Tropical Secant Graphs of Monomial Curves

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Joint work with Shaowei Lin

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Summary

- GOAL: Study the affine cone over the first secant variety of a monomial curve
$$t \mapsto (1 : t^{i_1} : t^{i_2} : \ldots : t^{i_n}).$$
- STRATEGY: Compute its tropicalization, which is a pure, balanced weighted polyhedral fan of dim. 4 in $\mathbb{R}^{n+1}$, with a 2-dimensional lineality space
$$\mathbb{R}\langle \mathbf{1}, (1, i_1, i_2, \ldots, i_n)\rangle.$$
We encode it as a *weighted graph* in an $(n-2)$-dim'l sphere.

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A *tropical approach* to the first secant of monomial curves

Let C be the monomial projective curve $(1 : t^{i_1} : \ldots : t^{i_n})$ parameterized by n coprime integers $0 < i_1 < \ldots < i_n$. By definition,

$$Sec^1(C) = \overline{\{a \cdot p + b \cdot q \,|\, (a : b) \in \mathbb{P}^1, p, q \in C\}} \subset \mathbb{T}^{n+1}.$$

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• Pick points $p = (1 : t^{i_1} : \ldots : t^{i_n})$, $q = (1 : s^{i_1} : \ldots : s^{i_n})$ in C. Use the monomial change of coordinates $b = -\lambda a$, $t = \omega s$, and rewrite $v = a \cdot p + b \cdot q$, as

$$v_k = \underbrace{as^{i_k}}_{\in \tilde{C}} \cdot \underbrace{(\omega^{i_k} - \lambda)}_{\in Z} \qquad \text{for all } k = 0, \ldots, n,$$

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Definition

Let $X, Y \subset \mathbb{T}^N$ be two subvarieties of tori. The Hadamard product of X and Y equals $X \centerdot Y = \overline{\{(x_0 y_0, \ldots, x_n y_n) \,|\, x \in X, y \in Y\}} \subset \mathbb{T}^N$.

Theorem ([C. - Tobis - Yu])

Let $X, Y \subset \mathbb{T}^N$ be closed subvarieties and consider their Hadamard product $X \centerdot Y \subset \mathbb{T}^N$. Then as **weighted sets***: $\mathcal{T}(X \centerdot Y) = \mathcal{T}X + \mathcal{T}Y$.*

Corollary ([C. - Lin])

Given a monomial curve C: $t \mapsto (1 : t^{i_1} : \ldots : t^{i_n})$, and the surface Z: $(\lambda, \omega) \mapsto (1 - \lambda, \omega^{i_1} - \lambda, \ldots, \omega^{i_n} - \lambda) \subset \mathbb{T}^{n+1}$. Then:

$$\mathcal{T}Sec^1(C) = \mathcal{T}Z + \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$$

where $\Lambda = \mathbb{Z}\langle \mathbf{1}, (0, i_1, \ldots, i_n) \rangle$ is the intrinsic lin. lattice of $\mathcal{T}Sec^1(C)$.

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- Construct the graph $\mathcal{T}Z$.

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Strategy

- Construct the graph $\mathcal{T}Z$. $\rightsquigarrow$ "Geometric tropicalization"
- Modify $\mathcal{T}Z$ to get a weighted graph (the tropical secant graph (TSG)) representing $\mathcal{T}Sec^1(C)$ as a *weighted set*.

Construction of $\mathcal{T}Z$

Theorem (Geometric Tropicalization [Hacking - Keel - Tevelev])

Consider $\mathbb{T}^N$ with coordinate functions $t_1, \ldots, t_N$, and let $Z \subset \mathbb{T}^N$ be a closed smooth surface. Suppose $\overline{Z} \supset Z$ is any compactification whose boundary D is a smooth divisor with C.N.C. Let $D_1, \ldots, D_m$ be the irred. comp. of D, and write Δ for the graph on $\{1, \ldots, m\}$ defined by

$$\{k_i, k_j\} \in \Delta \iff D_{k_i} \cap D_{k_j} \neq \emptyset.$$

Let $[D_k]{:=}(\mathit{val}_{D_k}(t_1), \ldots, \mathit{val}_{D_k}(t_N)){\in}\mathbb{Z}^N$, and $[\sigma] := \mathbb{Z}_{\geq 0}\langle [D_k] : k \in \sigma\rangle$, for $\sigma \in \Delta$. Then,

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Theorem ([C.])

$$m_w = \sum_{\sigma \in \Delta \ s.t. \ w \in \mathbb{R}_{\geq 0}[\sigma]} (D_{k_1} \cdot D_{k_2}) \ \mathit{index}\big((\mathbb{R} \otimes_{\mathbb{Z}} [\sigma]) \cap \mathbb{Z}^N : \mathbb{Z}[\sigma]\big)$$

- Today, $\beta = (f_0, f_{i_1}, \ldots, f_{i_n})\colon X \to Z \subset \mathbb{T}^{n+1}$, $f_{i_j}(w,\lambda) := w^{i_j} - \lambda$, and

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• Compactify X inside $\mathbb{P}^2$ and pick the map: $\beta \colon \mathbb{P}^2 \supset X \to \mathbb{T}^{n+1}$, where

$$\beta_j := f_{i_j}^h(\omega, \lambda, u) / u^{\deg f_{i_j}}.$$

Our boundary divisors in $\overline{X} \subset \mathbb{P}^2$ are $D_{i_j} = (f_{i_j}^h = 0)$, $D_\infty = (u = 0)$, and

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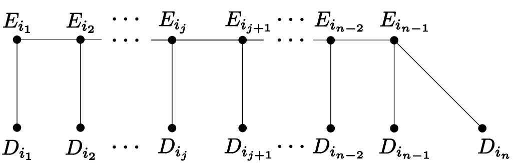
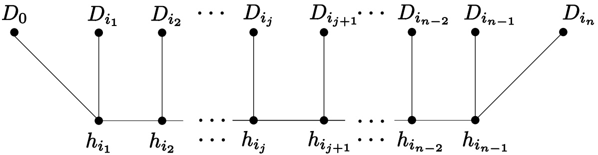
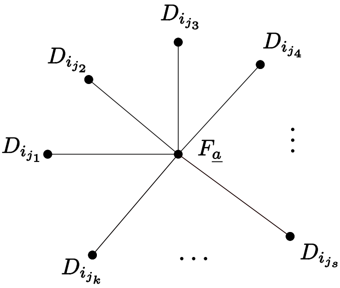
- The **resolution diagrams** come in three flavors: two caterpillar trees and families of star trees. We glue together these graphs along common nodes to obtain the abstract graph Δ from the theorem.
- We recover $\mathcal{T}Z$ from the abstract graph Δ using the map β.

Three flavors of resolution diagrams







$\underline{a} = \{i_{j_1}, \ldots, i_{j_k}\}$

for all subsets $\underline{a} \subseteq \{0, i_1, \ldots, i_n\}$ of size ≥ 2 obtained by intersecting an **arithmetic progression** in $\mathbb{Z}$ with the index set.

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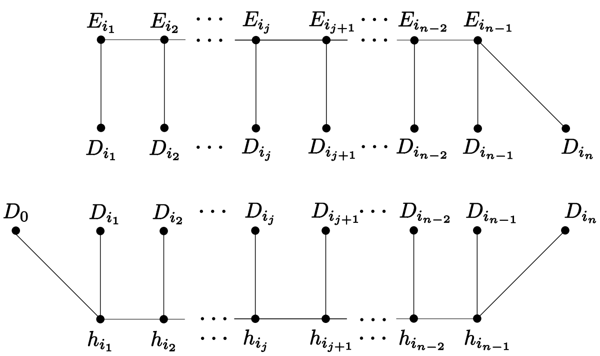

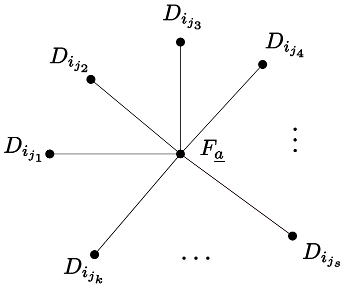

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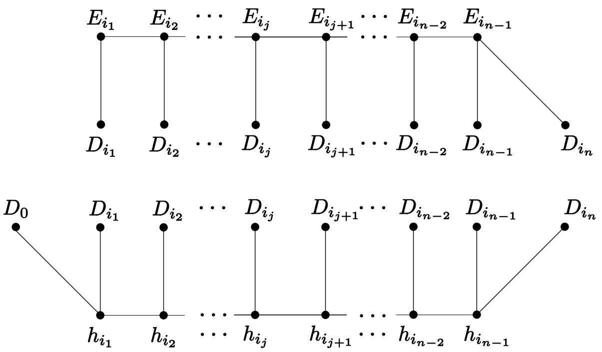

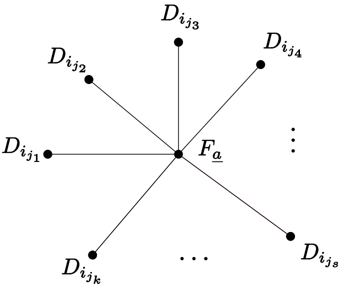

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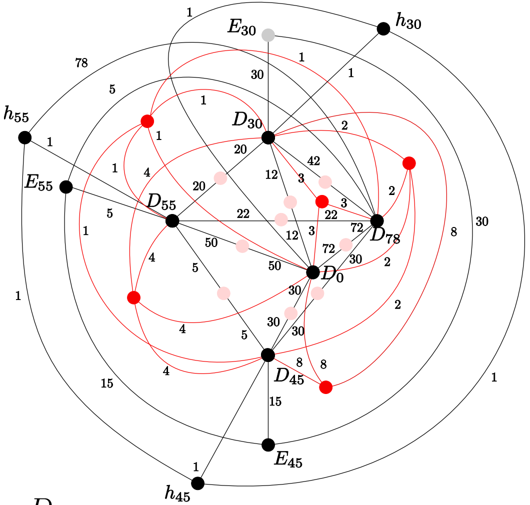
- Why $F_{\underline{a}}$?If $D_{i_{j_1}}, \ldots, D_{i_{j_k}}$ intersect at $p \in \mathbb{T}^2$ then $p = (\zeta, \zeta^{i_{j_1}})$ and ζ is a prim. qth-root of unity for some $q \mid \gcd(i_{j_2} - i_{j_1}, \ldots, i_{j_k} - i_{j_1})$. So

$$\underline{a} = \{i_{j_1}, \ldots, i_{j_k}\} \rightsquigarrow \sum_q \varphi(q) \text{ exc. divisors } F_{\underline{a},\zeta}, \text{ BUT } [F_{\underline{a},\zeta}] = [F_{\underline{a},\zeta'}] := F_{\underline{a}}.$$

Our favorite example: $\{0, 30, 45, 55, 78\}$ (K. Ranestad)



- $D_{i_k} = e_k$;
- $E_{i_j} = \quad (0, i_1, \ldots, i_j, i_j, \ldots, i_j)$
  $h_{i_j} = -(i_j, i_j, \ldots, i_j, i_{j+1}, \ldots, i_n)$

- 16 vertices (incl. bivalent node E_{30}), and 36 edges.

- *Explicit* combinatorial formula for all weights.

- Five red non-bivalent (unlabeled) nodes $F_{\underline{a}}$:

$$
\begin{aligned}
F_{0,30,45,55,78} &= (1,1,1,1,1),\\
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- $\begin{cases} F_{0,i_1,\ldots,i_n} = \mathbf{1} \in \mathbb{R}\otimes\Lambda \;\; ; \;\; E_{i_j} \equiv h_{i_j} (mod\, \mathbb{R}\otimes\Lambda) \\ E_{i_1} = i_1 \cdot F_{i_1,\ldots,i_n} \;\; ; \;\; E_{i_{n-1}} \equiv (i_n - i_{n-1}) \cdot F_{0,i_1,\ldots,i_{n-1}} (mod\, \mathbb{R}\otimes\Lambda) \end{cases}$

Reduction rules: from $\mathcal{T}Z$ to $\mathcal{T}Sec^1C = \mathcal{T}Z + \mathbb{R}\otimes\Lambda$

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  $\rightsquigarrow$ Eliminate all h_{i_j}, $F_{0,i_1,\ldots,i_n}$ and glue $F_{i_1,\ldots,i_n}$ with E_{i_1}, and $F_{0,i_1,\ldots,i_{n-1}}$ with $E_{i_{n-1}}$ in $\mathcal{T}Z$.

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- Eliminate all edges e in $\mathcal{T}Z$ s.t. $\mathbb{R}_{\geq 0}\langle e\rangle + \mathbb{R}\otimes\Lambda$ is not 4-dim'l.

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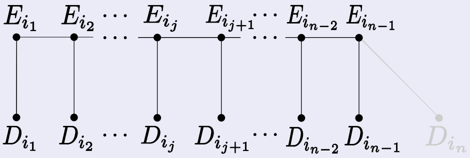
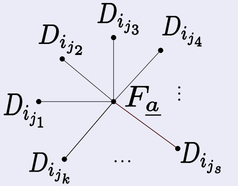
Theorem ([C. - Lin])

We describe $\mathcal{T}Sec^1C$ by a weighted graph obtained by gluing the graphs





$\underline{a} \neq \{0,\dots,i_n\}$

along all nodes D_{i_j}, and gluing together $E_{i_1} \equiv F_{i_1,\dots,i_n}$, $E_{i_{n-1}} \equiv F_{0,\dots,i_{n-1}}$.

The first secant of the curve $(1 : t^{30} : t^{45} : t^{55} : t^{78})$

- Known degree: 1 820 (K. Ranestad).

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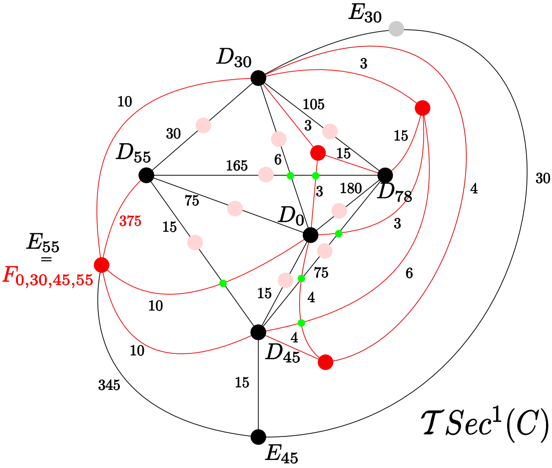
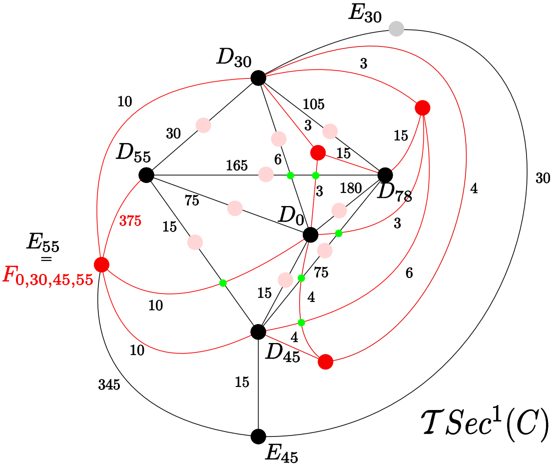
- Using out *tropical approach*:
  - multidegree w.r.t. Λ: (1 820, 76 950)
  - Newton polytope of $Sec^1(C)$.
  - f-vector$=(24, 38, 16)$.



- Known degree: 1 820 (K. Ranestad).
- Using out *tropical approach*:
  - multidegree w.r.t. Λ: (1 820, 76 950)
  - Newton polytope of $Sec^1(C)$.
  - f-vector$=(24, 38, 16)$.



Note: 6 green nodes $\leftrightarrow$ crossings of edges (hidden from us, but not an issue for tropical implicitization algorithms).