Implicitization of surfaces via Geometric Tropicalization

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CIAM-KTH Workshop
Feb 21st 2011

Three references:

Sturmfels, Tevelev, Yu: The Newton polytope of the implicit equation (2007)
Sturmfels, Tevelev: Elimination theory for tropical varieties (2008)
MAC: Tropical Implicitization (Ch. 5) (2010) $\rightsquigarrow$ arXiv soon!
(and many, many more!)

Implicitization problem: Classical vs. tropical approach

Input: Laurent polynomials $f_1, f_2, \ldots, f_n \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.

Algebraic Output: The *prime* ideal I defining the Zariski closure Y of the image of the map:

$$\mathbf{f} = (f_1, \ldots, f_n) \colon \mathbb{T}^d \dashrightarrow \mathbb{T}^n$$

The ideal I consists of all polynomial relations among $f_1, f_2, \ldots, f_n$.

Existing methods: Gröbner bases and resultants.

- GB: always applicable, but often too slow.
- Resultants: useful when $n = d + 1$ and I is *principal*, with limited use.

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Punchline: We can *effectively* compute them using tropical geometry.

TODAY: Study the case when $\mathbf{d = 2}$ and $\mathbf{Y}$ is a **surface**.

Example: parametric surface in $\mathbb{T}^3$

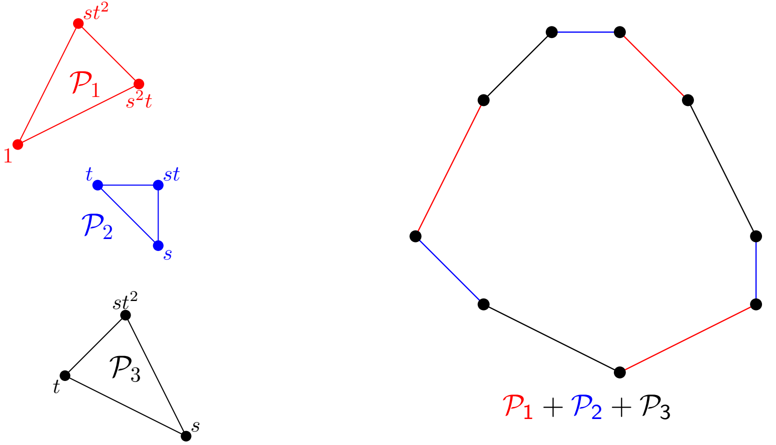
`Input`: Three Laurent polynomials in two unknowns:

$$\begin{cases} x = f_1(s,t) = 1 + s^3\,t + s\,t^2, \\ y = f_2(s,t) = 2s\,t + 3s + 5t, \\ z = f_3(s,t) = -t + s^2 + s\,t^2. \end{cases}$$

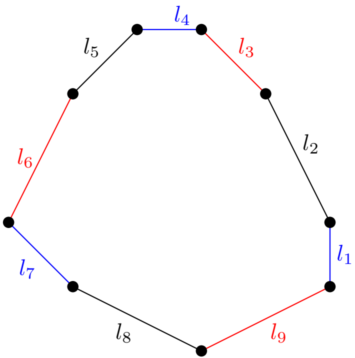
`Output`: The Newton polytope of the implicit equation $g(x,y,z)$.

The Newton polytope of g is the convex hull in $\mathbb{R}^3$ of all lattice points (i,j,k) such that $x^i\,y^j\,z^k$ appears with *nonzero* coefficient in $g(x,y,z)$.

STEP 1: Draw the three Newton polytopes and their Minkowski sum.



STEP 2: From these four polytopes, build the abstract graph Δ



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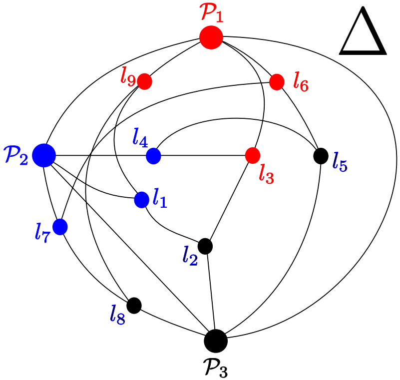
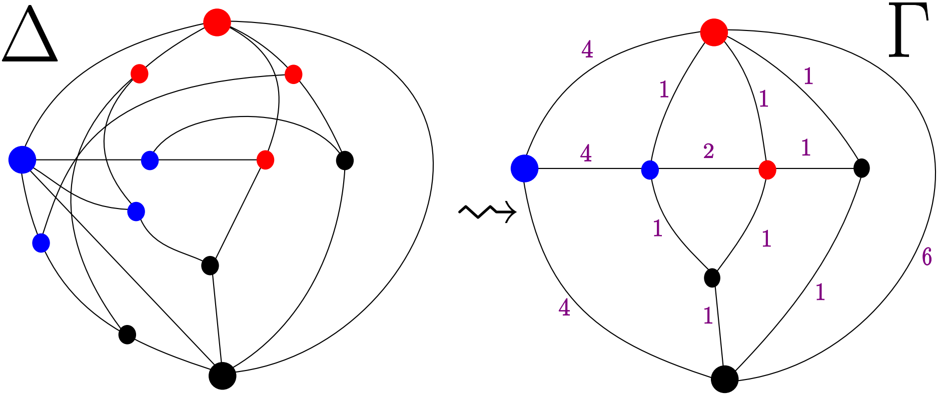


- Add one fat colored node per polytope $\mathcal{P}_i$ $(i = 1, 2, 3)$ and draw the **triangle** joining these three nodes.
- Add one skinny colored node per edge in the polytope $\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3$ and draw the corresponding polygon joining nodes of adjacent edges in the polytope (in this case, a **9-gon**).
- Join all skinny nodes to the corresponding fat node of the same color.
- In our example: 3 fat nodes, 9 skinny nodes and 21 edges.

STEP 3: Realize the abstract graph Δ as a weighted graph Γ in $\mathbb{S}^2$.



f-vector$(\Gamma) = (7, 13, 8)$.

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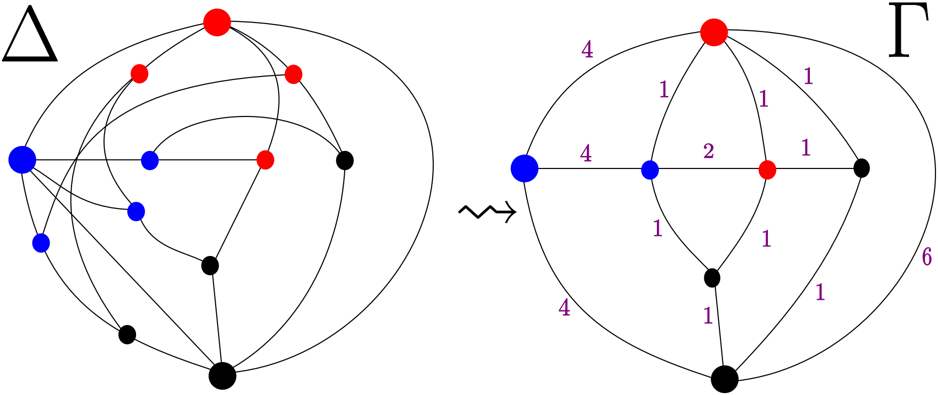

$$f\text{-vector}(\Gamma) = (7, 13, 8).$$

- Γ is a balanced weighted *planar* graph in $\mathbb{R}^3$. It is the tropical variety $\mathcal{T}(g(x, y, z))$, dual to the Newton polytope of g.
- We can recover $g(x, y, z)$ from Γ using *numerical linear algebra*.

What is Tropical Geometry?

Given a variety $X \subset \mathbb{T}^n$ with defining ideal $I \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the tropicalization of X equals:

$$\mathcal{T}X = \mathcal{T}I := \{w \in \mathbb{R}^n \,|\, \mathrm{in}_w(I) \text{ contains no monomial}\}.$$

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1. It is a rational polyhedral fan in $\mathbb{R}^n \rightsquigarrow \mathcal{T}X \cap \mathbb{S}^{n-1}$ is a spherical polyhedral complex.
2. If I is prime, then $\mathcal{T}X$ is pure of the same dimension as X.
3. Maximal cones have canonical multiplicities attached to them. With these multiplicities, $\mathcal{T}X$ satisfies the balancing condition.

Example (hypersurfaces):

- $\mathcal{T}(g)$ is the union of all codim. 1 cones in the (inner) normal fan of the Newton polytope $\mathrm{NP}(g)$.
- Maximal cones in $\mathcal{T}(g)$ are dual to edges in $\mathrm{NP}(g)$, and m_σ is the lattice length of the associated edge.
- Multiplicities are essential to recover $\mathrm{NP}(g)$ from $\mathcal{T}(g)$.

What is Geometric Tropicalization?

AIM: Given $Z \subset \mathbb{T}^N$ a **surface**, compute $\mathcal{T}Z$ from the *geometry* of Z.
KEY FACT: $\mathcal{T}Z$ can be characterized in terms of divisorial valuations.

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Theorem (Geometric Tropicalization [Hacking - Keel - Tevelev])

Consider $\mathbb{T}^N$ with coordinate functions $\chi_1, \ldots, \chi_N$, and let $Z \subset \mathbb{T}^N$ be a closed smooth **surface**. *Suppose $\overline{Z} \supset Z$ is any compactification, whose boundary divisor has m irreducible components $D_1, \ldots, D_m$ which are smooth and with no triple intersections (***C.N.C.***). Let Δ be the graph:*

$$V(\Delta) = \{1, \ldots, m\} \quad ; \quad (i,j) \in E(\Delta) \iff D_i \cap D_j \neq \emptyset.$$

Realize Δ *as a graph $\Gamma \subset \mathbb{R}^N$ by $[D_k] := (val_{D_k}(\chi_1), \ldots, val_{D_k}(\chi_N)) \in \mathbb{Z}^N$.*

Then, $\mathcal{T}Z$ is the **cone over the graph** Γ.

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Theorem (Combinatorial formula for multiplicities [C.])

$$m_{([D_i],[D_j])} = (D_i \cdot D_j)\,\left[\, (\mathbb{Z}\langle [D_i], [D_j] \rangle)^{sat} : \mathbb{Z}\langle [D_i], [D_j] \rangle \,\right]$$

QUESTION: How to compute $\mathcal{T}Y$ from a parameterization

$$\mathbf{f} = (f_1, \ldots, f_n)\colon \mathbb{T}^2 \dashrightarrow Y \subset \mathbb{T}^n \quad ?$$

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ANSWER: Compactify the domain $X = \mathbb{T}^2 \smallsetminus \bigcup_{i=1}^n (f_i = 0)$ and use the map $\mathbf{f}$ to translate back to Y.

Proposition

Given $\mathbf{f} : X \subset \mathbb{T}^2 \to Y \subset \mathbb{T}^n$ generically finite map of degree δ, let $\overline{X}$ be a smooth, CNC compactification with associated intersection complex Δ. Map each vertex D_k of Δ in $\mathbb{Z}^n$ to a vertex $\widetilde{D_k}$ of $\Gamma \subset \mathbb{R}^n$, where

$$[\widetilde{D_k}] = val_{D_k}(\chi \circ f) = f^{\#}([D_k]).$$

Then, $\mathcal{T}Y$ is the cone over the graph $\Gamma \subset \mathbb{R}^n$, with multiplicities

$$m_{([\widetilde{D_i}],[\widetilde{D_j}])} = \frac{1}{\delta} (D_i \cdot D_j) \left[(\mathbb{Z}\langle[\widetilde{D_i}],[\widetilde{D_j}]\rangle)^{sat} : \mathbb{Z}\langle[\widetilde{D_i}],[\widetilde{D_j}]\rangle \right].$$

Implicitization of *generic* surfaces

SETTING: Let $f = (f_1, \ldots, f_n)\colon \mathbb{T}^2 \dashrightarrow Y \subset \mathbb{T}^n$ of $\deg(f) = \delta$, where

- each $f_i \in \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ is irreducible and has **fixed Newton polytope**,
- we assume **generic coefficients**.

GOAL: Compute the graph Γ of $\mathcal{T}Y$ from the Newton polytopes $\{\mathcal{P}_i\}_{i=1}^n$.

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IDEA: Compactify X inside the proj. toric variety $\mathbb{P}(\mathscr{N})$, where $\mathscr{N}$ is the normal fan of $\sum_{i=1}^n \mathcal{P}_i$. *Generically*, $\overline{X}$ is smooth and has CNC boundary.

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The vertices and edges of the boundary intersection complex Δ are

$$V(\Delta) = \{E_i : \dim \mathcal{P}_i \neq 0, 1 \leq i \leq n\} \bigcup \{D_\rho : \rho \in \mathscr{N}^{[1]}\},$$

- $(D_\rho, D_{\rho'}) \in E(\Delta)$ iff ρ, ρ' are *consecutive* rays in $\mathscr{N}$.
- $(E_i, D_\rho) \in E(\Delta)$ iff $\rho \in \mathscr{N}(\mathcal{P}_i)$.
- $(E_i, E_j) \in E(\Delta)$ iff $(f_i = f_j = 0)$ has a solution in $\mathbb{T}^2$.

Then, Γ is the realization of Δ via

$$[E_i] := e_i \quad (1 \leq i \leq n) \quad , \quad [D_\rho] := \Big(\min_{\alpha \in \mathcal{P}_i}\{\alpha \cdot \eta_\rho\}\Big)_{i=1}^n \;\; \forall \rho \in \mathscr{N}^{[1]},$$

where η_ρ is the primitive lattice vector generating ρ.

Tropical implicitization of *generic* surfaces

Theorem (Sturmfels-Tevelev-Yu, C.)

The tropical variety $\mathcal{T}Y$ *is the cone over the graph* Γ*, with multiplicities*

- $m_{([D_\rho],[D_{\rho'}])} = \frac{1}{\delta}\frac{\gcd\{\textit{2-minors of } ([D_\rho]|[D_{\rho'}])\}}{|\det(\eta_\rho|\eta_{\rho'})|}$, *for* ρ, ρ' *consec. rays in* $\mathcal{N}$.
- $m_{(e_i,[D_\rho])} = \frac{1}{\delta}\left(|face_\rho \mathcal{P}_i \cap \mathbb{Z}^2| - 1\right) \gcd\{[D_\rho]_j : j \neq i\}$, *for* $\rho \in \mathcal{N}_i^{[1]}$.
- $m_{(e_i,e_j)} = \frac{1}{\delta} length\big((f_i = f_j = 0) \cap \mathbb{T}^2\big)$, *if* $\dim(\mathcal{P}_i + \mathcal{P}_j) = 2$.

Under further genericity assumptions,

$$length\big((f_i = f_j = 0) \cap \mathbb{T}^2\big) = MV(\mathcal{P}_i, \mathcal{P}_j).$$

Example (generic surface)

$$Y = \begin{cases} x = f_1(s,t) = a_1 + a_2\, s + a_3\, t, \\ y = f_2(s,t) = b_1 + b_2\, t + b_3\, s^2, \\ z = f_3(s,t) = c_1 + c_2\, st, \end{cases} \qquad a_1, \dots, c_2 \in \mathbb{C}^* \text{ are } \textit{generic}.$$

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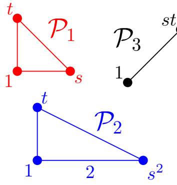

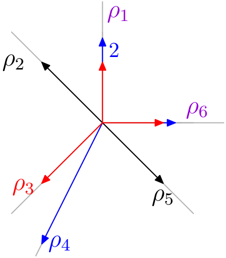

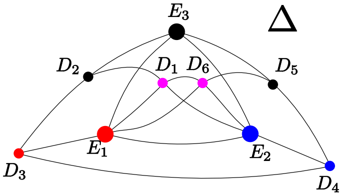

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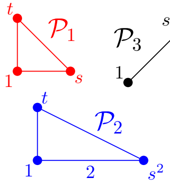
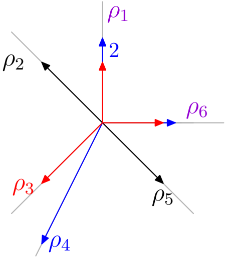
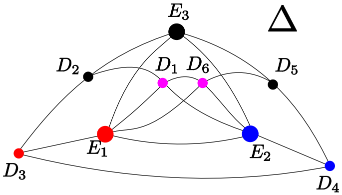
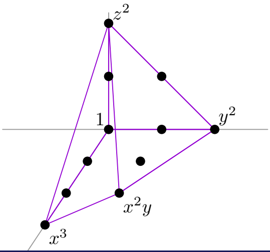


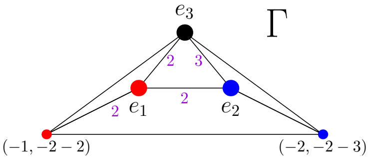






$\leftrightsquigarrow$

f-vector$= (5,\ 8,\ 5)$



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Non-genericity $\leftrightarrow$ CNC/smoothness condition is violated, i.e. triple intersections among:

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Solution 2:

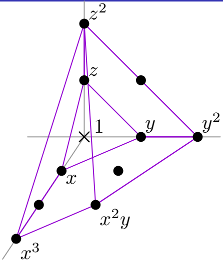
1. Embed X in $\mathbb{P}^2_{(s,t,u)} \rightsquigarrow n+1$ boundary divisors
$$E_i = (f_i = 0) \quad (1 \leq i \leq n), \quad E_\infty = (u = 0).$$
2. Resolve triple intersections and singularities by blow-ups $\pi\colon \tilde{X} \to X$, and read divisorial valuations by *columns*

$$(f \circ \pi)^*(\chi_i) = \pi^*(E_i - \deg(f_i)E_\infty) = E_i' - \deg(f_i)E_\infty' - \sum_{j=1}^{r} b_{ij} H_j \quad \forall i.$$

The graph Δ is obtained by gluing resolution diagrams and adding pairwise intersections.

Example (non-generic surface)

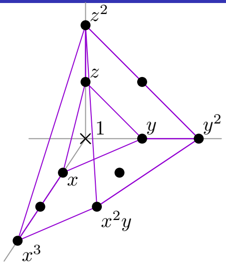
$$Y = \begin{cases} x = f_1(s,t) = s - t, \\ y = f_2(s,t) = t - s^2, \\ z = f_3(s,t) = -1 + s\,t, \end{cases} \quad \leftrightsquigarrow$$



$(7, 11, 6)$

Example (non-generic surface)

$$Y = \begin{cases} {\color{red} x = f_1(s,t) = s - t,} \\ {\color{blue} y = f_2(s,t) = t - s^2,} \\ z = f_3(s,t) = -1 + s\,t, \end{cases} \quad \leftrightsquigarrow \qquad (7, 11, 6)$$



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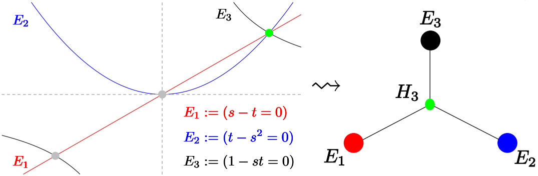

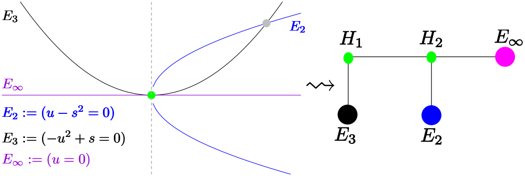

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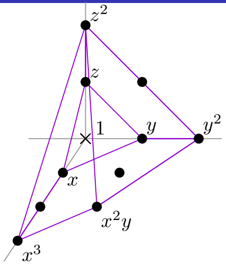

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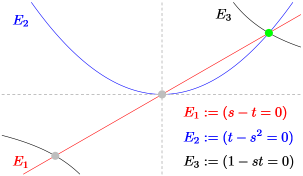

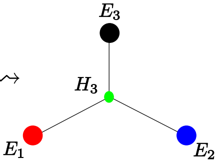

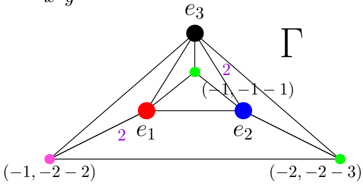

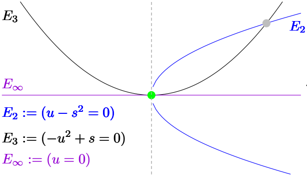

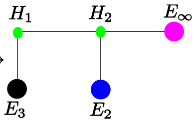

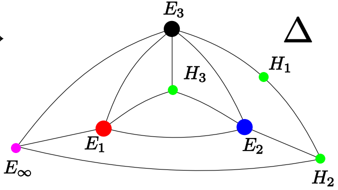

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2. Special surfaces are tropicalized via resolution of singularities, which is hard to do in practice.
   - Is there an alternative approach? $\rightsquigarrow$ combinatorial resolutions?
   - Can we predict the graph Γ from the topology/geometry of the singularities on the domain X? $\rightsquigarrow$ Enriques/dual diagrams, clusters of infinitely near points, ...

Further questions

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3. If $\dim Z > 2$, geometric tropicalization requires the boundary of a compactification $\overline{Z}$ to have **simple** *normal crossings*. Can we replace it with **combinatorial** *normal crossings*?