SOLUTIONS

Midterm 1
Calculus III Section 8 - Fall 2013

- The use of class notes, book, formulae sheet, calculator is **not permitted**.
- In order to get full credit, you **must**:
  a) get the **correct answer**, and
  b) **show all your work** and/or explain the reasoning that lead to that answer.
- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- Please make sure the solutions you hand in are **legible and lucid**. You may only use techniques we have developed in class.
- You have **one hour and fifteen minutes** to complete the exam.
- Do not forget to write your name and UNI in the space provided below and on the bottom of the last page.

Full Name (Print) ______________________________

UNI ______________________________

Enjoy the exam, and good luck!

Exercise 1. [15 points] Consider the planes π_1: $2x - y + 3z = 1$ and π_2: $x + 5y + z = 3$.

a) Show that the two planes are perpendicular to each other, i.e., their normal vectors are perpendicular.

Normal $n_1 = \langle 2, -1, 3 \rangle$

" $n_2 = \langle 1, 5, 1 \rangle$

$n_1 \perp n_2$ because $n_1 \cdot n_2 = 2 - 5 + 3 = 0$

b) Find the parametric equations of the line where the two planes intersect.

We put the two equations together to solve them at the same time:

$$\begin{cases} 2x - y + 3z = 1 & \to y = -1 + 2x + 3z \\ x + 5y + z = 3 & \to x + 5(-1 + 2x + 3z) + z = 11x + 16z - 5 = 3 \end{cases}$$

$$11x + 16z = +8$$

$$x = \frac{8 - 16z}{11}$$

$$\Rightarrow y = -1 + 2x + 3z = -1 + \frac{2}{11}(8 - 16z) + 3z = \frac{5}{11} + \frac{1}{11}z$$

Param eqn: $\boxed{x = \frac{8}{11} - \frac{16}{11}z, \quad y = \frac{5}{11} + \frac{1}{11}z, \quad z = z \quad (z \in \mathbb{R})}$

c) Find a plane π_3 that is perpendicular to both π_1 an π_2. passing through the point $(2, 1, 1)$. (*Hint:* Use the previous item).

$\pi_1 \perp \pi_2$ The direction of the line L from (b) is $\vec{r} = \langle -16, 1, 11 \rangle$

By construction $\vec{r} \perp \vec{n_1}$ and $\vec{r} \perp \vec{n_2}$

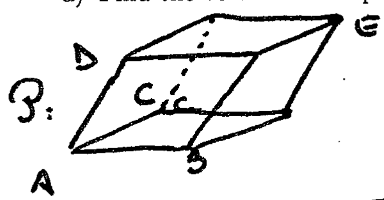
So the plane π_3 has normal $\vec{r}$ & passes through $(2, 1, 1)$.

It's equation is $-16(x-2) + (y-1) + 11(z-1) = 0$

$$\boxed{-16x + y + 11z = -18}$$

Exercise 2. [20 points] Let $A = (0, 1, -1)$, $B = (2, 3, 0)$ and $C = (2, 2, -2)$ and $D = (-3, -1, -1)$ be four points in $\mathbb{R}^3$.

a) Find the volume of the parallelopiped formed by edges AB, AC and AD.



$$\overline{AB} = \langle 2, 2, +1 \rangle$$
$$\overline{AC} = \langle 2, 1, -1 \rangle$$
$$\overline{AD} = \langle -3, -2, 0 \rangle$$

$$\mathrm{Vol}(P) = \left| \overline{AB} \cdot \left(\overline{AC} \times \overline{AD} \right) \right| = \left| \langle 2, 2, 1 \rangle \cdot \langle -2, 3, -1 \rangle \right| = |-4+6-1| = |1| = \boxed{1}$$

$$\overline{AC} \times \overline{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ -3 & -2 & 0 \end{vmatrix} = -2\hat{i} - (-3)\hat{j} + (-1)\hat{k} = \langle -2, 3, -1 \rangle$$

b) Find the coordinates of the point E opposite to A in this parallelopiped.

$$\overline{AE} = \overline{AB} + \overline{AC} + \overline{AD}$$
$$= \langle 1, 1, 0 \rangle$$
$$\Rightarrow E = \langle 1, 1, 0 \rangle + \langle 0, 1, -1 \rangle = \boxed{\langle 1, 2, -1 \rangle}$$

c) Find the angle $\angle CAE$.

$$\overline{AC} \cdot \overline{AE} = |\overline{AC}| \cdot |\overline{AE}| \cos(\angle CAE)$$

$$|\overline{AC}| = \sqrt{6}$$
$$|\overline{AE}| = \sqrt{2}$$
$$\overline{AC} \cdot \overline{AE} = 2 + 1 = 3$$

$$\Rightarrow \cos(\angle CAE) = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \angle CAE = 30^\circ = \frac{\pi}{6}.$$

d) Find the area of the triangle with vertices C, A and E.

$$2\,\text{Area}\,\Delta = \left| \overline{AC} \times \overline{AE} \right| = |\overline{AC}| \cdot |\overline{AE}| \underbrace{\sin 30^\circ}_{= \frac{1}{2}} = \sqrt{6}\sqrt{2}\,\frac{1}{2} = \sqrt{3}$$

$$\Rightarrow \boxed{\text{Area}\,\Delta = \frac{\sqrt{3}}{2}}$$

($2\,\text{Area}\,\Delta$ = area parallelogram with edges $\overline{AC}$ and $\overline{AE}$).

Exercise 3. [10 points]

Assume $\vec{v} \neq \vec{0}$.

a) Prove that $\mathrm{Proj}_{\vec{v}}(\vec{u}) = \vec{0}$ if and only if $\vec{u}$ is orthogonal to $\vec{v}$.

($\Rightarrow$) By def, $\mathrm{Proj}_{\vec{v}}(\vec{u}) = \left(\dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \vec{0}$.

Since $\vec{v} \neq \vec{0} \Rightarrow \vec{u}\cdot\vec{v} = 0$ and is orthogonal to $\vec{v}$

($\Leftarrow$) If $\vec{u}\cdot\vec{v} = 0$, then $\mathrm{Proj}_{\vec{v}}(\vec{u}) = 0\cdot\vec{v} = \vec{0}$

by the formula above.

b) Show that if $2\vec{u} - 2\vec{v}$ and $\vec{u} + \vec{v}$ are perpendicular, then the vectors $\vec{u}$ and $\vec{v}$ have the same length.

By def, $(2\vec{u} - 2\vec{v})\cdot(\vec{u} + \vec{v}) = 2\Big(|\vec{u}|^2 - |\vec{v}|^2 + \underbrace{\vec{u}\cdot\vec{v} - \vec{v}\cdot\vec{u}}_{=0}\Big) = 2(|\vec{u}|^2 - |\vec{v}|^2) = 0$

$\Rightarrow |\vec{u}|^2 = |\vec{v}|^2$

Both numbers are positive, so $|\vec{u}| = |\vec{v}|$.

Exercise 4. **[20 points]** True/False. Justify your answer with a proof if true, or a counterexample if false.

a) $|\vec{u} - 2\vec{v}| = |\vec{u}| + 4|\vec{v}|$.

FALSE $\quad \vec{u} = \langle 1\ 0\ 0 \rangle \qquad \vec{u} - 2\vec{v} = \langle -1, -2, 0 \rangle$

$\vec{v} = \langle 1\ 1\ 0 \rangle \qquad |\vec{v}| = \sqrt{2}, \quad |\vec{u}| = 1$

$|\vec{u} - 2\vec{v}| = \sqrt{5}$

and $\sqrt{5} \neq 1 + \sqrt{2}$.

b) $|\vec{u} \times \vec{w}|^2 = |\vec{w}|^2|\vec{u}|^2 - (\vec{w} \cdot \vec{u})^2$.

TRUE $\quad |\vec{u} \times \vec{w}|^2 = (\vec{u} \times \vec{w}) \cdot (\vec{u} \times \vec{w}) = \cancel{k} = |\vec{u}|^2 \cdot |\vec{w}|^2 \operatorname{sen}^2\theta$

where θ is angle between $\vec{u}$ & $\vec{w}$.

$|\vec{w} \cdot \vec{u}|^2 = |\vec{u}|^2 |\vec{v}|^2 \cos^2\theta$

$\Rightarrow |\vec{w}|^2 |\vec{u}|^2 - (\vec{w} \cdot \vec{u})^2 = |\vec{u}|^2 \cdot |\vec{w}|^2 (1 - \cos^2\theta) = |\vec{u}|^2 |\vec{w}|^2 \operatorname{sen}^2\theta$

$= |\vec{u} \times \vec{w}|^2$.

c) The surface $z - 3x = 2z + 4y$ is a cylindrical surface.

TRUE The surface is the plane: $3x + 4y + z = 0$, so it's a cylindrical surface: The line with direction $\langle 1, 0, -3 \rangle$ passing thought $(0,0,0)$ gives a ruling of the surface.

Also: All z-traces are lines parallel to $\begin{cases} 3x + 4y = 0 \\ z = 0 \end{cases} \Rightarrow x = -\frac{4}{3}y$ which is a line in $\mathbb{R}^3$.

d) The vectors $\vec{u} = -\hat{\imath} + 5\hat{\jmath} + 4\hat{k}$, $\vec{v} = 3\hat{\imath} - \hat{k}$ and $\vec{w} = -2\hat{\imath} - 7\hat{\jmath} + 3\hat{k}$ are coplanar.

FALSE It suffices to compute the volume of the parallelepiped $(\vec{u}, \vec{v}, \vec{w})$.

$\text{Vol} = |\vec{u} \cdot (\vec{v} \times \vec{w})| = 0 \iff$ the vectors are coplanar

$$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 0 & -1 \\ -2 & -7 & 3 \end{vmatrix} = -7\hat{\imath} - \underbrace{(9-2)}_{=7}\hat{\jmath} + (-21)\hat{k}$$

$$= 7\langle -1, -1, -3 \rangle$$

$\Rightarrow |\vec{u} \cdot (\vec{v} \times \vec{w})| = 7\,|\langle -1, 5, 4 \rangle \cdot \langle -1, -1, -3 \rangle|$

$= 7\,|1 - 5 - 12| = 7|-16| \neq 0$.

$\Rightarrow$ They are not coplanar.

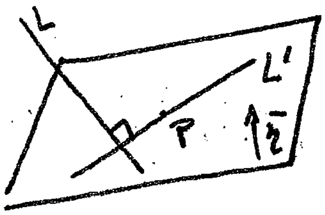
Exercise 5. [15 points] Given the plane π: $x - 2y - z = 4$, the point $P = (1, -2, 1)$ and the line

$$L: \qquad x = 2 + 2t, \quad y = -3 - 2t, \quad z = 3 + 4t \qquad (\text{with } t \in \mathbb{R})$$

Find the symmetric equations of line L' that satisfies the following three conditions:

- passes through the point P,
- L' lies in the plane π, and
- L' is perpendicular to L.

We draw a picture:



First, we check that $P \in \pi$ because $1 + 4 - 1 = 4$.

We need:

- The direction of L' is $\perp$ to the normal of π $\vec{\eta} = \langle 1, -2, -1 \rangle$
- The direction of L' is $\perp$ to the direction $\vec{v}$ of L.

We find $\vec{v}$ from the param. equations of L: $\vec{v} = \langle 2, -2, 4 \rangle$, can take $\langle 1, -1, 2 \rangle$.

$\Rightarrow$ The direction of L' is $\langle 1, -2, -1 \rangle \times \langle 1, -1, 2 \rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = -5\hat{i} - (3)\hat{j} + \hat{k} = \langle -5, -3, 1 \rangle$$

$$\Rightarrow \boxed{L': \quad \frac{x-1}{-5} = \frac{y+2}{-3} = \frac{z-1}{1} \qquad (\text{symm. eqns.})}$$

Need to check $L' \cap L \neq \emptyset$ (otherwise they'll be skew).

$L \cap L'$: Plug param. eqns. of L in L':

$$\frac{x-1}{-5} = \frac{2+2t-1}{-5} = \frac{1+2t}{-5}$$

$$\frac{y+2}{-3} = \frac{-3-2t+2}{-3} = \frac{-1-2t}{-3}$$

$$\Rightarrow -3(1+2t) = -5(-1-2t)$$

$$3 + 6t = -5 - 10t$$

$$8 = -16t \Rightarrow t = -1/2$$

$$z - 1 = 3 + 4t - 1 \Rightarrow z - 1 = 3 - \frac{4}{2} - 1 = 0$$

$$\frac{x-1}{-5} = \frac{1+2t}{-5} = \frac{1-1}{-5} = \frac{0}{5} = 0 \Rightarrow L \cap L' \neq \emptyset$$

In fact, it's the point $(1, -2, 1) = P$

Exercise 6. **[20 points]** Consider the equation $-4z^2 - \frac{x^2}{9} + y^2 - 2y - 3 = 0$.

a) Sketch the x- and y-traces.

First, we rewrite the equation: $-4z^2 - \frac{x^2}{9} + (y-1)^2 - 4 = 0$

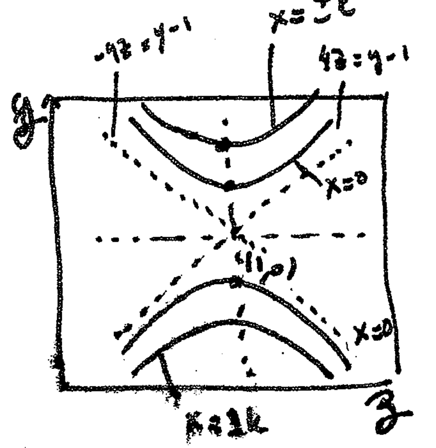
$-z^2 - \frac{x^2}{36} + \frac{(y-1)^2}{4} = 1$

x-traces:

$x=0$: $-z^2 + \frac{(y-1)^2}{4} = 1$ hyperbola shifted to $(1,0)$

$x = \pm k$ $(k \in \mathbb{R})$: $-z^2 + \frac{(y-1)^2}{4} = 1 + \frac{k^2}{36} > 0$ " " " $(1,0)$

Asymptotes: $\begin{cases} z = \pm\frac{(y-1)}{4} \\ \pm 4z = y-1 \end{cases}$



y-traces:

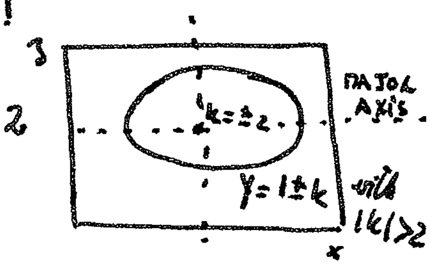
$y=1$: $-z^2 - \frac{x^2}{36} = 1 \to$ no soln!

$y = 1 \pm k$: $+z^2 + \frac{x^2}{36} = 1 + \frac{k^2}{4}$ $=0$ → pt $(x,z) = (0,0)$ with $k = \pm 2$ (ie $y = -1$ or 3)

<0 → no solution!

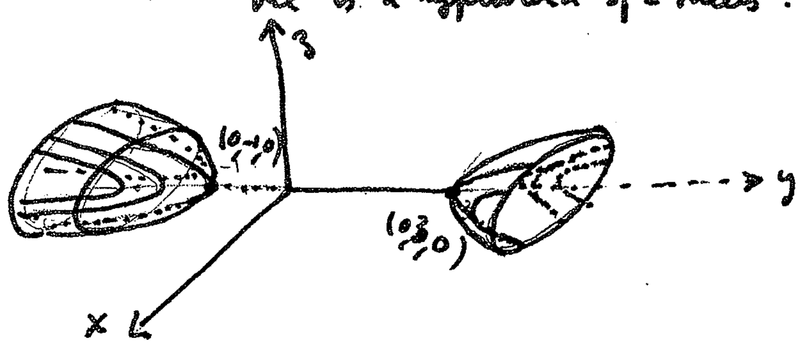
>0 → ellipse

(values: $k^2 > 4 \to |k| > 2$)



b) Use these traces to draw the surface.

The surface is a hyperboloid of 2 sheets.



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For Grader's use only:

1			2				3		4				5	6		TOTAL
a	b	c	a	b	c	d	a	b	a	b	c	d		a	b	