Tropical Secant Graphs of Monomial Curves

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Summary

- GOAL: Study the affine cone over the first secant variety of a monomial curve
$$t \mapsto (1 : t^{i_1} : t^{i_2} : \ldots : t^{i_n}).$$
- STRATEGY: Compute its tropicalization, which is a pure, weighted balanced rational polyhedral fan of dim. 4 in $\mathbb{R}^{n+1}$, with a 2-dimensional lineality space
$$\mathbb{R}\langle \mathbf{1}, (0, i_1, i_2, \ldots, i_n)\rangle.$$

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- **Why?** Given the tropicalization $\mathcal{T}X$ of a projective variety X, we can recover useful information about X. E.g.: its *Chow polytope* (hence, its *degree*, …).
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- **Main examples:** monomial curves C in $\mathbb{P}^4$. $\rightsquigarrow$ Compute *Newton polytope* of the defining equation of $Sec^1(C)$.

A *tropical approach* to the first secant of monomial curves

Let C be the monomial projective curve $(1 : t^{i_1} : \ldots : t^{i_n})$ parameterized by n coprime integers $0 < i_1 < \ldots < i_n$. By definition,

$$Sec^1(C) = \overline{\{a \cdot p + b \cdot q \,|\, (a : b) \in \mathbb{P}^1, p, q \in C\}} \subset (\mathbb{C}^*)^{n+1}.$$

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• Pick points $p = (1 : t^{i_1} : \ldots : t^{i_n})$, $q = (1 : s^{i_1} : \ldots : s^{i_n})$ in C. Use the monomial change of coordinates $b = -\lambda a$, $t = \omega s$, and rewrite $v = a \cdot p + b \cdot q$, as

$$v_k = \underbrace{as^{i_k}}_{\in C} \cdot \underbrace{(\omega^{i_k} - \lambda)}_{\in \text{ surface } Z} \qquad \text{for all } k = 0, \ldots, n.$$

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Definition

Let $X, Y \subset (\mathbb{C}^*)^N$ be two subvarieties of tori. The Hadamard product of X and Y equals $X \centerdot Y = \overline{\{(x_0y_0, \ldots, x_ny_n) \,|\, x \in X, y \in Y\}} \subset (\mathbb{C}^*)^N$.

Theorem ([C. - Tobis - Yu], [Allermann-Rau], ...)

Let $X, Y \subset (\mathbb{C}^*)^N$ *be closed subvarieties and consider their Hadamard product* $X \,.\, Y \subset (\mathbb{C}^*)^N$. *Then as* **sets**: $\mathcal{T}(X \,.\, Y) = \mathcal{T}X + \mathcal{T}Y$.

Corollary ([C. - Lin])

Given a monomial curve C: $t \mapsto (1 : t^{i_1} : \ldots : t^{i_n})$, *and the surface* Z: $(\lambda, \omega) \mapsto (1 - \lambda, \omega^{i_1} - \lambda, \ldots, \omega^{i_n} - \lambda) \subset (\mathbb{C}^*)^{n+1}$. *Then:*

$$\mathcal{T}Sec^1(C) = \mathcal{T}Z + \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$$

where $\Lambda = \mathbb{Z}\langle \mathbf{1}, (0, i_1, \ldots, i_n) \rangle$ *generates the lineality space of* $\mathcal{T}Sec^1(C)$.

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Strategy

- Construct the *weighted* graph $\mathcal{T}Z$.
- Modify $\mathcal{T}Z$ to get a weighted graph representing $\mathcal{T}Sec^1(C)$ as a *weighted set*.

Construction of $\mathcal{T}Z$

Theorem (Geometric Tropicalization [Hacking - Keel - Tevelev])

Consider $(\mathbb{C}^*)^N$ *with coordinate functions* $t_1, \ldots, t_N$*, and let* $Z \subset (\mathbb{C}^*)^N$ *be a closed smooth surface. Suppose* $\overline{Z} \supset Z$ *is any compactification whose boundary* D *is a smooth divisor with C.N.C. Let* $D_1, \ldots, D_m$ *be the irred. comp. of* D*, and write* Δ *for the graph on* $\{1, \ldots, m\}$ *defined by*

$$\{k_i, k_j\} \in \Delta \iff D_{k_i} \cap D_{k_j} \neq \emptyset.$$

We realize Δ *in* $\mathbb{R}^N$ *via* $\{k\} \mapsto [D_k] := (\mathit{val}_{D_k}(t_1), \ldots, \mathit{val}_{D_k}(t_N)) \in \mathbb{Z}^N$.

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Theorem ([C.])

Combinatorial formula for computing the weights of the edges of Δ.

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$$X = (\mathbb{C}^*)^2 \smallsetminus \bigcup_{j=0}^{n} (w^{i_j} - \lambda = 0).$$

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• Compactify X inside $\mathbb{P}^2$ and extend β to $\beta\colon \mathbb{P}^2 \supset X \hookrightarrow (\mathbb{C}^*)^{n+1}$.

Our boundary divisors in $\overline{X} \subset \mathbb{P}^2$ are $D_{i_j} = (w^{i_j} - \lambda = 0)$ $_{(j=0,\ldots,n)}$, D_∞.

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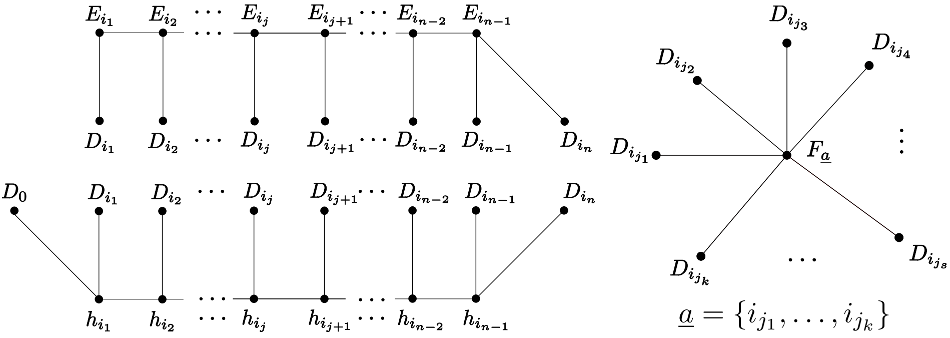
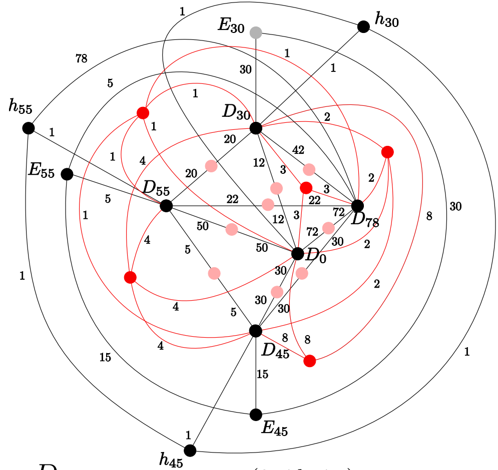
• The **resolution diagrams** come in three flavors: two caterpillar trees and families of star trees. We glue together these graphs along common nodes to obtain the *intersection complex* Δ from the theorem.

Three flavors of resolution diagrams



for all subsets $\underline{a} \subseteq \{0, i_1, \ldots, i_n\}$ of size ≥ 2 obtained by intersecting an arithmetic progression in $\mathbb{Z}$ with the index set.

- Glue together along common nodes D_{i_j}'s.



- $D_{i_k} = \; e_k \qquad (0 \leq k \leq n)$
- $E_{i_j} = \quad (0, i_1, \ldots, i_j, \;\; i_j \;\; , \ldots, i_j)$

  $h_{i_j} = -(i_j, i_j, \ldots, i_j, i_{j+1}, \ldots, i_n)$

  $(0 < j < n)$

- 15 vertices (excluding degree 2 nodes E_{30}, F_{i_j,i_k})

- Five red non-bivalent (unlabeled) nodes $F_{\underline{a}}$:

$$
\begin{aligned}
F_{0,30,45,55,78} &= (1,1,1,1,1),\\
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F_{0,30,45,55} &= (1,1,1,1,0),\\
F_{0,30,45} &= (1,1,1,0,0),\\
F_{0,30,78} &= (1,1,0,0,1).
\end{aligned}
$$

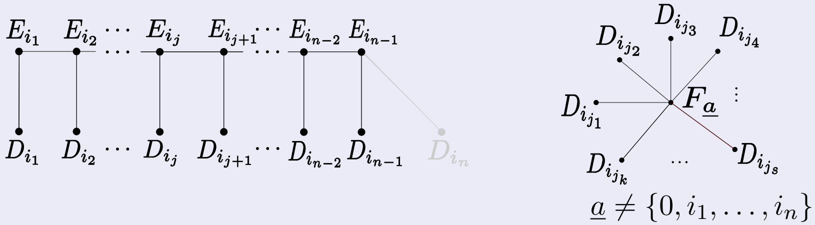
Reduction rules: from $\mathcal{T}Z$ to $\mathcal{T}Sec^1C = \mathcal{T}Z + \mathbb{R}\otimes\Lambda$

- $\begin{cases} F_{0,i_1,\ldots,i_n} = \mathbf{1} \in \mathbb{R}\otimes\Lambda \;\; ; \;\; E_{i_j} \equiv h_{i_j} (mod\, \mathbb{R}\otimes\Lambda) \\ E_{i_1} = i_1 \cdot F_{i_1,\ldots,i_n} \;\; ; \;\; E_{i_{n-1}} \equiv (i_n - i_{n-1}) \cdot F_{0,i_1,\ldots,i_{n-1}} (mod\, \mathbb{R}\otimes\Lambda) \end{cases}$

  $\rightsquigarrow$ Eliminate all h_{i_j}, $F_{0,i_1,\ldots,i_n}$; glue $F_{i_1,\ldots,i_n}$ with E_{i_1}, and $F_{0,i_1,\ldots,i_{n-1}}$ with $E_{i_{n-1}}$ in the graph of $\mathcal{T}Z$.
- Eliminate all edges σ in the graph of $\mathcal{T}Z$ s.t. $\mathbb{R}_{\geq 0}\langle\sigma\rangle + \mathbb{R}\otimes\Lambda$ is not 4-dim'l.

Theorem ([C. - Lin])

We describe $\mathcal{T}Sec^1C$ by a weighted graph obtained by gluing the graphs



along all nodes D_{i_j}, and gluing together $E_{i_1} \equiv F_{i_1,\ldots,i_n}$, $E_{i_{n-1}} \equiv F_{0,\ldots,i_{n-1}}$.

The first secant of the curve $(1 : t^{30} : t^{45} : t^{55} : t^{78})$

- Known degree: 1 820 (K. Ranestad).
- Using out *tropical approach*:
  - multidegree w.r.t. Λ: (1 820, 76 950)
  - Newton polytope of $Sec^1(C)$.
  - f-vector$=(24, 38, 16)$.