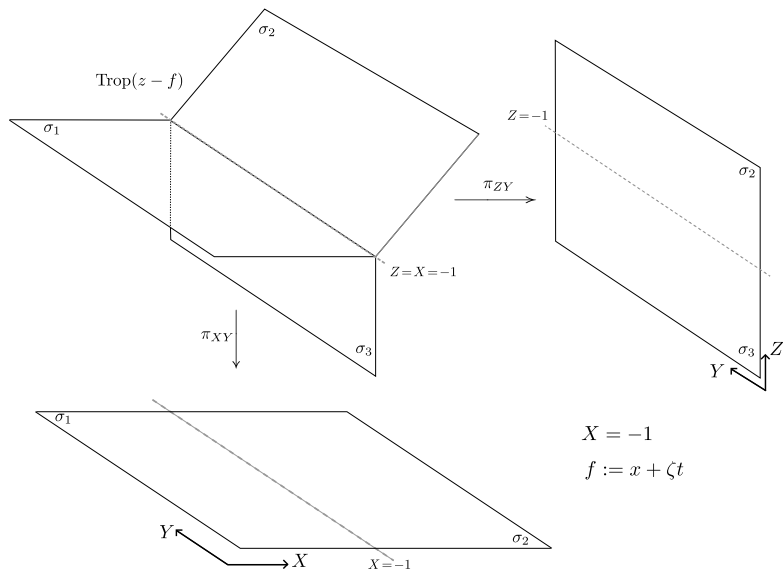


Linear trop. modification of \mathbb{R}^2 along $\{X = \ell\}$

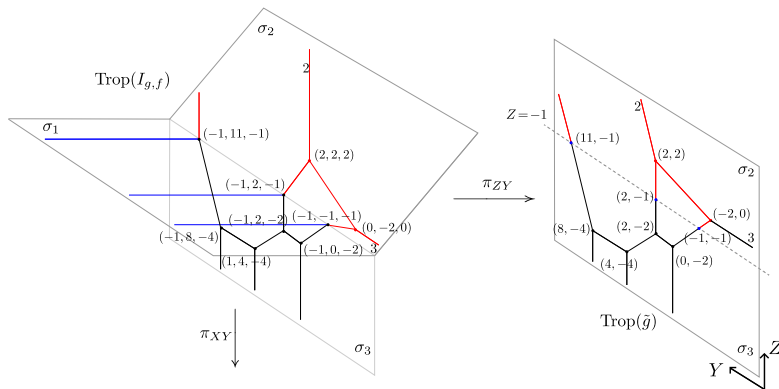


$$X = -1$$

$$f := x + \zeta t$$

$$\sigma_1 = \{X \leq \ell, Z = \ell\}, \quad \sigma_2 = \{X \geq \ell, Z = X\}, \quad \sigma_3 = \{X = \ell, Z \leq \ell\}.$$

Special modification of a plane cubic along $\{X = \ell\}$



$$X = -1$$

$$f := x + 1t$$

$$I_{g,f} := \langle g, z - f \rangle$$

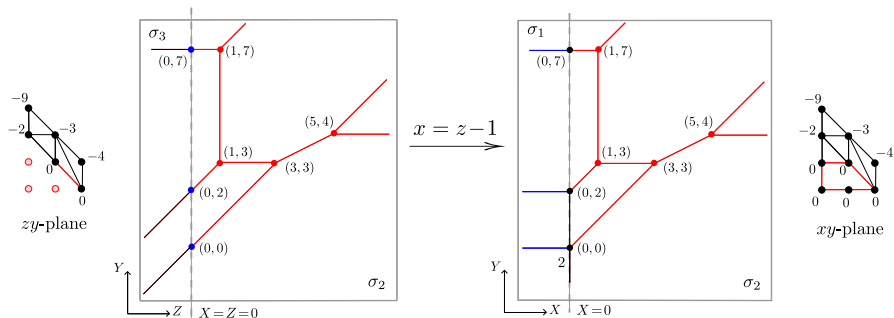
$$\sigma_1 = \{X \leq \ell, Z = \ell\}, \quad \sigma_2 = \{X \geq \ell, Z = X\}, \quad \sigma_3 = \{X = \ell, Z \leq \ell\}.$$

Example: Unfold a cycle of a rational cubic plane curve

$$g = t^4 x^2 y + 5t^3 xy^2 + t^9 y^3 + x^2 + 3xy + t^2 y^2 + 2x + (3 - t^4)y + 1$$

$$\Delta_{(0,0)}(\text{in}_{(0,0)}(g)) = c_{00}c_{11}^2 - c_{10}c_{01}c_{11} + c_{20}c_{01}^2 = 1 \cdot 3^2 - 2 \cdot 3 \cdot 3 + 1 \cdot 3^2 = 0.$$

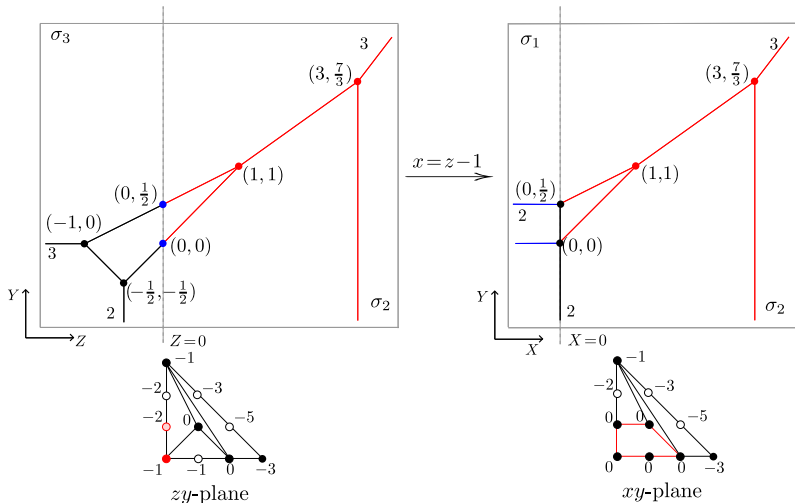
Want: $0 = c_{20} \text{in}(\zeta)^2 - c_{11} \text{in}(\zeta) + c_{00} = \text{in}(\zeta)^2 - 2 \text{in}(\zeta) + 1 = (\text{in}(\zeta) - 1)^2.$



$$\tilde{g} = g(z-1, y) = t^4 z^2 y + 5t^3 zy^2 + t^9 y^3 + z^2 + (3 - 2t^4)zy + (t^2 - 5t^3)y^2$$

Example: Repair a cycle on the visible side of a vert. line

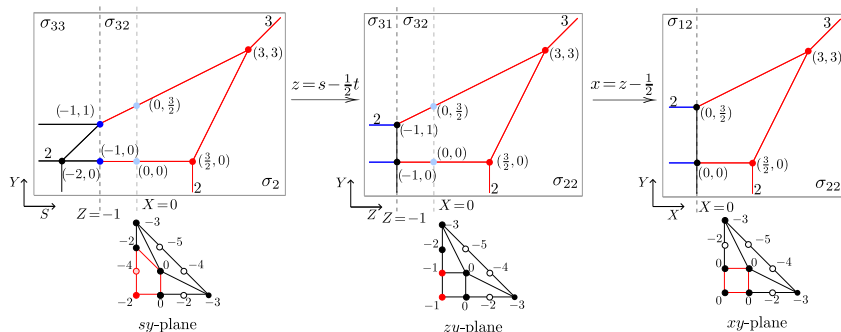
$$g(x, y) = t^3 x^3 + t^5 x^2 y + t^3 x y^2 + t y^3 + x^2 + 3xy + t^2 y^2 + (2 + 3t/2)x + (3 + t^2)y + 1.$$



$$\text{in}_{(0,0)}(g) = (1+x)^2 + 3(x+1)y = (x+1)((1+x)+y) \rightsquigarrow \zeta = 1.$$

Cycle visible and 2 iterations

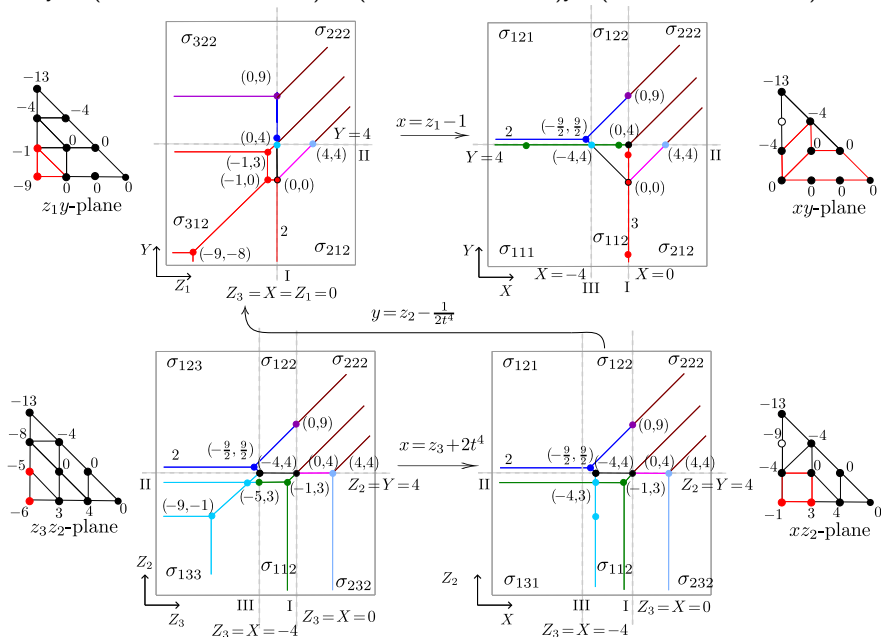
$$g = -t^3 x^3 + (t^4 + t^5)x^2y + (-t^5 + t^6)xy^2 + t^3 y^3 + (t^2 - t^3)x^2 + 4xy + (2t^2 + 3t^3)y^2 + 2x + (2 + 2t)y + (1 + t).$$



Re-embed by $l_{g, f_1, f_2} := \langle g, z - (x + 1/2), s - (z + t/2) \rangle$.

Enough: $g(s - (1/2 + t/2), y) \subset K[s, y]$.

$$g := x^3 + (1 - 9t^2)x^2y + 2t^4xy^2 + t^{20}y^3 + (1 - 24t^9 - t^{40})x^2 + (1 + 5t - 16t^9 + 144t^{11})xy + 8t^{67}y^2 + (1 - 16t^9 + t^{15} + 192t^{18})x + (2t^4 + 64t^{18} - 576t^{20})y + (1 - 8t^9 + 64t^{18} - 8t^{24}).$$

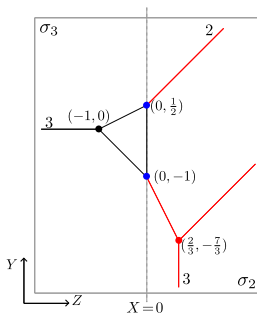
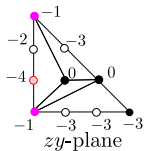
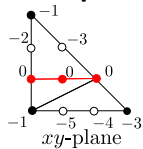


Re-embed by $l_{g, f_1, f_2, f_3} := \langle g(z_3 - 2t^4, y), z_2 - y - t^4/2, z_3 - z_1 + 1 + 2t^4 \rangle$.

Theorem (Repairing bounded edges with high multiplicity)

Let e be a vertical bounded edge of $\text{Trop}(g)$ of multiplicity $n \geq 2$ whose endpoints have valency 3. If Δ_{e^\vee} does **not** vanish at $\text{in}_e(g)$, then e witnesses a folding of edges. Moreover, we can unfold e and produce a cycle using the tropical modification along the line $\mathbb{R}\langle e \rangle$.

Example:



$$x = z - \frac{1 + \sqrt{-3}}{2}$$

