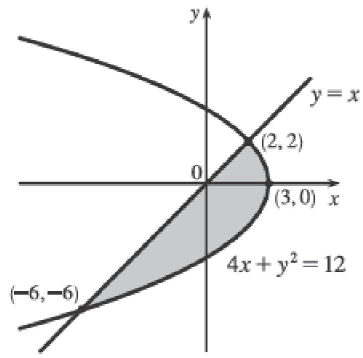
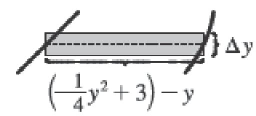
**12.** $4x + x^2 = 12 \quad \Leftrightarrow \quad (x+6)(x-2) = 0 \quad \Leftrightarrow$

$x = -6$ or $x = 2$, so $y = -6$ or $y = 2$ and

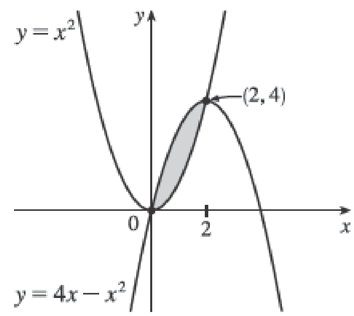
$$A = \int_{-6}^{2} \left[\left(-\tfrac{1}{4}y^2 + 3\right) - y\right] dy$$
$$= \left[-\tfrac{1}{12}y^3 - \tfrac{1}{2}y^2 + 3y\right]_{-6}^{2}$$
$$= \left(-\tfrac{2}{3} - 2 + 6\right) - (18 - 18 - 18)$$
$$= 22 - \tfrac{2}{3} = \tfrac{64}{3}$$



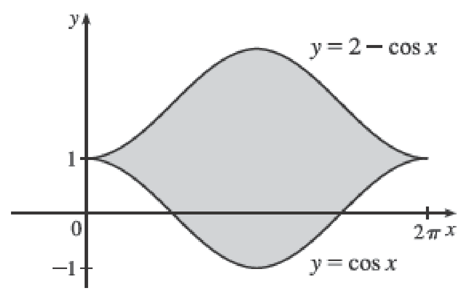




**14.** $x^2 = 4x - x^2 \quad \Leftrightarrow \quad 2x^2 - 4x = 0 \quad \Leftrightarrow \quad 2x(x-2) = 0 \quad \Leftrightarrow \quad x = 0$ or $2$, so

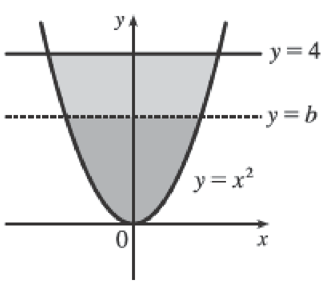
$$A = \int_0^2 \left[(4x - x^2) - x^2\right] dx$$
$$= \int_0^2 (4x - 2x^2)\, dx$$
$$= \left[2x^2 - \tfrac{2}{3}x^3\right]_0^2$$
$$= 8 - \tfrac{16}{3} = \tfrac{8}{3}$$




**16.** $$A = \int_0^{2\pi} \left[(2 - \cos x) - \cos x\right] dx$$
$$= \int_0^{2\pi} (2 - 2\cos x)\, dx$$
$$= \left[2x - 2\sin x\right]_0^{2\pi}$$
$$= (4\pi - 0) - 0 = 4\pi$$



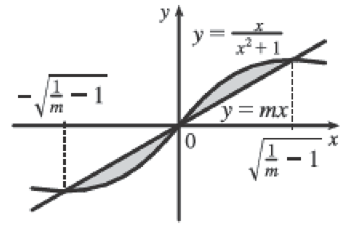

**51.**




By the symmetry of the problem, we consider only the first quadrant, where $y = x^2 \quad \Rightarrow \quad x = \sqrt{y}$. We are looking for a number $b$ such that

$$\int_0^b \sqrt{y}\, dy = \int_b^4 \sqrt{y}\, dy \quad \Rightarrow \quad \tfrac{2}{3}\left[y^{3/2}\right]_0^b = \tfrac{2}{3}\left[y^{3/2}\right]_b^4 \quad \Rightarrow$$

$b^{3/2} = 4^{3/2} - b^{3/2} \quad \Rightarrow \quad 2b^{3/2} = 8 \quad \Rightarrow \quad b^{3/2} = 4 \quad \Rightarrow \quad b = 4^{2/3} \approx 2.52.$




**55.** The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2+1) = mx \Rightarrow$ $x = x(mx^2+m) \Rightarrow x(mx^2+m) - x = 0 \Rightarrow$ $x(mx^2+m-1) = 0 \Rightarrow x = 0$ or $mx^2+m-1 = 0 \Rightarrow$ $x = 0$ or $x^2 = \dfrac{1-m}{m} \Rightarrow x = 0$ or $x = \pm\sqrt{\dfrac{1}{m}-1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2+1)$ at the origin is $y'(0) = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since $mx$ and $x/(x^2+1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m-1}\right]$. So the total area enclosed is

$$2\int_0^{\sqrt{1/m-1}}\left[\frac{x}{x^2+1} - mx\right]dx = 2\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}} = [\ln(1/m-1+1) - m(1/m-1)] - (\ln 1 - 0)$$

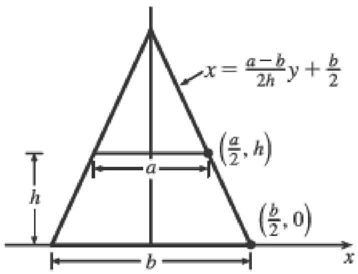
$$= \ln(1/m) - 1 + m = m - \ln m - 1$$

**39.** $\pi\int_0^{\pi}\sin x\,dx = \pi\int_0^{\pi}\left(\sqrt{\sin x}\right)^2 dx$ describes the volume of solid obtained by rotating the region $\mathcal{R} = \left\{(x,y) \mid 0 \le x \le \pi, 0 \le y \le \sqrt{\sin x}\right\}$ of the $xy$-plane about the $x$-axis.

**42.** $\pi\int_0^{\pi/2}\left[(1+\cos x)^2 - 1^2\right]dx$ describes the volume of the solid obtained by rotating the region $\mathcal{R} = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{2}, 1 \le y \le 1+\cos x\right\}$ of the $xy$-plane about the $x$-axis.

*Or:* The solid could be obtained by rotating the region $\mathcal{R}' = \left\{(x,y) \mid 0 \le x \le \frac{\pi}{2}, 0 \le y \le \cos x\right\}$ about the line $y = -1$.

**50.** An equation of the line is $x = \dfrac{\Delta x}{\Delta y}\,y + (x\text{-intercept}) = \dfrac{a/2 - b/2}{h - 0}\,y + \dfrac{b}{2} = \dfrac{a-b}{2h}\,y + \dfrac{b}{2}$.

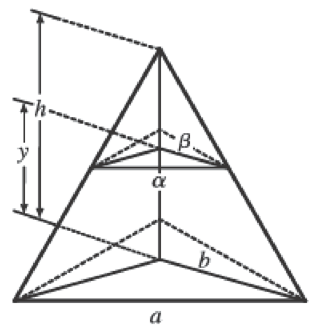



$$\begin{aligned}
V &= \int_0^h A(y)\,dy = \int_0^h (2x)^2\,dy \\
&= \int_0^h \left[2\left(\frac{a-b}{2h}y + \frac{b}{2}\right)\right]^2 dy = \int_0^h \left[\frac{a-b}{h}y + b\right]^2 dy \\
&= \int_0^h \left[\frac{(a-b)^2}{h^2}y^2 + \frac{2b(a-b)}{h}y + b^2\right] dy \\
&= \left[\frac{(a-b)^2}{3h^2}y^3 + \frac{b(a-b)}{h}y^2 + b^2 y\right]_0^h \\
&= \tfrac{1}{3}(a-b)^2 h + b(a-b)h + b^2 h = \tfrac{1}{3}\left(a^2 - 2ab + b^2 + 3ab\right)h \\
&= \tfrac{1}{3}\left(a^2 + ab + b^2\right)h
\end{aligned}$$

[Note that this can be written as $\frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\,\right)h$, as in Exercise 48.]

If $a = b$, we get a rectangular solid with volume $b^2 h$. If $a = 0$, we get a square pyramid with volume $\frac{1}{3}b^2 h$.

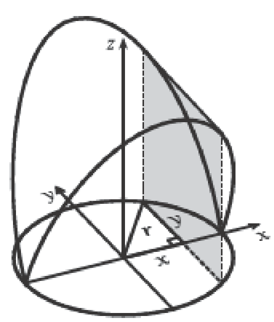
**52.** Consider the triangle consisting of two vertices of the base and the center of the base. This triangle is similar to the corresponding triangle at a height $y$, so $a/b = \alpha/\beta \quad \Rightarrow \quad \alpha = a\beta/b$. Also by similar triangles, $b/h = \beta/(h-y) \Rightarrow \beta = b(h-y)/h$. These two equations imply that $\alpha = a(1 - y/h)$, and since the cross-section is an equilateral triangle, it has area




$$A(y) = \frac{1}{2}\cdot\alpha\cdot\frac{\sqrt{3}}{2}\alpha = \frac{a^2(1-y/h)^2}{4}\sqrt{3}, \text{ so}$$

$$\begin{aligned}
V &= \int_0^h A(y)\,dy = \frac{a^2\sqrt{3}}{4}\int_0^h \left(1 - \frac{y}{h}\right)^2 dy \\
&= \frac{a^2\sqrt{3}}{4}\left[-\frac{h}{3}\left(1 - \frac{y}{h}\right)^3\right]_0^h = -\frac{\sqrt{3}}{12}a^2 h(-1) = \frac{\sqrt{3}}{12}a^2 h
\end{aligned}$$

**54.** A cross-section is shaded in the diagram.




$$A(x) = (2y)^2 = \left(2\sqrt{r^2 - x^2}\,\right)^2, \text{ so}$$

$$\begin{aligned}
V &= \textstyle\int_{-r}^{r} A(x)\,dx = 2\int_0^r 4(r^2 - x^2)\,dx \\
&= 8\left[r^2 x - \tfrac{1}{3}x^3\right]_0^r = 8\left(\tfrac{2}{3}r^3\right) = \tfrac{16}{3}r^3
\end{aligned}$$