1. $f(x) = 5 - 12x + 3x^2$, $[1, 3]$. Since f is a polynomial, it is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[1, 3]$ and differentiable on $(1, 3)$. Also $f(1) = -4 = f(3)$. $f'(c) = 0 \quad \Leftrightarrow \quad -12 + 6c = 0 \quad \Leftrightarrow \quad c = 2$, which is in the open interval $(1, 3)$, so $c = 2$ satisfies the conclusion of Rolle's Theorem.

10. $f(x) = x^3 - 3x + 2$, $[-2, 2]$. f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad 3c^2 - 3 = \dfrac{f(2) - f(-2)}{2 - (-2)} = \dfrac{4 - 0}{4} = 1 \quad \Leftrightarrow \quad 3c^2 = 4 \quad \Leftrightarrow$ $c^2 = \dfrac{4}{3} \quad \Leftrightarrow \quad c = \pm\dfrac{2}{\sqrt{3}}$, which are both in $(-2, 2)$.

17. Let $f(x) = 2x + \cos x$. Then $f(-\pi) = -2\pi - 1 < 0$ and $f(0) = 1 > 0$. Since f is the sum of the polynomial $2x$ and the trignometric function $\cos x$, f is continuous and differentiable for all x. By the Intermediate Value Theorem, there is a number c in $(-\pi, 0)$ such that $f(c) = 0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \sin r > 0$ since $\sin r \leq 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one root.

23. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4 - 1)$ for some $c \in (1, 4)$. But for every $c \in (1, 4)$ we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4 - 1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

36. Assume that f is differentiable (and hence continuous) on $\mathbb{R}$ and that $f'(x) \neq 1$ for all x. Suppose f has more than one fixed point. Then there are numbers a and b such that $a < b$, $f(a) = a$, and $f(b) = b$. Applying the Mean Value Theorem to the function f on $[a, b]$, we find that there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. But then $f'(c) = \dfrac{b - a}{b - a} = 1$, contradicting our assumption that $f'(x) \neq 1$ for every real number x. This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.

1. (a) f is increasing on $(1, 3)$ and $(4, 6)$.

(b) f is decreasing on $(0, 1)$ and $(3, 4)$.

(c) f is concave upward on $(0, 2)$.

(d) f is concave downward on $(2, 4)$ and $(4, 6)$.

(e) The point of inflection is $(2, 3)$.

10. (a) $f(x) = 4x^3 + 3x^2 - 6x + 1 \quad \Rightarrow \quad f'(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x-1)(x+1)$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $x > \frac{1}{2}$ and $f'(x) < 0 \quad \Leftrightarrow \quad -1 < x < \frac{1}{2}$. So f is increasing on $(-\infty, -1)$ and $\left(\frac{1}{2}, \infty\right)$ and f is decreasing on $\left(-1, \frac{1}{2}\right)$.

(b) f changes from increasing to decreasing at $x = -1$ and from decreasing to increasing at $x = \frac{1}{2}$. Thus, $f(-1) = 6$ is a local maximum value and $f\left(\frac{1}{2}\right) = -\frac{3}{4}$ is a local minimum value.

(c) $f''(x) = 24x + 6 = 6(4x + 1)$. $\quad f''(x) > 0 \quad \Leftrightarrow \quad x > -\frac{1}{4}$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -\frac{1}{4}$. Thus, f is concave upward on $\left(-\frac{1}{4}, \infty\right)$ and concave downward on $\left(-\infty, -\frac{1}{4}\right)$. There is an inflection point at $\left(-\frac{1}{4}, f\left(-\frac{1}{4}\right)\right) = \left(-\frac{1}{4}, \frac{21}{8}\right)$.

13. (a) $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi$. $\quad f'(x) = \cos x - \sin x = 0 \quad \Rightarrow \quad \cos x = \sin x \quad \Rightarrow \quad 1 = \dfrac{\sin x}{\cos x} \quad \Rightarrow$ $\tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad \cos x - \sin x > 0 \quad \Leftrightarrow \quad \cos x > \sin x \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{4}$ or $\frac{5\pi}{4} < x < 2\pi$ and $f'(x) < 0 \quad \Leftrightarrow \quad \cos x < \sin x \quad \Leftrightarrow \quad \frac{\pi}{4} < x < \frac{5\pi}{4}$. So f is increasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ and f is decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

(b) f changes from increasing to decreasing at $x = \frac{\pi}{4}$ and from decreasing to increasing at $x = \frac{5\pi}{4}$. Thus, $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ is a local maximum value and $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$ is a local minimum value.

(c) $f''(x) = -\sin x - \cos x = 0 \quad \Rightarrow \quad -\sin x = \cos x \quad \Rightarrow \quad \tan x = -1 \quad \Rightarrow \quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Divide the interval $(0, 2\pi)$ into subintervals with these numbers as endpoints and complete a second derivative chart.

Interval	$f''(x) = -\sin x - \cos x$	Concavity
$\left(0, \frac{3\pi}{4}\right)$	$f''\left(\frac{\pi}{2}\right) = -1 < 0$	downward
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	$f''(\pi) = 1 > 0$	upward
$\left(\frac{7\pi}{4}, 2\pi\right)$	$f''\left(\frac{11\pi}{6}\right) = \frac{1}{2} - \frac{1}{2}\sqrt{3} < 0$	downward

There are inflection points at $\left(\frac{3\pi}{4}, 0\right)$ and $\left(\frac{7\pi}{4}, 0\right)$.

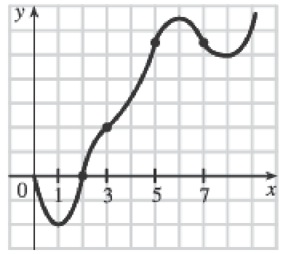
32. (a) f is increasing where f' is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where f' is negative, on $(0, 1)$ and $(6, 8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where f' is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e)

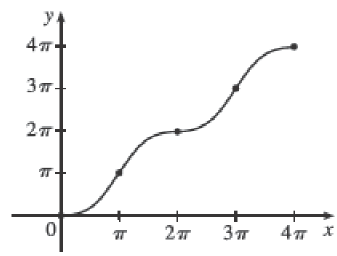


44. (a) $S(x) = x - \sin x$, $0 \le x \le 4\pi \quad \Rightarrow \quad S'(x) = 1 - \cos x$. $S'(x) = 0 \quad \Leftrightarrow \quad \cos x = 1 \quad \Leftrightarrow \quad x = 0$, 2π, and 4π. $S'(x) > 0 \quad \Leftrightarrow \quad \cos x < 1$, which is true for all x except integer multiples of 2π, so S is increasing on $(0, 4\pi)$ since $S'(2\pi) = 0$.

(b) There is no local maximum or minimum.

(c) $S''(x) = \sin x$. $S''(x) > 0$ if $0 < x < \pi$ or $2\pi < x < 3\pi$, and $S''(x) < 0$ if $\pi < x < 2\pi$ or $3\pi < x < 4\pi$. So S is CU on $(0, \pi)$ and $(2\pi, 3\pi)$, and S is CD on $(\pi, 2\pi)$ and $(3\pi, 4\pi)$. There are inflection points at (π, π), $(2\pi, 2\pi)$, and $(3\pi, 3\pi)$.

(d)



7. This limit has the form $\frac{0}{0}$. We can simply factor and simplify to evaluate the limit.

$$\lim_{x \to 1} \frac{x^2 - 1}{x^2 - x} = \lim_{x \to 1} \frac{(x+1)(x-1)}{x(x-1)} = \lim_{x \to 1} \frac{x+1}{x} = \frac{1+1}{1} = 2$$

12. This limit has the form $\frac{0}{0}$. $\displaystyle \lim_{x \to 0} \frac{\sin 4x}{\tan 5x} \overset{\text{H}}{=} \lim_{x \to 0} \frac{4 \cos 4x}{5 \sec^2(5x)} = \frac{4(1)}{5(1)^2} = \frac{4}{5}$

33. $\displaystyle \lim_{x \to 0} \frac{x + \sin x}{x + \cos x} = \frac{0+0}{0+1} = \frac{0}{1} = 0$. L'Hospital's Rule does not apply.

44. This limit has the form $0 \cdot (-\infty)$.

$$\lim_{x \to 0^+} \sin x \, \ln x = \lim_{x \to 0^+} \frac{\ln x}{\csc x} \overset{\text{H}}{=} \lim_{x \to 0^+} \frac{1/x}{-\csc x \, \cot x} = -\lim_{x \to 0^+} \left(\frac{\sin x}{x} \cdot \tan x \right) = -\left(\lim_{x \to 0^+} \frac{\sin x}{x} \right) \left(\lim_{x \to 0^+} \tan x \right)$$
$$= -1 \cdot 0 = 0$$

56. $y = (\tan 2x)^x \quad \Rightarrow \quad \ln y = x \cdot \ln \tan 2x$, so

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \cdot \ln \tan 2x = \lim_{x\to 0^+} \frac{\ln \tan 2x}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2} = \lim_{x\to 0^+} \frac{-2x^2 \cos 2x}{\sin 2x \, \cos^2 2x}$$

$$= \lim_{x\to 0^+} \frac{2x}{\sin 2x} \cdot \lim_{x\to 0^+} \frac{-x}{\cos 2x} = 1 \cdot 0 = 0 \quad \Rightarrow$$

$$\lim_{x\to 0^+} (\tan 2x)^x = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1.$$

62. $y = (e^x + x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x} \ln(e^x + x)$,

so $\displaystyle\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x + 1} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \quad \Rightarrow$

$$\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e.$$

63. $y = (4x+1)^{\cot x} \quad \Rightarrow \quad \ln y = \cot x \, \ln(4x+1)$, so $\displaystyle\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln(4x+1)}{\tan x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\frac{4}{4x+1}}{\sec^2 x} = 4 \quad \Rightarrow$

$$\lim_{x\to 0^+} (4x+1)^{\cot x} = \lim_{x\to 0^+} e^{\ln y} = e^4.$$

12. $y = f(x) = x/(x^2 - 9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$. **C.** $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin.

D. $\displaystyle\lim_{x\to\pm\infty} \frac{x}{x^2 - 9} = 0$, so $y = 0$ is a HA. $\displaystyle\lim_{x\to 3^+} \frac{x}{x^2 - 9} = \infty$, $\displaystyle\lim_{x\to 3^-} \frac{x}{x^2 - 9} = -\infty$, $\displaystyle\lim_{x\to -3^+} \frac{x}{x^2 - 9} = \infty$,

$\displaystyle\lim_{x\to -3^-} \frac{x}{x^2 - 9} = -\infty$, so $x = 3$ and $x = -3$ are VA. **E.** $f'(x) = \dfrac{(x^2 - 9) - x(2x)}{(x^2 - 9)^2} = -\dfrac{x^2 + 9}{(x^2 - 9)^2} < 0$ $[x \neq \pm 3]$

so f is decreasing on $(-\infty, -3)$, $(-3, 3)$, and $(3, \infty)$.

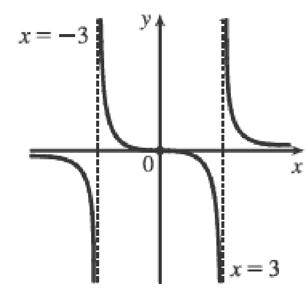
F. No extreme values

G. $f''(x) = -\dfrac{2x(x^2 - 9)^2 - (x^2 + 9) \cdot 2(x^2 - 9)(2x)}{(x^2 - 9)^4}$

$$= \frac{2x(x^2 + 27)}{(x^2 - 9)^3} > 0 \text{ when } -3 < x < 0 \text{ or } x > 3,$$

so f is CU on $(-3, 0)$ and $(3, \infty)$; CD on $(-\infty, -3)$ and $(0, 3)$. IP at $(0, 0)$

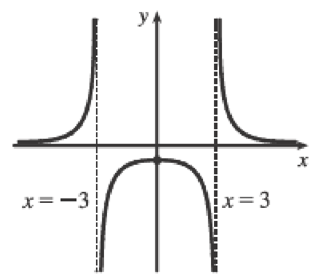
H.



13. $y = f(x) = 1/(x^2 - 9)$ **A.** $D = \{x \mid x \neq \pm 3\} = (-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **B.** y-intercept $= f(0) = -\frac{1}{9}$, no x-intercept **C.** $f(-x) = f(x) \Rightarrow f$ is even; the curve is symmetric about the y-axis. **D.** $\lim_{x \to \pm\infty} \frac{1}{x^2 - 9} = 0$, so $y = 0$ is a HA. $\lim_{x \to 3^-} \frac{1}{x^2 - 9} = -\infty$, $\lim_{x \to 3^+} \frac{1}{x^2 - 9} = \infty$, $\lim_{x \to -3^-} \frac{1}{x^2 - 9} = \infty$, $\lim_{x \to -3^+} \frac{1}{x^2 - 9} = -\infty$, so $x = 3$ and $x = -3$ are VA. **E.** $f'(x) = -\frac{2x}{(x^2 - 9)^2} > 0 \Leftrightarrow x < 0$ $(x \neq -3)$ so f is increasing on $(-\infty, -3)$ and $(-3, 0)$ and decreasing on $(0, 3)$ and $(3, \infty)$. **F.** Local maximum value $f(0) = -\frac{1}{9}$.

G. $y'' = \frac{-2(x^2 - 9)^2 + (2x)2(x^2 - 9)(2x)}{(x^2 - 9)^4} = \frac{6(x^2 + 3)}{(x^2 - 9)^3} > 0 \Leftrightarrow$ $x^2 > 9 \Leftrightarrow x > 3$ or $x < -3$, so f is CU on $(-\infty, -3)$ and $(3, \infty)$ and CD on $(-3, 3)$. No IP

H.

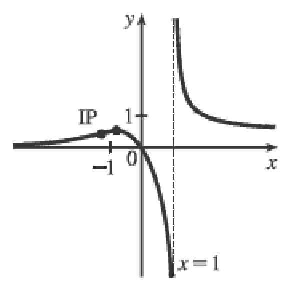


18. $y = f(x) = \frac{x}{x^3 - 1}$ **A.** $D = (-\infty, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$ **C.** No symmetry **D.** $\lim_{x \to \pm\infty} \frac{x}{x^3 - 1} = 0$, so $y = 0$ is a HA. $\lim_{x \to 1^-} f(x) = -\infty$ and $\lim_{x \to 1^+} f(x) = \infty$, so $x = 1$ is a VA. **E.** $f'(x) = \frac{(x^3 - 1)(1) - x(3x^2)}{(x^3 - 1)^2} = \frac{-2x^3 - 1}{(x^3 - 1)^2}$. $f'(x) = 0 \Rightarrow x = -\sqrt[3]{1/2}$. $f'(x) > 0 \Leftrightarrow x < -\sqrt[3]{1/2}$ and $f'(x) < 0 \Leftrightarrow -\sqrt[3]{1/2} < x < 1$ and $x > 1$, so f is increasing on $\left(-\infty, -\sqrt[3]{1/2}\right)$ and decreasing on $\left(-\sqrt[3]{1/2}, 1\right)$ and $(1, \infty)$. **F.** Local maximum value $f\left(-\sqrt[3]{1/2}\right) = \frac{2}{3}\sqrt[3]{1/2}$; no local minimum

G.
$$f''(x) = \frac{(x^3 - 1)^2(-6x^2) - (-2x^3 - 1)2(x^3 - 1)(3x^2)}{[(x^3 - 1)^2]^2}$$
$$= \frac{-6x^2(x^3 - 1)[(x^3 - 1) - (2x^3 + 1)]}{(x^3 - 1)^4} = \frac{6x^2(x^3 + 2)}{(x^3 - 1)^3}.$$

$f''(x) > 0 \Leftrightarrow x < -\sqrt[3]{2}$ and $x > 1$, $f''(x) < 0 \Leftrightarrow -\sqrt[3]{2} < x < 0$ and $0 < x < 1$, so f is CU on $\left(-\infty, -\sqrt[3]{2}\right)$ and $(1, \infty)$ and CD on $\left(-\sqrt[3]{2}, 1\right)$. IP at $\left(-\sqrt[3]{2}, \frac{1}{3}\sqrt[3]{2}\right)$

H.



25. $y = f(x) = x/\sqrt{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; x = 0$

C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+0}} = 1$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/x} = \lim_{x\to-\infty} \frac{x/x}{\sqrt{x^2+1}/\left(-\sqrt{x^2}\right)} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+1/x^2}}$$

$$= \frac{1}{-\sqrt{1+0}} = -1 \quad \text{so } y = \pm 1 \text{ are HA.}$$

No VA.

E. $\displaystyle f'(x) = \frac{\sqrt{x^2+1} - x \cdot \dfrac{2x}{2\sqrt{x^2+1}}}{[(x^2+1)^{1/2}]^2} = \frac{x^2+1-x^2}{(x^2+1)^{3/2}} = \frac{1}{(x^2+1)^{3/2}} > 0$ for all x, so f is increasing on $\mathbb{R}$.

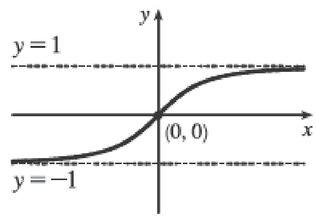
F. No extreme values

G. $\displaystyle f''(x) = -\tfrac{3}{2}(x^2+1)^{-5/2} \cdot 2x = \frac{-3x}{(x^2+1)^{5/2}}$, so $f''(x) > 0$ for $x < 0$

and $f''(x) < 0$ for $x > 0$. Thus, f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$.

IP at $(0, 0)$

H.



28. $y = f(x) = x/\sqrt{x^2-1}$ **A.** $D = (-\infty, -1) \cup (1, \infty)$ **B.** No intercepts **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = 1$ and $\displaystyle\lim_{x\to-\infty} \frac{x}{\sqrt{x^2-1}} = -1$, so $y = \pm 1$ are HA.

$\displaystyle\lim_{x\to 1^+} f(x) = +\infty$ and $\displaystyle\lim_{x\to -1^-} f(x) = -\infty$, so $x = \pm 1$ are VA.

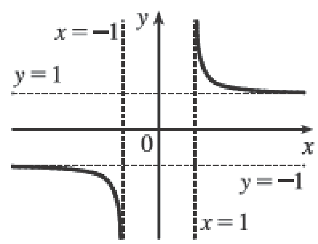
E. $\displaystyle f'(x) = \frac{\sqrt{x^2-1} - x \cdot \dfrac{x}{\sqrt{x^2-1}}}{[(x^2-1)^{1/2}]^2} = \frac{x^2-1-x^2}{(x^2-1)^{3/2}} = \frac{-1}{(x^2-1)^{3/2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No extreme values

G. $\displaystyle f''(x) = (-1)\left(-\tfrac{3}{2}\right)(x^2-1)^{-5/2} \cdot 2x = \frac{3x}{(x^2-1)^{5/2}}$.

$f''(x) < 0$ on $(-\infty, -1)$ and $f''(x) > 0$ on $(1, \infty)$, so f is CD on $(-\infty, -1)$ and CU on $(1, \infty)$. No IP

H.



46. $y = f(x) = e^{2x} - e^x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow e^{2x} = e^x \Rightarrow e^x = 1 \Rightarrow x = 0$. **C.** No symmetry **D.** $\lim\limits_{x \to -\infty} e^{2x} - e^x = 0$, so $y = 0$ is a HA. No VA. **E.** $f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)$, so $f'(x) > 0 \Leftrightarrow e^x > \frac{1}{2} \Leftrightarrow x > \ln\frac{1}{2} = -\ln 2$ and $f'(x) < 0 \Leftrightarrow e^x < \frac{1}{2} \Leftrightarrow x < \ln\frac{1}{2}$, so f is decreasing on $\left(-\infty, \ln\frac{1}{2}\right)$ and increasing on $\left(\ln\frac{1}{2}, \infty\right)$.

F. Local minimum value $f\left(\ln\frac{1}{2}\right) = e^{2\ln(1/2)} - e^{\ln(1/2)} = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$

G. $f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)$, so $f''(x) > 0 \Leftrightarrow e^x > \frac{1}{4} \Leftrightarrow x > \ln\frac{1}{4}$ and $f''(x) < 0 \Leftrightarrow x < \ln\frac{1}{4}$. Thus, f is CD on $\left(-\infty, \ln\frac{1}{4}\right)$ and CU on $\left(\ln\frac{1}{4}, \infty\right)$. IP at $\left(\ln\frac{1}{4}, \left(\frac{1}{4}\right)^2 - \frac{1}{4}\right) = \left(\ln\frac{1}{4}, -\frac{3}{16}\right)$

H.