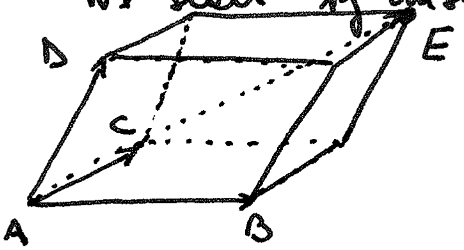
Solutions for the practice midterm 1

Exercise 1: We start by constructing the parallelepiped:



↘ opposite to A.

1) We compute the volume with the formula

$$\mathrm{Vol}(\vec{AB}, \vec{AC}, \vec{AD}) = |\vec{AB} \cdot (\vec{AC} \times \vec{AD})|$$

$\vec{AC} = \langle 2,2,1 \rangle$

$\vec{AB} = \langle 2,1,-1 \rangle$

$\vec{AD} = \langle -3,-2,0 \rangle$

$$\Rightarrow \quad \vec{AC} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ -3 & -2 & 0 \end{vmatrix} = \underbrace{\begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix}}_{=2} \hat{i} - \underbrace{\begin{vmatrix} 2 & 1 \\ -3 & 0 \end{vmatrix}}_{=3} \hat{j} + \underbrace{\begin{vmatrix} 2 & 2 \\ -3 & -2 \end{vmatrix}}_{=2} \hat{k}$$

$$= 2\hat{i} - 3\hat{j} + 2\hat{k}.$$

$$\Rightarrow \mathrm{Vol} = |(2,1,-1)\cdot(2,-3,2)| = |4-3-2| = |-1| = 1.$$

2) $\vec{AE} = \vec{AB} + \vec{AC} + \vec{AD} = \langle 1, 1, 0 \rangle$

$$\Rightarrow E = (1\ 1\ 0) + (1\ 0\ 1) = \boxed{(2\ 1\ +1)}.$$

3) How do we find the angle?

$$A \xrightarrow{\ \theta\ } E, \quad A \to B \qquad 0 \le \theta \le 180^\circ$$

Answer Use the dot product

$$\underbrace{\underset{\langle 1\ 1\ 0\rangle}{\vec{AE}} \cdot \underset{\langle 2\ 1\ -1\rangle}{\vec{AB}}}_{=\ 2+1=3} = |\vec{AE}| \cdot |\vec{AB}| \cos\theta = \sqrt{2}\sqrt{6}\cos\theta = 2\sqrt{3}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

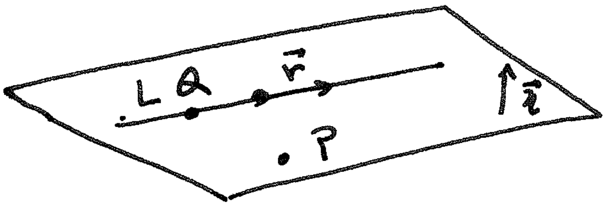
$|\vec{AE}| = \sqrt{2}$

$|\vec{AB}| = \sqrt{6}$

$$\Rightarrow \theta = \arccos\frac{\sqrt{3}}{2} = 30^\circ \quad \left(= \frac{\pi}{6}\right)$$

$$\boxed{\angle EAB = 30^\circ}$$

Exercise 2: We need to find the normal direction $\vec{n}$ to the plane π.



Note that $P \notin L$ (it doesn't satisfy the equations of L because $\begin{cases} x-1=5 \\ y-2=5 \\ z/2=0 \end{cases}$ are $\neq$)

The direction of L gives one condition: $\vec{v} = \langle 1,1,2 \rangle$ is $\perp$ to $\vec{n}$

To find another condition, $\vec{PQ} \perp \vec{n}$ for $Q \in L$. Pick $Q=(1,2,0)$

$\vec{PQ} = (-5,-5\ 0) \Rightarrow \vec{n} \perp \vec{PQ}$, $\vec{n} \perp \vec{v}$, so we can take $\vec{n} = \vec{PQ} \times \vec{v}$

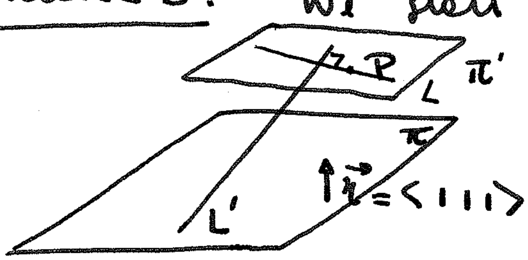
$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2 \\ -5 & -5 & 0 \end{vmatrix} = 10\hat{i} - (10)\hat{j} + 0\hat{k} = 10\hat{i} - 10\hat{j} = \langle 10,-10,0 \rangle$$

$\Rightarrow$ Take $\vec{n} = \langle 1,-1,0 \rangle$

$$\Rightarrow \pi : 1(x-6) - 1(y-7) + 0(z-0) = 0$$

$$\boxed{x - y + 1 = 0}$$

Exercise 3: We start by drawing a picture:



$L = \vec{OQ} = \langle 0\,1\,2 \rangle + t\vec{v}$ ($t \in \mathbb{R}$) with $\vec{v}$ to be determined

$P = (0,1,2)$

Does $P \in \pi$? $0+1+2 = 3 \neq 2 \Rightarrow$ no

$L \in \pi'$ because it is $\parallel$ to π.

. Draw the line $L' = x-1 = 1-y = 2z$.

. Does $P \in L'$? $0-1 \overset{?}{=} 1-1 \overset{?}{=} 4$ NO. $\Rightarrow P \notin L'$.

Since L lies in π', we know $\vec{v} \perp \vec{n}$

Since $L \perp L'$, " " $\vec{v} \perp \langle 1,-1,\tfrac{1}{2} \rangle$ = direction of L'

Answer: Take $\vec{v} = \vec{n} \times \langle 1,-1,\tfrac{1}{2} \rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & \tfrac{1}{2} \end{vmatrix} = \frac{3}{2}\hat{i} - \left(-\frac{1}{2}\right)\hat{j} + (-2)\hat{k} = \frac{3}{2}\hat{i} + \frac{1}{2}\hat{j} - 2\hat{k}.$$

Better: $\vec{v} = \langle 3,1,-4 \rangle$. parallel to

$\Rightarrow \boxed{L: \quad x = 3t, \quad y = 1+t, \quad z = 2-4t \quad \text{with } t \in \mathbb{R}}$

Exercise 4

(a) T. $|\vec{v} \times \vec{u}|^2 = |\vec{u}|^2 |\vec{v}|^2 \sin^2\theta$

$(\vec{u}\cdot\vec{v})^2 = |\vec{u}|^2|\vec{v}|^2 \cos^2\theta$

$$\Rightarrow |\vec{u}|^2|\vec{v}|^2 - (\vec{u}\cdot\vec{v})^2 = |\vec{u}|^2|\vec{v}|^2 \underbrace{(1-\cos^2\theta)}_{=\sin^2\theta} = |\vec{v}\times\vec{u}|^2 \checkmark$$

(b) F Pick $\vec{a} = (1,1)$, $\vec{b} = (-1,0)$

$|\vec{a}+\vec{b}| = |(0,1)| = 1$

$|\vec{a}| = \sqrt{2}$, $|\vec{b}| = 1$

and $1 \neq 1+\sqrt{2}$.

(c) T Use properties:

$$(\vec{u}-\vec{w}) \times (\vec{u}+\vec{w}) = \underbrace{\vec{u}\times(\vec{u}+\vec{w})}_{\vec{0}+\vec{u}\times\vec{w}} - \underbrace{\vec{w}\times(\vec{u}+\vec{w})}_{\vec{w}\times\vec{u}+\vec{0}}$$

$$= \vec{u}\times\vec{w} \underbrace{- \vec{w}\times\vec{u}}_{=+\vec{u}\times\vec{w}} = 2\,\vec{u}\times\vec{w}.$$

(d) F By definition, skew lines cannot lie in the same plane (otherwise, we would have skew lines in $\mathbb{R}^2$!)

Example: L: x-axis → direction $\langle 1,0,0\rangle$

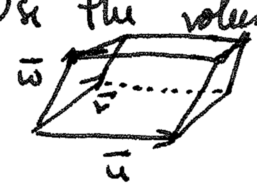
L': y-axis shifted to $(0,0,1)$

$\begin{cases} x=0 \\ z=1 \end{cases}$ equations. → direction $\langle 0\ 1\ 0\rangle$

⇒ Only one possible normal direction $= \langle 0,0,1\rangle$ but the two lines don't lie in the same plane parallel to $z=0$.

(L lies in $(z=0)$, L' lies in $(z=1)$)

(e) T Use the volume formula:



$$\text{Vol} = |\vec{u}\cdot(\vec{v}\times\vec{w})| = \langle 1\ 5\ -2\rangle\cdot\langle 4,12,32\rangle = 4+60-64 = 0$$

$$\vec{v}\times\vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = 4\hat{\imath} + 12\hat{\jmath} + 32\hat{k}$$

So they are coplanar

(f) T The surface is the plane $x-3y=z+4y$, which we write $x-7y-z=0$. It's ruling is given by any line in the plane, for example $\begin{cases} x-7y=0 \\ z=0 \end{cases} \Rightarrow L: \begin{cases} x=7y \\ \cancel{y=y} \\ z=0 \end{cases}$

All cross sections along z are lines parallel to L, namely $\begin{cases} x-7y=k \\ z=k \end{cases}$

Exercise 5

(a) $a=-1$: $\quad -2z = -x^2+y^2$

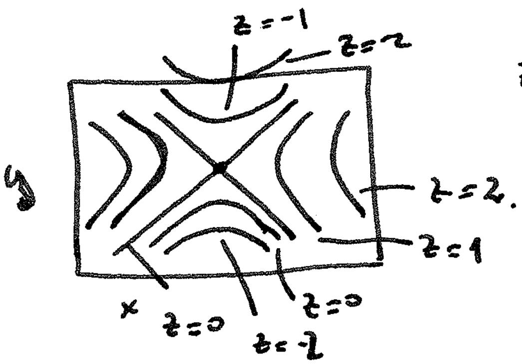
z-traces:

$z=0 \qquad 0 = y^2-x^2 \Rightarrow x=\pm y$ 2 lines.

$z=k<0 \qquad (-2k) = y^2-x^2 \Rightarrow$ hyperbola ($0<$)

$z=k>0 \qquad -2k = -x^2+y^2$

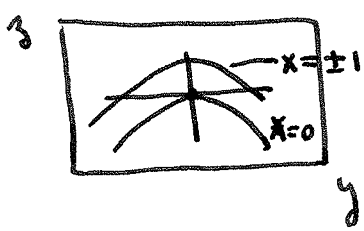
$0<2k = x^2-y^2 \rightarrow$ hyperbola.



x-traces:

$x=0 \qquad -2z=y^2 \rightarrow$ parabola $\quad z=-\frac{1}{2}y^2$

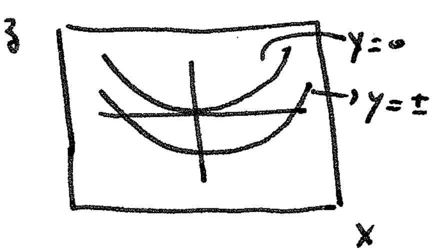
$x=\pm 1 \qquad -2z = -1+y^2 \rightarrow$ " $\quad z-\frac{1}{2} = -\frac{1}{2}y^2$



y-traces:

$y=0 \qquad -2z=-x^2 \Rightarrow \frac{1}{2}x^2 = z$ parabola

$y=\pm 1 \qquad -2z=-x^2+1 \Rightarrow \frac{1}{2}x^2 = z+\frac{1}{2}$



$\underline{\underline{a=1}}$ $\quad 0 = x^2+y^2 \longrightarrow$ point. $\underline{\underline{\forall z}}$ $(0,0,z)$ is z-axis!

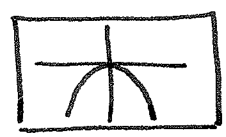
x-Traces: empty unless $x=0$ — get the z-axis.

y-Traces: " " $y=0$ — . . . .

All z-traces: are $(0,0)$.

$\underline{\underline{a=0}}$ $\quad (\cancel{a}-1)z = 0x^2+y^2 = y^2. \longrightarrow$ cylinder. ruling: x-axis

x-Traces: $(\cancel{a}-1)z = y^2$ — parabola. $\boxed{z=-y^2}$



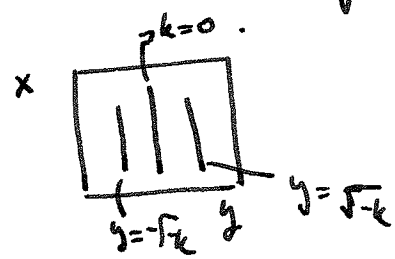
for $\underline{\underline{\text{all}}}$ x.

y-traces ($y=\pm k$): $-z = k^2 \Rightarrow$ pt. $(\pm k, -k^2)$ in the parabola

z-traces $z=k$: $-k = y^2$ → for $k > 0$ no solution!

for $k \leq 0$ → 2 points $y = \pm\sqrt{-k}$.

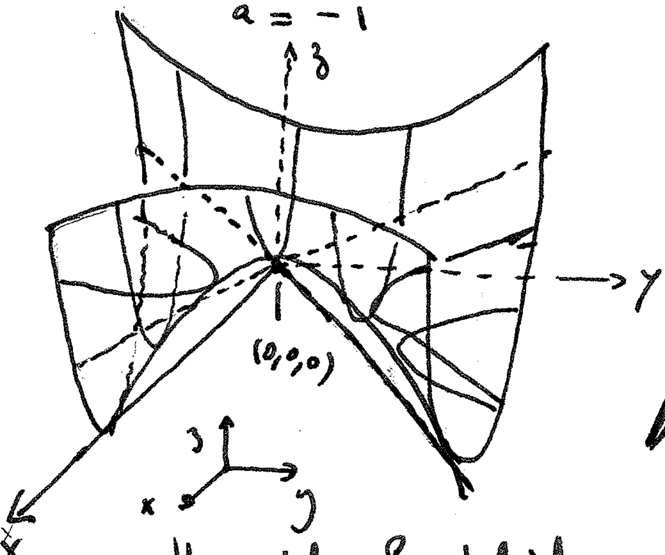
(1 point if $k=0$)



x moves freely so we get 1 or 2 lines.

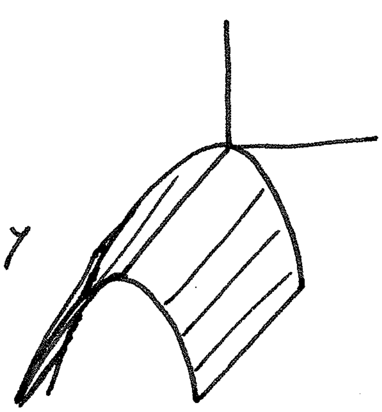
b) Surface sketches:

$a=-1$



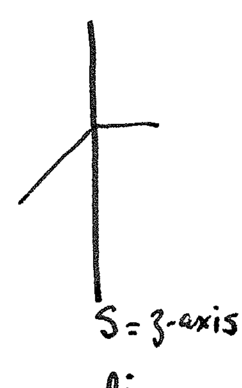
Hyperbolic Paraboloid (Saddle)

$a=0$



cylinder.

$a=1$



$S = z$-axis line.

Exercise 6:

(a) $(\vec{u} + \vec{v}) \perp (\vec{u} - \vec{v}) \iff (\vec{u} + \vec{v}) \cdot (\vec{u} - \vec{v}) = 0$

$$|\vec{u}|^2 \underbrace{- \vec{u} \cdot \vec{v} + \vec{v}\,\vec{u}}_{=0} - |\vec{v}|^2$$

so $0 = |\vec{u}|^2 - |\vec{v}|^2$, and $\bar{u}$ & $\bar{v}$ have the same length.

(b) $\text{Orth}_{\vec{w}}(\vec{z}) = \vec{0} \iff \vec{z} - \text{proj}_{\vec{w}}\, \vec{z} = 0$

so $\vec{z} = \text{proj}_{\vec{w}}\, \vec{z}$.

That means $\vec{z} = \dfrac{\vec{z} \cdot \vec{w}}{|\vec{w}|^2} \cdot \vec{w} \Rightarrow \vec{z}$ and $\vec{w}$ are parallel.

Conversely, $\bar{z} \parallel \vec{w}$, so $\bar{z} = a\bar{w}$ for some a.

$$\text{proj}_{\vec{w}}\, \vec{z} = \frac{\vec{z} \cdot \vec{w}}{|\vec{w}|^2}\, \vec{w} = \frac{a\,|w|^2}{|\vec{w}|^2} \cdot \vec{w} = a\vec{w} = \vec{z},$$

so then $\text{Orth}_{\vec{w}}(\vec{z}) = \vec{0}$