



# Tropical secant graphs of monomial curves

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## Abstract

TROPICAL ALGEBRAIC GEOMETRY provides new tools to study elimination theory. Given a monomial curve  $t \mapsto (1 : t^{i_1} : \dots : t^{i_n})$  in  $\mathbb{P}^n$  parameterized by a sequence of  $n$  coprime integers  $i_1 < i_2 < \dots < i_n$ , we wish to study its first secant variety.

The goal of this project is to effectively calculate the TROPICALIZATION of the first secant variety of any monomial curve in  $\mathbb{P}^n$ . Using methods from Tropical Implicitization (Dickenstein, Feichtner and Sturmfels (2007) ; Sturmfels, Tevelev and Yu (2007)) this information will allow us to construct its Chow polytope. In the hypersurface case ( $n = 4$ ), the Chow polytope is the Newton polytope and interpolation techniques (i.e. linear algebra) can give us the defining equation.

The main characters in our story are the rank-2 lattice  $\Lambda = \langle \mathbf{1}, (0, i_1, \dots, i_n) \rangle \subset \mathbb{Z}^{n+1}$  and a surface in  $\mathbb{T}^{n+1}$  parameterized by binomials. This surface is associated to a certain dehomogenization of our three-fold. We build its tropicalization as a weighted polyhedral complex in the  $n$ -sphere using the theory of Geometric Tropicalization developed by Hacking, Keel and Tevelev. In particular we enrich this theory providing a formula to compute multiplicities of regular points (equiv. of maximal cones).

This is joint work with Shaowei Lin (UC Berkeley).

## 1. What is ... Geometric Tropicalization?

**Definition 1.** Let  $Y \subset \mathbb{T}^n = (\mathbb{C}^*)^n$  be an algebraic variety with defining ideal  $I = I(Y) \subset \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ . The tropicalization of  $Y$  or  $I$  is defined as:

$$TY = TI = \{w \in \mathbb{R}^{n+1} \mid \text{in}_w(I) \neq \emptyset\},$$

where  $\text{in}_w(I) = \langle \text{in}_w(f) : f \in I \rangle$ , and  $\text{in}_w(f)$  is the sum of all nonzero terms of  $f = \sum_{\alpha} c_{\alpha} x^{\alpha}$  such that  $\alpha \cdot w$  is minimum.

TROPICALIZATIONS of closed subvarieties of tori are rational polyhedral fans. If  $Y$  is irreducible, then  $TY$  is a pure fan of dimension equal to  $\dim Y$ . Regular points (equiv. maximal cones) in  $TY$  can be endowed with positive integer weights called multiplicities. With these weights, the fan satisfies the so called "balancing condition."

The aim of Geometric Tropicalization consists of computing  $TY$  without knowing its defining ideal, by using a parameterization of the variety. The key-step behind Geometric tropicalization is given by the valuative definition of  $TY$ . Hacking, Keel and Tevelev (2009) showed that we only need to consider divisorial valuations. This description of  $TY$  becomes absolutely explicit if  $Y$  is smooth and has a known compactification with normally crossing boundary. In case  $Y$  is a surface, we can weaken this condition to combinatorial normal crossing boundary (CNC), i.e. no three boundary divisors intersect at a point. For simplicity, we state the results in the surface case.

**Theorem 2** (Hacking, Keel, Tevelev). Assume  $Y \subset \mathbb{T}^N$  and let  $\bar{Y}$  be any compactification whose boundary  $D$  is a divisor with CNC. Let  $D_1, \dots, D_m$  denote the irreducible components of  $D$ , and write  $\Delta_{Y,D}$  for the simplicial complex on  $\{1, \dots, m\}$  defined by

$$\{k_i, k_j\} \in \Delta_{Y,D} \iff D_{k_i} \cap D_{k_j} \neq \emptyset.$$

(i.e.,  $\Delta_{Y,D}$  is the intersection complex of the boundary divisor  $D$ .)

Define the integer vectors  $[D_k] = [\text{val}_{D_k}] \in \mathbb{Z}^N$ , and for any  $\sigma \in \Delta_{Y,D}$ , let  $[\sigma]$  be the cone in  $\mathbb{Z}^N$  spanned by  $\{[D_k] : k \in \sigma\}$  and let  $\mathbb{Q}_{\geq 0}[\sigma]$  be the cone in  $\mathbb{Q}^N$  spanned by the same integer vectors. Then,

$$TY = \bigcup_{\sigma \in \Delta_{Y,D}} \mathbb{Q}_{\geq 0}[\sigma].$$

We complement the previous result with a formula giving the multiplicities of regular points in  $TY$ :

**Theorem 3** (—). In the notation of Theorem 2, the multiplicity of a regular point  $w \in TY$  in the tropical surface equals:

$$m_w = \sum_{\substack{\sigma \in \Delta_{Y,D} \\ w \in \mathbb{Q}_{\geq 0}[\sigma]}} (D_{k_1} \cdot D_{k_2}) \text{index}((\mathbb{Q} \otimes_{\mathbb{Z}} [\sigma]) \cap \mathbb{Z}^N : \mathbb{Z}[\sigma]),$$

where  $D_{k_1} \cdot D_{k_2}$  denotes the intersection number of these divisors and we sum over all 2 dimensional cones  $\sigma$  whose associated rational cone  $\mathbb{Q}_{\geq 0}[\sigma]$  contains the point  $w$ .

• **QUESTION:** How to obtain a compactification of a surface in  $\mathbb{T}^N$  with CNC?

• **TWO ANSWERS:**

1. Compactify the surface  $Y$  by  $\bar{Y} \subset \mathbb{P}^{N-1}$  and resolve the isolated singularities at the boundary by blow-ups of points on curves.

2. If  $f : \mathbb{T}^2 \rightarrow \bar{Y} \subset \mathbb{C}^N$  is a polynomial parameterization:

**Step 1:** Define  $X := \mathbb{T}^2 \setminus \bigcup_{k=1}^N (f_k = 0)$  and extend  $f$  to  $\tilde{f} : X \subset \mathbb{P}^2 \dashrightarrow \mathbb{T}^N$ .

**Step 2:** Resolve  $X$  by  $\pi : \tilde{X} \rightarrow X$  to obtain a compactification of  $\tilde{X}$  with CNC boundary.

**Step 3:** Use the map  $\tilde{f} \circ \pi$  to push-forward the construction of  $\tilde{X}$  and  $\Delta_{\tilde{X}, \tilde{D}}$  to a nice compactification  $\bar{Z}$  and its intersection complex  $\Delta_{\bar{Z}, \partial \bar{Z}}$ .

## 2. The first secant is a Hadamard product

WE parameterize the first secant of the curve  $C \subset \mathbb{P}^n$  via the secant map:

$$\phi : \mathbb{P}^1 \times \mathbb{C}^2 \dashrightarrow \mathbb{P}^n \quad ((a : b), (p, q)) \mapsto a \cdot p + b \cdot q.$$

Pick  $p = (1 : t^{i_1} : \dots : t^{i_n})$  and  $q = (1 : s^{i_1} : \dots : s^{i_n})$ . By using the monomial change of coordinates  $b = -\lambda a$ ,  $t = \omega s$  we get a new parameterization:

$$\tilde{\phi}(a, s, \omega, \lambda) = a s^{i_k} \cdot (\omega^{i_k} - \lambda) \quad \text{for all } k = 0, \dots, n. \quad (1)$$

**Definition 4.** We let  $Y$  be the surface parameterized by  $(\omega, \lambda) \mapsto (1 - \lambda, \omega^{i_1} - \lambda, \dots, \omega^{i_n} - \lambda)$ .

From (1) it is natural to consider the Hadamard product of subvarieties of tori.

**Definition 5.** Let  $X, Y \subset \mathbb{T}^N$  be subvarieties of tori. The Hadamard product of  $X$  and  $Y$  is

$$X \cdot Y = \overline{\{(x_0 y_0, \dots, x_n y_n) \mid x \in X, y \in Y\}} \subset \mathbb{T}^N.$$

**Theorem 6** (—, Lin). The affine cone over the first secant variety of the curve  $C$  equals the Hadamard product  $(C' \cap \mathbb{T}^{n+1}) \cdot Y$ , where  $C'$  is the affine cone over the curve  $C$ .

Hadamard products interplay nicely with tropicalization (Cueto, Tobis, Yu (2009)), thus:

**Corollary 7** (—, Lin). As sets:  $TSec^1(C) = TC' + TY = \mathbb{R} \otimes_{\mathbb{Z}} \Lambda + TY \subset \mathbb{R}^{n+1}$ , where  $\Lambda = \langle \mathbf{1}, (0, i_1, \dots, i_n) \rangle \subset \mathbb{Z}^{n+1}$ . We provide explicit formulas to compute multiplicities of regular points in  $TSec^1(C)$  from multiplicities in  $TY$ .

**Definition 8.** By identifying nodes in a graph encoding  $TY$  (the master graph) via a combinatorial criterion we construct a weighted graph called the tropical secant graph.

**Theorem 9** (—, Lin). The cone over the tropical secant graph describes  $TSec^1(C)$  as a collection of 3-dimensional cones with multiplicities.

## 3. Computing the Master Graph $TY$

THE master graph will be constructed combinatorially by gluing two caterpillar graphs and a family of star graphs. These graphs represent resolution diagrams of the points  $(0 : 0 : 1)$  and  $(0 : 1 : 0)$  ("origin" and "infinity") and at families of points in  $\mathbb{T}^2$  associated to subsets  $\underline{a} \subset \{0, i_1, \dots, i_n\}$  of at least three boundary divisors  $(\omega^{i_j} - \lambda = 0)$  intersecting in  $\mathbb{T}^2$ . Combinatorially, the subsets  $\underline{a}$  are obtained by intersecting an arithmetic progression in  $\mathbb{Z}$  with the index set  $\{0, i_1, \dots, i_n\}$ .

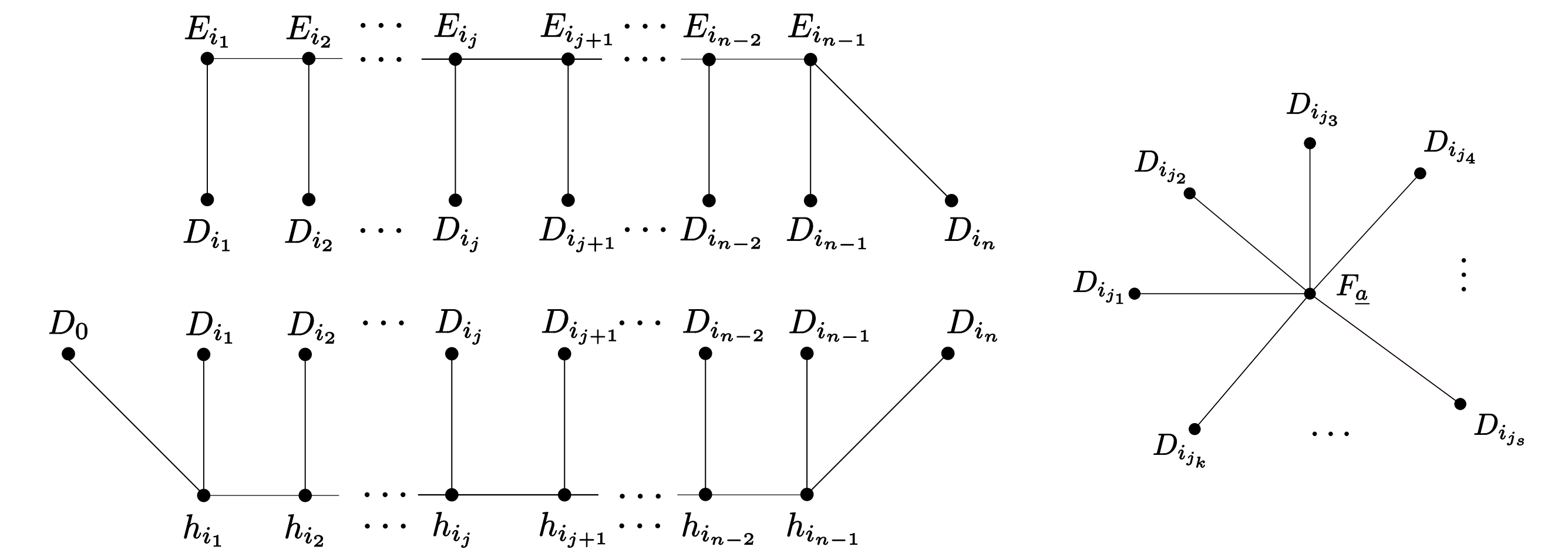


Figure 1: The graphs glue together along common nodes to form the master graph.

We embed this graph in  $\mathbb{R}^{n+1}$ :

$$D_{i_j} := e_j \quad (0 \leq j \leq n) ; \quad E_{i_j} := (0, i_1, \dots, i_{j-1}, i_j, i_j, \dots, i_j) \quad (1 \leq j \leq n-1);$$

$$F_{\underline{a}} := \sum_{i_j \in \underline{a}} e_j ; \quad h_{i_j} := (-i_j, -i_j, \dots, -i_j, -i_j, -i_{j+1}, \dots, -i_n) \quad (1 \leq j \leq n-1);$$

and we give arithmetic formulas for the weights at each edge.

**Theorem 10** (—, Lin). The tropical surface  $TY \subset \mathbb{R}^{n+1}$  coincides with the cone over the master graph as a collection of weighted polyhedral cones.

## 4. Example: the hypersurface case

CONSIDER the curve  $(1 : t^{30} : t^{45} : t^{55} : t^{78})$  studied by Ranestad (2006). We build the master graph and the tropical secant graph for this curve, as shown in Figure 2.

The eleven nodes in the tropical secant graph are labeled  $D_0, D_{30}, D_{45}, D_{55}, D_{78}, E_{45}, E_{55}$  and  $F_{0,30,78}, F_{0,30,45,78}, F_{0,30,45}$ . The unlabeled nodes represent nodes  $F_{\underline{a}}$ , where the subset  $\underline{a}$  consists of the indices of nodes  $D_{i_j}$  adjacent this unlabeled node.

After removing the bivalent node  $E_{30}$ , we have a non-planar graph with 10 nodes and 23 edges. The complement of the graph in  $\mathbb{S}^2$  has 24 connected components. After adding the 6 missing intersection points, the graph becomes planar:

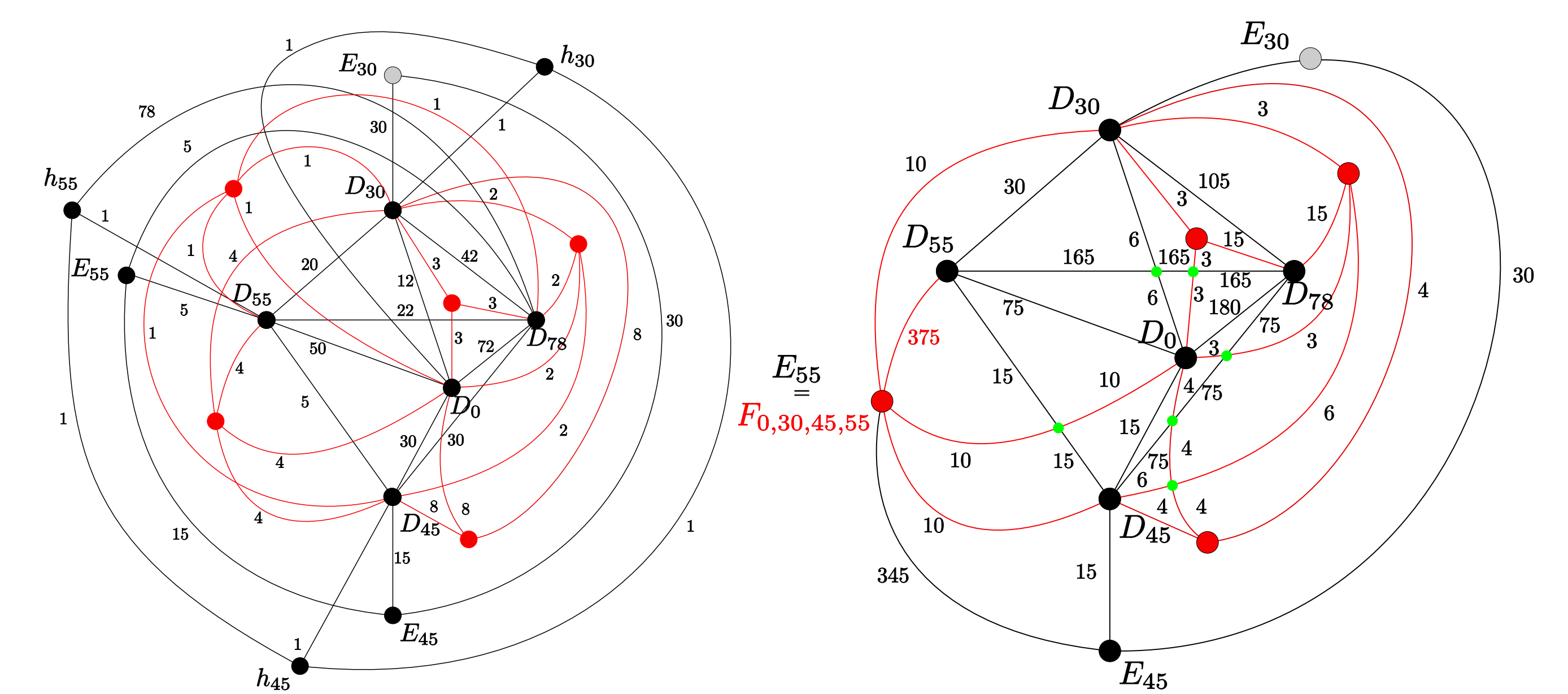


Figure 2: The master graph and the tropical secant graph of  $(1 : t^{30} : t^{45} : t^{55} : t^{78})$ .

Using methods from Tropical Implicitization we conclude:

**Proposition 11.** The multidegree of the secant variety of the curve  $(1 : t^{30} : t^{45} : t^{55} : t^{78})$  w.r.t. the rank-2 lattice  $\Lambda = \langle \mathbf{1}, (0, 30, 45, 55, 78) \rangle$  is  $(1820, 76950)$ . The Newton polytope has  $f$ -vector  $(24, 28, 16)$ . Its normal fan corresponds to the polyhedral complex on the (RHS) of Figure 2.