SOLUTIONS

Midterm 2

Calculus III Section 8 - Fall 2013

- The use of class notes, book, formulae sheet, calculator is **not permitted**.
- In order to get full credit, you **must**:
  a) get the **correct answer**, and
  b) **show all your work** and/or explain the reasoning that lead to that answer.

- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- Please make sure the solutions you hand in are **legible and lucid**. You may only use techniques we have developed in class.
- You have **one hour and fifteen minutes** to complete the exam.
- Do not forget to write your name and UNI in the space provided below and on the bottom of the last page.

Full Name (Print) ____________________

UNI ____________________

Happy Halloween!

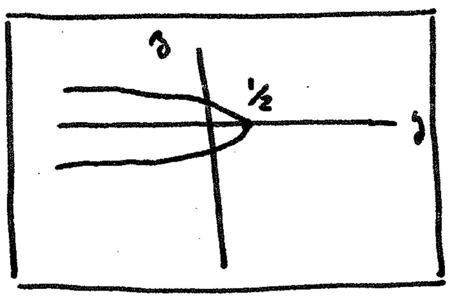
Enjoy the exam, and good luck!

Exercise 1. [**16 points**] The cone $y^2 + 3z^2 - x^2 = 0$ intersects the plane $y - x = 1$ in a parabola.

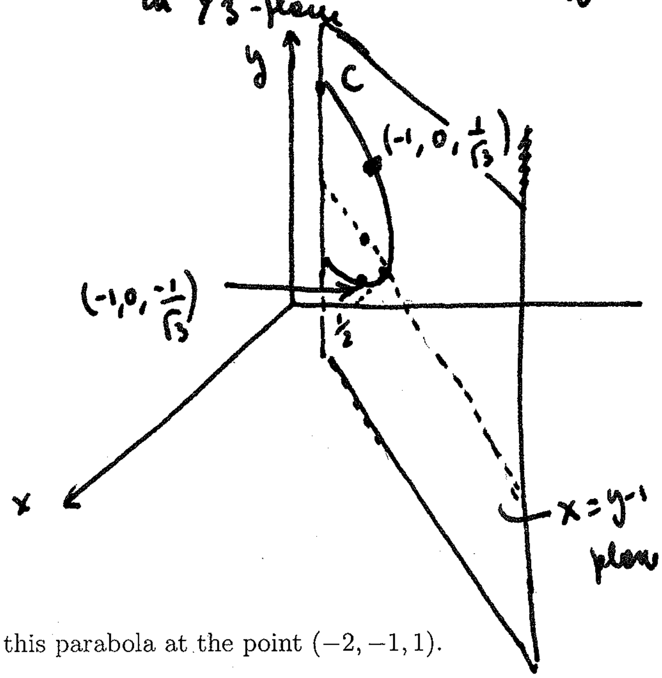
a) Draw the corresponding parabola in 3-space.

We replace the value of $x = y - 1$ in the equation of the cone:

$0 = y^2 + 3z^2 - (y-1)^2 = 3z^2 + 2y - 1 \Rightarrow$ parabola: $y - \frac{1}{2} = -\frac{3}{2} z^2$ in yz-plane



Put this parabola in the plane $y - x = 1$



b) Find the parametric equations of the tangent line to this parabola at the point $(-2, -1, 1)$.

We use (a) to write the parametric form of the curve. The free variable is z

$\vec{r}(z) = \langle -\frac{1}{2} - \frac{3}{2} z^2, \frac{1}{2} - \frac{3}{2} z^2, z \rangle \rightarrow \vec{r}(z_0) = \langle -2, -1, 1 \rangle$

$x = y - 1 = -1 + \left(\frac{1}{2} - \frac{3}{2} z^2\right) = -\frac{1}{2} - \frac{3}{2} z^2 \qquad \Rightarrow \boxed{z_0 = 1}$

$\Rightarrow r'(z) = \langle -3z, -3z, 1 \rangle \Rightarrow r'(z_0) = \langle -3, -3, 1 \rangle$

$\Rightarrow$ Tangent Line $\langle x, y, z \rangle = \langle -2, -1, 1 \rangle + s \langle -3, -3, 1 \rangle \quad s \in \mathbb{R}$

<u>ANSWER</u> $\begin{cases} x = -2 - 3s \\ y = -1 - 3s \\ z = 1 + s \end{cases} \quad s \in \mathbb{R}$

Exercise 2. [**16 points**] Consider the following function in three variables

$$f(x, y, z) = \ln(x)\sqrt{y^2 + z^2} + 5x.$$

a) Find the domain of f.

$\ln x$ is defined when $x > 0$

$\sqrt{y^2+z^2}$ " " " $y^2+z^2 \geq 0$ which happens for all y, z

So: Domain of $f = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$.

b) Compute the differential of f.

We use the formula:

$df = f_x\, dx + f_y\, dy + f_z\, dz = \left(\frac{\sqrt{y^2+z^2}}{x} + 5\right) dx + \frac{y \ln x}{\sqrt{y^2+z^2}}\, dy + \frac{z \ln x}{\sqrt{y^2+z^2}}\, dz$

$f_x = \frac{1}{x}\sqrt{y^2+z^2} + 5$

$f_y = \frac{\ln x \cdot y}{\sqrt{y^2+z^2}}$

$f_z = \ln x \frac{\cdot z}{\sqrt{y^2+z^2}}$

c) Find the linear approximation of f at $(1, 4, -3)$.

The linear approx is $L(x,y,z) = f(1,4,-3) + f_x(1,4,-3)(x-1) + f_y(1,4,-3)(y-4) + f_z(1,4,-3)(z+3)$

$f_x(1\,4\,-3) = 5 + 5 = 10$

$f_y(1\,4\,-3) = 0 = f_z(1,4,-3)$

$f(1\,4\,-3) = 5$

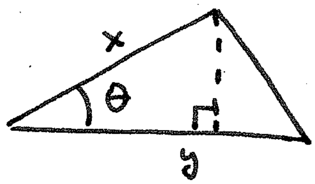
$\Rightarrow L(x,y,z) = 5 + 10(x-1) = 10x - 5$

d) Use the linear approximation to find an approximate value of $f(0.99, 3.98, -3.005)$.

Since $P = (0.99, 3.98, -3.005)$ is close to $(1,4,-3)$, we use the formula

$f(P) \approx 10(0.99) - 5 = 9.9 - 5 = \boxed{4.9}$

Exercise 3. [14 points] One side of a triangle is increasing at a rate of 3 cm/sec and a second side is decreasing at a rate of 2 cm/sec. Assume that the area of the triangle remains unchanged. At what rate is the angle between the two sides changing, when the first side is 40 cm, the second side is 50 cm and the angle is $\pi/6$?



$x'(t_0) = 3 \frac{cm}{sec}$.

$y'(t_0) = -2 \frac{cm}{sec}$. decreasing

$\theta'(t_0) = ?$

We know $x(t_0) = 40$ cm & $A'(t) = 0$ (area remains unchanged)

$y(t_0) = 50$ cm

$\theta(t_0) = \pi/6$

We write the formula for $A(x, y, \theta) = \frac{1}{2} y \cdot x \sin\theta$.

We use the chain rule to find $\theta'(t_0)$:

$$A(t) = A(x(t), y(t), \theta(t))$$

$$0 = A'(t_0) = A_x(40, 50, \pi/6) \cdot \underset{3}{\underset{\parallel}{x'(t_0)}} + A_y(40, 50, \pi/6) \cdot \underset{-2}{\underset{\parallel}{y'(t_0)}} + A_\theta(40, 50, \pi/6)\, \theta'(t_0)$$

$$A_x = \frac{1}{2} y \sin\theta \Rightarrow A_x(40, 50, \pi/6) = \frac{1}{2} 50 \cdot \frac{1}{2} = \frac{50}{4} = \frac{25}{2}$$

$$A_y = \frac{1}{2} x \sin\theta \Rightarrow A_y(40, 50, \pi/6) = 20 \cdot \frac{1}{2} = 10$$

$$A_\theta = \frac{1}{2} xy \cos\theta \Rightarrow A_\theta(40, 50, \pi/6) = \frac{1}{2} 40 \cdot 50 \frac{\sqrt{3}}{2} = 500\sqrt{3}$$

$$\Rightarrow 0 = \underbrace{\frac{25}{2} \cdot 3 - 2 \cdot 10}_{\frac{75-40}{2}} + 500\sqrt{3}\, \theta'(t_0)$$

$$0 = 35 + 250\sqrt{3}\, \theta'(t_0) \Rightarrow \boxed{\theta'(t_0) = \frac{-7}{50\sqrt{3}}}$$

$$0 = 7 + 50\sqrt{3}\, \theta'(t_0)$$

The angle θ is decreasing at a rate of $\frac{7}{50\sqrt{3}}$ rad/sec.

Exercise 4. [**20 points**] True/False. Justify your answer with a proof if true, or a counterexample if false.

a) FALSE
$$\lim_{(x,y)\to(0,0)} \frac{e^{(x^2+y^2)^2}-1}{(x^2+y^2)^{3/2}} = 1.$$

Use Polar coordinates $r^2 = x^2+y^2$

$\Rightarrow \lim = \lim\limits_{r\to 0^+} \frac{e^{r^4}-1}{r^3}$ which is of the form $\frac{0}{0}$. We use L'Hôpital's Rule

$$\lim_{r\to 0^+} \frac{e^{r^4}-1}{r^3} = \lim_{r\to 0^+} \frac{4r^3 e^{r^4}}{3r^2} = \lim_{r\to 0^+} \frac{4}{3} r e^{r^4} = 0 \neq 1.$$

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b) The domain of the function
$$h(s,t) = \frac{(1-s)}{1+t^2}\cos^{-1}(s^2+\frac{t^2}{9}) + \frac{ln(2+sin(t^2))}{st}$$

equals the bounded region enclosed by the ellipse $9s^2+t^2=9$ after removing from it the two axes of the ellipse.

We study the restrictions of each function:

- $1+t^2 \neq 0$ always.
- $\cos^{-1}(s^2+\frac{t^2}{9})$ is defined only when $-1 \leq \underbrace{s^2+\frac{t^2}{9}}_{\text{always} \geq 0} \leq 1$

  $\Rightarrow 0 \leq s^2+\frac{t^2}{9} \leq 1$

  $\boxed{9s^2+t^2 \leq 9}$ is the interior of the ellipse $9s^2+t^2=9$.
- $\ln(2+\sin(t^2)) \to$ need $2+\underbrace{\sin(t^2)}_{\geq -1} > 0 \Rightarrow$ always >0 (no restriction)
- $st \neq 0 \rightarrow$ we must remove the axes $s=0$ & $t=0$

  These are the axes of the ellipse above

Answer : Domain is $\{9s^2+t^2 \leq 9\} \setminus 2$ axes ✓

F c) There exists a differentiable function $f\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying that $f_{\cancel{x}} = x + y^2$ and $f_y = x - 1$.

We compute the cross derivatives & compare them:

$$\left.\begin{aligned} \cdot\ f_{yx} &= \frac{\partial(x-1)}{\partial x} = 1 \\ \cdot\ f_{xy} &= \frac{\partial(x+y^2)}{\partial y} = 2y \end{aligned}\right\}$$ these two functions are continuous

Since f is also differentiable, the cross derivative criterion ensures that $f_{xy} = f_{yx}$, which does not happen.

<u>Conclusion</u>: No such f can exist

d) FALSE

$$\frac{d|\vec{r}(t)|}{dt} = \left|\frac{d(\vec{r}(t))}{dt}\right|$$

We recall $|\vec{r}(t)| = \sqrt{r(t)\cdot r(t)}$.

$\Rightarrow$ We use the chain rule:

(LHS) $$\frac{d|\vec{r}(t)|}{dt} = \frac{1}{2\sqrt{r(t)\cdot r(t)}}\cdot\frac{d}{dt}\big(r(t)\cdot r(t)\big) = \frac{1}{2|r(t)|}\left(\vec{r}\,'(t)\cdot\vec{r}(t) + r(t)\cdot\vec{r}\,'(t)\right)$$

$$= \frac{1}{2|r(t)|}\left(2\vec{r}\,'(t)\cdot r(t)\right) = \frac{\vec{r}\,'(t)\cdot r(t)}{|r(t)|}$$

(RHS) : $= \sqrt{r'(t)\cdot r'(t)}$

Pick an example: $\vec{r}(t) = (0, t+1, t^2) \to |r(t)| = \sqrt{(t+1)^2 + t^2}$

$\vec{r}\,'(t) = (0, 1, 2t) \Rightarrow |\vec{r}\,'(t)| = \sqrt{1+4t^2}$

$\vec{r}\,'(t)\cdot r(t) = (t+1) + 2t^3$

For <u>t=0</u>: (LHS) $= \dfrac{1}{\sqrt{2}}$, (RHS) $= \sqrt{1} = 1$

The expressions are different for this example $\Rightarrow$ FALSE

Exercise 5. **[20 points]** Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by the formula

$$f(x,y) = \begin{cases} 1 & \text{if } x \leq 0 \quad \text{or} \quad x \geq y^4, \\ 0 & \text{if } 0 < x < y^4. \end{cases}$$

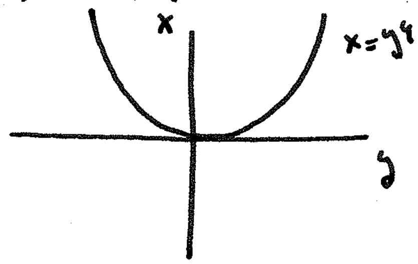
a) Show that $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along any line through $(0,0)$.

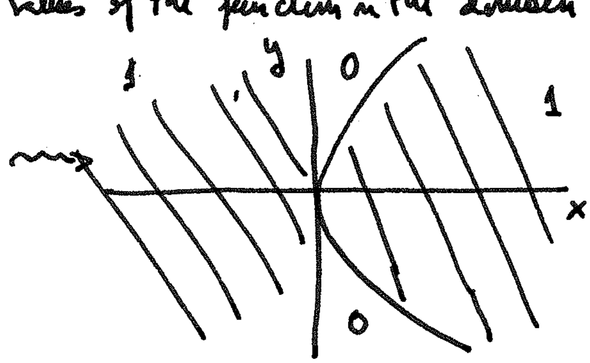
b) Despite part (a), show that f is discontinuous at $(0,0)$.

c) Find all points of $\mathbb{R}^3$ where the function f is discontinuous.

d) Show that all points in (c) lie in the union of two parametric curves and write down the corresponding parameterization of each of these curves.

(a) we write down the ~~domain~~ values of the function on the domain:





We have 3 types lines to check:

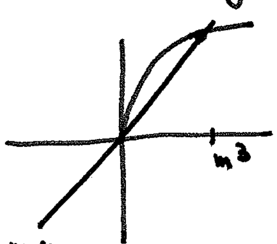
• Horiz: L: $x=0$ ⇝ points in L are of the form $(0,y)$ & $f(0,y)=1$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = \boxed{1}.$$

• Vertical L: $y=0$ ⇝ points in L are of the form $(x,0)$ & $f(x,0)=1$

$$\lim_{\substack{(x,y)\to(0,0)\\(x,y)\in L}} f(x,y) = \lim_{x\to 0} f(x,0) = \boxed{1}$$

• None of those $x = my$ for $m \neq 0$. The picture is symmetric, so we only need to do the case $m>0$ (the case $m<0$ will follow in a similar way)



$x=my$

Fix $m>0$. We claim that for $0<y<m^3$

$my \leq y^4$ ⇒ the line is above the curve $x=y^4$:

$$\Rightarrow \begin{cases} f(my,y) = 1 & \text{if } 0<y<m^3 \\ f(my,y) = 1 & \text{if } y<0 \end{cases}$$ (because $m \geq 0$ so $my<0$)

$$\Rightarrow \lim_{y\to 0^+} f(my,y) = \lim_{y\to 0^-} f(my,y) = 1$$ so lim along this line is $\boxed{1}$

(b) To show that f is discontinuous, we write down a new curve through $(0,0)$ that sits ~~above the~~ on the region where f has value 0.

<u>Example</u> $\quad C(y) = (\frac{1}{2}y^4, y) \xrightarrow[y \to 0]{} (0,0)$

$$\lim_{y \to 0} \underbrace{f(\tfrac{1}{2}y^4, y)}_{=0} = \boxed{0}$$

$=0$ because $0 < \frac{1}{2}y^4 < y^4$ when $y \neq 0$.

Since ~~the~~ limit along lines was 1, then the limit at $(0,0)$ does not exist.

(c) The points P outside the lines $x=0$ and ~~the~~ curve $(x=y^4)$ satisfy that f is locally constant around P (as seen in the diagram) so f is cont. at those points.

- By (b) we know f is discont. at $(0,0)$
- <u>Claim</u> f is discontinuous along the line $x=0$:

Reason: The points in the line (different from $(0,0)$) are of the form $(0, b)$ and $b \neq 0$. We take limits along horizontal line $y=b$ and look at the left & right limits

$$\begin{cases} \lim_{x \to 0^+} \underbrace{f(x,b)}_{=0} = 0 \quad (b \neq 0) \\ \lim_{x \to 0^-} \underbrace{f(x,b)}_{=1} = 1 \end{cases}$$

$\rightarrow$ They are different so the limit at $(0,b)$ doesn't exist

- Similarly, f is discontinuous along the curve $x=y^4$. Again, we only need to check points where $y \neq 0$, so they are of the form (b^4, b) $b \neq 0$

Again we take limits along horiz. lines $y=b$

$$\lim_{x \to \sqrt[4]{b}^+} \underbrace{f(x,b)}_{=1} = 1 \neq 0 = \lim_{x \to \sqrt[4]{b}^-} \underbrace{f(x,b)}_{=0 \ (b \neq 0)} \Rightarrow$$ there is no limit along the line.

(d) The line has param $\vec{r}(y) = \langle 0, y \rangle : \mathbb{R} \to \mathbb{R}^2$

" curve $(x=y^4)$ has param $\vec{r}(y) = \langle y^4, y \rangle : \mathbb{R} \to \mathbb{R}^2$

By (c) the union of these 2 curves is the set of points where f is discontinuous.

$= P = (a,b,c)$

Exercise 6. **[14 points]** Find the points of the hyperboloid $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to the plane $2x + 2y + z = 5$.

To use the formula for tangent plane to the surface, we need to see the the equation of the surface as the graph of a function $\Rightarrow$ we must use the implicit function theorem. Since the plane has normal $= \langle 2,2,1 \rangle$, that means that we can write $z = z(x,y)$. [the tangent plane will have normal with last component $\neq 0$.

We call $f_{(x,y,z)} = x^2+4y^2-z^2$. The Imp Func. Thm says:

$$\left.\begin{aligned} f_x(P) &= 2a \\ f_y(P) &= 8b \\ f_z(P) &= -2c \end{aligned}\right\} \text{cont. } \checkmark$$

If $c \neq 0$, we have $z = z(x,y)$ differentiable and $z_x(a,b) = \dfrac{-f_x(P)}{f_z(P)} = \dfrac{a}{c}$, $z(a,b) = c$

$z_y(a,b) = \dfrac{-f_y(P)}{f_z(P)} = 4\dfrac{b}{c}$

Now, the tangent plane to the graph of z at (a,b,c) has equation:

$$(*) \qquad z - \underbrace{z(a,b)}_{= c} = \underbrace{z_x(a,b)}_{= a/c}(x-a) + \underbrace{z_y(a,b)}_{= 4b/c}(y-b)$$

$\Rightarrow$ normal $= \langle -\frac{a}{c}, -\frac{4b}{c}, 1 \rangle$ is proportional to $\langle 2,2,1 \rangle$

$\Rightarrow 2 = -\frac{a}{c}$, $2 = -\frac{4b}{c}$ $\Rightarrow a = -2c$ & $b = -\frac{1}{2}c$

To find (a,b,c) we remember that $f(a,b,c) = 4$. So:

$4 = (-2c)^2 + 4(-\frac{1}{2}c)^2 - c^2 = 4c^2 + c^2 - c^2 \Rightarrow c^2 = 1 \Rightarrow c = \pm 1$

Conclusion: $P = (-2, -\frac{1}{2}, 1)$ & $(2, \frac{1}{2}, -1)$.

ALTERNATIVE solution: Solve the eq $f(x,y,z) = 4$ for z using the quadratic formula: $z = \pm\sqrt{x^2+4y^2-4}$ $\leadsto$ compute the tangent plane with the formula $(*)$ $\to$ 2 functions with give 1 point each according to the sign of the third component.

For Grader's use only:

1		2				3	4				5				6	TOTAL
a	b	a	b	c	d		a	b	c	d	a	b	c	d		