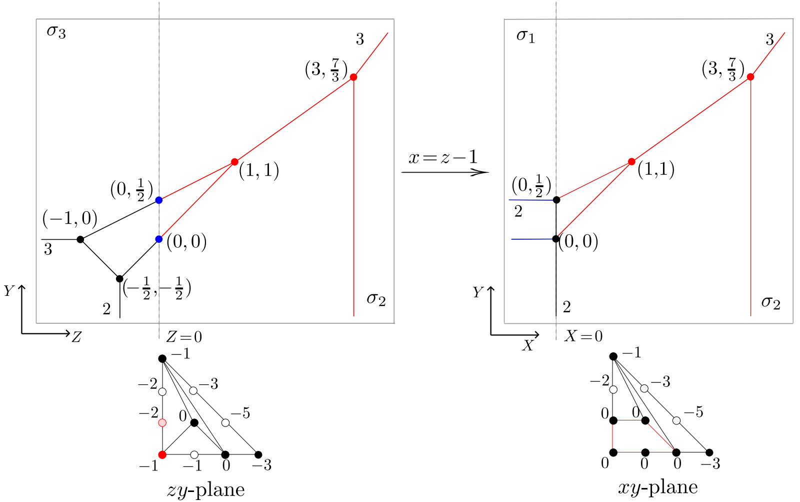
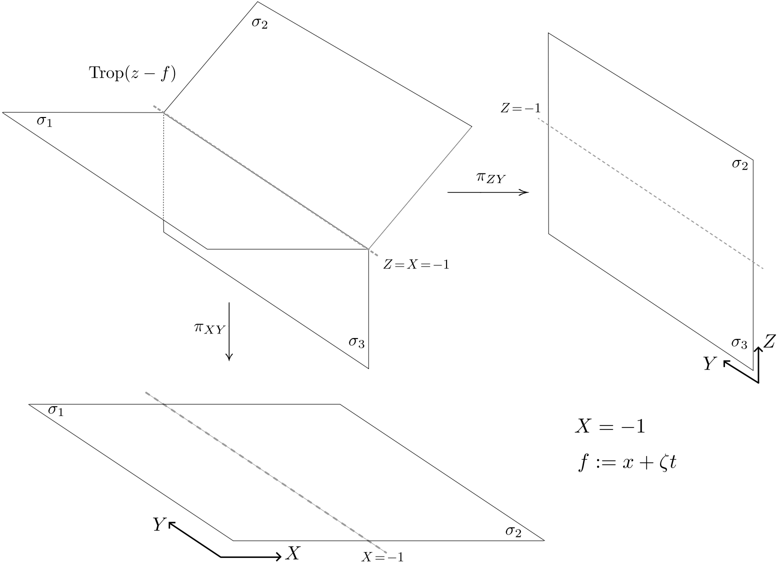
Example: Repair a loop on the visible side of a vert. line.

$$g(x,y) = t^3x^3 + t^5x^2y + t^3xy^2 + ty^3 + x^2 + 3xy + t^2y^2 + (2+3t/2)x + (3+t^2)y + 1.$$



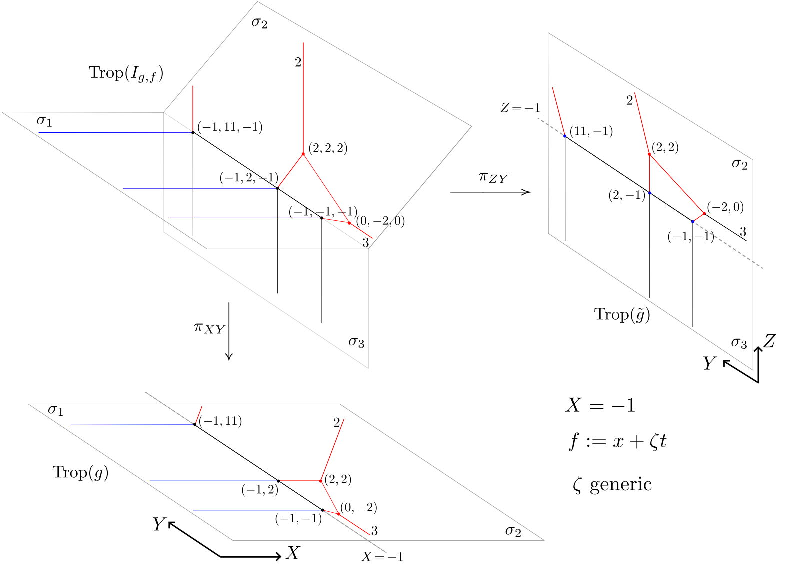
$$\mathrm{in}_{(0,0)}(g) = (1+x)^2 + 3(x+1)y = (x+1)((1+x)+y)) \rightsquigarrow \ \zeta = 1.$$

Linear trop. modification of $\mathbb{R}^2$ along $\{X = \ell\}$



$$\sigma_1 = \{X \leqslant \ell, Z = \ell\},\ \sigma_2 = \{X \geqslant \ell, Z = X\},\ \sigma_3 = \{X = \ell, Z \leqslant \ell\}.$$

Generic modification of a plane cubic along $\{X = \ell\}$



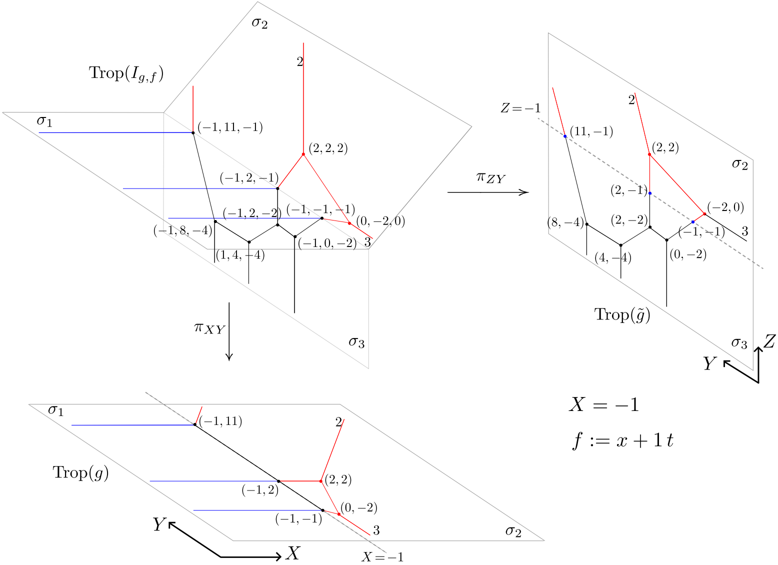
$X = -1$

$f := x + \zeta t$

ζ generic

$$\sigma_1 = \{X \leqslant \ell, Z = \ell\},\ \sigma_2 = \{X \geqslant \ell, Z = X\},\ \sigma_3 = \{X = \ell, Z \leqslant \ell\}.$$

Special modification of a plane cubic along $\{X = \ell\}$



$X = -1$

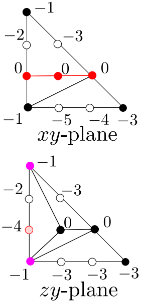
$f := x + 1\,t$

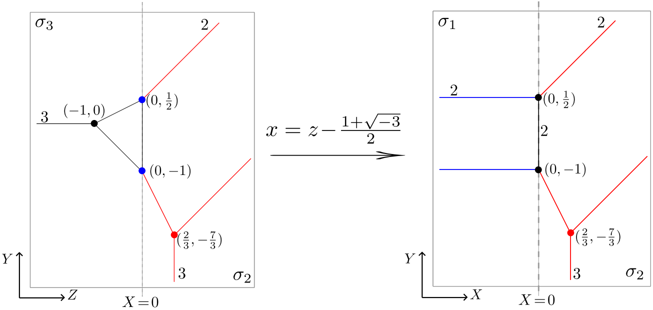
$$\sigma_1 = \{X \leqslant \ell, Z = \ell\},\ \sigma_2 = \{X \geqslant \ell, Z = X\},\ \sigma_3 = \{X = \ell, Z \leqslant \ell\}.$$

Theorem (Repairing bounded edges with high multiplicity)

Let e be a vertical bounded edge of $\operatorname{Trop}(g)$ *of multiplicity $n \geqslant 2$ whose endpoints have valency 3. If $\Delta_{e^\vee}$ does* **not** *vanish at* $\operatorname{in}_e(g)$*, then e witnesses a folding of edges. Moreover, we can unfold e and produce a loop using the tropical modification along the line $\mathbb{R}\langle e\rangle$.*

Example:



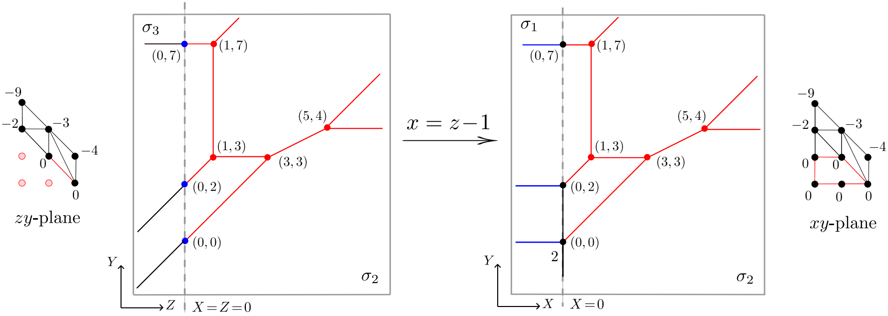


Example: Unfold a loop of a rational cubic plane curve.

$$g = t^4\, x^2 y + 5t^3\, xy^2 + t^9\, y^3 + x^2 + 3xy + t^2\, y^2 + 2x + (3 - t^4)y + 1$$

$$\Delta_{(0,0)}(\mathrm{in}_{(0,0)}(g)) = c_{00}\, c_{11}^2 - c_{10}\, c_{01}\, c_{11} + c_{20}\, c_{01}^2 = 1 \cdot 3^2 - 2 \cdot 3 \cdot 3 + 1 \cdot 3^2 = 0.$$

Want: $0 = c_{20}\, \mathrm{in}(\zeta)^2 - c_{11}\, \mathrm{in}_\zeta + c_{00} = \mathrm{in}(\zeta)^2 - 2\, \mathrm{in}(\zeta) + 1 = (\mathrm{in}(\zeta) - 1)^2.$



$$\tilde{g} = g(z-1, y) = t^4\, z^2 y + 5t^3\, zy^2 + t^9\, y^3 + z^2 + (3 - 2t^4)zy + (t^2 - 5t^3)y^2$$

Proof ideas: Repairing the loop of a tropical plane cubic.

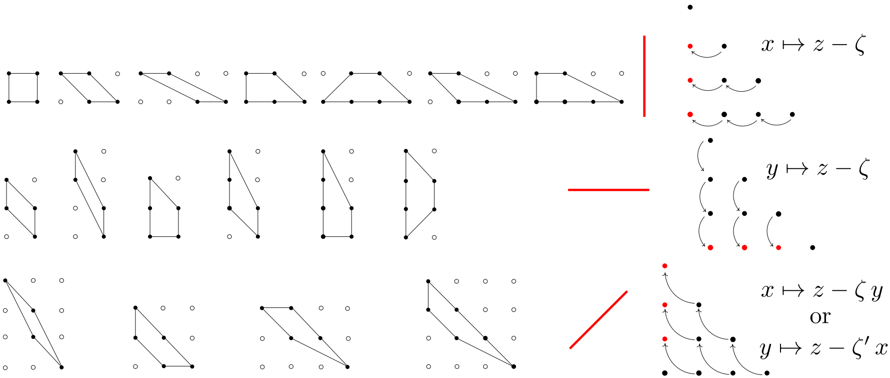
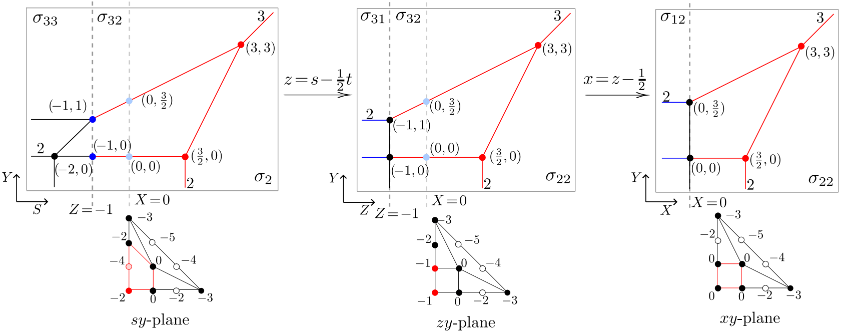


Figure: Locally reducible vertices on a cubic and feeding process

Case 1: Visible loop and 2 iterations.

$\mathbf{g} = -t^3\,x^3 + (t^4 + t^5)x^2y + (-t^5 + t^6)xy^2 + t^3\,y^3 + (t^2 - t^3)x^2 + 4xy + (2t^2 + 3t^3)y^2 + 2x + (2 + 2t)y + (1 + t).$



Re-embed by $I_{g,f_1,f_2} :=< g, z-(x+1/2), s-(z+t/2)\rangle$.

Enough: $g(s-(1/2+t/2), y) \subset K[s,y]$.

Case 2: loc. red. vertices with vanishing discrim. on the top left (v_2) and bottom right (v_1) and loop not visible.

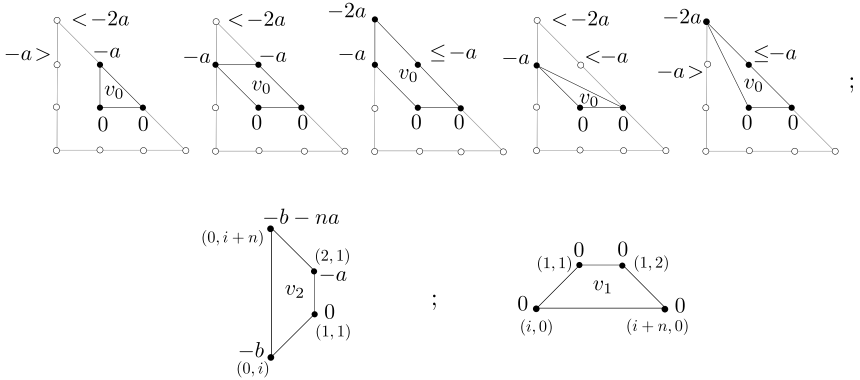
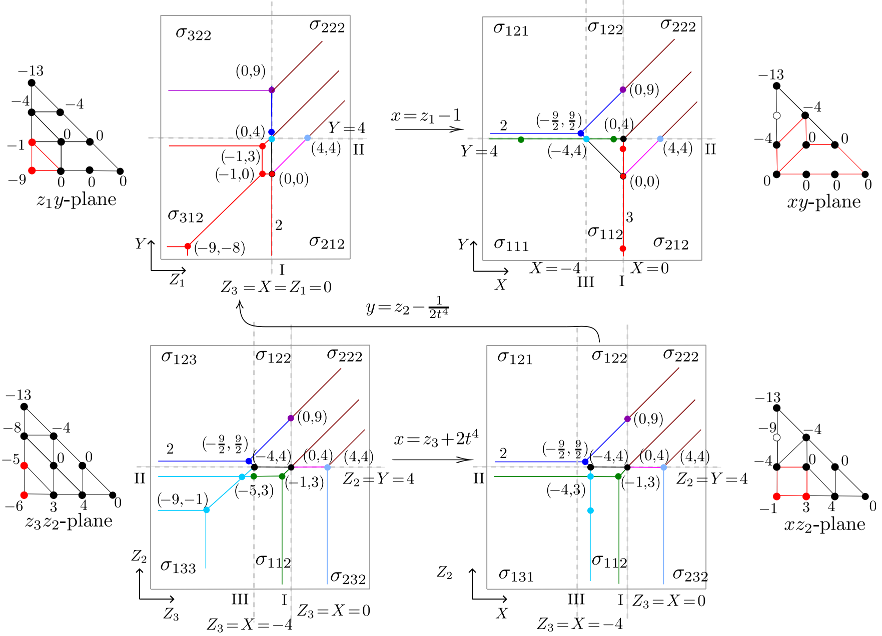


Figure: Combinatorics and heights of dual cells to distinguished vertices when no linear tropical modification keeps the loop of Trop(g) on its visible side. We disallow $i = n = 1$.

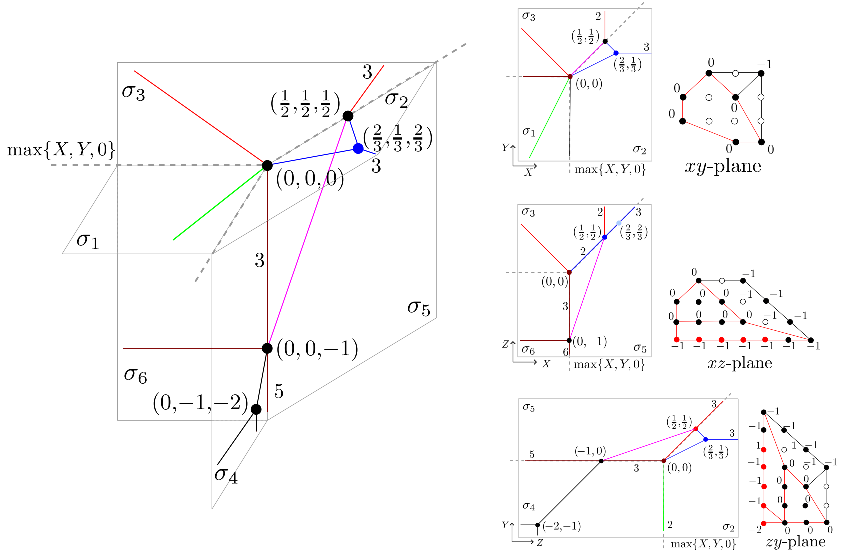
$\mathbf{g} := x^3 + (1-9t^2)x^2y + 2t^4xy^2 + t^{20}y^3 + (1-24t^9 - t^{40})x^2 + (1+5t-16t^9+144t^{11})xy + 8t^{67}y^2 + (1-16t^9+t^{15}+192t^{18})x + (2t^4+64t^{18}-576t^{20})y + (1-8t^9+64t^{18}-8t^{24})$.



Re-embed by $I_{g,f_1,f_2,f_3} := \langle g(z_3 - 2t^4, y), z_2 - y - t^{-4}/2, z_3 - z_1 + 1 + 2t^4\rangle$.

E.g.: Modif. along the trop. line $F := \max\{X, Y, 0\}$.

$$\mathbf{g} := (t+2t^2)x^3y^3 + (1-t^3)x^2y^2 + (1+t)xy^3 + (1-t^4)x^3 + (1+3t^2)x^2 + (1+6t)y^2 + (1+t)y.$$



Re-embed by $I_{g,f} := \langle g, z - (x+y+1)\rangle$.