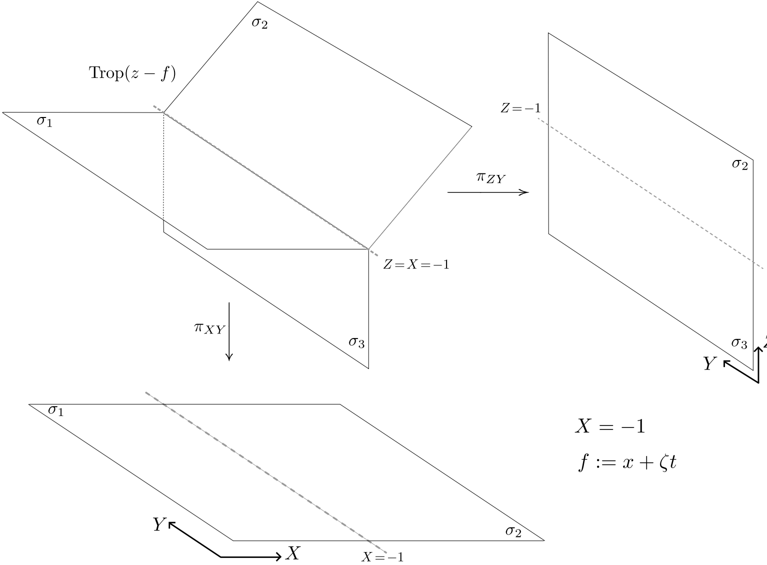
Linear trop. modification of $\mathbb{R}^2$ along $\{X = l\}$

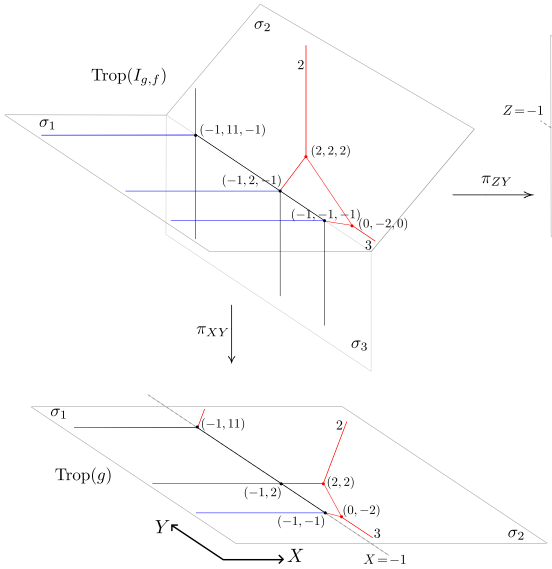


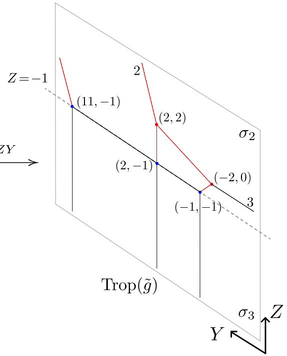
$X = -1$

$f := x + \zeta t$

$$\sigma_1 = \{X \leqslant l, Z = l\},\ \sigma_2 = \{X \geqslant l, Z = X\},\ \sigma_3 = \{X = l, Z \leqslant l\}.$$

Generic modification of a plane cubic along $\{X = l\}$





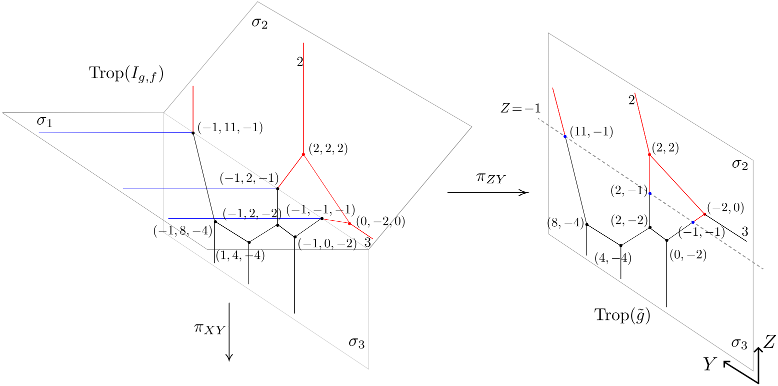
$X = -1$

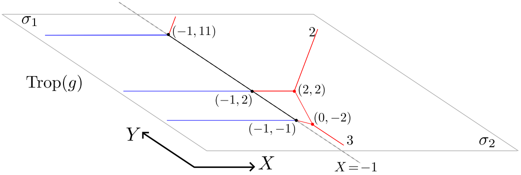
$f := x + \zeta t$

ζ generic

$$\sigma_1 = \{X \leqslant l, Z = l\},\ \sigma_2 = \{X \geqslant l, Z = X\},\ \sigma_3 = \{X = l, Z \leqslant l\}.$$

Special modification of a plane cubic along $\{X = l\}$





$$X = -1$$

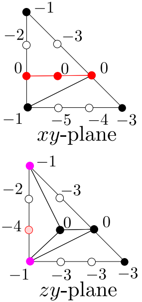
$$f := x + 1\,t$$

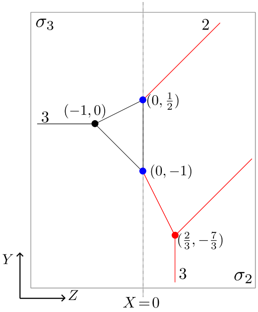
$$\sigma_1 = \{X \leqslant l, Z = l\},\ \sigma_2 = \{X \geqslant l, Z = X\},\ \sigma_3 = \{X = l, Z \leqslant l\}.$$

Theorem (Repairing bounded edges with high multiplicity)

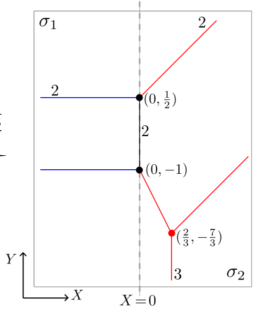
Let e be a vertical bounded edge of $\operatorname{Trop}(g)$ *of multiplicity $n \geqslant 2$ whose endpoints have valency 3. If $\Delta_{e^\vee}$ does* **not** *vanish at* $\operatorname{in}_e(g)$, *then e witnesses a folding of edges. Moreover, we can unfold e and produce a cycle using the tropical modification along the line $\mathbb{R}\langle e\rangle$.*

Example:





$$x = z - \frac{1+\sqrt{-3}}{2} \quad \longrightarrow$$

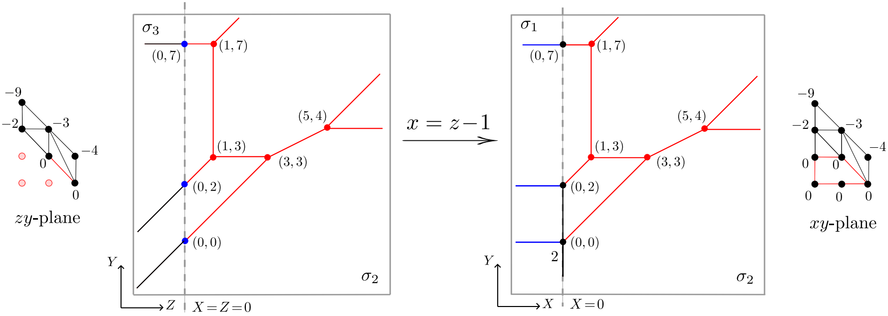


Example: Unfold a cycle of a rational cubic plane curve

$$g = t^4\,x^2y + 5t^3\,xy^2 + t^9\,y^3 + x^2 + 3xy + t^2\,y^2 + 2x + (3-t^4)y + 1$$

$$\Delta_{(0,0)}(\mathrm{in}_{(0,0)}(g)) = c_{00}c_{11}^2 - c_{10}c_{01}c_{11} + c_{20}c_{01}^2 = 1\cdot 3^2 - 2\cdot 3\cdot 3 + 1\cdot 3^2 = 0.$$

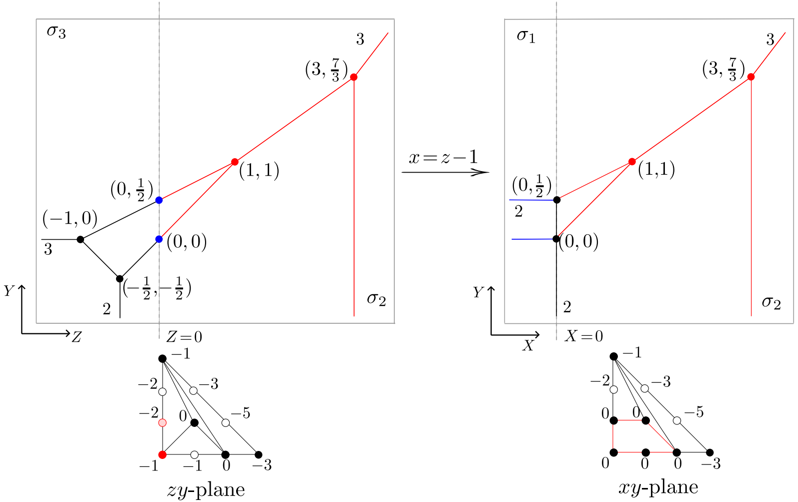
Want: $0 = c_{20}\,\mathrm{in}(\zeta)^2 - c_{11}\,\mathrm{in}_\zeta + c_{00} = \mathrm{in}(\zeta)^2 - 2\,\mathrm{in}(\zeta) + 1 = (\mathrm{in}(\zeta)-1)^2.$



$$\tilde{g} = g(z-1, y) = t^4\,z^2y + 5t^3\,zy^2 + t^9\,y^3 + z^2 + (3-2t^4)zy + (t^2-5t^3)y^2$$

Example: Repair a cycle on the visible side of a vert. line

$$g(x,y) = t^3x^3 + t^5x^2y + t^3xy^2 + ty^3 + x^2 + 3xy + t^2y^2 + (2+3t/2)x + (3+t^2)y + 1.$$



$$\mathrm{in}_{(0,0)}(g) = (1+x)^2 + 3(x+1)y = (x+1)((1+x)+y)) \rightsquigarrow \zeta = 1.$$

Proof ideas: Repairing the cycle of a tropical plane cubic.

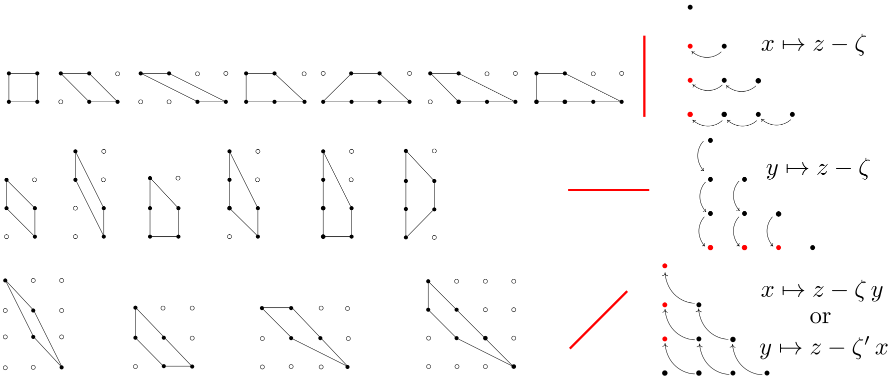
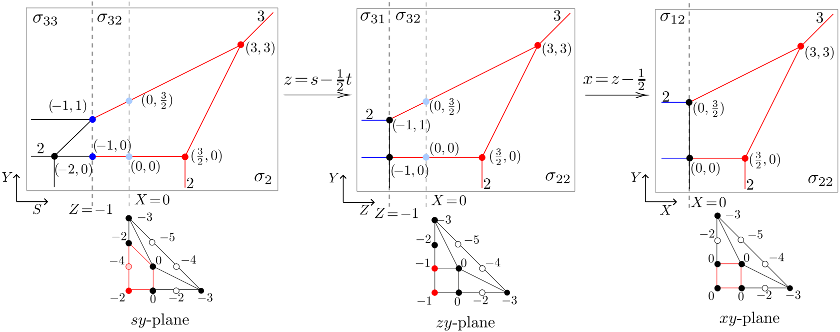


Figure: Locally reducible vertices on a cubic and feeding process

Case 1: Cycle visible and 2 iterations

$g = -t^3 x^3 + (t^4 + t^5)x^2 y + (-t^5 + t^6)xy^2 + t^3 y^3 + (t^2 - t^3)x^2 + 4xy + (2t^2 + 3t^3)y^2 + 2x + (2 + 2t)y + (1 + t)$.



Re-embed by $I_{g,f_1,f_2} := \langle g, z - (x + 1/2), s - (z + t/2)\rangle$.

Enough: $g(s - (1/2 + t/2), y) \subset K[s, y]$.

Case 2: loc. red. vertices with vanishing discrim. on the top left (v_2) and bottom right (v_1) and cycle not visible.

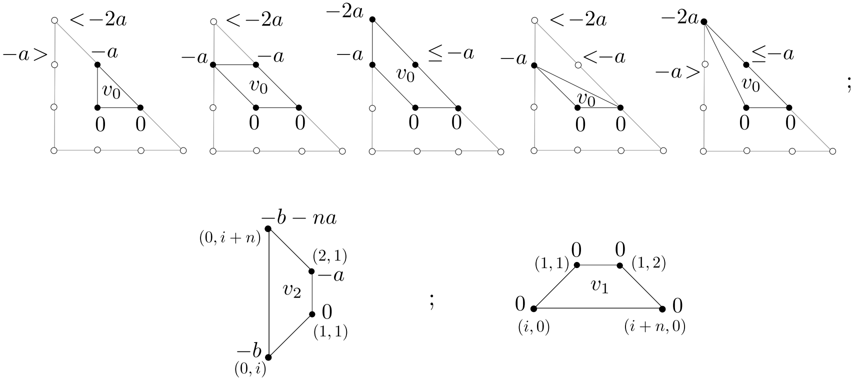
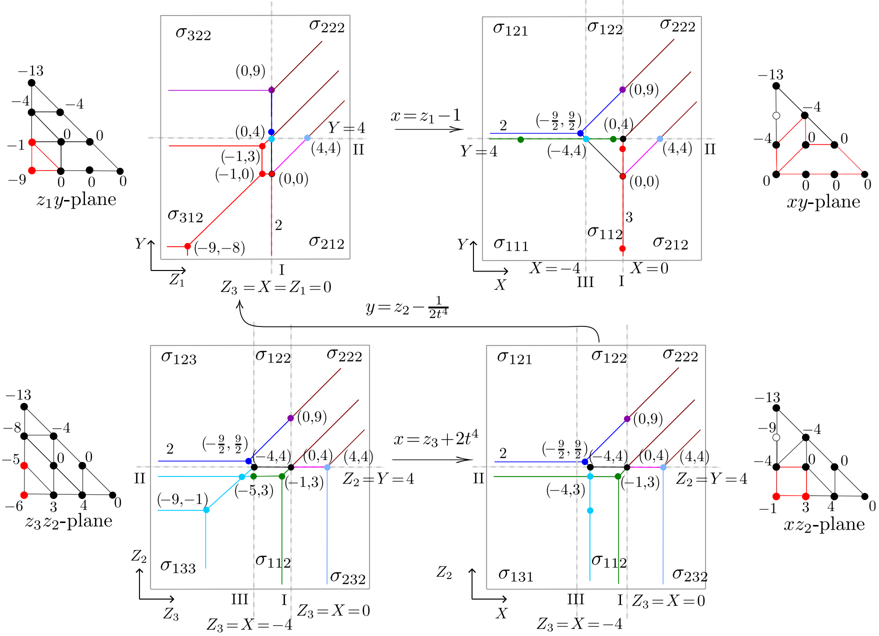


Figure: Combinatorics and heights of dual cells to distinguished vertices when no linear tropical modification keeps the cycle of Trop(g) on its visible side. We disallow $i = n = 1$.

$\mathbf{g} := x^3 + (1-9t^2)x^2y + 2t^4xy^2 + t^{20}y^3 + (1-24t^9 - t^{40})x^2 + (1+5t-16t^9+144t^{11})xy + 8t^{67}y^2 + (1-16t^9+t^{15}+192t^{18})x + (2t^4+64t^{18}-576t^{20})y + (1-8t^9+64t^{18}-8t^{24}).$



Re-embed by $I_{g,f_1,f_2,f_3} := \langle g(z_3 - 2t^4, y), z_2 - y - t^{-4}/2, z_3 - z_1 + 1 + 2t^4 \rangle$.