\text{Gr}(2, n) \to \text{Gr}(2, n)^{\text{an}}$ *to* trop *maps a pt.* x *to the unique Shilov boundary point in* $\text{trop}^{-1}(x)$ *and all trop. mult. are 1. The image of* σ *is a candidate canonical polyhedron.*

Grassmannian of 2-planes in $\mathbb{A}^n$ and the space of trees

- The Plücker map φ embeds $\mathrm{Gr}(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ by the list of 2×2-minors:

$$\varphi(X) = [p_{ij} := \det(X^{(i,j)})]_{i<j} \qquad \forall\, X \in \mathrm{Gr}(2,n) := \mathbb{A}^{2\times n}_{\mathrm{rk}\,2} / \,\mathrm{GL}(2).$$

Its Plücker ideal $I_{2,n}$ is generated by the 3-term (quadratic) Plücker eqns:

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \qquad (1 \leqslant i < j < k < l \leqslant n).$$

Note: $\mathbb{G}_m^n/\mathbb{G}_m$ acts on $\mathrm{Gr}(2,n)$ via $t * (p_{ij}) = t_i t_j\, p_{ij}$.

- Write $\mathrm{Gr}_0(2,n) := \varphi^{-1}(\mathbb{G}_m^{\binom{n}{2}}/\mathbb{G}_m)$ (proj. dim $= 2(n-2)$).

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Theorem (Speyer-Sturmfels)

The (open) tropical Grassmannian $\mathrm{Trop}(\mathrm{Gr}_0(2,n))$ *in* $\mathbb{R}^{\binom{n}{2}}/\mathbb{R}\cdot\mathbf{1}$ *is the space of phylogenetic trees on n leaves:*

- *all leaves are labeled 1 through n (no repetitions);*
- *weights on all edges (non-negative weights for internal edges).*

It is cut out by the tropical Plücker equations. The lineality space is generated by the n cut-metrics $\ell_i = \sum_{j\neq i} e_{ij}$, *modulo* $\mathbb{R}\cdot\mathbf{1}$.

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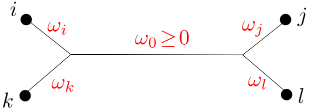
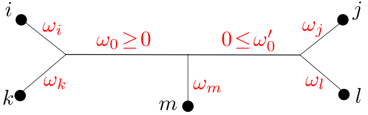


$(ij|kl)$

$$\begin{cases} x_{ik} = \omega_i + \omega_k, \\ x_{ij} = \omega_i + \omega_0 + \omega_j, \ldots \end{cases}$$



$(ik|jl) \cap (im|jl) \cap (km|jl) \cap \ldots$

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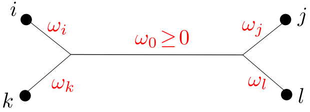

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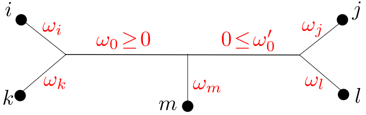

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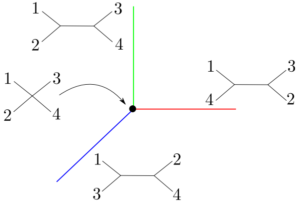
Claim: $(T, \omega) \xleftrightarrow{1-\text{to}-1}$ $\mathbf{x}$ satisfying Tropical Plücker eqns.

Why? (1) $\max\{x_{ik} + x_{jl}, \underline{x_{ij} + x_{kl}}, \underline{x_{il} + x_{jk}}\} \iff$ quartet $(ik|jl)$.

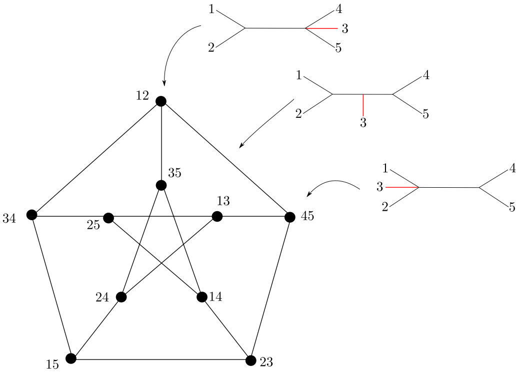
(2) tree T is reconstructed form the list of quartets,

(3) linear algebra recovers the weight function ω from T and $\mathbf{x}$.

Examples:



$\mathcal{T}_4/\mathbb{R}^3$ has f-vector $(1, 3)$.



$\mathcal{T}_5/\mathbb{R}^4$ is the cone over the Petersen graph.
f-vector $= (1, 10, 15)$.

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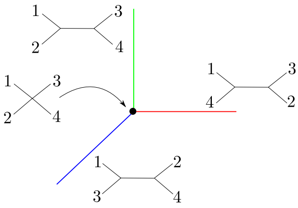

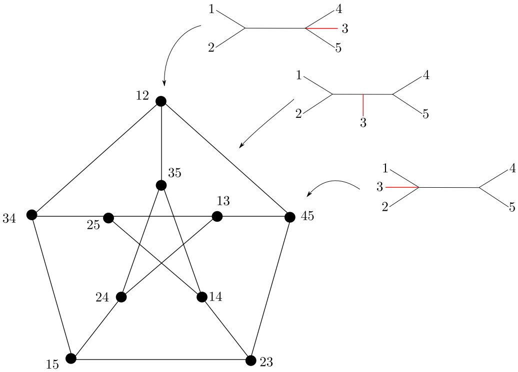

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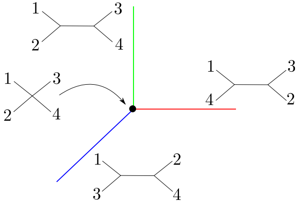

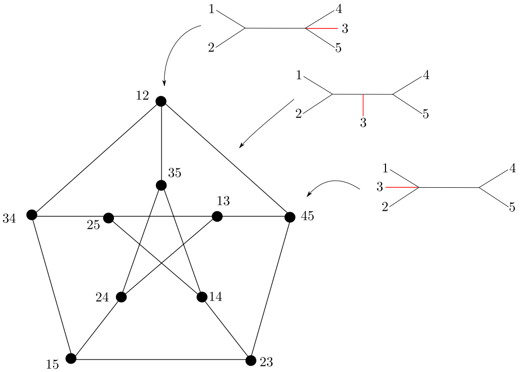

$\mathcal{T}_4/\mathbb{R}^3$ has f-vector $(1, 3)$. $\mathcal{T}_5/\mathbb{R}^4$ is the cone over the Petersen graph. f-vector $= (1, 10, 15)$.

Question: What is the boundary structure on $\operatorname{Trop}\operatorname{Gr}(2, n) = \bigcup_{T \in \mathcal{T}_n} \overline{\mathscr{C}_T}$?

Answer: Use the matroid stratification of $\operatorname{Gr}(2, n)$ [GGMS]:

$$\operatorname{Gr}_J(2, n) := \varphi^{-1}(\{p_{ij} = 0 \iff ij \in J\}) \quad \text{for } J \subset \binom{[n]}{2}$$

We can view $\operatorname{Gr}_J(2, n) \subset \mathbb{G}_m^{\binom{n}{2} - |J|}/\mathbb{G}_m$.

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- **Why?** $U_{ij} = \operatorname{Spec} R_{ij}$ for $R_{ij} := K[u_{ik}, u_{jk} : k \neq i,j] \rightsquigarrow \exists$ skeleton norms!

Idea: Adapt skeleton norm of U_{ij} to (T, J) given $x \in \overline{\mathscr{C}_T} \cap \operatorname{TropGr}_J(2,n)$. Concretely, for any $x \in \operatorname{Trop} U_{ij}$, find $I = I(ij, T, J) \subset \binom{[n]}{2}$ giving $2(n-2)$ alg. indep. coords. on $R(ij)$ and $\forall\, k \neq i,j$ either ik or $jk \in I$. Write

$$K[u_{kl} : kl \in I] \hookrightarrow R(ij) \xhookrightarrow{\theta} K[u_{kl} : kl \in I][u_{kl}^{-1} : kl \in H \smallsetminus J],$$

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Maximality prop: $\sigma^{(ij)}(x)(f) \geqslant \rho(f)\ \forall f \in R(ij)$ and $\rho \in \operatorname{trop}^{-1}(x)$.

Cell structure on the compact Grassmannian

Use the matroid stratification $\{\mathrm{Gr}_J(2,n) : J \subset \binom{[n]}{2}\}$:
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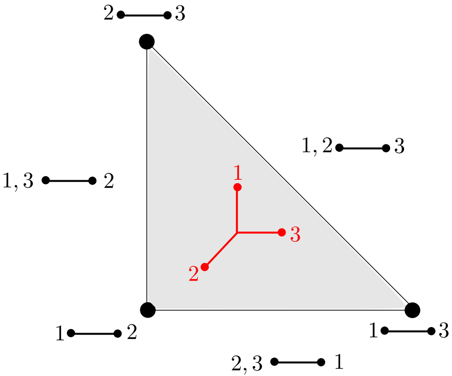
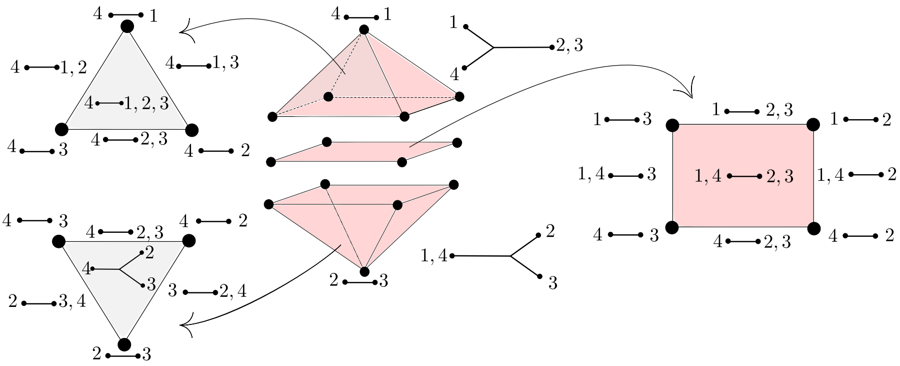
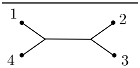


$\overline{\mathcal{T}_3}$

f-vector $(3,3,1)$



Boundary cells in $\overline{(14|23)}$ = 

f-vector of $\overline{\mathcal{T}_4} = (6, 12, 11, 7, 3)$.