Tropical Secant Graphs of Monomial Curves

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Combinatorics Seminar - UC Berkeley

Outline

1. Introducing our favorite graphs: the abstract tropical secant *surface* graph and the *master graph*.
2. What is behind these graphs? $\rightsquigarrow$ A *surface* in $\mathbb{P}^n$ parameterized by binomials, and its tropicalization.
3. Geometric tropicalization by example (with lots of blow-ups!)
4. Towards the first secant of monomial curves in $\mathbb{P}^n$ $\rightsquigarrow$ Our other two favorite graphs: the tropical secant graph and its planar buddy, the *Gröbner* tropical secant graph.
5. The hypersurface case: from the tropical secant graph to the Newton polytope.

The abstract tropical secant surface graph

- Fix $n \geq 4$ and n coprime distinct integers $I := \{0 = i_0 < i_1 < \ldots < i_n\}$.
- Consider all sequences $\underline{a} \subset I$ arising from **arith. prog.** in $\mathbb{Z}$, with $|\underline{a}| \geq 2$.

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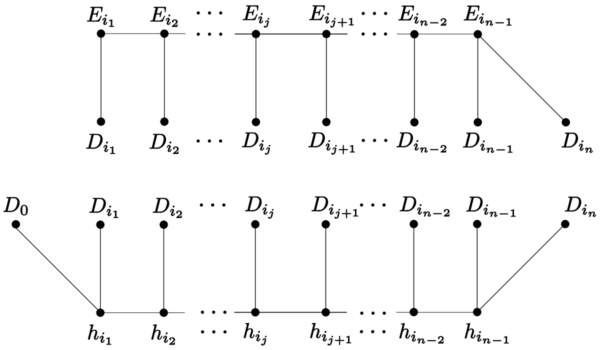
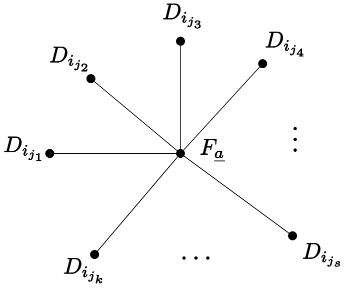
- Build two caterpillar trees $G_{E,D}, G_{h,D}$ and a family of star trees $G_{F_{\underline{a}},D}$:





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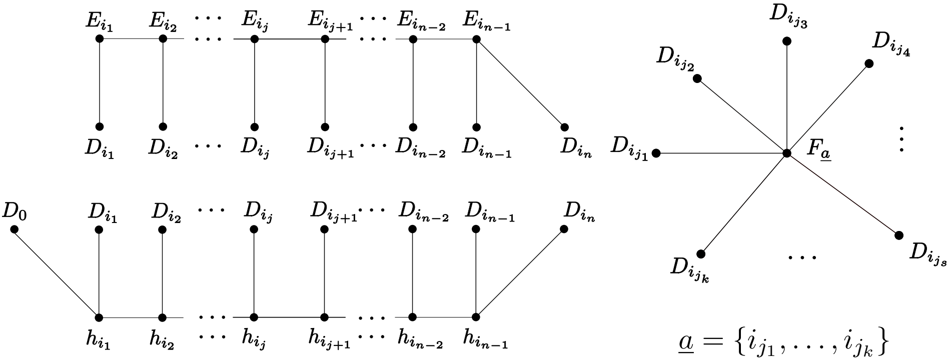


- Glue the graphs $G_{E,D}$, $G_{h,D}$ and $G_{F_{\underline{a}},D}$ along common nodes to form the abstract tropical secant surface graph.

The master graph (a.k.a. the tropical secant *surface* graph)

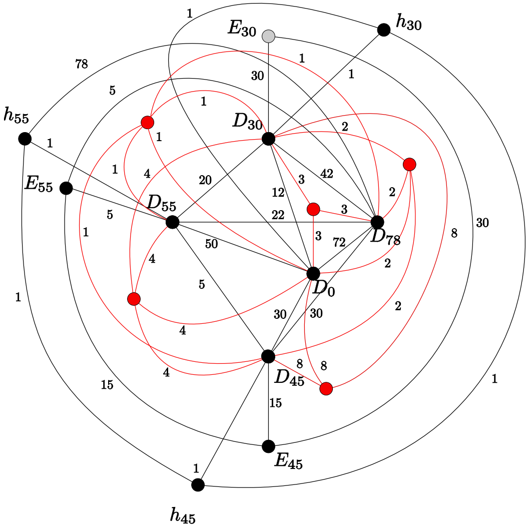
It is defined by a weighted embedding of the abstract graph in $\mathbb{R}^{n+1}$.

Definition (*master graph*)

1. $D_{i_j} = e_j := (0, \dots, 0, 1, 0, \dots, 0) \qquad (0 \le j \le n)$,
2. $E_{i_j} = (0, i_1, \dots, i_{j-1}, i_j, \dots, i_j) \qquad (1 \le j \le n-1)$,
3. $h_{i_j} = (-i_j, -i_j, \dots, -i_j, -i_{j+1}, \dots, -i_n) \qquad (1 \le j \le n-1)$,
4. $F_{\underline{a}} = \sum_{i_j \in \underline{a}} e_j$ for $\underline{a} \subseteq \{0, i_1, \dots, i_n\}$ arith. progr., $|\underline{a}| \ge 2$.

The edges have weights:

1. $m_{D_{i_j}, E_{i_j}} = \gcd(i_1, \dots, i_j)$, $m_{D_{i_j}, h_{i_j}} = \gcd(i_j, \dots, i_n)$,
2. $m_{D_{i_0}, h_{i_1}} = 1$, $m_{D_{i_n}, E_{i_{n-1}}} = \gcd(i_1, \dots, i_{n-1})$, $m_{D_{i_n}, h_{i_{n-1}}} = i_n$,
3. $m_{E_{i_j}, E_{i_{j+1}}} = \gcd(i_1, \dots, i_j)$, $m_{h_{i_j}, h_{i_{j+1}}} = \gcd(i_{j+1}, \dots, i_n)$,
4. $m_{F_{\underline{a}}, D_{i_j}} = \sum_r \varphi(r)$, where we sum over all common diff. r of all possible arith. prog. containing i_j and giving $\underline{a}$. Here, φ is *Euler's phi*.



- 9 bivalent nodes F_{i_j,i_k} are eliminated from the picture and replace its two adjacent edges by edge $D_{i_j}D_{i_k}$.
- 16 vertices (incl. bivalent node E_{30}), and 36 edges.
- Five red non-bivalent (unlabeled) nodes $F_{\underline{a}}$:

$$\begin{aligned} F_{0,30,45,55,78} &= (1,1,1,1,1), \\ F_{0,30,45,78} &= (1,1,1,0,1), \\ F_{0,30,45,55} &= (1,1,1,1,0), \\ F_{0,30,45} &= (1,1,1,0,0), \\ F_{0,30,78} &= (1,1,0,0,1). \end{aligned}$$

Remark (Disclaimer)

1. *If* $\underline{a} = \{i_j, i_k\}$, *we eliminate the bivalent node* $F_{\underline{a}}$, *replacing its two adj. edges by a single edge* $D_{i_j}D_{i_k}$, *with the inherited weight.*
2. E_{i_1} *is bivalent node, but we keep this one to simplify notation.*
3. $F_{i_{j_1},\ldots,i_{j_k}}$ *is a node* $\iff \gcd(i_{j_k} - i_{j_1}, \ldots, i_{j_2} - i_{j_1}) \neq 1$,
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Theorem (— - Lin)

The *master graph* *satisfies the* *balancing condition.*

The master graph is a tropical surface

Definition

For an irreducible algebraic variety $X \subset \mathbb{T}^N = (\mathbb{C}^*)^N$ with defining ideal $I = I(X) \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$, the tropicalization of X or I is defined as:

$$\mathcal{T}(X) = \mathcal{T}(I) = \{w \in \mathbb{Q}^N \,|\, 1 \notin \mathrm{in}_w(I)\}$$

where $\mathrm{in}_w(I) = \langle \mathrm{in}_w(f) : f \in I \rangle$, and $\mathrm{in}_w(f)$ is the sum of all *nonzero* terms of $f = \sum_\alpha c_\alpha x^\alpha$ such that $\alpha \cdot w$ is **minimum**.

Remark

1. *$\mathcal{T}(X)$ is a pure $\dim X$-dim'l poly. subfan of the Gröbner fan of $I(X)$.*
2. *The lineality space of the fan $\mathcal{T}(X)$ is the set*

$$L = \{w \in \mathcal{T}X : in_w(I) = I\}.$$

It describes action of the maximal torus acting on X (by the lattice $\Lambda := L \cap \mathbb{Z}^{n+1}$).

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It describes action of the maximal torus acting on X (by the lattice $\Lambda := L \cap \mathbb{Z}^{n+1}$).$\rightsquigarrow$ View $\mathcal{T}X$ in the $(N - rk\ \Lambda - 1)$-sphere of $\mathbb{R}^N/L$.

• A point $w \in \mathcal{T}X$ is *regular* if $\mathcal{T}X$ is a linear space locally near w.

• We can assing a positive multiplicity to every maximal cones in $\mathcal{T}X$, and give regular points the multiplicity of the corresp. mxl. cone.

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• Main tool: "Geometric Tropicalization" (Hacking-Keel-Tevelev)

Geometric tropicalization for SURFACES: an overview

- **IDEA:** Given $\beta : \mathbb{T}^2 \supset X \hookrightarrow \mathbb{T}^N$, compute $\mathcal{T}\overline{\beta(X)}$ from the geometry of X.

Theorem (Geometric Tropicalization [Hacking-Keel-Tevelev])

Consider $\mathbb{T}^N$ with coordinate functions $t_1, \ldots, t_N$, and let $Y \subset \mathbb{T}^N$ be a closed smooth surface. Suppose $\bar{Y} \supset Y$ is any compactification whose boundary D is a smooth divisor with C.N.C. Let $D_1, \ldots, D_m$ be the irred. comp. of D, and write Δ for the graph on $\{1, \ldots, m\}$ defined by

$$\{k_i, k_j\} \in \Delta \iff D_{k_i} \cap D_{k_j} \neq \emptyset.$$

Let $[D_k] := (\mathrm{val}_{D_k}(t_1), \ldots, \mathrm{val}_{D_k}(t_N)) \in \mathbb{Z}^N$, and $[\sigma] := \mathbb{N}_0\langle [D_k] : k \in \sigma\rangle$, for $\sigma \in \Delta$. Then,

$$\mathcal{T}Y = \bigcup_{\sigma \in \Delta} \mathbb{Q}_{\geq 0}[\sigma].$$

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Our contribution

$$m_w=\textstyle\sum_{\sigma\in\Delta\ s.t.\ w\in\mathbb{Q}_{\geq0}[\sigma]}(D_{k_1}\cdot D_{k_2})\ \ \mathrm{index}\Big((\mathbb{Q}\otimes_{\mathbb{Z}}[\sigma])\cap\mathbb{Z}^N:\mathbb{Z}[\sigma]\Big)$$

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$$\beta_j := f^h_{i_j}(\omega, \lambda, u)/u^{\deg f_{i_j}} = \big(u^{\deg f_{i_j}} f_{i_j}(\omega/u, \lambda/u)\big)/u^{\deg f_{i_j}}.$$

Our boundary divisors in $\bar{X}$ are $D_{i_j} = (f^h_{i_j} = 0)$, $D_\infty = (u = 0)$, and

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• The **resolution diagrams** at each one of these singularities are the three types of **subgraphs** of our original abstract graph, after contracting bivalent exc. divisors. The exceptional divisors will give us the nodes E_{i_j}, h_{i_j} or $F_{\underline{a}}$ resp. and the graph Δ is our abstract graph.

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• Why $F_{\underline{a}}$? If $D_{i_{j_1}}, \ldots, D_{i_{j_k}}$ intersect at $p \in \mathbb{T}^2$ then $p = (\zeta, \zeta^{i_{j_1}})$ and ζ is a prim. qth-root of unity for some $q \mid \gcd(i_{j_2} - i_{j_1}, \ldots, i_{j_k} - i_{j_1})$. So

$$\underline{a} = \{i_{j_1}, \ldots, i_{j_k}\} \rightsquigarrow \sum_q \varphi(q) \text{ exc. divisors } F_{\underline{a},\zeta}, \text{ BUT } [F_{\underline{a},\zeta}] = [F_{\underline{a},\zeta'}] := F_{\underline{a}}.$$

From the master graph to the secant of monomial curves

Let C be the monomial projective curve $(1 : t^{i_1} : \ldots : t^{i_n})$ parameterized by n coprime integers $0 < i_1 < \ldots < i_n$. By def.

$$Sec^1(C) = \overline{\{a \cdot p(t) + b \cdot p(s) : (a : b) \in \mathbb{P}^1, p(t), p(s) \in C\}} \subset \mathbb{T}^{n+1}.$$

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• Use the monomial change of coordinates $b = -\lambda a$, $t = \omega s$, and rewrite $v = a \cdot p(t) + b \cdot p(s)$ as

$$v_k = \underbrace{as^{i_k}}_{\in \tilde{C}} \cdot \underbrace{(\omega^{i_k} - \lambda)}_{\in Z} \qquad \text{for all } k = 0, \ldots, n,$$

where $\tilde{C}$ is the cone in $\mathbb{T}^{n+1}$ over the curve C.

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Definition

Let $X, Y \subset \mathbb{T}^N$ be two subvarieties of tori. The Hadamard product of X and Y equals $X \centerdot Y = \overline{\{(x_0 y_0, \ldots, x_n y_n) \,|\, x \in X, y \in Y\}} \subset \mathbb{T}^N$.

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• Hadamard products have nice tropicalizations...

Theorem (— -Tobis-Yu, Fink)

Let $X, Y \subset \mathbb{T}^N$ *closed subvarieties and consider their Hadamard product* $X \centerdot Y \subset \mathbb{T}^N$. *Then as* **weighted sets**: $\mathcal{T}(X \centerdot Y) = \mathcal{T}X + \mathcal{T}Y$.

Corollary (— - Lin)

Given a monomial curve C: $t \mapsto (1 : t^{i_1} : \ldots : t^{i_n})$, *and the surface* Z: $(\lambda, \omega) \mapsto (1 - \lambda, \omega^{i_1} - \lambda, \ldots, \omega^{i_n} - \lambda) \subset \mathbb{T}^{n+1}$. *Then:*

$$\mathcal{T}(Sec^1(C)) = \mathcal{T}Z + \mathcal{T}\tilde{C} = \mathcal{T}Z + \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$$

where $\Lambda = \mathbb{Z}\langle \mathbf{1}, (0, i_1, \ldots, i_n) \rangle$ *is the intrinsic lin. lattice of* $\mathcal{T}(Sec^1(C))$.

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Corollary

Modify the master graph ($\mathcal{T}Z$) *to get a weighted graph representing* ($\mathcal{T}(Sec^1(C))$) *as a set. We call it the tropical secant graph (TSG).*

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- **Question:** How to compute weights/multiplicities?

Theorem (Sturmfels-Tevelev-Yu)

Let $A \in \mathbb{Z}^{d \times N}$, defining a monomial map $\alpha \colon \mathbb{T}^N \to \mathbb{T}^d$ and a canonical linear map $A \colon \mathbb{R}^N \to \mathbb{R}^d$. Let $V \subset \mathbb{T}^N$ be a subvariety. Then

$$\mathcal{T}(\alpha(V)) \quad = \quad A(\mathcal{T}(V)).$$

Moreover, if α induces a generically finite morphism on V of degree δ, the multiplicity of $\mathcal{T}(\alpha(V))$ at a regular point w equals

$$m_w = \frac{1}{\delta} \cdot \sum_v m_v \cdot \ \textit{index}\, (\mathbb{L}_w \cap \mathbb{Z}^d : A(\mathbb{L}_v \cap \mathbb{Z}^N)),$$

where the sum is over all points $v \in \mathcal{T}(V)$ with $Av = w$. We also assume that the number of such v's is finite, all of them are regular in $\mathcal{T}(V)$, and $\mathbb{L}_v, \mathbb{L}_w$ are linear spans of nbd. of $v \in \mathcal{T}(V)$ and $w \in A(\mathcal{T}(V))$ resp.

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$$m_w = \frac{1}{\delta}\cdot\sum_v m_v \cdot \ \textit{index}\,(\mathbb{L}_w \cap \mathbb{Z}^d : A(\mathbb{L}_v \cap \mathbb{Z}^N)),$$

where the sum is over all points $v \in \mathcal{T}(V)$ *with* $Av = w$*. We also assume that the number of such* v*'s is finite, all of them are regular in* $\mathcal{T}(V)$*, and* $\mathbb{L}_v, \mathbb{L}_w$ *are linear spans of nbd. of* $v \in \mathcal{T}(V)$ *and* $w \in A(\mathcal{T}(V))$ *resp.*

In our case: $V = \tilde{C} \times Z$ and α is the monomial map associated to the matrix $(Id_{n+1} \mid Id_{n+1})$. Here $v = (c, z)$ and $m_v = m_c \cdot m_z = m_z$.

Theorem (Sturmfels-Tevelev-Yu)

Let $A \in \mathbb{Z}^{d\times N}$*, defining a monomial map* $\alpha\colon \mathbb{T}^N \to \mathbb{T}^d$ *and a canonical linear map* $A\colon \mathbb{R}^N \to \mathbb{R}^d$*. Let* $V \subset \mathbb{T}^N$ *be a subvariety. Then*

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Proposition

The generic fiber of $\alpha_{|_V}$ *has size* $\delta = 2$.

(Reason: Almost all points in $Sec^1(C)$ lie in a *unique* secant line.)

Lemma (Which edges σ of the master graph survive in the tropical secant graph TSG and what are their fibers)

1. *The points* $F_{0,i_1,\ldots,i_n}$, $D_0 + h_{i_1}$, $(i_n - i_{n-1})D_{i_n} + E_{i_{n-1}}$ *and* $(i_n - i_{n-1})D_{i_n} + h_{i_{n-1}} \in \Lambda$, *so the corresp. edges dissapear in TSG.*
2. $E_{i_j} \equiv h_{i_j}$ *modulo the lattice* Λ, *so all nodes* h_{i_j} *dissapear in TSG.*
3. *The fibers of* A *at points in the cones* $\langle F_{\underline{a}} D_{i_j}\rangle + \mathbb{R}\otimes\Lambda$ $(\underline{a} \neq \underline{b}, \underline{e})$ *and* $\langle D_{i_j} D_{i_k}\rangle + \mathbb{R}\otimes\Lambda$ *have size 1 (*$\underline{e} = I \smallsetminus \{0\}$, $\underline{b} = I \smallsetminus \{i_n\}$*.)*
4. $i_1 F_{\underline{e}} = E_{i_1}$. *Hence the fiber of* A *at pts. in* $\langle E_{i_1}, D_{i_1}\rangle + R\otimes\Lambda$ *has size* 2 *(if* $\exists F_{\underline{e}}$*) or 1 (if* $\nexists F_{\underline{e}}$*). The edges* $F_{\underline{e}} D_{i_1}$ *and* $D_{i_1} E_{i_1}$ *coincide in the TSG.*
5. $F_{\underline{b}} \equiv E_{i_{n-1}}$ *mod* Λ. *Hence, the fiber of* A *at pts in* $\langle E_{i_{n-1}}, D_{i_{n-1}}\rangle + \mathbb{R}\otimes\Lambda$ *has size 2 (if* $\exists F_{\underline{b}}$*) or* 1 *(if* $\nexists F_{\underline{b}}$*). The edges* $F_{\underline{b}} D_{i_{n-1}}$ *and* $E_{i_{n-1}} D_{i_{n-1}}$ *coincide in the TSG.*
6. *All other fibers have size one and the edges survive in the TSG.*

Theorem (— - Lin)

- *Complete description of the tropical secant graph:*

$$Nodes(TSG) := \{D_0, D_{i_n}\} \bigcup \{D_{i_j}, E_{i_j} : 1 \leq j \leq n-1\} \bigcup \{F_{\underline{a}} : \underline{a}\},$$

where $\underline{a} \subsetneq \{0, i_1, \ldots, i_n\}$ *varies among all proper maximal arithmetic progression containing at least two elements and such that* $\underline{a} \neq \underline{b}, \underline{e}$.

$$Edges(TSG) := \{E_{i_j} E_{i_{j+1}}\}_{1 \leq j \leq n-2} \bigcup \{D_{i_j} E_{i_j}\}_{1 \leq j \leq n-1} \bigcup \{F_{\underline{a}} D_{i_j} | i_j \in \underline{a}\},$$

plus the sets $\{E_{i_{n-1}} D_{i_j}\}_{0 \leq j \leq n-2}$ *(if* $\exists F_{\underline{b}}$*) and/or* $\{E_{i_1} D_{i_j}\}_{2 \leq j \leq n}$ *(if* $\exists F_{\underline{e}}$*).*

- *We give explicit formulas to compute all multiplicities.*

The first secant of the curve $(1 : t^{30} : t^{45} : t^{55} : t^{78})$

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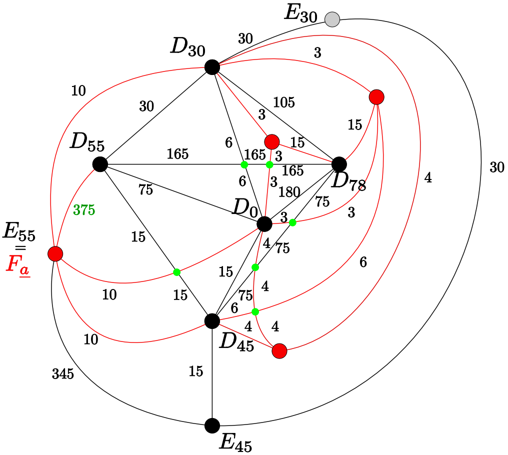

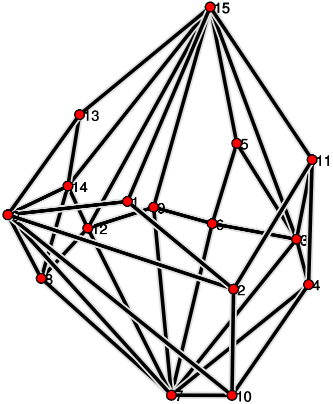

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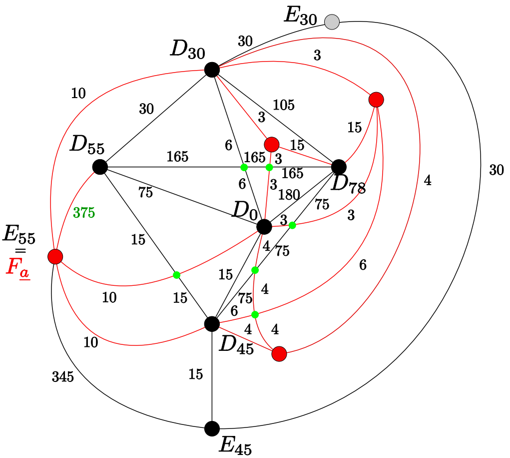

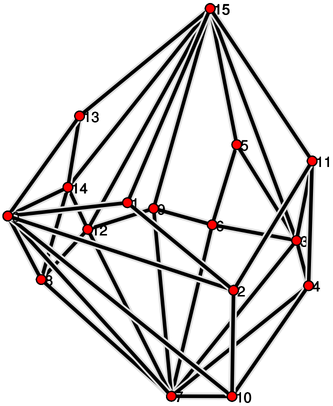

Note: 6 green nodes $\leftrightarrow$ crossings of edges in TSG. (hidden from us!)