Practice Problems Final (Fall 2011)
Calculus I - Sections 7 & 8

EXERCISE 1 (a) We start by analyzing the 7 points we saw in class

(1) Domain:

We need $x^2+x \geq 0 \iff x(x+1) \geq 0$

So $\underbrace{x \geq 0 \ \& \ x+1 \geq 0}_{x \geq 0}$ or $\underbrace{x \leq 0 \ \& \ x+1 \leq 0 \quad x \leq -1}_{x \leq -1}$

Hence, the domain is $(-\infty,-1] \cup [0,+\infty)$.

(2) Asymptotes:

(a) No Vertical Asymptotes because the two parts of the domain include the endpoints (also the function $g(y)=\sqrt{y}$ has no V.A.)

(b) Horizontal Asymptotes:

- $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt{x^2+x} - x \underset{x>0}{=} \lim_{x\to+\infty} x\sqrt{1+\frac{1}{x^2}} - x =$

$= \lim_{x\to+\infty} \underbrace{x}_{+\infty}\Big(\underbrace{\sqrt{1+\frac{1}{x^2}}-1}_{\to 0}\Big)$ so undefined.

$= \lim_{x\to+\infty} \frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty} \frac{x^2+x-x^2}{\underset{x\ (x>0)}{|x|}\left(\sqrt{1+\frac{1}{x^2}}+1\right)}$

$= \lim_{x\to\infty} \frac{x}{x\Big(\underbrace{\sqrt{1+\frac{1}{x^2}}+1}_{\to 2}\Big)} = \frac{1}{2}$.

$\rightsquigarrow$ $y=\frac{1}{2}$ is horiz asymp

• $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(|x| \sqrt{1+\frac{1}{x^2}} - x \right) = \lim_{x \to -\infty} \underbrace{|x|}_{+\infty} \underbrace{\left(\sqrt{1+\frac{1}{x^2}} + 1 \right)}_{2} = +\infty$ ②

(c) Slant Asymptotes:

• From the previous calculation, we see that $f(x)$ behaves like

$$\frac{x^2}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)} = x \frac{1}{\left(\sqrt{1+\frac{1}{x^2}}+1\right)} \quad \text{near } x = +\infty.$$

and $\lim_{x \to +\infty} \sqrt{1+\frac{1}{x^2}} + 1 = 2$, so we have a slant asymptote

$y = \frac{1}{2} x$ at $x \to +\infty$.

• Likewise, when $x \to -\infty$, the function behaves like

$$f(x) = |x| \left(\sqrt{1+\frac{1}{x^2}} + 1 \right) = -x \left(\underbrace{\sqrt{1+\frac{1}{x^2}} + 1}_{\to 2} \right)$$

so when $x \to -\infty$ we have the slant asymptote:

$y = -2x$.

(3) f is continuous and differentiable on its domain.

~~Increasing/Decreasing Intervals~~

x-intercepts: $f(x) = 0 \iff \sqrt{x^2+x} - x = 0$

$$\sqrt{x^2+x} = x \geq 0.$$

$$x^2 + x = x^2$$

$$\boxed{x=0}$$

y-intercept $f(0) = 0$ ✓

So we get the point $(0,0)$.

(4) Increasing / Decreasing Intervals:

We find them with f':

$$f'_{(x)} = \frac{1}{2}\frac{1}{\sqrt{x^2+x}}(2x+1) - 1 = \frac{2x+1}{2\sqrt{x^2+x}} - 1.$$

- $f'_{(x)} \geq 0 \iff \frac{2x+1}{2\sqrt{x^2+x}} > 1$

$$2x+1 > 2\sqrt{x^2+x} \quad \text{(because } \sqrt{x^2+x} > 0\text{)} \ (x \neq 0, -1).$$

so $2x+1>0$ & $(2x+1)^2 > 4(x^2+x)$

$$4x^2+1+4x > 4x^2+4x \quad \text{holds always. if}$$

$$2x+1>0$$
$$2x>-1$$
$$x>-\tfrac{1}{2}$$

So f is increasing on $[0,+\infty)$

- The same calculation shows that:

$f'_{(x)} < 0$ if $2x+1<0$ (in fact $f'_{(x)} < -1$ in that case)

so f is decreasing on $(-\infty, -1]$

- Critical points: when $f'_{(x)} = 0$ or $f'_{(x)}$ does not exist.

  - $f'_{(x)} = 0$ never (from the calculation above)
  - $f'_{(x)}$ does not exist when $x^2+x=0$ (the denominator vanishes)

$$\boxed{x = 0 \text{ or } -1}$$

So critical points: $x=0$ & $x=-1$.

(5) Concavity: We find it via f''.

$$f''(x) = \frac{2\, 2\sqrt{x^2+x} - (2x+1)\, 2\cdot\frac{1}{2}\frac{1}{\sqrt{x^2+x}}(2x+1)}{4(x^2+x)}$$

quotient rule

$$= \frac{4(x^2+x) - (2x+1)^2}{4(x^2+x)^{3/2}} = \frac{4x^2+4x-4x^2-1-4x}{4(x^2+x)^{3/2}}$$

$$= -\frac{1}{4}(x^2+x)^{-3/2}$$ ⇝ this is well defined ~~outside~~ when $x \neq 0$ & $x \neq -1$

$$f''(x) = \frac{-1}{4}\,\frac{1}{(x^2+x)\,(x^2+x)^{1/2}} = -\frac{1}{4\left((x^2+x)^{\frac{1}{2}}\right)^3} < 0$$

$(x^2+x)^{\frac{1}{2}}$ is positive in the domain of f.

So f is concave downwards <u>always</u>

In addition, there are no inflection points.

(6) <u>Local max / min</u>:

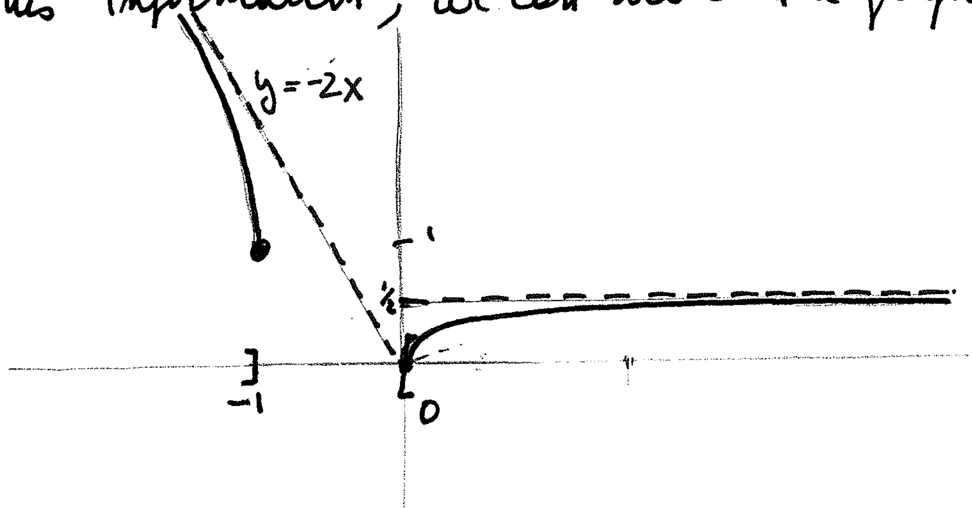
- Critical points $x = 0, -1$.

sign of f'	$-$	-1	undef	0	$+$

⇒ -1 we have a min
0 " " " max

but the function is not defined on $(-1, 0)$, so we don't have local max / min values.

- With this information, we can draw the graph:

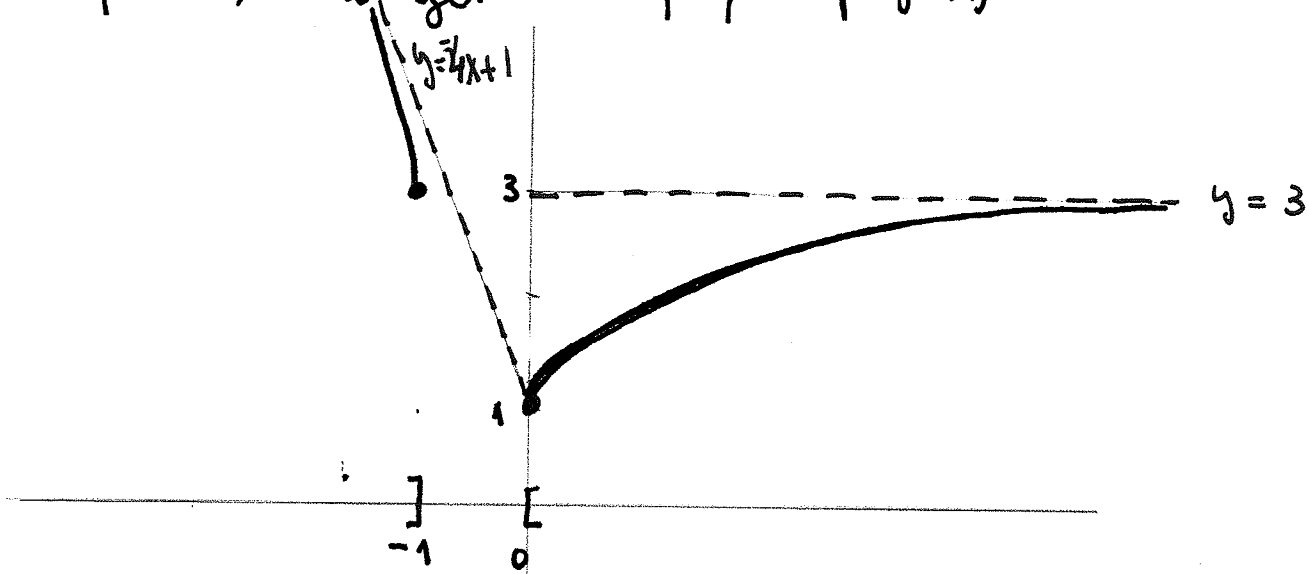


$f(-1) = 1$
$f(0) = 0$

(b) Given $f(x)$ we want to sketch the graph of

$$g(x) = 2\left(\sqrt{x^2+x} - x\right) + 1 = 2f(x) + 1.$$

~~So~~ So we can use the rules from Chapter 1 to modify the graph of $f(x)$ to get the graph of $g(x)$:



Dilate by 2 and add 1 to the y-axis

Asymptotes become $\quad y = 2(-2x)+1 = -4x+1.$

$$y = 2\left(\tfrac{1}{2}\right) + 1 = 3$$

EXERCISE 2:

$$P(t) = \sqrt{b^2+c^2t^2} \qquad t \geq 0 \qquad , b, c > 0$$

$P(0)$

1. • Velocity = derivative of the position , Acceleration = derivative of $v(t)$.

$$v(t) = P'(t) = \frac{1}{2\sqrt{b^2+c^2t^2}}(2tc^2) = c^2\frac{t}{(b^2+c^2t^2)^{1/2}}.$$

• $$a(t) = v'(t) = \frac{c^2\left((b^2+c^2t^2)^{1/2} - t\,\frac{1}{2}(b^2+c^2t^2)^{-1/2}\,2c^2t\right)}{(b^2+c^2t^2)}$$

$$= \frac{c^2\left(-c^2t^2 + b^2 + c^2t^2\right)}{(b^2+c^2t^2)^{3/2}} = \frac{c^2b^2}{(b^2+c^2t^2)^{3/2}}.$$

2. To show that the particle moves in the positive direction, it suffices to show that $v(t) > 0$.

$$v(t) = \underbrace{c^2}_{+}\ \underbrace{t}_{+}\ \frac{1}{\underbrace{(b^2+c^2t^2)^{1/2}}_{+}} > 0.$$

EXERCISE 3 :

We need to express the volume as a function of the radius r (This is a relative rates problem).

$$V(r) = \frac{4}{3}\pi r^3$$

But $r = r(t)$ because the radius is a function of time.

So $$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt} = 12\pi r^2 \underbrace{\frac{dr}{dt}}_{= 4\frac{mm}{s} \text{ at } t=t_0}$$

$$r = \frac{\text{diameter}}{2} = \frac{80}{2} mm = 40 mm \quad \text{at } t=t_0.$$

So $$\frac{dV}{dt}(t_0) = 12\pi (40)^2\, 4 \frac{mm^3}{s} = \boxed{76800\pi \frac{mm^3}{s}}$$

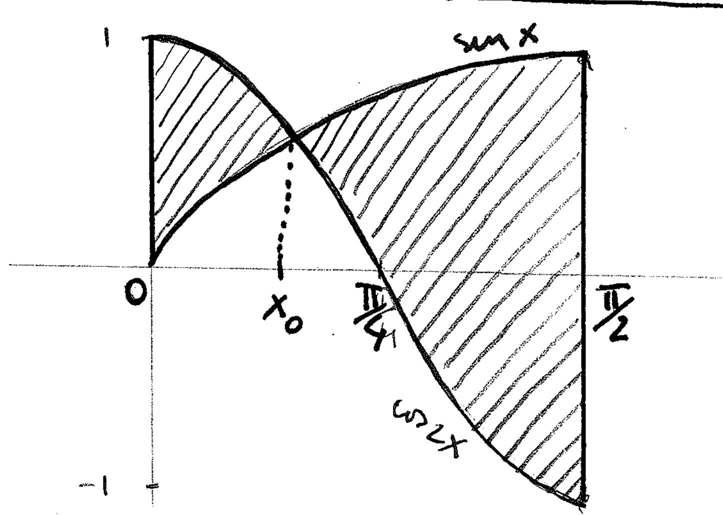
EXERCISE 4 :

$y = \sin x \rightsquigarrow f(x)$

$y = \cos 2x \rightsquigarrow g(x)$

$x = 0 \rightsquigarrow a$

$x = \pi/2 \rightsquigarrow b$



We need to find the intersection pts between $f(x)$ & $g(x)$ indicated by x_0 in the previous picture, $0 \le x_0 \le \frac{\pi}{2}$.

$$f(x) = \sin x = \cos(2x) = g(x)$$

$$\sin x = \cos^2 x - \sin^2 x = (1-\sin^2 x) - \sin^2 x$$

$$\Rightarrow \sin x = 1 - 2\sin^2 x$$

$\Rightarrow \sin x$ verifies $2\sin^2 x + \sin x - 1 = 0$,

which is a ~~linear~~ quadratic equation, so we use the quadratic formula:

$$\sin x = \frac{-1 \pm \sqrt{1^2 + 4 \cdot 2}}{2 \cdot 2} = \frac{-1 \pm 3}{4} \quad \begin{cases} \sin x = -1 \\ \sin x = \frac{1}{2} \end{cases}$$

Since $\sin x_0 > 0$ in the picture, this is $\sin x_0 = \frac{1}{2}$,

so $\boxed{x_0 = \frac{\pi}{6}}$

$\Rightarrow$ Area () $= \displaystyle\int_0^{\pi/6} (\cos 2x - \sin x)\,dx + \int_{\pi/6}^{\pi/2} (\sin x - \cos 2x)\,dx$

because $\cos 2x > \sin x$ $\qquad$ $\sin x > \cos 2x$.

$$= \left(\frac{\sin 2x}{2} + \cos x\right)\Big|_0^{\pi/6} - \left(\frac{\sin 2x}{2} + \cos x\right)\Big|_{\pi/6}^{\pi/2}$$

$$= \left(\frac{\sin\frac{\pi}{3}}{2} + \cos\frac{\pi}{6}\right) - (0+1) - \left(0 + 0 - \left(\frac{\sin \pi/3}{2} + \cos\frac{\pi}{6}\right)\right)$$

$$= \frac{\sqrt{3}/2}{2} + \frac{\sqrt{3}}{2} - 1 + \frac{\frac{\sqrt{3}}{2}}{2} + \frac{\sqrt{3}}{2} = \left(\frac{\sqrt{3}}{2} + \sqrt{3} - 1\right) = \boxed{\frac{3\sqrt{3}-1}{2}}$$

EXERCISE 5

(a) $\lim\limits_{x\to 0} \dfrac{e^x-1}{\tan x} = ?$ Substitution of numerator & denom. gives $\dfrac{0}{0}$, indeterminate!

We can use L'Hospital's Rule.

$$\lim_{x\to 0}\frac{e^x-1}{\tan x} = \lim_{x\to 0}\frac{(e^x-1)'}{(\tan x)'} = \lim_{x\to 0}\frac{e^x}{\frac{1}{\cos^2 x}} \underset{\text{substitute}}{=} \frac{e^0}{1/\cos^2 0} = \frac{1}{1/1^2} = \boxed{1}$$

(b) $\lim\limits_{x\to 0}\dfrac{\tan 4x}{x+\sin 2x}$ Substitution gives $\dfrac{0}{0+0} = \dfrac{0}{0}$,

so we use L'Hospital:

$$\lim_{x\to 0}\frac{\tan 4x}{x+\sin 2x} = \lim_{x\to 0}\frac{\frac{1}{\cos^2 4x}\cdot 4}{1+(\cos 2x)\cdot 2} \underset{\text{subst}}{=} \frac{\frac{4}{1^2}}{1+2} = \boxed{\frac{4}{3}}$$

(c) $\lim\limits_{x\to 1^-}\left(\dfrac{1}{x-1}+\dfrac{1}{x^2-3x+2}\right)$

$x^2-3x+2 = (x-1)(x-2)$ (use the quadratic formula)

$$\frac{1}{x-1}+\frac{1}{x^2-3x+2} = \frac{1}{x-1}+\frac{1}{(x-1)(x-2)} = \frac{x-2+1}{(x-1)(x-2)} =$$

$$= \frac{x-1}{(x-1)(x-2)} = \frac{1}{x-2}$$

$$\Rightarrow \lim_{x\to 1^-}\frac{1}{x-1}+\frac{1}{x^2-3x+2} = \lim_{x\to 1^-}\frac{1}{x-2} \underset{\text{substitute}}{=} \frac{1}{1-2} = \boxed{-1}$$

EXERCISE 6 :

We put the integrals on one side of the equations, and the rest on the other:

$$\int_1^x f(t)\,dt - \int_1^x e^{-t} f(t)\,dt = (x-1)e^{2x}$$

$$\int_1^x \left(f(t) - e^{-t} f(t)\right) dt = (x-1)e^{2x}$$

$$(*) \quad \int_1^x f(t)\left(1-e^{-t}\right) dt = (x-1)e^{2x}$$

The function under the integral symbol is continuous, so by the FTC, the derivative of the left-hand side is $f(x)\left(1-e^{-x}\right)$.

So we differentiate each side of $(*)$ gives:

$$f(x)\left(1-e^{-x}\right) = (x-1)e^{2x}$$

$$\boxed{f(x) = \frac{(x-1)e^{2x}}{1-e^{-x}}} \qquad (x \neq 0)$$

because the denominator vanishes when $x=0$.

EXERCISE 7 :

(1) $\int \sqrt{1+x^2}\, x^3\, dx = ?$ We use substitution $u = x^2$, $du = 2x\,dx$

$\Rightarrow \sqrt{1+x^2}\, x^3\, dx = \sqrt{1+u}\; u\, du$ We substitute $v = 1+u$, $dv = du$

$= \sqrt{v}\,(v-1)\,dv$

$$\int \sqrt{1+x^2}\, x^3\, dx = \int \sqrt{v}\,(v-1)\, dv = \int v^{3/2} - v^{1/2}\, dv$$

$$= \frac{v^{5/2}}{5/2} - \frac{v^{3/2}}{3/2} + C = \frac{2}{5} v^{5/2} - \frac{2}{3} v^{3/2} + C$$

$$\underset{\uparrow}{=} \frac{2}{5}(1+x^2)^{5/2} - \frac{2}{3}(1+x^2)^{3/2} + C \quad (C \text{ constant})$$

$v = 1+u = 1+x^2$

(2) $$\int_{-1}^{3} (31 + x^2x^5)\, dx = \int_{-1}^{3} (31 + x^7)\, dx = 31(4) + \int_{-1}^{3} x^7\, dx$$

$$= 124 + \frac{x^8}{8}\Big|_{-1}^{3} = 124 \;\; \frac{3^8 - 1}{8} = \frac{31}{2}(3^8 - 1).$$

(3) $$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$ → substitution

$u = \cos x$

$du = -\sin x\, dx$

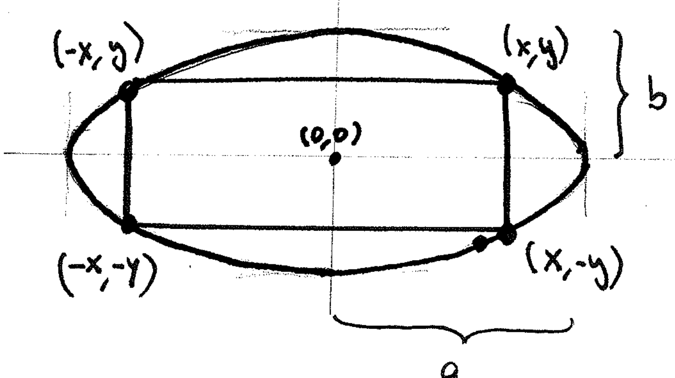
$$= \int \frac{-1}{u}\, du = -\int \frac{1}{u}\, du = -\ln|u| + C$$

$$= -\ln|\cos x| + C \quad (C \text{ constant})$$

(4) $$\int_{0}^{\pi/2} \tan x\, dx \underset{\uparrow \text{ use (3)}}{=} -\ln|\cos x|\Big|_{0}^{\pi/2} = \underbrace{-\ln 0}_{=-\infty} - \underbrace{\ln 1}_{=0} = +\infty.$$

EXERCISE 8

This is an optimization problem.



By symmetry, the ~~points~~ vertices of the rectangle will be (x,y), $(x,-y)$, $(-x,y)$ & $(-x,-y)$ where all these points are in the ellipse.

base of the rectangle $= 2x$, height $= 2y$. Note: $0 \leq y \leq b$

Area (rectangle) $= 2x \cdot 2y = 4xy$.

We have 2 parameters, and we want to eliminate one:

$(x,y) \in$ ellipse so $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\Rightarrow \quad x^2 = a^2\left(1 - \frac{y^2}{b^2}\right) \underset{a,b,x,y>0}{\Rightarrow} \quad x = \frac{a}{b}\sqrt{b^2 - y^2}.$$

Now we replace x into the area of the rectangle:

$$\text{Area}(y) = 4\,\frac{a}{b}\sqrt{b^2-y^2} \cdot y$$

We want to <u>maximize</u> Area(y) subject to the condition $0 \le y \le b$

$\Rightarrow$ We use the Close Interval Method

- Find critical points & evaluate A at those points:

$$A'(y) = \frac{4a}{b}\left(\sqrt{b^2-y^2} + y\,\frac{1}{2\sqrt{b^2-y^2}}(-2y)\right)$$

$$= \frac{4a}{b}\left(\frac{b^2-y^2 - y^2}{\sqrt{b^2-y^2}}\right) = \frac{4a(b^2-2y^2)}{b\sqrt{b^2-y^2}}$$

So critical <u>values</u>: $\begin{cases} \bullet \text{ where } A' \text{ is not defined: } y = b. \\ \bullet\ A'(y) = 0 \text{ when } y = b/\sqrt{2}. \end{cases}$

$\Rightarrow A(b) = 0$, $A(b/\sqrt{2}) = 4\,\frac{a}{b}\sqrt{b^2 - \frac{b^2}{2}} \cdot \frac{b}{\sqrt{2}} = \boxed{2ab}$

- Evaluate A at the end points: $A(0) = 0$, $A(b) = 0$
- Compare the 4 values ~~and~~ pick the maximum.

$\Rightarrow A = 2ab$, $y = b/\sqrt{2}$, $x = \frac{a}{b}\sqrt{b^2 - \frac{b^2}{2}} = \frac{a}{\sqrt{2}} \Rightarrow$ height $= \frac{2b}{\sqrt{2}}$, base: $2a/\sqrt{2}$

EXERCISE 9:

(1) $\int_0^1 \frac{e^t+1}{e^t+t}\,dt$ Use substitution. $u = u(t)$

$\frac{e^t+1}{e^t+t} = (e^t+1)\,\frac{1}{e^t+t}$ & one of these factors is $u'(t)$

$\Rightarrow u'(t) = e^t+1$ or $u'(t) = \frac{1}{e^t+t}$

Let's see if the first one works.

$u'(t) = e^t+1 \Rightarrow u'(t) = e^t+t+C$ for some C (possibly $C=0$)

$du = (e^t+1)\,dt$

$\frac{e^t+1}{e^t+t}\,dt = \frac{du}{u}$ if $C=0$.

$t=0 \to u(0) = e^0+0 = 1$

$t=1 \to u(1) = e^1+1 = e+1$

$$\int_0^1 \frac{e^t+1}{e^t+t}\,dt = \int_1^{e+1} \frac{du}{u} = \ln|u|\,\Big|_1^{e+1} = \ln(\underbrace{e+1}_{>0}) - \underbrace{\ln(\underbrace{1}_{>0})}_{=0}$$

$$= \boxed{\ln(e+1)}$$

(2) Now, we use this to solve the other integral. They have in common many things.

- Two functions share the denominator
- " " " " 1 in the numerator
- The endpoints of the integrals are the same.

Idea: Change $\frac{1-t}{e^t+t}$ to get $\frac{e^t+1}{e^t+t}$.

We work with the numerator:

$$1-t = \underbrace{(e^t-e^t)}_{=0}+1-t = e^t+1-e^t-t = (e^t+1)-(e^t+t)$$

add and substract e^t reorder

$$\int_0^1 \frac{1-t}{e^t+t}\,dt = \int_0^1 \frac{e^t+1-(e^t+t)}{e^t+t}\,dt = \int_0^1 \left(\frac{e^t+1}{e^t+t} - \underbrace{\frac{e^t+t}{e^t+t}}_{=1}\right)dt$$

$$= \int_0^1 \frac{e^t+1}{e^t+t}\,dt - \int_0^1 1\,dt = \ln(e+1) - 1\cdot 1 =$$

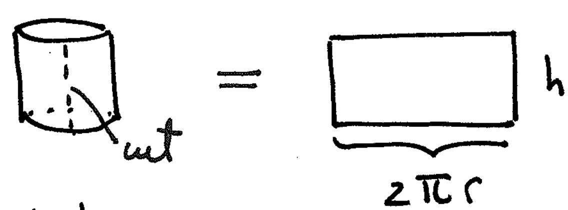
$$= \boxed{\ln(e+1) - 1}$$

EXERCISE 10 :

can To minimize the cost is to minimize the amount of metal.

Metal : . bottom : 

. side



(*) Metal $= 2\pi r^2 + h\,2\pi r = (2\pi r)(r+h) \rightarrow$ variables r & h

The information of the volume will allow us to eliminate one of them.

$$\mathrm{Vol}(h,r) = \mathrm{Vol}(\text{cylinder}) = (\underset{\substack{\uparrow \\ \text{area of the base}}}{\pi r^2})\ \underset{\substack{\uparrow \\ \text{height}}}{h} = V$$

$$\Rightarrow \boxed{h = \frac{V}{\pi r^2}}$$

Now, we plug in (*) :

$$\text{Metal}(r) = 2\pi(r^2 + hr) = 2\pi\left(r^2 + \frac{rV}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{2V}{r}.$$

Here we have $0 < r$ (it's the radius of a circle).

So we want to minimize the function

$$\Pi(r) = 2\pi r^2 + \frac{2V}{r}$$

subject to the constraint $0 < r$.

A minimum will be a local minimum, hence a critical point:

$$\Pi'(r) = 4\pi r - \frac{2V}{r^2} = \frac{4\pi r^3 - 2V}{r^2}$$

Since $r > 0$, $\Pi'(r) = 0 \Leftrightarrow 4\pi r^3 = 2V$

$$r^3 = \frac{V}{2\pi}$$

$$r = \sqrt[3]{\frac{V}{2\pi}}$$

$\Rightarrow$

$$\Pi(r) = 2\pi\left(\frac{V}{2\pi}\right)^{3/2} + 2V\left(\frac{V}{2\pi}\right)^{-1/3}..$$

<u>Dimensions</u>: $r = \sqrt[3]{\frac{V}{2\pi}}$, $h = \frac{V}{\pi\left(\frac{V}{2\pi}\right)^{2/3}} = 2\left(\frac{V}{\pi}\right)^{1-\frac{2}{3}} = 2\left(\frac{V}{2\pi}\right)^{1/3}$

Since we only have 1 critical value, the dimension we just obtained have to give us the minimum.

<u>EXERCISE 12</u>

We need to find the velocity, which is the antiderivative of the acceleration $a(t) = 3t - 5$

$$v(t) = \frac{3t^2}{2} - 5t + C \quad \text{for } C \text{ an arbitrary constant}$$

The extra condition $v(0) = \frac{8}{3}\frac{m}{s}$ help us find what is the value of C.

$$\frac{8}{3} = v(0) = 0 + C \Rightarrow C = \frac{8}{3}.$$

$$\Rightarrow v(t) = \frac{3t^2}{2} - 5t + \frac{8}{3}.$$

- **Displacement**: $\displaystyle\int_0^3 v(t)\,dt = \int_0^3 \left(\frac{3t^2}{2} - 5t + \frac{8}{3}\right)dt$

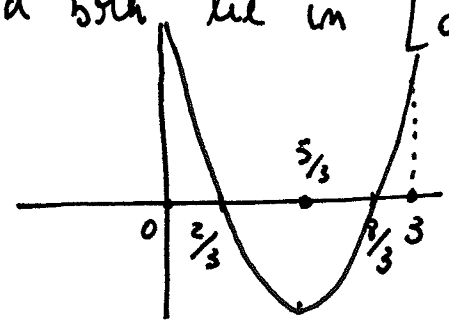
$$= \frac{t^3}{2} - \frac{5t^2}{2} + \frac{8}{3}t \Big|_0^3 = \left(\frac{3^3}{2} - \frac{5\cdot 9}{2} + 8\right) - 0 = \frac{-2}{2} = \boxed{-1}$$

Displ $= \underline{-1\,m}$

- **Distance**: $\displaystyle\int_0^3 |v(t)|\,dt = ?$ We need to study the sign of $v(t)$ on the interval $[0,3]$. Since v is a parabola facing upwards (because the coefficient of t^2 is $\frac{3}{2} > 0$), the easiest thing is the find the zeros of v. We use the quadratic formula:

$$t = \frac{5 \pm \sqrt{25 - 4\cdot\frac{3}{2}\frac{8}{3}}}{2\cdot 3/2} = \frac{5 \pm \sqrt{25-16}}{3} = \frac{5 \pm 3}{3} \begin{cases} \frac{8}{3} \\ \frac{2}{3} \end{cases}$$

and both lie in $[0,3]$:



(vertex of the parabola $\left(\frac{5}{3}, v\left(\frac{5}{3}\right)\right)$
$\left(\frac{8}{3} + \frac{2}{3}\right)/2 = 5/3$

$v(t) > 0$ on $\left[0, \frac{2}{3}\right]$
$v(t) < 0$ on $\left[\frac{2}{3}, \frac{8}{3}\right]$
$v(t) > 0$ on $\left[\frac{8}{3}, 3\right]$

$$\Rightarrow \int_0^3 |v(t)|\,dt = \int_0^{2/3}\left(\frac{3}{2}t^2 - 5t + \frac{8}{3}\right)dt + \int_{2/3}^{8/3} -\left(\frac{3}{2}t^2 - 5t + \frac{8}{3}\right)dt + \int_{8/3}^{3}\left(\frac{3}{2}t^2 - 5t + \frac{8}{3}\right)dt$$

$$= \underbrace{\left(\frac{t^3}{2} - \frac{5t^2}{2} + \frac{8}{3}t\right)}_{= f(t)}\Bigg|_0^{2/3} - \left(g(t)\right)\Bigg|_{2/3}^{8/3} + \left(g(t)\right)\Bigg|_{8/3}^{3}$$

$$= \frac{44}{2\cdot 27} - 0 - \left(-\frac{64}{2\cdot 27} - \frac{44}{2\cdot 27}\right) + \left(-1 + \frac{64}{2\cdot 27}\right)$$

$$= \frac{44}{27} + \frac{64}{27} - 1 = \frac{108}{27} - 1 = 4 - 1 = \boxed{3}$$

EXERCISE 12:

(1) To see if f is continuous, it suffices to show that the functions on each piece agree on the end points. Since each function is continuous, we can evaluate and see if we get the same numbers:

(i) $\underline{x=2}$: $1 \overset{?}{=} (2-1) = 1$ ✓

(ii) $\underline{x=4}$ $4-1 = 3 \overset{?}{=} -2\cdot 4 + 13 = 5$ ✗

(iii) $x=5$ $(-2\cdot 5+13) \overset{?}{=} 3 \overset{?}{=} -5+8 = 3$ ✓

(iv) $x=9$ $(-9+8) = -1 \overset{?}{=} -1$ ✓

(v) $x=10$ $(10-11) = -1 \overset{?}{=} -1$ ✓

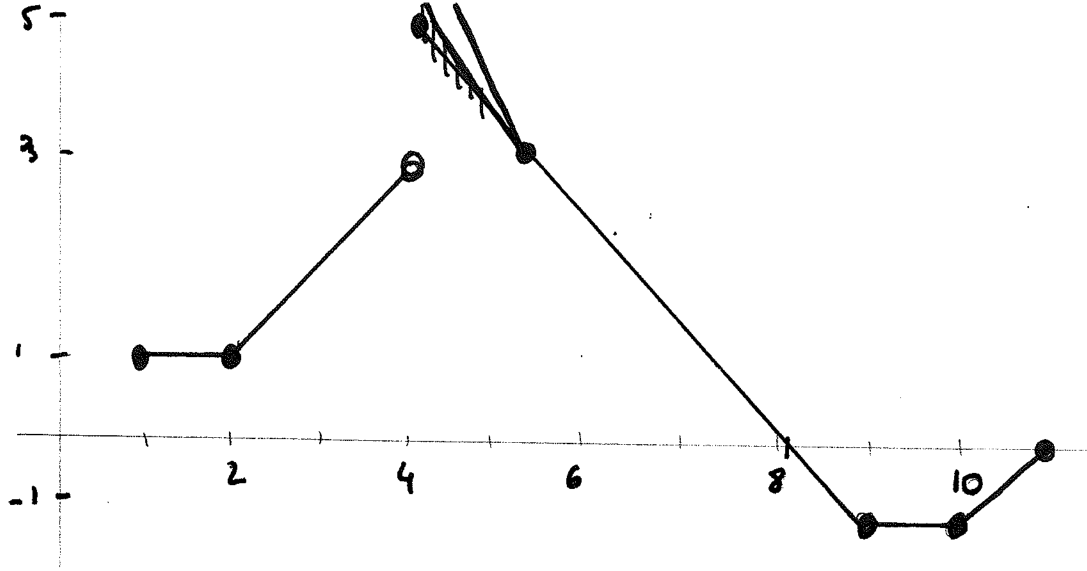
So f is cont everywhere except at $x=4$.

(2) The same idea holds for the derivatives: We can take the derivative on each piece and see if they agree on the end points

$$f'(x) = \begin{cases} 1 & 1<x<2 \\ 1 & 2<x<4 \\ -2 & 4<x<5 \\ -1 & 5<x<9 \\ -1 & 9<x<10 \\ 1 & 10<x<11 \end{cases} \Rightarrow f' \text{ is not defined at } x = 4, 5, 10.$$

$\Rightarrow$ f is differentiable everywhere except at $x=4, 5, 10$.

(3)



(4) $g(1) = \int_1^1 f(t)\,dt = 0$

$g(2) = \int_1^2 1\,dt = 1$

$g(3) = \int_1^3 f(t)\,dt = \underbrace{\int_1^2 1\,d\cancel{x}}_{=1} + \int_2^3 (x-1)\,dx = 1 + \left(\frac{x^2}{2} - x\right)\Big|_2^3 = 1 + (3-0)$
$= 4$

$g(4) = g(3) + \int_3^4 x-1\,dx = 4 + \left(\frac{x^2}{2} - x\right)\Big|_3^4 = 4 + (4-3) = 5$

$g(5) = g(4) + \int_4^5 -2x+13\,dx = 5 + \left(-x^2 + 13x\right)\Big|_4^5$
$= 5 + (40-36) = 9$

$g(x) = g(5) + \int_5^x (-t+8)\,dt = 9 + \left(-\frac{t^2}{2} + 8t\right)\Big|_5^x = 9 + t\left(8 - \frac{t}{2}\right)\Big|_5^x$

$5 < x < 9$

$= 9 + \left(\frac{x}{2}(16-x) - 5\left(8 - \frac{5}{2}\right)\right)$

$= -\frac{37}{2} + 8x - \frac{x^2}{2}.$

$\Rightarrow$ We obtain $g(6), g(7), g(8), g(9)$. , $g(9) = 13$

$g(10) = g(9) + \int_9^{10} -1\,dt = 13 - 1 = 12$

$g(11) = 12 + \int_{10}^{11} x - 11\,dt = 12 + \left(\frac{x^2}{2} - 11x\right)\Big|_{10}^{11}$

$= 12 + 11\left(\frac{11}{2} - 1\right) - 10\,(5 - 11)$

$= \frac{243}{2}.$

(5.) To find out where g is increasing, we use the first derivative test, so we need to find g'.

Since f is not continuous everywhere, we cannot use the FTC where $x=4$ lies in the interval where we integrate.

- If $x<4$, we know $f(t)$ is cont, so

$$g'(x) = f(x).$$

- If $x \geq 4$, we write $g(x) = \int_1^x f(t)\,dt = g(4) + \int_4^x f(t)\,dt$ and now f is cont in $[4,11]$ so by the FTC

$$g'(x) = 0 + \left(\int_4^x f(t)\,dt\right)' = f(x).$$

So we conclude that $g' = f$ on $[1,11]$.

Thus, from (3) we know the sign of f. if we find the x-intercept between $x=5$ & $x=9$.

$$-x+8 = 0 \Rightarrow x = 8 \checkmark$$

$$\Rightarrow \begin{cases} \cdot\ g \text{ is } \underline{\text{increasing}} \text{ on } [1,8] \\ \cdot\ g \text{ is decreasing on } [8,11] \end{cases}$$

- For the concavity, we need to find g''. Since we know that $g' = f$, then

$$g''(x) = \begin{cases} 1 & 1<x<4 \\ -2 & 4<x<5 \\ -1 & 9 \leq x \leq 10 \\ 1 & 10 \leq x \leq 11 \end{cases}$$

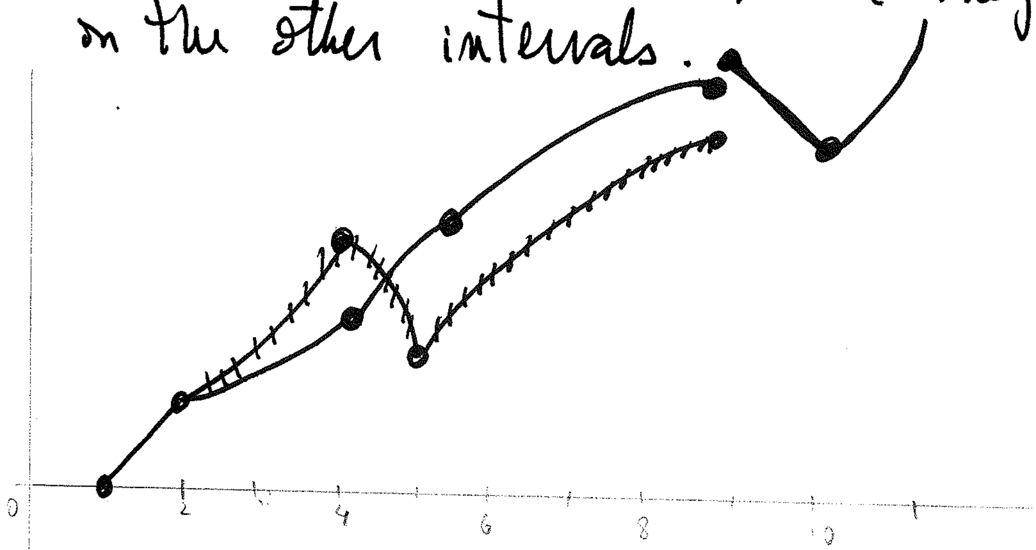
(see page 16)

so g is CU on $(1,4) \cup (10,11)$
g is CD on $(4,10)$ by the Concavity Test

- Critical points : $x = 8, 11$ (see the graph of f)
- Inflection points : $x = 4, 10$ (pts where the concavity changes)

(6) From the construction we see that g is <u>linear</u> on $(1,2) \cup (9,10)$ & it is a quadratic function on the rest of the domain, and it has $\neq$ formulas on the intervals: $(2,4)$, $(4,5)$, $(5,9)$ & $(10,11)$.

The parabolas are facing up in $(2,4) \cup (10,11)$, because the coefficient would be 1, and they are facing down on the other intervals.



$g(1) = 0$

$g(x) = x - 1 \qquad 1 < x < 2$

$g(x) = 1 + \frac{x^2}{2} - x \qquad 2 < x < 4$

$g(x) = -31 - x^2 + 13x \qquad 4 < x < 5$

$g(x) = \frac{-37}{2} + 8x - \frac{x^2}{2} \qquad 5 < x < 9$

$g(x) = 13 - (x - 9) =$
$= 22 - x \qquad 9 < x < 10$

$g(x) = 72 + \frac{x^2}{2} - 11 \cdot x \qquad 10 < x < 11$

EXERCISE 14:

We need to find g''. For this we use the FTC to find g'. We can do so because t^2+t+2 does not vanish in $\mathbb{R}$.

$$g'(x) = \frac{x^2}{x^2+x+2} \Rightarrow g''(x) = \frac{2x(x^2+x+2)-x^2(2x+1)}{(x^2+x+2)^2}$$

We need to solve $g''(x) < 0$.

Since the denominator is >0 we need the numerator to be <0, hence $2x^3+2x^2+4-2x^3-x^2 = 3x^2+4<0$ and this never happens.

Conclusion: g is CU always.

EXERCISE 15:

For this we use implicit differentiation

$y = f(x)$. $\quad 2(x^2+f_{(x)}^2)^2 = 25(x^2-f_{(x)}^2)$

Differentiate: $2\cdot 2(x^2+f_{(x)}^2)\cdot(2x+2f_{(x)}\cdot f'_{(x)}) =$

$$= 25(2x-2f_{(x)}f'_{(x)}).$$

Solve for f': $8(x^2+f_{(x)}^2)(x+f_{(x)}f'_{(x)}) = 50\cdot(x-f_{(x)}f'_{(x)})$

$$\Rightarrow f_{(x)}f'_{(x)}(8(x^2+f_{(x)}^2)+50) = 50x-8x(x^2+f_{(x)}^2)$$

$$f'(x) = \frac{x(50-8(x^2+f^2))}{f_{(x)}\cdot(50+8(x^2+f_{(x)}^2))}.$$

We want to find the tangent at $(3,1)=(x,y)$, so we substitute

$$f'(3) = \frac{3\left(50-8(1+9)\right)}{1\cdot\left(50+8(10)\right)} = \frac{3\,(50-80)}{1\,(50+80)} = \frac{-9}{13}$$

$\Rightarrow$ Equation $\boxed{y-1=\left(\frac{-9}{13}\right)(x-3)}$

EXERCISE 16 :

Horizontal tangent means slope $=0=f'(x)$.

$\Rightarrow f'(x) = e^x - 2 = 0 \Rightarrow e^x = 2$

$\boxed{x=\ln 2}$.

EXERCISE 17

(1). $f(x) = \ln(x-1)-1$ is defined only if $x-1>0$

$\boxed{x>1}$

.Range : is the range of ln since $x-1$ varies among all points in $(0,+\infty)$.

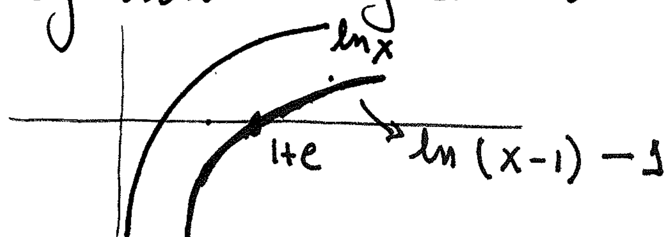
So $\text{Range}(f)=\mathbb{R}$.

(2) x-intercept : $\ln(x-1)-1=0$. $\ln(x-1)=1$

$x-1=e$

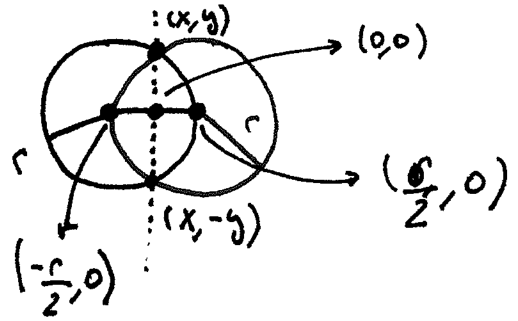
$\boxed{x=1+e}$

(3) The graph of f is build from the graph of ln, by translating down and right by 1 the graph of ln



EXERCISE 14

As a warmup, we solve the analogous question for areas & circles



By symmetry, it is not hard to see that $x=0$.

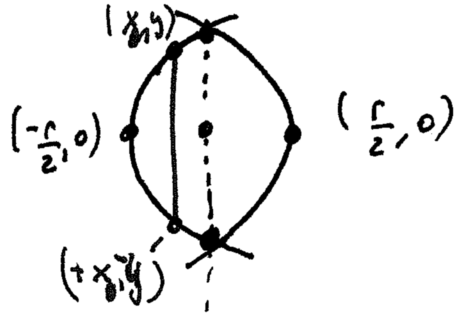
The equation of the circle on the right is $\left(x-\frac{r}{2}\right)^2+y^2=r^2$

(because the center of the circle is $\left(\frac{r}{2},0\right)$).

In particular (x,y) satisfies the equation:

$$\left(\frac{-r}{2}\right)^2+y^2=\frac{r^2}{4}+y^2=r^2 \Rightarrow \boxed{y=\frac{\sqrt{3}}{2}r}$$

Now, we see that the region we are interested is can be sliced with vertical lines.



The line $\{(x,y),(-x,y)\}$ is obtained by intersecting the circle with the line $(x=x_0)$ $x_0\leq 0$.

$$\left(x_0-\frac{r}{2}\right)^2+y^2=\left(r\right)^2$$

$$y=\sqrt{\left(r\right)^2-\left(x_0-\frac{r}{2}\right)^2} \quad \text{if } x_0\leq 0$$

So Area $= \displaystyle\int_{-\frac{r}{2}}^{r/2}$ length of vertical segment w/ $x=t$ dt $=$ $\to$ the picture is symmetric

$$= 2\int_{-r/2}^{0} \text{length of V segment } (x=t)\, dt = 2\int_{-r/2}^{0} 2\sqrt{\left(r\right)^2-\left(t-\frac{r}{2}\right)^2}\, dt$$

$$= 4\int_{-r/2}^{0}\sqrt{\left(r\right)^2-\left(t-\frac{r}{2}\right)^2}\, dt = 4r\int_{-r/2}^{0}\sqrt{1-\left(\frac{t-r/2}{r}\right)^2}\, dt$$

substitution $u = \dfrac{t - \frac{r}{2}}{r}$ $\quad du = \dfrac{1}{r}\,dt$

$t = -\frac{r}{2} \to u = -1, \quad t = 0 \to u = \frac{-1}{2}$

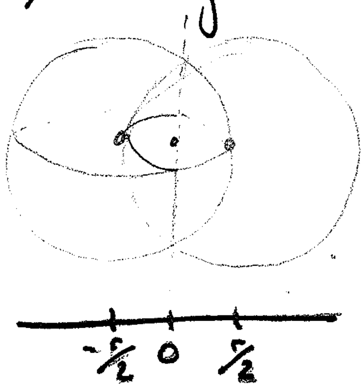
$$= 4r \int_{-1}^{-1/2} \sqrt{1-u^2}\; r\,du = 4r^2 \int_{-1}^{-1/2} \sqrt{1-u^2}\,du$$

$$= 4r^2 \left(\frac{u}{2}\sqrt{1-u^2} + \frac{1}{2}\arcsin u \right)\Bigg|_{-1}^{-1/2} = 4r^2\left(\frac{-1}{4}\frac{\sqrt{3}}{2} + \frac{1}{2}\left(\frac{-\pi}{6}\right) - \left(\frac{-1}{2}\cdot 0 + \frac{1}{2}\left(\frac{-\pi}{2}\right)\right)\right)$$

↑ Table in the book

$$= 4r^2\left(\frac{-\sqrt{3}}{8} - \frac{\pi}{12} + \frac{\pi}{4} \right) = r^2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{8} \right)$$

Now, we try to solve the case of spheres:



The region is bounded by parts of the two spheres, as it happened with the circles.

To find the volume it suffices to use cross-sections as we vary x ~~between~~ in $[-\frac{r}{2}, 0]$, because of symmetry.

The surface consists of 2 parts, one corresponding to the sphere on the right if $-\frac{r}{2} \le x \le 0$ & the other one corresponding to the sphere on the left if $0 \le x \le \frac{r}{2}$.

- Equation of sphere on the right $\quad \left(x - \frac{r}{2}\right)^2 + y^2 + z^2 = r^2$
- " " " " left $\quad \left(x + \frac{r}{2}\right)^2 + y^2 + z^2 = r^2$

They intersect when $x = 0$ & $y^2 + z^2 + \left(\frac{r}{2}\right)^2 = r^2$

$$y^2 + z^2 = r^2 - \left(\frac{r}{2}\right)^2 = \left(\frac{\sqrt{3}\,r}{2}\right)^2.$$

The cross-sections are circles:

Cross-section at $x=t$, where $-\frac{r}{2} \leq t \leq 0$

$$\left(t-\frac{r}{2}\right)^2 + y^2 + z^2 = r^2$$

$$y^2 + z^2 = r^2 - \left(t-\frac{r}{2}\right)^2 = \left(\sqrt{r^2-\left(t-\frac{r}{2}\right)^2}\right)^2$$

radius of circle

$$\Rightarrow \text{Vol} = 2\int_{-r/2}^{0} \text{Area}\left(\text{cross section for } x=t\right) dt$$

$$= 2\int_{-r/2}^{0} \pi\left(r^2-\left(t-\frac{r}{2}\right)^2\right) dt$$

$$= 2\pi\left(\underbrace{\int_{-r/2}^{0} r^2\,dt}_{=r^2\frac{r}{2}} - \int_{-r/2}^{0} \underbrace{\left(t-\frac{r}{2}\right)}_{=u}{}^2\,dt\right) \quad \text{substitution}$$

$$= 2\pi\left(\frac{r^3}{2} - \int_{-r}^{-r/2} u^2\,du\right) = 2\pi\left(\frac{r^3}{2} - \frac{u^3}{3}\Big|_{-r}^{-r/2}\right)$$

$$= 2\pi\left(\frac{r^3}{2} - \left(\frac{-r^3}{24} - \left(\frac{-r^3}{3}\right)\right)\right) = r^3\,2\pi\left(\underbrace{\frac{1}{2}-\frac{1}{3}}_{=1/6}+\frac{1}{24}\right)$$

$$= \boxed{\frac{5\pi r^3}{12}}$$