Combinatorial Aspects of Tropical Geometry

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What is tropical geometry?

- Trop. semiring $\overline{\mathbb{R}}_{\text{tr}} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$, $a \oplus b = \max\{a,b\}$, $a \odot b = a + b$.
- Fix $K = \mathbb{C}\{\{t\}\}$ field of Puiseux series, with valuation given by **lowest exponent**, e.g. $\text{val}(t^{-4/3} + 1 + t + \ldots) = -4/3$, $\text{val}(0) = \infty$.

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$$\boxed{F(\mathbf{x}) \text{ in } K[x_1^{\pm}, \dots, x_n^{\pm}] \rightsquigarrow \operatorname{Trop}(F)(\boldsymbol{\omega}) \text{ in } \overline{\mathbb{R}}_{\text{tr}}[\omega_1^{\odot\pm}, \dots, \omega_n^{\odot\pm}]}$$

$$F := \sum_{\alpha} c_\alpha \mathbf{x}^\alpha \mapsto \operatorname{Trop}(F)(\boldsymbol{\omega}) := \bigoplus_{\alpha} -\operatorname{val}(c_\alpha) \odot \boldsymbol{\omega}^{\odot\alpha} = \max_{\alpha}\{-\operatorname{val}(c_\alpha) + \langle \alpha, \boldsymbol{\omega} \rangle\}$$

$(F = 0)$ in $(K^*)^n \rightsquigarrow \operatorname{Trop}(F) = \{\boldsymbol{\omega} \in \mathbb{R}^n : \max \text{ in } \operatorname{Trop}(F)(\boldsymbol{\omega}) \text{ is } \underline{\text{not}} \text{ unique}\}$

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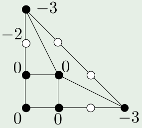
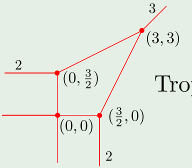
Example: $g = -t^3 x^3 + t^3 y^3 + t^2 y^2 + (4 + t^5)xy + 2x + 7y + (1 + t)$.

Newton subdivision of g



height of $(i, j) = -\text{val}(c_{i,j})$

⟷



Trop(g)

Tropical Geometry is a combinatorial shadow of algebraic geometry

Input: $X \subset (K^*)^n$ irred. of dim d defined by an ideal $I \subset K[x_1^{\pm}, \ldots, x_n^{\pm}]$.
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- **Gröbner theory:** $\operatorname{Trop}(I) = \{\omega \in \mathbb{R}^n \,|\, \operatorname{in}_\omega(I) \neq 1\}$.

Weight of $\omega \in$ mxl cone $= \#\{$ components of $\operatorname{in}_\omega(I)\}$ (with mult.)
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- $(K^*)^r$ action on X via $A \in \mathbb{Z}^{r \times n} \rightsquigarrow$ Row span (A) in all cones of $\text{Trop}(I)$.

$\rightsquigarrow$ Mod. out $\text{Trop}(I)$ by this lineality space preserves the combinatorics.

- The ends of a curve $\text{Trop}(X)$ in $\mathbb{R}^2$ give an ambient toric variety $\supset \overline{X}$.

Conclusion: $\text{Trop}(I)$ sees dimension, torus actions, initial degenerations, compactifications and other *geometric invariants* of X (e.g. degree)

Notice: $\text{Trop}(X)$ is highly sensitive to the embedding of X

Grassmannian of lines in $\mathbb{P}^{n-1}$ and the space of trees

Definition: $\mathrm{Gr}(2,n) = \{\text{lines in } \mathbb{P}^{n-1}\} := K^{2\times n}_{\mathrm{rk}\,2}/\,\mathrm{GL}_2$ $\qquad$ ($\dim = 2(n-2)$).

The Plücker map embeds $\mathrm{Gr}(2,n) \hookrightarrow \mathbb{P}^{\binom{n}{2}-1}$ by the list of 2×2-minors:

$$\varphi(X) = [p_{ij} := \det(X^{(i,j)})]_{i<j} \qquad \forall\, X \in K^{2\times n}.$$

Its Plücker ideal $I_{2,n}$ is generated by the 3-term (quadratic) Plücker eqns:

$$p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} \qquad (1 \leqslant i < j < k < l \leqslant n).$$

Note: $(K^*)^n/K^*$ acts on $\mathrm{Gr}(2,n)$ via $t * (p_{ij}) = t_i t_j\, p_{ij}$.

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$\rightsquigarrow$ Tropical Plücker eqns: $\max\{x_{ij}+x_{kl}, x_{ik}+x_{jl}, x_{il}+x_{jl}\}$.

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Theorem (Speyer-Sturmfels)

The tropical Grassmannian $\mathrm{Trop}(\mathrm{Gr}(2,n) \cap ((K^*)^{\binom{n}{2}}/K^*))$ *in* $\mathbb{R}^{\binom{n}{2}}/\mathbb{R}\cdot\mathbf{1}$ *is the space of phylogenetic trees on n leaves:*

- *all leaves are labeled 1 through n (no repetitions);*
- *weights on all edges (non-negative weights for internal edges).*

It is cut out by the tropical Plücker equations. The lineality space is generated by the n cut-metrics $\ell_i = \sum_{j\neq i} e_{ij}$, *modulo* $\mathbb{R}\cdot\mathbf{1}$.

The space of phylogenetic trees $\mathcal{T}_n$ on n leaves

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From the data (T, ω), we construct $\mathbf{x} \in \mathbb{R}^{\binom{n}{2}}$ by $x_{pq} = \sum_{e \in p \to q} \omega(e)$:

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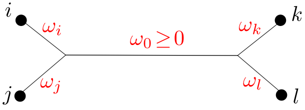
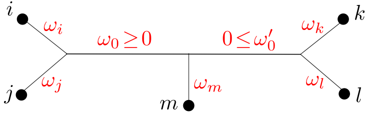


$(ij|kl)$

$$\begin{cases} x_{ij} = \omega_i + \omega_j, \\ x_{ik} = \omega_i + \omega_0 + \omega_k, \dots \end{cases}$$



$(ij|kl) \cap (im|kl) \cap (jm|kl) \cap \dots$

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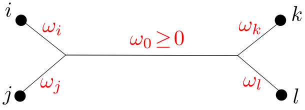

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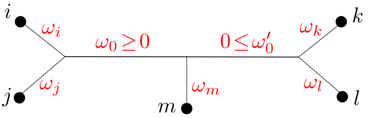


$(ij|kl) \cap (im|kl) \cap (jm|kl) \cap \ldots$

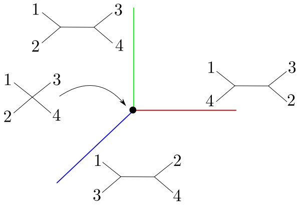
Claim: $(T, \omega) \xleftrightarrow{1-\text{to}-1} \mathbf{x}$ satisfying Tropical Plücker eqns.

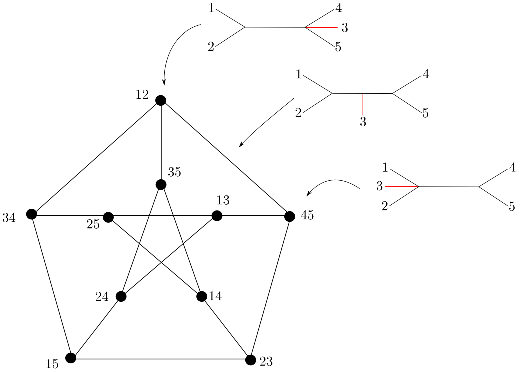
Why? (1) $\max\{x_{ij} + x_{kl}, \underline{x_{ik} + x_{jl}}, \underline{x_{il} + x_{jk}}\} \iff$ quartet $(ij|kl)$.

(2) tree T is reconstructed form the list of quartets,

(3) linear algebra recovers the weight function ω from T and $\mathbf{x}$.

Examples:





$\mathcal{T}_4/\mathbb{R}^3$ has f-vector $(1, 3)$.

$\mathcal{T}_5/\mathbb{R}^4$ is the cone over the Petersen graph. f-vector $= (1, 10, 15)$.

$$\dim \mathrm{Gr}(2, n) = \dim(\mathrm{Trop}(Gr(2, n) \cap \mathbb{R}^{\binom{n}{2}-1}) = 2(n-2).$$

How to compactify $\mathcal{T}_n$?

- Write $\mathbb{TP}^{\binom{n}{2}-1} := (\mathbb{R} \cup \{-\infty\})^{\binom{n}{2}} \smallsetminus (-\infty, \ldots, -\infty))/\mathbb{R}\cdot(1, \ldots, 1)$
- Compactify $\mathcal{T}_n$ using $\text{Trop}(\text{Gr}(2, n)) \subset \mathbb{TP}^{\binom{n}{2}-1}$.
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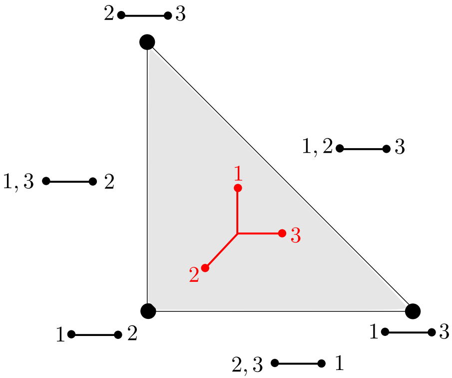

$\overline{\mathcal{T}_3}$

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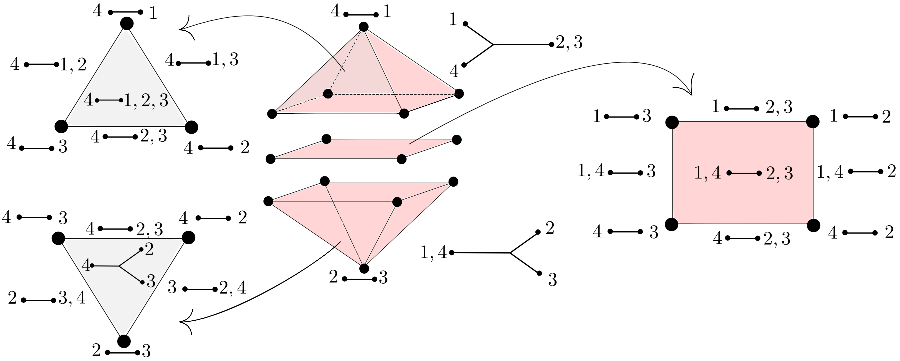
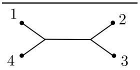


Boundary cells in $\overline{(14|23)}$ =



Problem: find nice embeddings or repair bad ones

GOAL 1: Find embeddings of a plane curve $\mathscr{C}$ into nice toric varieties such that Trop($\mathscr{C}$) better reflects the geometry of $\mathscr{C}$.

GOAL 2: Given a bad embedding of $\mathscr{C}$, repair it by effective methods.

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- **Thm** [Katz–Markwig[2]]: Trop(E) has a cycle of length $\leqslant -\text{val}(j(E))$, and have equality for *generic* coefficients with fixed Trop(g) (g =cubic eqn).
- If E is given in Weierstrass form $y^2 = (x^3 + ax + b) \Rightarrow$ no cycle at all!
- If the cycle is shorter than expected, can we find a re-embedding that prolongs it, without changing the structure of the curve?

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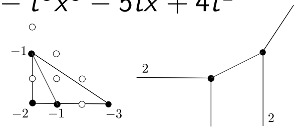
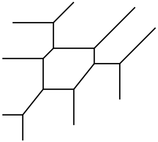
[Chan-Sturmfels] Any plane elliptic cubic admits a honeycomb form in $\mathbb{R}^2$.

Example: $g = y^2 - t^3x^3 - 5tx + 4t^2$



Weierstrass form

$\xrightarrow{\text{[Ch-St]}}$



honeycomb form

Re-embeddings via linear tropical modifications

We construct the modification of $\mathbb{R}^2$ along a linear tropical polynomial F:

1. Fix $F = \max\{A, B + X, C + Y\} = A \oplus B \odot X \oplus C \odot Y$ a linear tropical polynomial in $\mathbb{R}^2$, with $A, B, C \in \mathbb{R} \cup \{-\infty\}$.
2. Take the graph of F in $\mathbb{R}^3$: it has at most three linear pieces.
3. At each break-line, we attach two-dimensional cells spanned by the vector $(0, 0, -1)$ and assign mult 1 to it ($\rightsquigarrow$ balanced fan!).

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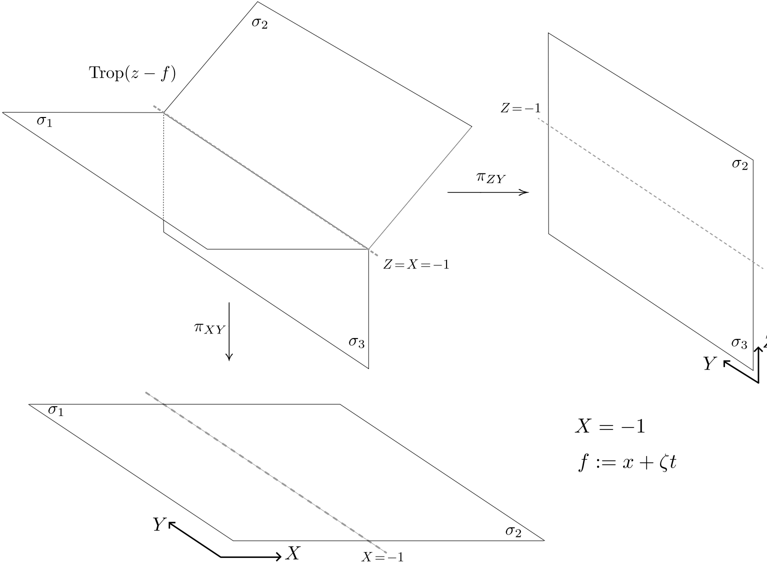
• Given a plane curve $\mathscr{C}$ defined by a polynomial $g \in K[x, y]$, we define a new linear re-embedding of $\mathscr{C}$ by the ideal

$$I_{g,f} := \langle g, z - f \rangle \subset K[x, y, z],$$

where $f = a + bx + cy \in K[x, y]$ be a Puiseux series lift of F, i.e. $-\operatorname{val}(a) = A$, $-\operatorname{val}(b) = B$ and $-\operatorname{val}(c) = C$.

Notice: The curve $\operatorname{Trop}(I_{g,f})$ lies in the tropical plane $\operatorname{Trop}(z - f)$.
The projection π_{XY} gives $\operatorname{Trop}(g)$.

Linear tropical modification of $\mathbb{R}^2$ along $\{X = \ell\}$

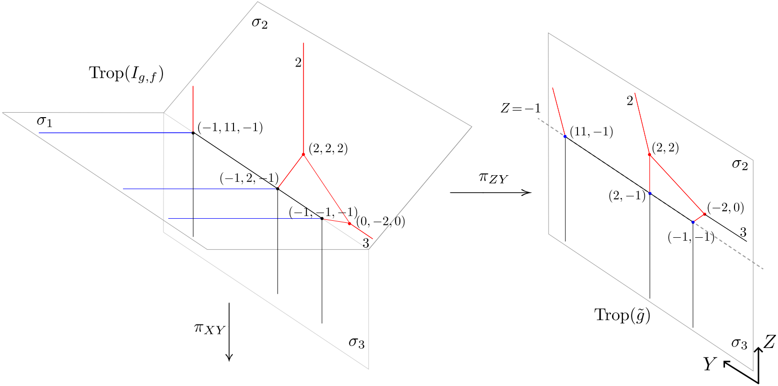


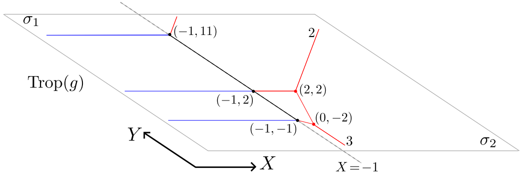
$X = -1$

$f := x + \zeta t$

$\sigma_1 = \{X \leqslant \ell, Z = \ell\},\ \sigma_2 = \{X \geqslant \ell, Z = X\},\ \sigma_3 = \{X = \ell, Z \leqslant \ell\}.$

Generic modification of a plane cubic along $\{X = \ell\}$





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ζ generic

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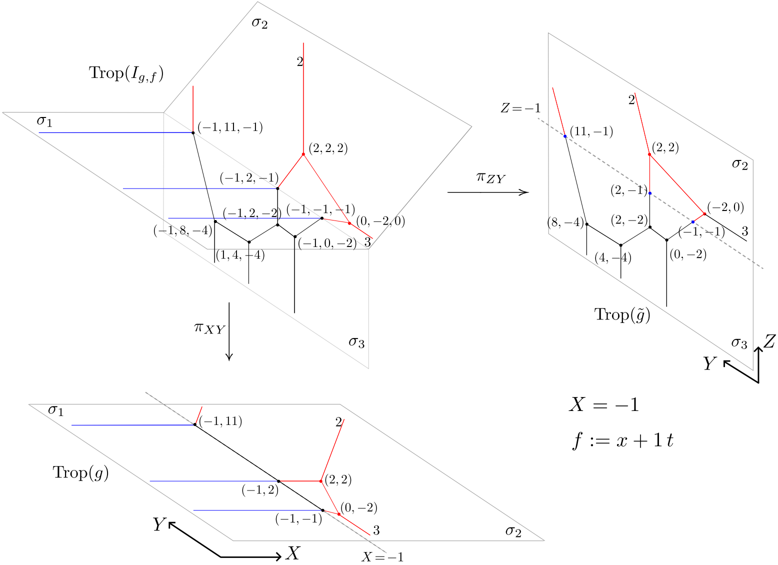
Special modification of a plane cubic along $\{X = \ell\}$



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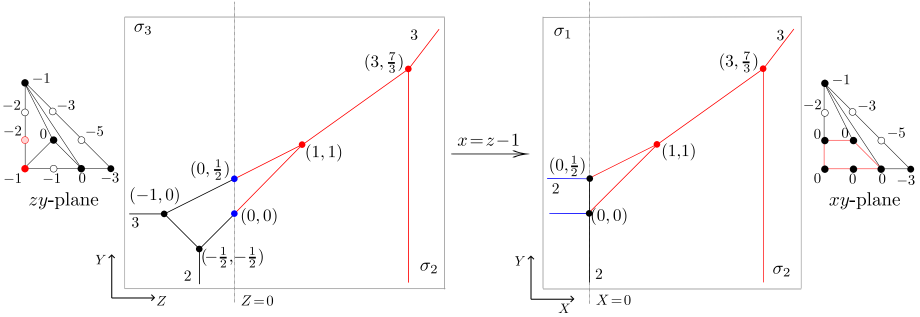
The cycle of a smooth tropical plane elliptic cubic

Theorem (C.–Markwig)

Let g define a plane elliptic cubic where the cycle of $\mathrm{Trop}(g)$ *has length* $< -\mathrm{val}(j(g))$*. Then, we can recursively repair it (in dim. 4) with linear tropical modifications along straight lines.*

Example: $g := t^3x^3 + t^5x^2y + t^3xy^2 + ty^3 + x^2 + 3xy + t^2y^2 + (2+\frac{3}{2}t)x + (3+t^2)y + 1$



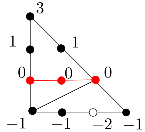
$$\mathrm{in}_{(0,0)}(g) = (1+x)^2 + 3(x+1)y = (x+1)((1+x)+3y)) \rightsquigarrow \zeta = 1.$$

Make a cycle appear from a high multiplicity edge

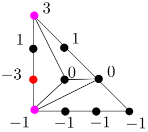
Theorem (C.–Markwig)

Let g be a plane elliptic cubic where $\operatorname{Trop}(g)$ *has no cycle but it contains a vertical bounded edge e of multiplicity $n \geqslant 2$ with trivalent endpoints. If* $\operatorname{in}_e(g)$ *has n components then we can unfold this edge into a cycle using the tropical modification along the line $\mathbb{R}\langle e\rangle$.*

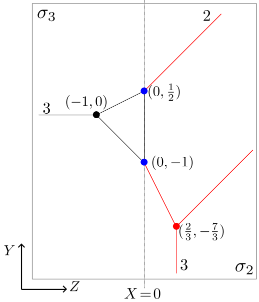
Example: $g = t^3x^3 + x^2y + t^3xy^2 + ty^3 + t^4x^2 + (1+t^2)xy + t^2y^2 + t^5x + (1+t)y + t$



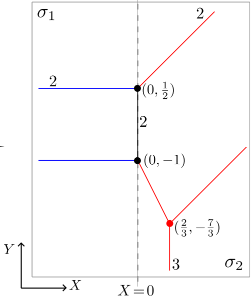
xy-plane



zy-plane



$x = z - \frac{1+\sqrt{-3}}{2}$



$\Delta_e = c_{1,1}^2 - 4c_{1,2}c_{1,0} = -3$; $\operatorname{in}_e(g) = y(1 + x + x^2) \rightsquigarrow \zeta = \frac{1 \pm \sqrt{-3}}{2}$.