4. $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \ln\cos t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \dfrac{-\sin t}{\cos t}\,\mathbf{k} = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} - \tan t\,\mathbf{k}$,

$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + (-\tan t)^2} = \sqrt{1+\tan^2 t} = \sqrt{\sec^2 t} = |\sec t|$. Since $\sec t > 0$ for $0 \le t \le \pi/4$, here we can say $|\mathbf{r}'(t)| = \sec t$. Then

$$L = \int_0^{\pi/4} \sec t\,dt = \Big[\ln|\sec t + \tan t|\Big]_0^{\pi/4} = \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0|$$
$$= \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \ln(\sqrt{2}+1).$$

5. $\mathbf{r}(t) = \mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{4t^2+9t^4} = t\sqrt{4+9t^2}$ [since $t \ge 0$].

Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 t\sqrt{4+9t^2}\,dt = \frac{1}{18}\cdot\frac{2}{3}(4+9t^2)^{3/2}\Big]_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{1}{27}(13^{3/2} - 8)$.

11. The projection of the curve C onto the xy-plane is the curve $x^2 = 2y$ or $y = \frac{1}{2}x^2$, $z = 0$. Then we can choose the parameter $x = t \quad\Rightarrow\quad y = \frac{1}{2}t^2$. Since C also lies on the surface $3z = xy$, we have $z = \frac{1}{3}xy = \frac{1}{3}(t)(\frac{1}{2}t^2) = \frac{1}{6}t^3$. Then parametric equations for C are $x = t$, $y = \frac{1}{2}t^2$, $z = \frac{1}{6}t^3$ and the corresponding vector equation is $\mathbf{r}(t) = \left\langle t, \frac{1}{2}t^2, \frac{1}{6}t^3\right\rangle$. The origin corresponds to $t = 0$ and the point $(6, 18, 36)$ corresponds to $t = 6$, so

$$L = \int_0^6 |\mathbf{r}'(t)|\,dt = \int_0^6 \left|\left\langle 1, t, \tfrac{1}{2}t^2\right\rangle\right|\,dt = \int_0^6 \sqrt{1^2 + t^2 + \left(\tfrac{1}{2}t^2\right)^2}\,dt = \int_0^6 \sqrt{1 + t^2 + \tfrac{1}{4}t^4}\,dt$$
$$= \int_0^6 \sqrt{(1+\tfrac{1}{2}t^2)^2}\,dt = \int_0^6 (1+\tfrac{1}{2}t^2)\,dt = \left[t + \tfrac{1}{6}t^3\right]_0^6 = 6 + 36 = 42$$

13. $\mathbf{r}(t) = 2t\,\mathbf{i} + (1-3t)\,\mathbf{j} + (5+4t)\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4+9+16} = \sqrt{29}$. Then $s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{29}\,du = \sqrt{29}\,t$. Therefore, $t = \frac{1}{\sqrt{29}}s$, and substituting for t in the original equation, we have $\mathbf{r}(t(s)) = \frac{2}{\sqrt{29}}s\,\mathbf{i} + \left(1 - \frac{3}{\sqrt{29}}s\right)\mathbf{j} + \left(5 + \frac{4}{\sqrt{29}}s\right)\mathbf{k}$.

17. (a) $\mathbf{r}(t) = \langle t, 3\cos t, 3\sin t\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle 1, -3\sin t, 3\cos t\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{1 + 9\sin^2 t + 9\cos^2 t} = \sqrt{10}$.

Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{10}}\langle 1, -3\sin t, 3\cos t\rangle$ or $\left\langle \frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\sin t, \frac{3}{\sqrt{10}}\cos t\right\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle \quad\Rightarrow\quad |\mathbf{T}'(t)| = \frac{1}{\sqrt{10}}\sqrt{0 + 9\cos^2 t + 9\sin^2 t} = \frac{3}{\sqrt{10}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{10}}{3/\sqrt{10}}\langle 0, -3\cos t, -3\sin t\rangle = \langle 0, -\cos t, -\sin t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{3/\sqrt{10}}{\sqrt{10}} = \dfrac{3}{10}$

25. $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle \quad \Rightarrow \quad \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$. The point $(1, 1, 1)$ corresponds to $t = 1$, and $\mathbf{r}'(1) = \langle 1, 2, 3 \rangle \quad \Rightarrow$ $|\mathbf{r}'(1)| = \sqrt{1+4+9} = \sqrt{14}$. $\mathbf{r}''(t) = \langle 0, 2, 6t \rangle \quad \Rightarrow \quad \mathbf{r}''(1) = \langle 0, 2, 6 \rangle$. $\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 6, -6, 2 \rangle$, so $|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{36+36+4} = \sqrt{76}$. Then $\kappa(1) = \dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{76}}{\sqrt{14}^3} = \dfrac{1}{7}\sqrt{\dfrac{19}{14}}$.

30. $y' = \dfrac{1}{x}, \quad y'' = -\dfrac{1}{x^2}$,

$$\kappa(x) = \frac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}} = \left|\frac{-1}{x^2}\right| \frac{1}{(1 + 1/x^2)^{3/2}} = \frac{1}{x^2}\frac{(x^2)^{3/2}}{(x^2+1)^{3/2}} = \frac{|x|}{(x^2+1)^{3/2}} = \frac{x}{(x^2+1)^{3/2}} \quad [\text{since } x > 0].$$

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = \frac{(x^2+1)^{3/2} - x\left(\frac{3}{2}\right)(x^2+1)^{1/2}(2x)}{[(x^2+1)^{3/2}]^2} = \frac{(x^2+1)^{1/2}[(x^2+1) - 3x^2]}{(x^2+1)^3} = \frac{1-2x^2}{(x^2+1)^{5/2}};$$

$\kappa'(x) = 0 \quad \Rightarrow \quad 1 - 2x^2 = 0$, so the only critical number in the domain is $x = \frac{1}{\sqrt{2}}$. Since $\kappa'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$ and $\kappa'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x = \frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}}, \ln\frac{1}{\sqrt{2}}\right)$.

Since $\displaystyle\lim_{x\to\infty} \frac{x}{(x^2+1)^{3/2}} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

32.
We can take the parabola as having its vertex at the origin and opening upward, so the equation is $f(x) = ax^2, a > 0$. Then by Equation 11, $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \dfrac{|2a|}{[1 + (2ax)^2]^{3/2}} = \dfrac{2a}{(1 + 4a^2x^2)^{3/2}}$, thus $\kappa(0) = 2a$. We want $\kappa(0) = 4$, so $a = 2$ and the equation is $y = 2x^2$.

38.
Notice that the curve a is highest for the same x-values at which curve b is turning more sharply, and a is 0 or near 0 where b is nearly straight. So, a must be the graph of $y = \kappa(x)$, and b is the graph of $y = f(x)$.

46.

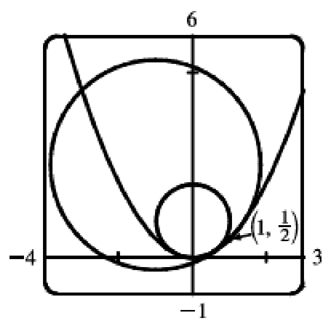
$f(x) = e^{cx}$, $f'(x) = ce^{cx}$, $f''(x) = c^2 e^{cx}$. Using Formula 11 we have

$\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \dfrac{|c^2 e^{cx}|}{[1 + (ce^{cx})^2]^{3/2}} = \dfrac{c^2 e^{cx}}{(1 + c^2 e^{2cx})^{3/2}}$ so the curvature at $x = 0$ is

$\kappa(0) = \dfrac{c^2}{(1 + c^2)^{3/2}}$. To determine the maximum value for $\kappa(0)$, let $f(c) = \dfrac{c^2}{(1 + c^2)^{3/2}}$. Then

$f'(c) = \dfrac{2c \cdot (1 + c^2)^{3/2} - c^2 \cdot \frac{3}{2}(1 + c^2)^{1/2}(2c)}{[(1 + c^2)^{3/2}]^2} = \dfrac{(1 + c^2)^{1/2}\left[2c(1 + c^2) - 3c^3\right]}{(1 + c^2)^3} = \dfrac{(2c - c^3)}{(1 + c^2)^{5/2}}$. We have a critical number when $2c - c^3 = 0 \Rightarrow c(2 - c^2) = 0 \Rightarrow c = 0$ or $c = \pm\sqrt{2}$. $f'(c)$ is positive for $c < -\sqrt{2}$, $0 < c < \sqrt{2}$ and negative elsewhere, so f achieves its maximum value when $c = \sqrt{2}$ or $-\sqrt{2}$. In either case, $\kappa(0) = \dfrac{2}{3^{3/2}}$, so the members of the family with the largest value of $\kappa(0)$ are $f(x) = e^{\sqrt{2}x}$ and $f(x) = e^{-\sqrt{2}x}$.

52. $y = \frac{1}{2}x^2 \Rightarrow y' = x$ and $y'' = 1$, so Formula 11 gives $\kappa(x) = \dfrac{1}{(1 + x^2)^{3/2}}$. So the curvature at $(0, 0)$ is $\kappa(0) = 1$ and the osculating circle has radius 1 and center $(0, 1)$, and hence equation $x^2 + (y - 1)^2 = 1$. The curvature at $\left(1, \frac{1}{2}\right)$ is $\kappa(1) = \dfrac{1}{(1 + 1^2)^{3/2}} = \dfrac{1}{2\sqrt{2}}$. The tangent line to the parabola at $\left(1, \frac{1}{2}\right)$ has slope 1, so the normal line has slope -1. Thus the center of the osculating circle lies in the direction of the unit vector $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. The circle has radius $2\sqrt{2}$, so its center has position vector $\left\langle 1, \frac{1}{2} \right\rangle + 2\sqrt{2}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -1, \frac{5}{2} \right\rangle$. So the equation of the circle is $(x + 1)^2 + \left(y - \frac{5}{2}\right)^2 = 8$.



6. $f(x, y) = e^{xy}$, $g(x, y) = x^3 + y^3 = 16$, and $\nabla f = \lambda \nabla g \Rightarrow \langle ye^{xy}, xe^{xy} \rangle = \langle 3\lambda x^2, 3\lambda y^2 \rangle$, so $ye^{xy} = 3\lambda x^2$ and $xe^{xy} = 3\lambda y^2$. Note that $x = 0 \Leftrightarrow y = 0$ which contradicts $x^3 + y^3 = 16$, so we may assume $x \neq 0$, $y \neq 0$, and then $\lambda = ye^{xy}/(3x^2) = xe^{xy}/(3y^2) \Rightarrow x^3 = y^3 \Rightarrow x = y$. But $x^3 + y^3 = 16$, so $2x^3 = 16 \Rightarrow x = 2 = y$. Here there is no minimum value, since we can choose points satisfying the constraint $x^3 + y^3 = 16$ that make $f(x, y) = e^{xy}$ arbitrarily close to 0 (but never equal to 0). The maximum value is $f(2, 2) = e^4$.

9.
$f(x,y,z)=xyz,\ g(x,y,z)=x^2+2y^2+3z^2=6$. $\nabla f=\lambda\nabla g \quad\Rightarrow\quad \langle yz,xz,xy\rangle=\lambda\langle 2x,4y,6z\rangle$. If any of x, y, or z is zero then $x=y=z=0$ which contradicts $x^2+2y^2+3z^2=6$. Then $\lambda=(yz)/(2x)=(xz)/(4y)=(xy)/(6z)$ or $x^2=2y^2$ and $z^2=\frac{2}{3}y^2$. Thus $x^2+2y^2+3z^2=6$ implies $6y^2=6$ or $y=\pm 1$. Then the possible points are $\left(\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

10. $f(x,y,z)=x^2y^2z^2,\ g(x,y,z)=x^2+y^2+z^2=1 \quad\Rightarrow\quad \nabla f=\langle 2xy^2z^2,2yx^2z^2,2zx^2y^2\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$. Then $\nabla f=\lambda\nabla g$ implies **(1)** $\lambda=y^2z^2=x^2z^2=x^2y^2$ and $\lambda\neq 0$, or **(2)** $\lambda=0$ and one or two (but not three) of the coordinates are 0. If **(1)** then $x^2=y^2=z^2=\frac{1}{3}$. The minimum value of f on the sphere occurs in case **(2)** with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from **(1)**, that is, the points $\left(\pm\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$.

14. $f(x_1,x_2,\ldots,x_n)=x_1+x_2+\cdots+x_n,\ g(x_1,x_2,\ldots,x_n)=x_1^2+x_2^2+\cdots+x_n^2=1 \quad\Rightarrow$
$\langle 1,1,\ldots,1\rangle=\langle 2\lambda x_1,2\lambda x_2,\ldots,2\lambda x_n\rangle$, so $\lambda=1/(2x_1)=1/(2x_2)=\cdots=1/(2x_n)$ and $x_1=x_2=\cdots=x_n$.
But $x_1^2+x_2^2+\cdots+x_n^2=1$, so $x_i=\pm 1/\sqrt{n}$ for $i=1,\ldots,n$. Thus the maximum value of f is $f(1/\sqrt{n},1/\sqrt{n},\ \ldots,1/\sqrt{n})=\sqrt{n}$ and the minimum value is $f(-1/\sqrt{n},-1/\sqrt{n},\ \ldots,\ -1/\sqrt{n})=-\sqrt{n}$.

15. $f(x,y,z)=x+2y,\ g(x,y,z)=x+y+z=1,\ h(x,y,z)=y^2+z^2=4 \quad\Rightarrow\quad \nabla f=\langle 1,2,0\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$ and $\mu\nabla h=\langle 0,2\mu y,2\mu z\rangle$. Then $1=\lambda$, $2=\lambda+2\mu y$ and $0=\lambda+2\mu z$ so $\mu y=\frac{1}{2}=-\mu z$ or $y=1/(2\mu)$, $z=-1/(2\mu)$. Thus $x+y+z=1$ implies $x=1$ and $y^2+z^2=4$ implies $\mu=\pm\frac{1}{2\sqrt{2}}$. Then the possible points are $\left(1,\pm\sqrt{2},\mp\sqrt{2}\right)$ and the maximum value is $f\left(1,\sqrt{2},-\sqrt{2}\right)=1+2\sqrt{2}$ and the minimum value is $f\left(1,-\sqrt{2},\sqrt{2}\right)=1-2\sqrt{2}$.

18.

$f(x,y,z) = x^2 + y^2 + z^2,\ g(x,y,z) = x - y = 1,\ h(x,y,z) = y^2 - z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 2x, 2y, 2z \rangle$,

$\lambda \nabla g = \langle \lambda, -\lambda, 0 \rangle$, and $\mu \nabla h = \langle 0, 2\mu y, -2\mu z \rangle$. Then $2x = \lambda$, $2y = -\lambda + 2\mu y$, and $2z = -2\mu z \quad \Rightarrow \quad z = 0$ or $\mu = -1$.

If $z = 0$ then $y^2 - z^2 = 1$ implies $y^2 = 1 \quad \Rightarrow \quad y = \pm 1$. If $y = 1$, $x - y = 1$ implies $x = 2$, and if $y = -1$ we have $x = 0$, so possible points are $(2, 1, 0)$ and $(0, -1, 0)$. If $\mu = -1$ then $2y = -\lambda + 2\mu y$ implies $4y = -\lambda$, but $\lambda = 2x$ so $4y = -2x \quad \Rightarrow \quad x = -2y$ and $x - y = 1$ implies $-3y = 1 \quad \Rightarrow \quad y = -\frac{1}{3}$. But then $y^2 - z^2 = 1$ implies $z^2 = -\frac{8}{9}$, an impossibility. Thus the maximum value of f subject to the constraints is $f(2,1,0) = 5$ and the minimum is $f(0,-1,0) = 1$.

Note: Since $x - y = 1 \quad \Rightarrow \quad x = y + 1$ is one of the constraints we could have solved the problem by solving $f(y,z) = (y+1)^2 + y^2 + z^2$ subject to $y^2 - z^2 = 1$.

20.

$f(x,y) = 2x^2 + 3y^2 - 4x - 5 \quad \Rightarrow \quad \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad x = 1, y = 0$. Thus $(1, 0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \quad \Rightarrow \quad \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \quad \Rightarrow \quad$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \quad \Rightarrow \quad x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \le 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.

23.

(a) $f(x,y) = x, \quad g(x,y) = y^2 + x^4 - x^3 = 0 \quad \Rightarrow \quad \nabla f = \langle 1, 0 \rangle = \lambda \nabla g = \lambda \left\langle 4x^3 - 3x^2, 2y \right\rangle$. Then

$1 = \lambda(4x^3 - 3x^2)$ **(1)** and $0 = 2\lambda y$ **(2)**. We have $\lambda \ne 0$ from **(1)**, so **(2)** gives $y = 0$. Then, from the constraint equation, $x^4 - x^3 = 0 \quad \Rightarrow \quad x^3(x - 1) = 0 \quad \Rightarrow \quad x = 0$ or $x = 1$. But $x = 0$ contradicts **(1)**, so the only possible extreme value subject to the constraint is $f(1,0) = 1$. (The question remains whether this is indeed the minimum of f.)

(b) The constraint is $y^2 + x^4 - x^3 = 0 \quad \Leftrightarrow \quad y^2 = x^3 - x^4$. The left side is non-negative, so we must have $x^3 - x^4 \ge 0$ which is true only for $0 \le x \le 1$. Therefore the minimum possible value for $f(x,y) = x$ is 0 which occurs for $x = y = 0$. However, $\lambda \nabla g(0,0) = \lambda \langle 0 - 0, 0 \rangle = \langle 0, 0 \rangle$ and $\nabla f(0,0) = \langle 1, 0 \rangle$, so $\nabla f(0,0) \ne \lambda \nabla g(0,0)$ for all values of λ.

(c) Here $\nabla g(0,0) = \mathbf{0}$ but the method of Lagrange multipliers requires that $\nabla g \ne \mathbf{0}$ everywhere on the constraint curve.

36.

If the dimensions of the box are x, y, and z then minimize $f(x,y,z) = 2xy + 2xz + 2yz$ subject to $g(x,y,z) = xyz = 1000$ $(x > 0, y > 0, z > 0)$. Then $\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 2y + 2z, 2x + 2z, 2x + 2y \rangle = \lambda \langle yz, xz, xy \rangle \quad \Rightarrow \quad 2y + 2z = \lambda yz$, $2x + 2z = \lambda xz$, $2x + 2y = \lambda xy$. Solving for λ in each equation gives $\lambda = \dfrac{2}{z} + \dfrac{2}{y} = \dfrac{2}{z} + \dfrac{2}{x} = \dfrac{2}{y} + \dfrac{2}{x} \quad \Rightarrow \quad x = y = z$.

From $xyz = 1000$ we have $x^3 = 1000 \quad \Rightarrow \quad x = 10$ and the dimensions of the box are $x = y = z = 10$ cm.

42.

Let the dimensions of the box be x, y, and z, so its volume is $f(x,y,z)=xyz$, its surface area is $2xy+2yz+2xz=1500$ and its total edge length is $4x+4y+4z=200$. We find the extreme values of $f(x,y,z)$ subject to the constraints $g(x,y,z)=xy+yz+xz=750$ and $h(x,y,z)=x+y+z=50$. Then $\nabla f=\langle yz,xz,xy\rangle=\lambda\nabla g+\mu\nabla h=\langle\lambda(y+z),\lambda(x+z),\lambda(x+y)\rangle+\langle\mu,\mu,\mu\rangle$. So $yz=\lambda(y+z)+\mu$ **(1)**, $xz=\lambda(x+z)+\mu$ **(2)**, and $xy=\lambda(x+y)+\mu$ **(3)**. Notice that the box can't be a cube or else $x=y=z=\frac{50}{3}$ but then $xy+yz+xz=\frac{2500}{3}\neq 750$. Assume x is the distinct side, that is, $x\neq y$, $x\neq z$. Then **(1)** minus **(2)** implies $z\,(y-x)=\lambda(y-x)$ or $\lambda=z$, and **(1)** minus **(3)** implies $y(z-x)=\lambda(z-x)$ or $\lambda=y$. So $y=z=\lambda$ and $x+y+z=50$ implies $x=50-2\lambda$; also $xy+yz+xz=750$ implies $x(2\lambda)+\lambda^2=750$. Hence $50-2\lambda=\dfrac{750-\lambda^2}{2\lambda}$ or

$3\lambda^2-100\lambda+750=0$ and $\lambda=\dfrac{50\pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}\left(50\mp 10\sqrt{10}\right),\frac{1}{3}\left(50\pm 5\sqrt{10}\right),\frac{1}{3}\left(50\pm 5\sqrt{10}\right)\right)$.

Thus the minimum of f is $f\left(\frac{1}{3}\left(50-10\sqrt{3}\right),\frac{1}{3}\left(50+5\sqrt{10}\right),\frac{1}{3}\left(50+5\sqrt{10}\right)\right)=\frac{1}{27}\left(87{,}500-2500\sqrt{10}\right)$, and its maximum is $f\left(\frac{1}{3}\left(50+10\sqrt{10}\right),\frac{1}{3}\left(50-5\sqrt{10}\right),\frac{1}{3}\left(50-5\sqrt{10}\right)\right)=\frac{1}{27}\left(87{,}500+2500\sqrt{10}\right)$.

Note: If either y or z is the distinct side, then symmetry gives the same result.