Combinatorial Aspects of Tropical Geometry and its interactions with phylogenetics

María Angélica Cueto

Department of Mathematics
Columbia University

Rabadan Lab Metting
Columbia University College of Physicians and Surgeons

May 4th 2015

What is tropical geometry?

- Trop. semiring $\overline{\mathbb{R}}_{\text{tr}} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, $a \oplus b = \max\{a,b\}$, $a \odot b = a + b$.
- Fix $K = \mathbb{C}\{\{t\}\}$ field of Puiseux series, with valuation given by **lowest exponent**, e.g. $\text{val}(t^{-4/3} + 1 + t + \dots) = -4/3$, $\text{val}(0) = \infty$.

What is tropical geometry?

- Trop. semiring $\overline{\mathbb{R}}_{\text{tr}} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, $a \oplus b = \max\{a, b\}$, $a \odot b = a + b$.
- Fix $K = \mathbb{C}\{\{t\}\}$ field of Puiseux series, with valuation given by **lowest exponent**, e.g. $\text{val}(t^{-4/3} + 1 + t + \dots) = -4/3$, $\text{val}(0) = \infty$.

$$\boxed{F(\mathbf{x}) \text{ in } K[x_1^{\pm}, \dots, x_n^{\pm}] \rightsquigarrow \text{Trop}(F)(\boldsymbol{\omega}) \text{ in } \overline{\mathbb{R}}_{\text{tr}}[\omega_1^{\odot\pm}, \dots, \omega_n^{\odot\pm}]}$$

$$F := \sum_{\alpha} c_\alpha \mathbf{x}^\alpha \mapsto \text{Trop}(F)(\boldsymbol{\omega}) := \bigoplus_{\alpha} -\text{val}(c_\alpha) \odot \boldsymbol{\omega}^{\odot\alpha} = \max_{\alpha}\{-\text{val}(c_\alpha) + \langle \alpha, \boldsymbol{\omega} \rangle\}$$

$(F = 0)$ in $(K^*)^n \rightsquigarrow \text{Trop}(F) = \{\boldsymbol{\omega} \in \mathbb{R}^n : \text{max in Trop}(F)(\boldsymbol{\omega}) \text{ is } \underline{\text{not}} \text{ unique}\}$

What is tropical geometry?

- Trop. semiring $\overline{\mathbb{R}}_{\text{tr}} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, $a \oplus b = \max\{a,b\}$, $a \odot b = a+b$.
- Fix $K = \mathbb{C}\{\{t\}\}$ field of Puiseux series, with valuation given by **lowest exponent**, e.g. $\text{val}(t^{-4/3} + 1 + t + \dots) = -4/3$, $\text{val}(0) = \infty$.

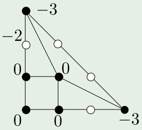
$$\boxed{F(\mathbf{x}) \text{ in } K[x_1^{\pm}, \dots, x_n^{\pm}] \rightsquigarrow \text{Trop}(F)(\boldsymbol{\omega}) \text{ in } \overline{\mathbb{R}}_{\text{tr}}[\omega_1^{\odot\pm}, \dots, \omega_n^{\odot\pm}]}$$

$$F := \sum_{\alpha} c_\alpha \mathbf{x}^\alpha \mapsto \text{Trop}(F)(\boldsymbol{\omega}) := \bigoplus_{\alpha} -\text{val}(c_\alpha) \odot \boldsymbol{\omega}^{\odot \alpha} = \max_{\alpha}\{-\text{val}(c_\alpha) + \langle \alpha, \boldsymbol{\omega} \rangle\}$$

$(F = 0)$ in $(K^*)^n \rightsquigarrow \text{Trop}(F) = \{\boldsymbol{\omega} \in \mathbb{R}^n : \text{max in Trop}(F)(\boldsymbol{\omega}) \text{ is } \underline{\text{not}} \text{ unique}\}$

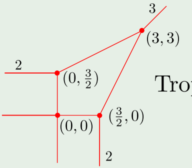
Example: $g = -t^3 x^3 + t^3 y^3 + t^2 y^2 + (4 + t^5) xy + 2x + 7y + (1 + t)$.

Newton subdivision of g



height of $(i,j) = -\text{val}(c_{i,j})$

$\longleftrightarrow$



Trop(g)

Tropical Geometry is a combinatorial shadow of algebraic geometry

Input: $X \subset (K^*)^n$ irred. of dim d defined by an ideal $I \subset K[x_1^{\pm}, \ldots, x_n^{\pm}]$.
Output: Its tropicalization $\operatorname{Trop}(I) = \bigcap_{f \in I} \operatorname{Trop}(f) \subset \mathbb{R}^n$.

Tropical Geometry is a combinatorial shadow of algebraic geometry

Input: $X \subset (K^*)^n$ irred. of dim d defined by an ideal $I \subset K[x_1^{\pm}, \ldots, x_n^{\pm}]$.
Output: Its tropicalization $\text{Trop}(I) = \bigcap_{f \in I} \text{Trop}(f) \subset \mathbb{R}^n$.

- $\text{Trop}(I)$ is a polyhedral complex of *pure* dim. d & connected in codim. 1.
- **Gröbner theory:** $\text{Trop}(I) = \{\omega \in \mathbb{R}^n | \, \text{in}_\omega(I) \neq 1\}$.

Weight of $\omega \in$ mxl cone $= \#\{$ components of $\text{in}_\omega(I)\}$ (with mult.)
With these weights, $\text{Trop}(I)$ is a balanced complex (0-tension condition)

- **Fund. Thm. Trop. Geom.:** $\text{Trop}(I) = \overline{\{(-\text{val}(x_i))_{i=1}^n : x \in X\}}$.

Tropical Geometry is a combinatorial shadow of algebraic geometry

Input: $X \subset (K^*)^n$ irred. of dim d defined by an ideal $I \subset K[x_1^{\pm}, \ldots, x_n^{\pm}]$.
Output: Its tropicalization $\operatorname{Trop}(I) = \bigcap_{f \in I} \operatorname{Trop}(f) \subset \mathbb{R}^n$.

- $\operatorname{Trop}(I)$ is a polyhedral complex of *pure* dim. d & connected in codim. 1.
- **Gröbner theory:** $\operatorname{Trop}(I) = \{\omega \in \mathbb{R}^n \,|\, \operatorname{in}_\omega(I) \neq 1\}$.

Weight of $\omega \in$ mxl cone $= \#\{$ components of $\operatorname{in}_\omega(I)\}$ (with mult.)
With these weights, $\operatorname{Trop}(I)$ is a balanced complex (0-tension condition)

- **Fund. Thm. Trop. Geom.:** $\operatorname{Trop}(I) = \overline{\{(-\operatorname{val}(x_i))_{i=1}^n : x \in X\}}$.

- $(K^*)^r$ action on X via $A \in \mathbb{Z}^{r \times n} \rightsquigarrow$ Row span (A) in all cones of $\operatorname{Trop}(I)$.
$\rightsquigarrow$ Mod. out $\operatorname{Trop}(I)$ by this lineality space preserves the combinatorics.
- The ends of a curve $\operatorname{Trop}(X)$ in $\mathbb{R}^2$ give a compact toric variety $\supset \overline{X}$.

Conclusion: $\operatorname{Trop}(I)$ sees dimension, torus actions, initial degenerations, compactifications and other *geometric invariants* of X (e.g. degree)

Notice: $\operatorname{Trop}(X)$ is highly sensitive to the embedding of X.

Grassmannian of lines in $\mathbb{P}^{n-1}$ and the space of trees

Definition: $\mathrm{Gr}(2,n) = \{\text{lines in } \mathbb{P}^{n-1}\} := K^{2\times n}_{\mathrm{rk}\,2}/\,\mathrm{GL}_2$ $\qquad (\dim = 2(n-2))$.

The Plücker map embeds $\mathrm{Gr}(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ by the list of 2×2-minors:

$$\varphi(X) = [p_{ij} := \det(X^{(i,j)})]_{i<j} \qquad \forall\, X \in K^{2\times n}.$$

Its Plücker ideal $I_{2,n}$ is generated by the 3-term (quadratic) Plücker eqns:

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \qquad (1 \leqslant i < j < k < l \leqslant n).$$

Note: $(K^*)^n/K^*$ acts on $\mathrm{Gr}(2,n)$ via $t * (p_{ij}) = t_i t_j\, p_{ij}$.

Grassmannian of lines in $\mathbb{P}^{n-1}$ and the space of trees

Definition: $\mathrm{Gr}(2,n) = \{\text{lines in } \mathbb{P}^{n-1}\} := K^{2\times n}_{\mathrm{rk}\,2} / \mathrm{GL}_2 \qquad (\dim = 2(n-2))$.

The Plücker map embeds $\mathrm{Gr}(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ by the list of 2×2-minors:

$$\varphi(X) = [p_{ij} := \det(X^{(i,j)})]_{i<j} \qquad \forall\, X \in K^{2\times n}.$$

Its Plücker ideal $I_{2,n}$ is generated by the 3-term (quadratic) Plücker eqns:

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \qquad (1 \leqslant i < j < k < l \leqslant n).$$

Note: $(K^*)^n/K^*$ acts on $\mathrm{Gr}(2,n)$ via $t * (p_{ij}) = t_i t_j\, p_{ij}$.

$\rightsquigarrow$ Tropical Plücker eqns: $\max\{x_{ij} + x_{kl}, x_{ik} + x_{jl}, x_{il} + x_{jl}\}$.

Grassmannian of lines in $\mathbb{P}^{n-1}$ and the space of trees

Definition: $\mathrm{Gr}(2,n) = \{\text{lines in } \mathbb{P}^{n-1}\} := K^{2\times n}_{\mathrm{rk}\,2} / \mathrm{GL}_2$ $\quad (\dim = 2(n-2))$.

The Plücker map embeds $\mathrm{Gr}(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ by the list of 2×2-minors:

$$\varphi(X) = [p_{ij} := \det(X^{(i,j)})]_{i<j} \qquad \forall\, X \in K^{2\times n}.$$

Its Plücker ideal $I_{2,n}$ is generated by the 3-term (quadratic) Plücker eqns:

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \qquad (1 \leqslant i < j < k < l \leqslant n).$$

Note: $(K^*)^n/K^*$ acts on $\mathrm{Gr}(2,n)$ via $t * (p_{ij}) = t_i t_j\, p_{ij}$.

$\rightsquigarrow$ Tropical Plücker eqns: $\max\{x_{ij}+x_{kl}, x_{ik}+x_{jl}, x_{il}+x_{jl}\}$.

Theorem (Speyer-Sturmfels)

The tropical Grassmannian $\mathrm{Trop}(\mathrm{Gr}(2,n) \cap ((K^*)^{\binom{n}{2}}/K^*))$ *in* $\mathbb{R}^{\binom{n}{2}}/\mathbb{R}\cdot\mathbf{1}$ *is the space of phylogenetic trees on n leaves:*

- *all leaves are labeled* 1 *through n (no repetitions);*
- *weights on all edges (non-negative weights for internal edges).*

It is cut out by the tropical Plücker equations. The lineality space is generated by the n cut-metrics $\ell_i = \sum_{j\neq i} e_{ij}$, *modulo* $\mathbb{R}\cdot\mathbf{1}$.

The space of phylogenetic trees $\mathcal{T}_n$ on n leaves

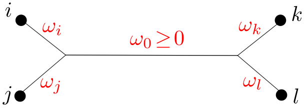
- all leaves are labeled 1 through n (no repetitions);
- weights on all edges (non-negative weights for internal edges).

From the data (T, ω), we construct $\mathbf{x} \in \mathbb{R}^{\binom{n}{2}}$ by $x_{pq} = \sum_{e \in p \to q} \omega(e)$:

The space of phylogenetic trees $\mathcal{T}_n$ on n leaves

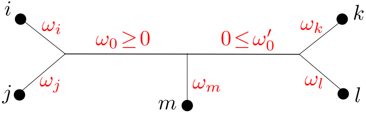
- all leaves are labeled 1 through n (no repetitions);
- weights on all edges (non-negative weights for internal edges).

From the data (T, ω), we construct $\mathbf{x} \in \mathbb{R}^{\binom{n}{2}}$ by $x_{pq} = \sum_{e \in p \to q} \omega(e)$:



$(ij|kl)$

$$\begin{cases} x_{ij} = \omega_i + \omega_j, \\ x_{ik} = \omega_i + \omega_0 + \omega_k, \dots \end{cases}$$

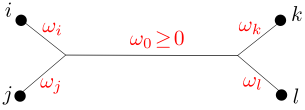


$(ij|kl) \cap (im|kl) \cap (jm|kl) \cap \dots$

The space of phylogenetic trees $\mathcal{T}_n$ on n leaves

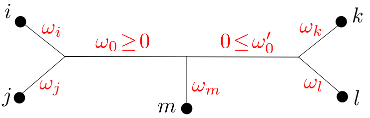
- all leaves are labeled 1 through n (no repetitions);
- weights on all edges (non-negative weights for internal edges).

From the data (T, ω), we construct $\mathbf{x} \in \mathbb{R}^{\binom{n}{2}}$ by $x_{pq} = \sum_{e \in p \to q} \omega(e)$:



$(ij|kl)$

$$\begin{cases} x_{ij} = \omega_i + \omega_j, \\ x_{ik} = \omega_i + \omega_0 + \omega_k, \dots \end{cases}$$



$(ij|kl) \cap (im|kl) \cap (jm|kl) \cap \dots$

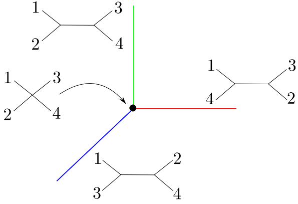
Claim: $(T, \omega) \xleftrightarrow{1-\text{to}-1}$ $\mathbf{x}$ satisfying Tropical Plücker eqns.

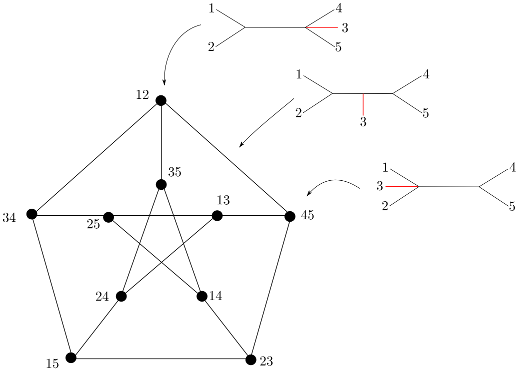
Why? (1) $\max\{x_{ij} + x_{kl}, \underline{x_{ik} + x_{jl}}, \underline{x_{il} + x_{jk}}\} \iff$ quartet $(ij|kl)$.

(2) tree T is reconstructed form the list of quartets,

(3) linear algebra recovers the weight function ω from T and $\mathbf{x}$.

Examples:





$\mathcal{T}_4/\mathbb{R}^3$ has f-vector $(1, 3)$.

$\mathcal{T}_5/\mathbb{R}^4$ is the cone over the Petersen graph.
f-vector $= (1, 10, 15)$.

$$\dim \mathrm{Gr}(2, n) = \dim(\mathrm{Trop}(Gr(2, n) \cap \mathbb{R}^{\binom{n}{2}-1}) = 2(n-2).$$

Constructing nice coordinates for $\mathrm{Gr}(2, n)$ from tree space

• We stratify the classical Grassmannian by collecting points according to the vanishing of prescribed coordinates:

$$\mathrm{Gr}_J(2, n) = \{\mathbf{p} \in \mathbb{P}^{\binom{n}{2}-1} : \mathbf{p}_{kl} = 0 \iff kl \in J\} \qquad \text{for } J \in \binom{[n]}{2}.$$

Example: For $J = \emptyset$ we get $\mathrm{Gr}_\emptyset(2, n) = \mathrm{Gr}(2, n) \cap ((K^*)^{\binom{n}{2}}/K^*)$.

Constructing nice coordinates for Gr$(2, n)$ from tree space

• We stratify the classical Grassmannian by collecting points according to the vanishing of prescribed coordinates:

$$\mathrm{Gr}_J(2,n) = \{\mathbf{p} \in \mathbb{P}^{\binom{n}{2}-1} : \mathbf{p}_{kl} = 0 \iff kl \in J\} \qquad \text{for } J \in \binom{[n]}{2}.$$

Example: For $J = \emptyset$ we get $\mathrm{Gr}_\emptyset(2,n) = \mathrm{Gr}(2,n) \cap ((K^*)^{\binom{n}{2}}/K^*)$.

Remark: Most J will give $\mathrm{Gr}_J(2,n) = \emptyset$. Meaningful J's determine m blocks (of the rank-2 matrix in $K^{2\times n}$) of maximal linear independent columns and a (possibly empty) block of $(0,0)$ columns:

$$\mathrm{Gr}_J(2,n) \ni \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \end{pmatrix} \equiv \left(\, B_1 \,\middle|\, \dots \,\middle|\, B_n \,\middle|\, \mathbf{0} \quad \dots \quad \mathbf{0} \, \right)$$

We identify it with a point in $\mathrm{Gr}_\emptyset(2,m)$ (pick one column per block!).

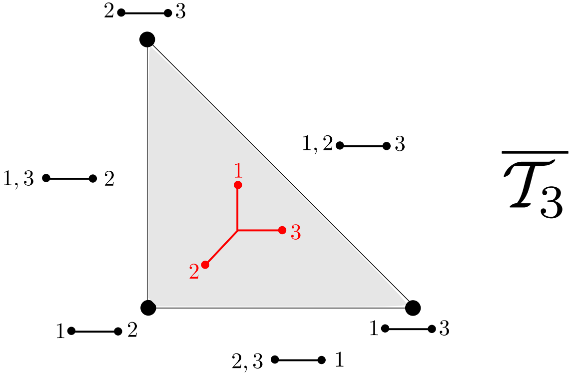
Proposition [C.]: $\mathrm{Trop}(Gr_J(2,n)) = \mathcal{T}_m$ with leaves labeled by $B_1, \dots, B_m$.

How to compactify $\mathcal{T}_n$?

- Write $\mathbb{TP}^{\binom{n}{2}-1} := (\mathbb{R} \cup \{-\infty\})^{\binom{n}{2}} \smallsetminus (-\infty, \ldots, -\infty))/\mathbb{R}\cdot(1, \ldots, 1)$
- Compactify $\mathcal{T}_n$ using $\text{Trop}(\text{Gr}(2, n)) \subset \mathbb{TP}^{\binom{n}{2}-1}$.
- Cell structure? Generalized space of phylogenetic trees [C.].

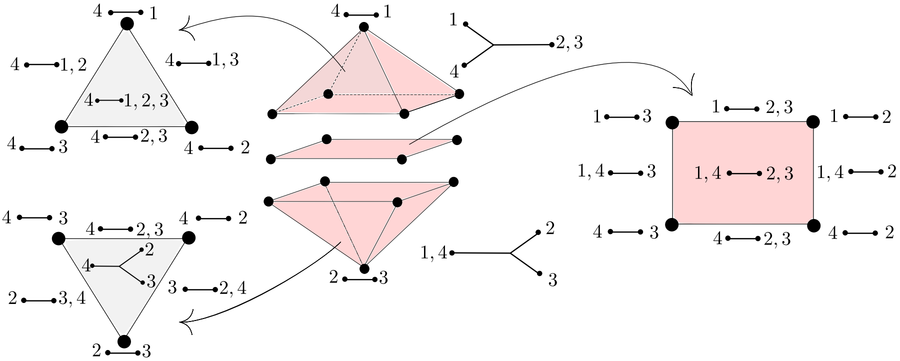
How to compactify $\mathcal{T}_n$?

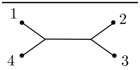
- Write $\mathbb{TP}^{\binom{n}{2}-1} := (\mathbb{R} \cup \{-\infty\})^{\binom{n}{2}} \smallsetminus (-\infty, \dots, -\infty))/\mathbb{R}\cdot(1, \dots, 1)$
- Compactify $\mathcal{T}_n$ using $\text{Trop}(\text{Gr}(2, n)) \subset \mathbb{TP}^{\binom{n}{2}-1}$.
- Cell structure? Generalized space of phylogenetic trees [C.].



How to compactify $\mathcal{T}_n$?

- Write $\mathbb{TP}^{\binom{n}{2}-1} := (\mathbb{R} \cup \{-\infty\})^{\binom{n}{2}} \smallsetminus (-\infty, \dots, -\infty))/\mathbb{R}\cdot(1, \dots, 1)$
- Compactify $\mathcal{T}_n$ using $\text{Trop}(\text{Gr}(2, n)) \subset \mathbb{TP}^{\binom{n}{2}-1}$.
- Cell structure? Generalized space of phylogenetic trees [C.].



Boundary cells in $\overline{(14|23)}$ = 

Choosing coordinates for Gr(2, n): from tropical to classical

Write: $\mathrm{Gr}(2,n) = \bigcup_{i<j} U_{ij}$, where $U_{ij} = \{p \in \mathrm{Gr}(2,n) : p_{ij} \neq 0\}$.

Can fix $p_{ij} = 1$, so

$$\mathrm{Trop}(U_{ij}) = \{x \in \mathrm{Trop}(\mathrm{Gr}(2,n)) : x_{ij} = 0\} \in \overline{\mathbb{R}}^{\binom{n}{2}-1}.$$

Now change coordinates to $\boxed{u_{kl} := p_{kl}/p_{ij}}$ for $kl \neq ij$. The Plücker eqns

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \qquad (1 \leqslant i < j < k < l \leqslant n).$$

yield the dependency $\boxed{u_{kl} = u_{ik}u_{jl} - u_{il}u_{jk}}$.

Conclusion: We parameterize U_{ij} by the $2(n-2)$ coordinates

$$\{u_{ik}, u_{jk} : k \neq i, j\}.$$

Choosing coordinates for Gr(2, n): from tropical to classical

Write: $\mathrm{Gr}(2,n) = \bigcup_{i<j} U_{ij}$, where $U_{ij} = \{p \in \mathrm{Gr}(2,n) : p_{ij} \neq 0\}$.

Can fix $p_{ij} = 1$, so

$$\mathrm{Trop}(U_{ij}) = \{x \in \mathrm{Trop}(\mathrm{Gr}(2,n)) : x_{ij} = 0\} \in \overline{\mathbb{R}}^{\binom{n}{2}-1}.$$

Now change coordinates to $\boxed{u_{kl} := p_{kl}/p_{ij}}$ for $kl \neq ij$. The Plücker eqns

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \qquad (1 \leqslant i < j < k < l \leqslant n).$$

yield the dependency $\boxed{u_{kl} = u_{ik}u_{jl} - u_{il}u_{jk}}$.

Conclusion: We parameterize U_{ij} by the $2(n-2)$ coordinates

$$\{u_{ik}, u_{jk} : k \neq i, j\}.$$

BIG ISSUE: these coordinates are not well adapted to the tree space.

- We view $\text{Trop}(\text{Gr}(2, n)) = \bigcup_{T \in \mathcal{T}_n} \overline{\mathscr{C}_T}$ inside $\mathbb{T}^{\binom{n}{2}-1}$.

Remark: A pt. lies in $\overline{\mathscr{C}_T}$ **if and only if** it satisfies the 4-pt conds. for T.

- We view $\text{Trop}(\text{Gr}(2,n)) = \bigcup_{T \in \mathcal{T}_n} \overline{\mathscr{C}_T}$ inside $\mathbb{T}^{\binom{n}{2}-1}$.

Remark: A pt. lies in $\overline{\mathscr{C}_T}$ **if and only if** it satisfies the 4-pt conds. for T.

For each valid J, we pick $ij \notin J$ and view each tree in "caterpillar form"

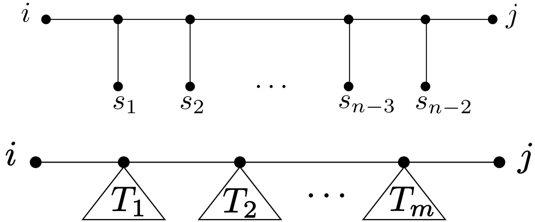


Figure: From left to right: the caterpillar tree on n leaves with endpoint leaves i and j, and the path from leaf i to j on a tree arranged in caterpillar-like form. The labeled triangles indicate subtrees of the original tree. The backbone of the caterpillar tree is the chain graph with $m+2$ nodes given by the horizontal path from i to j. The trees $T_1, \ldots, T_m$ need not be trivalent.

GOAL: Adapt our choice of $2(n-2)$ coords. $I \in \binom{[n]}{2}$ for $U_{ij} \subset \mathrm{Gr}_J(2,n)$ to:

(1) the indexing pair ij,

(2) the tree T and,

(3) a vanishing set J, with $ij \notin J$.

GOAL: Adapt our choice of $2(n-2)$ coords. $I \in \binom{[n]}{2}$ for $U_{ij} \subset \mathrm{Gr}_J(2, n)$ to:

(1) the indexing pair ij,

(2) the tree T and,

(3) a vanishing set J, with $ij \notin J$.

We do so by first constructing a suitable partial order $\preceq$ on $[n] \smallsetminus \{i, j\}$:

Definition

Let i, j be a pair of indices, and let $\preceq$ be a partial order on the set $[n] \smallsetminus \{i, j\}$. Let T be a tree on n leaves arranged in caterpillar form with backbone i–j. We say that $\preceq$ has the *cherry property on T* with respect to i and j if the following conditions hold:

(i) Two leaves of different subtrees T_a and T_b can't be compared by $\preceq$.

(ii) The partial order $\preceq$ restricts to a total order on the leaf set of each T_a, $a = 1, \ldots, m$.

(iii) If $k \prec l \prec v$, then either $\{k, l\}$ or $\{l, v\}$ is a cherry of the quartet $\{i, k, l, v\}$ (and hence also of $\{j, k, l, v\}$).

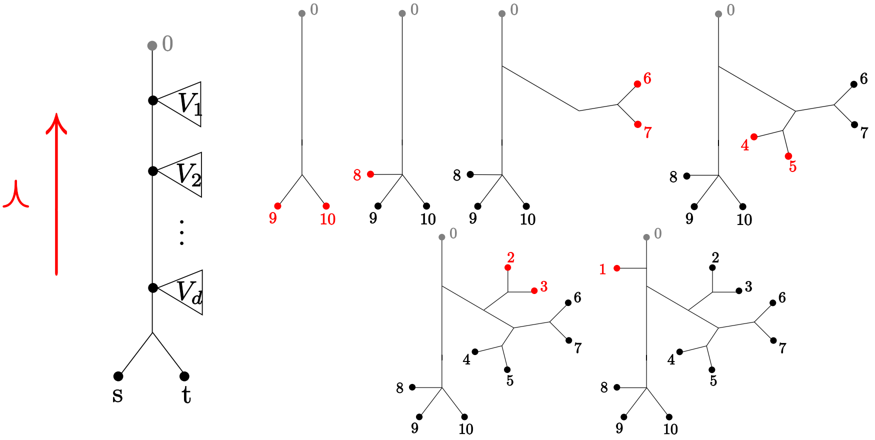
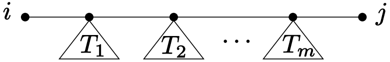


Figure: Inductive definition of the order $\preceq_a$ on the leaves of the subtree T_a ($s \prec_a t$, t is maximal) by example. We add one leaf or one cherry at a time so that the corresponding new leaf or leaves are smaller than the previous ones in the order $\preceq_a$. When adding a cherry, we arbitrarily order its two leaves as well. The grey dot with label 0 in T_a is internal in T. Broken leaf edges, such as the one in the third tree from the left, should be thought of as straight edges. The edge adjacent to the grey node with label 0 could be contracted.

• Fix two indices $\{i,j\}$, a "caterpillar like" tree T with backbone i–j, and a vanishing set J. Fix a partial order $\preceq$ on $[n] \smallsetminus \{i,j\}$ having the cherry property on T. Let $I \subset \binom{[n]}{2}$ be a set of size $2(n-2)$ not containing ij.

• $J(ij) := J \cap \{ik, jk : k \neq i,j\}$, and



Definition

We say that I is *compatible with* $\preceq$ *and* $J(ij)$ if for each index $a = 1, \ldots, m$ and each leaf $k \in T_a$, exactly one of the following condition holds:

(i) ik and $jk \in I$, and for all $l \prec k$ we have il or $jl \in J(ij)$; or

(ii) $ik \notin I$, $jl \in I$ for all $l \in T_a$, and there exists $t \prec k$ in T_a where $it, jt \notin J(ij)$. If t is the maximal element with this property, then $kt \in I$; or

(iii) $jk \notin I$, $il \in I$ for all $l \in T_a$ and there exists $t \prec k$ in T_a where $it, jt \notin J(ij)$. If t is the maximal element with this property, then $kt \in I$.

Theorem [C.-Häbich-Werner]: The coordinates I are well adapted as liftings of points from $\mathscr{C}_T \cap \mathrm{Trop}(\mathrm{Gr}_J(2,n))$ to $\mathrm{Gr}_J(2,n)$.

Why pick these compatibility properties? Fix T as in the figure. For each $a = 1, 2, 3$, we let $I_a := \{kl \in I : k \text{ or } l \in T_a\}$. Thus, $I = I_1 \sqcup I_2 \sqcup I_3$.

- $|T_1| = 1$, so $I_1 = \{i1, j1\}$ independently of J.
- If $i2$ or $j2 \in J$, then $I_3 = \{i2, j2, i3, j3\}$ in agreement with condition *(i)*.

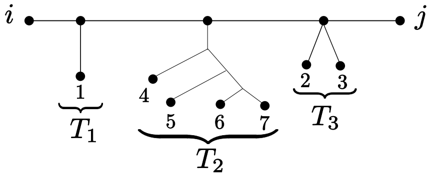
On the contrary, if $i2, j2 \notin J$ then we can choose between $I_3 = \{i2, j2, j3, 32\}$ (since *(ii)* is satisfied) or $I_3 = \{i2, j2, i3, 32\}$ (by *(iii)*).

- Choice of I_2, depends on $J_2(ij) := \{ik \in J : k \in T_2\} \cup \{jk \in J : k \in T_2\}$.

Example 1: If $\emptyset \neq J_2(ij) \subseteq \{i4, j4\}$, then we can take either $I_2 = \{i4, j4, i5, j5, i6, 65, i7, 76\}$ or $I_2 = \{i4, j4, i5, j5, j6, 65, j7, 76\}$.
Notice that in both cases $i5, j5 \in I_2$ by condition *(i)*.

Example 2: If $\emptyset \neq J_2(ij) \subseteq \{i7, j7\}$, we can take either $I_2 = \{i4, j4, i5, 54, i6, 65, i7, 76\}$ or $I_2 = \{i4, j4, j5, 54, j6, 65, j7, 76\}$.

Example 3: Finally, assume $J_2(ij) = \{j5, j6\}$. Then, we may choose $I_2 = \{i4, j4, i5, 54, i6, 64, i7, 74\}$ or $I_2 = \{i4, j4, j5, 54, j6, 64, j7, 74\}$.



- $i = 1, j = 2$,
- $\prec$ is the natural order on $\{3, \ldots, 7\}$.

Example:Coordinate changes when $n = 4$ and $i = 1, j = 2$.

• If T is the quartet $(13|24)$ or $(14|23)$, we pick our coordinates to be $u_{13}, u_{23}, u_{14}, u_{24}$. We derive the value of u_{34} from $u_{34} = u_{13}u_{24} - u_{14}u_{23}$.

• If T is the quartet $(12|34)$, then the choice of coordinates depends on J. We choose the order $3 \prec 4$:

(1) If $13, 23 \notin J$, we take $I = \{13, 23, 34, 14\}$. The expression for u_{24} is

$$u_{24} = u_{13}^{-1}(u_{34} + u_{14}u_{23})$$

Note: we must have $u_{13} \neq 0$ (this follows from $13 \notin J$).

(2) If 13 or $23 \in J$, then $I = \{13, 23, 14, 24\}$ and $u_{34} = u_{13}u_{24} - u_{14}u_{23}$.

References:

• C., Häbich, Werner: "Faithful tropicalization of the Grassmannian of planes." Math. Ann. **360**(1–2): 391–437, 2014.

• Speyer, Sturmfels: "The Tropical Grassmannian." Adv. Geom. **4**(3): 389–411, 2004.