Solutions Practice Midterm II

Exercise 1 We write the area function $A = \frac{1}{2} x y \sin(a)$

and use the general form of the chain rule : $\left.\begin{array}{l} x = x(t) \\ y = y(t) \\ a = a(t) \end{array}\right\}$ vary with time.

$$\frac{dA}{dt} = \frac{1}{2} y \sin a \frac{dx}{dt} + \frac{1}{2} x \sin a \frac{dy}{dt} + \frac{1}{2} x y \cos a \frac{da}{dt}$$

At $t = t_0$: $\quad x(t_0) = 40\text{ cm}, \quad y(t_0) = 50\text{ cm}, \quad a(t_0) = \frac{\pi}{6}\text{ rad}$

$x'(t_0) = 3 \frac{\text{cm}}{\text{sec}}, \quad y'(t_0) = \underset{\text{decreasing}}{-2 \frac{\text{cm}}{\text{sec}}}, \quad a'(t_0) = 0.5 \frac{\text{rad}}{\text{sec}}$

$$\Rightarrow \frac{dA}{dt}(t_0) = \frac{1}{2}\left[50 \cdot \frac{1}{2} \cdot 3 + 40\left(\frac{1}{2}\right)(-2) + 40 \cdot 50 \cdot \frac{\sqrt{3}}{2}(0.5)\right]$$
$$= \frac{1}{2}\left(75 - 40 + 500\sqrt{3}\right)$$
$$= \boxed{\frac{1}{2}\left(35 + 500\sqrt{3}\right) \frac{\text{cm}^2}{\text{sec}}}$$

→ The area A is increasing at this rate of change.

Exercise 2

(a) Since $\lim_{y \to 0} \frac{\sin y}{y} = 1$, we have

$$\lim_{(x,y)\to(0,0)} \frac{2x \sin y}{x^2 + 2y^2} = \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2} \cdot \underbrace{\frac{\sin y}{y}}_{\to 1} = \lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+2y^2}$$

- Take $C_1 = x = 0$: $\quad \lim_{y \to 0} 0 = 0$
- Take $C_2 : x = y$: $\quad \lim_{x \to 0} \frac{2x^2}{x^2 + 2x^2} = \lim_{x \to 0} \frac{2}{3} = \frac{2}{3}$

Since the limits along 2 curves through $(0,0)$ are different, the original limit <u>does not exist.</u>

(b) Use polar coordinates $x^2+y^2=r^2$ ②

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{r\to 0^+} \underbrace{r^2}_{0}\,\underbrace{\ln r^2}_{-\infty},$$ so we can use

L'Hôpital's Rule: $$\lim_{r\to 0^+} \frac{\ln(r^2)}{\frac{1}{r^2}} = \lim_{r\to 0^+} \frac{\frac{1}{r^2}\cdot 2r}{\frac{-2}{r^3}} = \lim_{r\to 0} -r^2 = 0$$

($\ln(r^2)$ & $\frac{1}{r^3}$ are differentiable around $r=0$)

$\Rightarrow$ The limit equals $\boxed{0}$.

(c) Note that we have a limit of the form $\frac{0}{0}$, so we cannot evaluate

We shift x to $x+1$. and change the limit pt to $(0,0)$

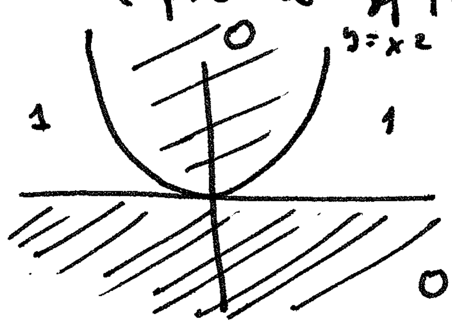
$$\lim_{(x,y)\to(0,0)} \frac{e^y(xy)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \overbrace{e^y}^{1} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

Again, we use polar coordinates $\sqrt{x^2+y^2}=r$, $xy=r^2\cos\theta\sin\theta$

$$= \lim_{r\to 0^+} \frac{r^2\cos\theta\sin\theta}{r} = \lim_{r\to 0^+} \underbrace{r}_{0}\,\underbrace{(\cos\theta\sin\theta)}_{\text{bounded}} = \boxed{0}$$

Exercise 3

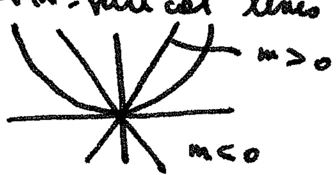
We draw a picture of the domains of definition of f



In the shaded parts, the function has value 0

(a) We have 2 cases to analyze:

- Vertical line $x=0$ $f(0,y)=0$, so $\lim_{y\to 0} f(0,y)=0$
- Non-vertical lines: $y=mx$ with $m=$ slope.



Our claim is that if $y\geq 0$ and $y=mx$ then for values of x close to 0, we have $mx > x^2$. Then,

$\lim_{x\to 0^-} f(x, mx) = 0$ since when $y \le 0$ $f(x,y) = 0$.
$\qquad\qquad\qquad \overset{\shortparallel}{0}$

(b) If we take the curve $y = x^4$, then $y \geq 0$ and $y < x^2$ ~~and x>~~ when $0 < x < 1$.

So $\lim_{x\to 0^+} f(x, x^4) = \lim_{x \to 0^+} 1 = 1$

Since the limit along lines is 0 by part (a), we conclude that the limit does not exist.

(c) The function is continuous away from the curves $y=0$ (horiz line) and $y = x^2$ (parabola), since the function is locally constant away from these curves.

. f Discontinuous on $y=0$, because if $C_a = \{x = a\}$, for any value of $a \neq 0$ (C_a is vertical line through $(a,0)$), then

$$\lim_{y\to 0^+} f(a,y) = \lim_{y\to 0^+} 1 = 1, \quad \underbrace{\lim_{y\to 0^-} f(a,y)}_{0} = 0,$$

so $\lim_{(x,y)\to(a,0)} f(x,y)$ does not exist for any value of $a \neq 0$

Since we know it's also discontinuous at $(0,0)$, by part (b), we conclude it's discontinuous on the entire horiz line.

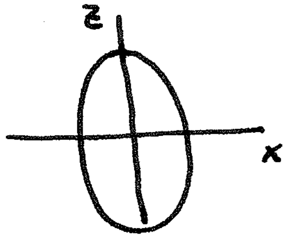
• f is discontinuous along the parabola: Pick any point (a,a^2) with $a \neq 0$ (we know it's discontinuous at $(0,0)$ by part (b)). Then we take the horiz. line $y = a^2$, and conclude

$$\underbrace{\lim_{x\to a^-} f(x,a^2)}_{1} = 1 \quad \text{but} \quad \underbrace{\lim_{x\to a^+} f(x,a^2)}_{0} = 0$$

[see picture on the previous page.]

Exercise 4 We write down the ellipse:

$$4x^2 + 3\cdot 4 + 2z^2 = 16$$
$$4x^2 + 2z^2 = 4$$



$$2x^2 + z^2 = 2 \quad \longrightarrow \quad \begin{cases} x^2 + \left(\frac{z}{\sqrt{2}}\right)^2 = 1 \\ y = 2 \end{cases}$$

~~we~~ $\cancel{\left(\frac{z}{\sqrt{2}}\right)^2 + x^2 = 1}$

We parametrize this curve in polar coordinates $(\rho\, x, z)$

$$x = \cos\theta, \quad z = \sqrt{2}\sin\theta \qquad y = 2$$

$$\Rightarrow \vec{r}(\theta) = \langle \cos\theta, 2, \sqrt{2}\sin\theta \rangle$$

When $\vec{r}(\theta) = (1, 2, 0)$ we take $\theta = 0$.

To find the tangent, we complete $\vec{r}\,'(0)$.

$$\vec{r}\,'(\theta) = \langle -\sin\theta, 0, \sqrt{2}\cos\theta \rangle \Rightarrow \vec{r}\,'(0) = \langle 0, 0, \sqrt{2} \rangle$$

$\Rightarrow$ Tangent line $\langle x, y, z \rangle = \langle 1, 2, 0 \rangle + t\langle 0, 0, \sqrt{2} \rangle$

$$\begin{cases} x = 1 \\ y = 2 \\ z = \sqrt{2}\, t \end{cases} \quad t \in \mathbb{R} \quad \text{are parametric equations.}$$

Exercise 5 We use the implicit differentiation method.

(a) We write $z = z(x, y)$ and differentiate the function $f(x, y, z) = yz + x\ln y - z^2$ describing the surface S

$$\Rightarrow 0 = \frac{\partial f(x, y, z(x,y))}{\partial x} = y\frac{\partial z}{\partial x} + \ln y - 2z\frac{\partial z}{\partial x} = (y - 2z)\frac{\partial z}{\partial x} + \ln y$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-\ln y}{y - 2z} \quad \text{whenever } y \neq 2z.$$

Similarly: $$\frac{\partial z}{\partial y} = -\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}} = \frac{-\left(z + \frac{x}{y}\right)}{y - 2z} \quad \text{whenever } y \neq 0, \; y \neq 2z$$

These two formulas are valid over the domain of $f = \mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{R} = \{(x, y, z) : y > 0\}$ whenever f_x, f_y, f_z are continuous and $f_z \neq 0$.

By construction f_x, f_y, f_z are continuous everywhere.

In conclusion, they hold in $D = \{(x,y,z) \in S \mid y > 0 \text{ and } z \neq \frac{y}{2}\}$

(b) To solve this, we find the formula for z by using $F_{(x,y,z)}$

$yz + x\ln y - z^2 = 0$ is quadratic in z, so there are 2 solutions:

$$z = \frac{-y \pm \sqrt{y^2 + 4x\ln y}}{2(-1)} = \frac{y \pm \sqrt{y^2+4x\ln y}}{2}$$

We know these solutions exist when $z \neq \frac{y}{2}$ by part (a), but in this case, the domain of z is $\{(x,y) \mid y^2 + 4x\ln y \geq 0\}$

The 2 solutions are continuous in this set, but (we don't need to exclude $y^2+4x\ln y = 0$) they are differentiable in $\{(x,y) \mid y^2+4x\ln y > 0\}$.

[Equivalently, the domain of z is $\{(x,y)$: there exists z with (x,y,z) in $S\}$
& z is differentiable at the points $= \{(x,y)$: " " " " " " and $z \neq \frac{y}{2}\}$.

Exercise 6

(a) $\underline{\underline{F}}$ Write $\vec{r}_1(t) = \langle a_1(t), b_1(t), c_1(t)\rangle$
$\vec{r}_2(t) = \langle a_2(t), b_2(t), c_2(t)\rangle$

$$(\vec{r}_1 \times \vec{r}_2)_{(t)} = \langle b_1c_2 - c_1b_2, -(a_1c_2 - c_1a_2), a_1b_2 - b_1a_2\rangle_{(t)}$$

$$\Rightarrow \frac{d(\vec{r}_1\times\vec{r}_2)(t)}{dt} = \langle b_1'c_2 - c_1'b_2 + b_1c_2' - c_1b_2', -(a_1'c_2 - c_1'a_2 + a_1c_2' - c_1a_2'), (a_1'b_2 - b_1'a_2) + (a_1b_2' - b_1a_2')\rangle$$

Take deriv. componentwise, use product rule & rearrange

$$= \langle b_1'c_2 - c_1'b_2, -(a_1'c_2 - c_1'a_2), (a_1'b_2 - b_1'a_2)\rangle + \langle b_1c_2' - c_1b_2', -(a_1c_2' - c_1a_2'), (a_1b_2' - b_1a_2')\rangle$$

$$= \vec{r}_1{}'_{(t)} \times \vec{r}_2{}_{(t)} + \vec{r}_1{}_{(t)} \times \vec{r}_2{}'(t)$$

Example $\vec{r}_1 = \langle t, t, t \rangle \longrightarrow \vec{r}_1' = \langle 1, 1, 1 \rangle$

$\vec{r}_2 = \langle 0, 1, -1 \rangle \longrightarrow \vec{r}_2' = \langle 0, 0, 0 \rangle$

$\vec{r}_1 \times \vec{r}_2 = \langle -2t, t, t \rangle$

$\frac{d(\vec{r}_1 \times \vec{r}_2)}{dt} = \langle -2, 1, 1 \rangle$ but $\vec{r}_1' \times \vec{r}_2' = \langle 0, 0, 0 \rangle$.

(b) $\underline{\underline{F}}$ We use the formula for linear approximation if f is differentiable at $(0,0)$

$$L(x,y) = \underbrace{f(0,0)}_{=1} + \underbrace{f_x(0,0)}_{=0}\, x + \underbrace{f_y(0,0)}_{=\frac{1}{2}}\, y = 1 + \frac{y}{2}$$

- $f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\cos h - \overset{\cos 0}{1}}{h} = \cos'(0) = \sin(0) = 0$

~~Any other reason~~: In general: $f_x(x,y) = \frac{1}{2\sqrt{y + \cos^2 x}} \cdot 2\cos x \sin x$ cont. at $(0,0)$

$$f_x(0,0) = 0 \quad \checkmark$$

- $f_y(x,y) = \frac{1}{2\sqrt{y + \cos^2 x}}$ is cont at $(0,0)$ and $f_y(x,y) = \frac{1}{2}$

Since $L(x,y) \neq 1 + \frac{y}{3}$, the function in the statement is not a linear approximation.

(c) $\underline{\underline{T}}$ The domain of f is the intersection of

- $-1 \leq x^2 + y^2 \leq 1$ (arcsin of sin defined on $[-1, 1]$)
- $1 + x^2 > 0$ $\checkmark$
- $x + y - 1 \neq 0$

So it's the intersection of the disc of radius 1 centered at $(0,0)$ and the complement of the line $x = 1 - y$, as stated.

(d) $\underline{\underline{F}}$ We use the Fundamental Theorem of Calculus:

Write $F = \left(\int_{\beta}^{\alpha} \sqrt{1+t^3}\, dt \right) \circ (\alpha^2, \beta)$ and use chain rule

⑦

$$F_\alpha = \left(\sqrt{1+t^3}\right)_{(\alpha^2)} \cdot 2\alpha = 2\alpha\sqrt{1+\alpha^6}$$

$$\Rightarrow F_{\alpha\alpha} = 2\left(\sqrt{1+\alpha^6} + \frac{\alpha}{2}\frac{6\alpha^5}{\sqrt{1+\alpha^6}}\right) = \frac{1}{\sqrt{1+\alpha^6}}\left(2(1+\alpha^6)+6\alpha^6\right)$$

$$= \frac{8\alpha^6+2}{\sqrt{1+\alpha^6}}$$, which is NOT the value they gave us.

(e) $\underline{\underline{T}}$ The function is a product of differentiable functions, with all higher partials that are continuous. In particular f has continuous cross partials f_{xy} & f_{yx}, so by the cross derivatives condition Theorem, they agree

(f) $\underline{\underline{F}}$ We compute the tangent vectors of $\vec{r}_1'$, $\vec{r}_2$:

$\vec{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$

$\vec{r}_2'(t) = \langle 2\cos 2t, 4\cos 4t, 2 \rangle = 2\langle \cos 2t, 2\cos 4t, 1 \rangle$

The point of intersection satisfies:

$$\begin{cases} s = \sin 2t \\ s^2 = \sin 4t = \sin(2\cdot 2t) = 2\underbrace{(\sin 2t)}_{s}\cos 2t \\ s^3 = 2t \end{cases} \Rightarrow \text{Either } s=0 \text{ or } \underbrace{s}_{\sin(2t)} = 2\cos(2t)$$

$s=0$ is a solution, the other solutions have $s^3 = 2t$

$S = \sin 2t = 2\cos 2t$

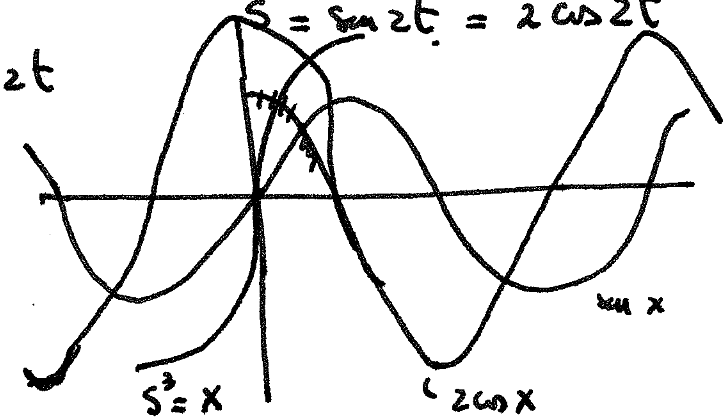
call $x = 2t$

$\Rightarrow r_1'(0) = \langle 1, 0, 0 \rangle$

$r_2'(0) = 2\langle 1, 2, 1 \rangle$

$\vec{r}_1'(0) \cdot \vec{r}_2'(0) = 2(1+0) = 2 \neq 0$

so the angle is $\underline{\text{not}}$ $\frac{\pi}{2}$



there is no triple solution. other than $\underline{s=x=0}$