Tropical secant graphs of monomial curves

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Abstract

TROPICAL ALGEBRAIC GEOMETRY *provides new tools to study elimination theory. Given a monomial curve $t \mapsto (1 : t^{i_1} : \ldots : t^{i_n})$ in $\mathbb{P}^n$ parameterized by a sequence of n coprime integers $i_1 < i_2 < \ldots < i_n$, we wish to study its first secant variety.*

The goal of this project is to effectively calculate the* TROPICALIZATION *of the first secant variety of any monomial curve in $\mathbb{P}^n$. Using methods from* Tropical Implicitization *(Dickenstein, Feichtner and Sturmfels (2007) ; Sturmfels, Tevelev and Yu (2007)) this information will allow is to construct its* Chow polytope*. In the hypersurface case ($n = 4$), the Chow polytope is the* Newton polytope *and interpolation techniques (i.e. linear algebra) can give us the defining equation.

***The main characters in our story are the rank-2 lattice** $\Lambda = \langle \mathbf{1}, (0, i_1, \ldots, i_n)\rangle \subset \mathbb{Z}^{n+1}$ **and a surface in** $\mathbb{T}^{n+1}$ **parameterized by** binomials. This surface is associated to a certain dehomogeneization of our three-fold. We build its tropicalization as a* weighted polyhedral complex *in the n-sphere using the theory of **Geometric Tropicalization** developed by Hacking, Keel and Tevelev. In particular we enrich this theory providing a formula to compute multiplicities of regular points (equiv. of maximal cones).*

This is joint work with Shaowei Lin (UC Berkeley).

1. What is... Geometric Tropicalization?

Definition 1. Let $Y \subset \mathbb{T}^n = (\mathbb{C}^*)^n$ be an algebraic variety with defining ideal $I = I(Y) \subset \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm}]$. The *tropicalization* of Y or I is defined as:

$$\mathcal{T}Y = \mathcal{T}I = \{w \in \mathbb{R}^{n+1} \,|\, \text{in}_w(I) \neq 1\},$$

where $\text{in}_w(I) = \langle \text{in}_w(f) : f \in I\rangle$, and $\text{in}_w(f)$ is the sum of all *nonzero* terms of $f = \sum_\alpha c_\alpha x^\alpha$ such that $\alpha \cdot w$ is **minimum**.

TROPICALIZATIONS of closed subvarieties of tori are rational polyhedral fans. If Y is irreducible, then $\mathcal{T}Y$ is a *pure* fan of dimension equal to $\dim Y$. *Regular points* (equiv. maximal cones) in $\mathcal{T}Y$ can be endowed with positive integer weights called **multiplicities**. With these weights, the fan satisfies the so called *"balancing condition."*

The aim of Geometric Tropicalization consists of computing $\mathcal{T}Y$ *without* knowing its defining ideal, by using a parameterization of the variety. The key-step behind Geometric tropicalization is given by the *valuative definition* of $\mathcal{T}Y$. Hacking, Keel and Tevelev (2009) showed that we only need to consider **divisorial valuations**. This description of $\mathcal{T}Y$ becomes absolutely explicit if Y is *smooth and has a known compactification with normally crossing boundary*. **In case Y is a surface, we can weaken this condition to *combinatorial normal crossing boudary (CNC)*, i.e. no three boundary divisors intersect at a point.** For simplicity, we state the results in the surface case.

Theorem 2 (Hacking, Keel, Tevelev). *Assume $Y \subset \mathbb{T}^N$ and let $\bar{Y}$ be any compactification whose boundary D is a divisor with CNC. Let $D_1, \ldots, D_m$ denote the irreducible components of D, and write $\Delta_{Y,D}$ for the simplicial complex on $\{1, \ldots, m\}$ defined by*

$$\{k_i, k_j\} \in \Delta_{Y,D} \iff D_{k_i} \cap D_{k_j} \neq \emptyset.$$

(i.e., $\Delta_{Y,D}$ is the intersection complex of the boundary divisor D.)

Define the integer vectors $[D_k] = [\text{val}_{D_k}] \in \mathbb{Z}^N$, and for any $\sigma \in \Delta_{Y,D}$, let $[\sigma]$ be the cone in $\mathbb{Z}^N$ spanned by $\{[D_k] : k \in \sigma\}$ and let $\mathbb{Q}_{\geq 0}[\sigma]$ be the cone in $\mathbb{Q}^N$ spanned by the same integer vectors. Then,

$$\mathcal{T}Y = \bigcup_{\sigma \in \Delta_{Y,D}} \mathbb{Q}_{\geq 0}[\sigma].$$

We complement the previous result with a **formula giving the multiplicities of regular points in $\mathcal{T}Y$**:

Theorem 3 (—). *In the notation of Theorem 2, the multiplicity of a regular point $w \in \mathcal{T}Y$ in the tropical surface equals:*

$$m_w = \sum_{\substack{\sigma \in \Delta_{Y,D} \\ w \in \mathbb{Q}_{\geq 0}[\sigma]}} (D_{k_1} \cdot D_{k_2})\; \textit{index}\big((\mathbb{Q} \otimes_{\mathbb{Z}} [\sigma]) \cap \mathbb{Z}^N : \mathbb{Z}[\sigma]\big),$$

where $D_{k_1} \cdot D_{k_2}$ denotes the intersection number of these divisors and we sum over all 2 dimensional cones σ whose associated rational cone $\mathbb{Q}_{\geq 0}[\sigma]$ contains the point w.

- **QUESTION: How to obtain a compactification of a surface in $\mathbb{T}^N$ with CNC?**
- **TWO ANSWERS:**

1. **Compactify the surface Y by $\overline{Y} \subset \mathbb{P}^{N-1}$ and resolve the isolated singularities at the boundary by blow-ups of points on curves.**
2. If $\mathbf{f}\colon \mathbb{T}^2 \to \overline{Y} \subset \mathbb{C}^N$ is a polynomial parameterization:

   Step 1: Define $X := \mathbb{T}^2 \smallsetminus \bigcup_{k=1}^{N} (f_k = 0)$ and extend $\mathbf{f}$ to $\tilde{\mathbf{f}}\colon \overline{X} \subset \mathbb{P}^2 \dashrightarrow \mathbb{T}^N$.

   Step 2: Resolve X by $\pi\colon \tilde{X} \to X$ to obtain a compactification of $\tilde{X}$ with CNC boundary.

   Step 3: Use the map $\tilde{\mathbf{f}} \circ \pi$ to push-forward the construction of $\overline{\tilde{X}}$ and $\Delta_{\tilde{X},\tilde{D}}$ to a nice compactification $\overline{Z}$ and its intersection complex $\Delta_{Z,\partial Z}$.

2. The first secant is a Hadamard product

WE parameterize the first secant of the curve $C \subset \mathbb{P}^n$ via the **secant map**:

$$\phi\colon \mathbb{P}^1 \times C^2 \dashrightarrow \mathbb{P}^n \qquad ((a:b),(p,q)) \mapsto a \cdot p + b \cdot q.$$

Pick $p = (1 : t^{i_1} : \ldots : t^{i_n})$ and $q = (1 : s^{i_1} : \ldots : s^{i_n})$. By using the *monomial change of coordinates* $b = -\lambda a$, $t = \omega s$ we get a new parameterization:

$$\tilde{\phi}(a, s, \omega, \lambda) = a s^{i_k} \cdot (\omega^{i_k} - \lambda) \qquad \text{for all } k = 0, \ldots, n. \tag{1}$$

Definition 4. We let Y be the **surface parameterized by** $(\omega, \lambda) \mapsto (1-\lambda, \omega^{i_1}-\lambda, \ldots, \omega^{i_n}-\lambda)$.

From (1) it is natural to consider the Hadamard product of subvarieties of tori.

Definition 5. Let $X, Y \subset \mathbb{T}^N$ be subvarieties of tori. The *Hadamard product* of X and Y is

$$X \centerdot Y = \overline{\{(x_0 y_0, \ldots, x_n y_n) \,|\, x \in X, y \in Y\}} \subset \mathbb{T}^N.$$

Theorem 6 (—, Lin). *The affine cone over the first secant variety of the curve C equals the* Hadamard product $(C' \cap \mathbb{T}^{n+1}) \centerdot Y$*, where C' is the affine cone over the curve C.*

Hadamard products interplay nicely with tropicalization (Cueto, Tobis, Yu (2009)), thus:

Corollary 7 (—, Lin). *As sets:* $\mathcal{T}Sec^1(C) = \mathcal{T}C' + \mathcal{T}Y = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda + \mathcal{T}Y \subset \mathbb{R}^{n+1}$, *where $\Lambda = \langle \mathbf{1}, (0, i_1, \ldots, i_n)\rangle \subset \mathbb{Z}^{n+1}$. We provide* explicit formulas *to compute multiplicities of regular points in $\mathcal{T}Sec^1(C)$ from multiplicities in $\mathcal{T}Y$.*

Definition 8. By identifying nodes in a graph encoding $\mathcal{T}Y$ (the master graph) via a *combinatorial criterion* we construct a weighted graph called the **tropical secant graph**.

Theorem 9 (—, Lin). *The cone over the tropical secant graph describes $\mathcal{T}Sec^1(C)$ as a collection of 3-dimensional cones with multiplicities.*

3. Computing the Master Graph $\mathcal{T}Y$

THE **master graph** will be constructed combinatorially by gluing two caterpillar graphs and a family of star graphs. These graph represent *resolution diagrams* of a the points $(0:0:1)$ and $(0:1:0)$ ("origin" and "infinity") and at at families of points in $\mathbb{T}^2$ associated to subsets $\underline{a} \subset \{0, i_1, \ldots, i_n\}$ of at least three boundary divisors $(\omega^{i_j} - \lambda = 0)$ intersecting in $\mathbb{T}^2$. Combinatorially, the subsets $\underline{a}$ are obtained by intersecting an arithmetic progression in $\mathbb{Z}$ with the index set $\{0, i_1, \ldots, i_n\}$.

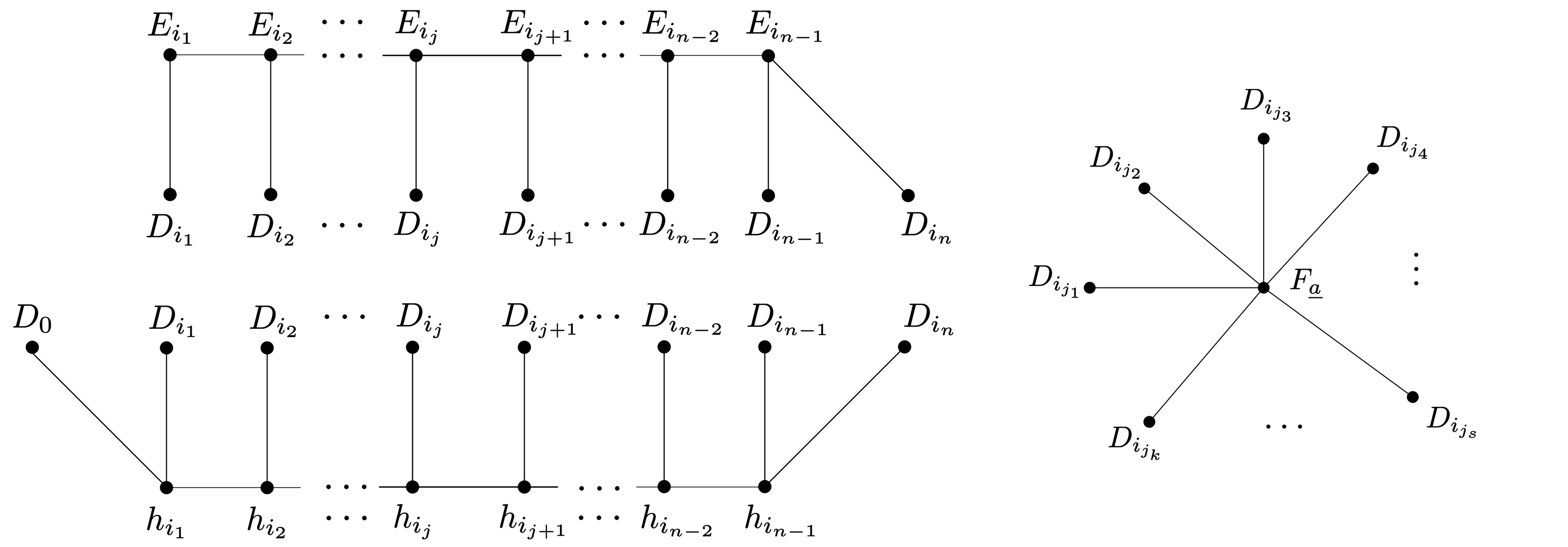

Figure 1: *The graphs glue together along common nodes to form the* master graph.

We embed this graph in $\mathbb{R}^{n+1}$:

$$D_{i_j} := e_j \quad (0 \leq j \leq n) \;;\; E_{i_j} := (\;0,\; i_1, \ldots, i_{j-1},\; i_j,\; i_j,\; \ldots,\; i_j) \; (1 \leq j \leq n-1);$$

$$F_{\underline{a}} := \sum_{i_j \in \underline{a}} e_j \;;\; h_{i_j} := (-i_j, -i_j, \ldots, -i_j, -i_j, -i_{j+1}, \ldots, -i_n) \; (1 \leq j \leq n-1);$$

and we give arithmetic formulas for the weights at each edge.

Theorem 10 (—, Lin). *The tropical surface $\mathcal{T}Y \subset \mathbb{R}^{n+1}$ coincides with the cone over the master graph as a collection of weighted polyhedral cones.*

4. Example: the hypersurface case

CONSIDER the curve $(1 : t^{30} : t^{45} : t^{55} : t^{78})$ studied by Ranestad (2006). We build the *master graph* and the *tropical secant graph* for this curve, as shown in Figure 2.

The eleven nodes in the tropical secant graph are labeled $D_0, D_{30}, D_{45}, D_{55}, D_{78}, E_{45}, E_{55}$ and $F_{0,30,78}, F_{0,30,45,78}, F_{0,30,45}$. The unlabeled nodes represent nodes $F_{\underline{a}}$, where the subset $\underline{a}$ consists of the indices of nodes D_{i_j} adjacent this unlabeled node.

After removing the bivalent node E_{30}, we have a *non-planar* graph with 10 nodes and 23 edges. The complement of the graph in $\mathbb{S}^2$ has 24 connected components. After adding the 6 missing intersection points, the graph becomes planar:

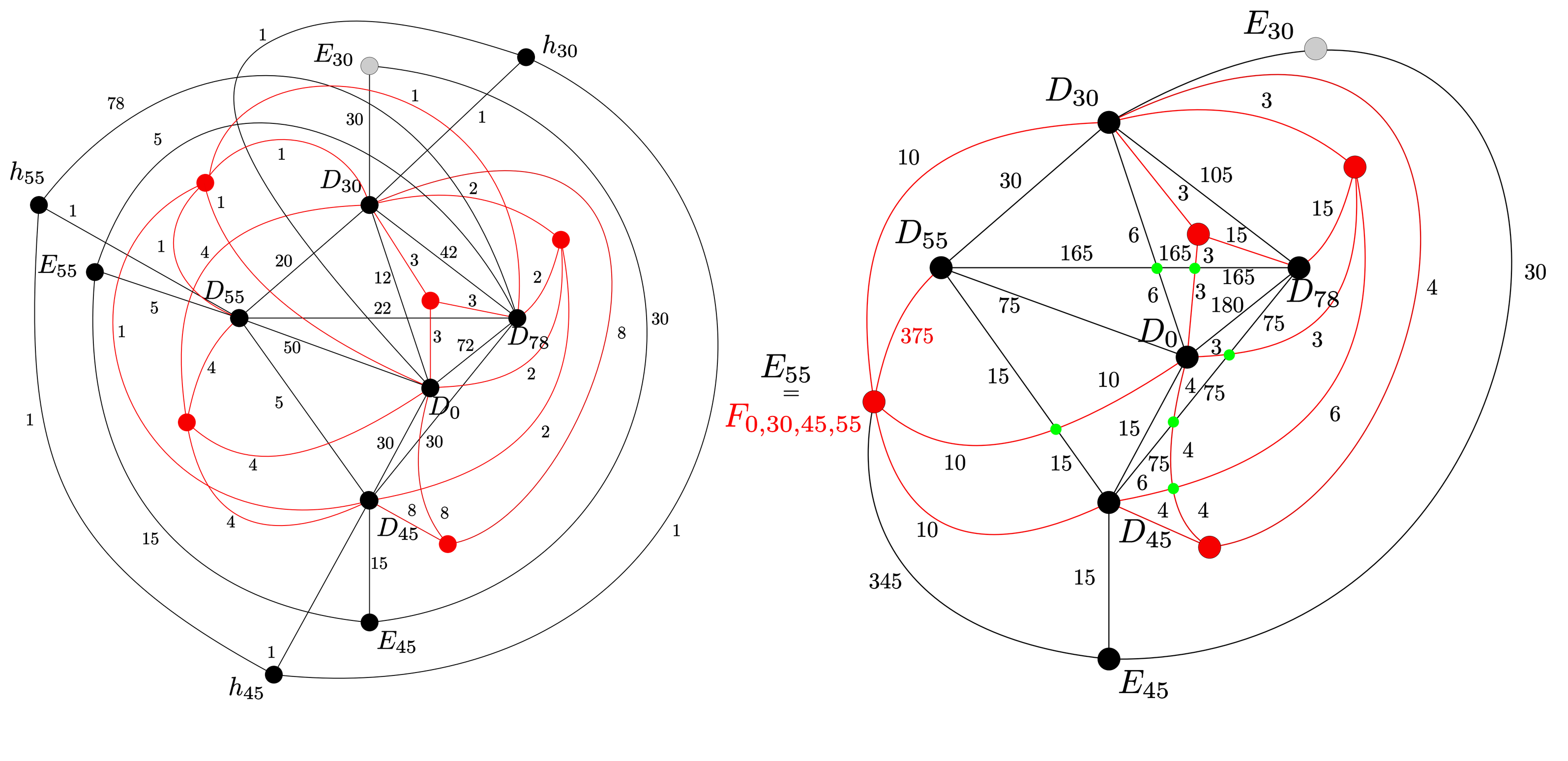

Figure 2: *The* master graph *and the* tropical secant graph *of* $(1 : t^{30} : t^{45} : t^{55} : t^{78})$.

Using methods from ***Tropical Implicitization*** we conclude:

Proposition 11. *The multidegree of the secant variety of the curve $(1 : t^{30} : t^{45} : t^{55} : t^{78})$ w.r.t. the rank-2 lattice $\Lambda = \langle \mathbf{1}, (0, 30, 45, 55, 78)\rangle$ is $(1\,820, 76\,950)$. The Newton polytope has f-vector $(24, 28, 16)$. Its normal fan corresponds to the polyhedral complex on the (RHS) of Figure 2.*

40 Years and Counting: AWM's Celebration of Women in Mathematics. ICERM, Brown University, Providence, RI, Sept. 17-18, 2011.