

# SOLUTIONS

## Midterm 1

Math 1181H - Honors Calculus (Section 110)

Prof. Cueto

Tuesday Sept. 19th 2017

- The use of class notes, book, formulae sheet or calculator is **not permitted**.
- In order to get full credit, you **must**:
  - a) get the **correct answer**, and
  - b) **show all your work** and/or explain the reasoning that leads to that answer.
- Answer the questions **in the spaces provided** on the question sheets. If you run out of room for an answer, continue on the back of the page.
- You will find a sheet of Scratch paper at the end. You can take it out if you wish.
- Please make sure the solutions you hand in are **legible and lucid**.
- You have **fifty-five (55) minutes** to complete the exam.
- Do not forget to write your full name (in PRINT) in the space provided below and on the bottom of the last page.

Full Name (Print): \_\_\_\_\_

**Good luck!**

**Exercise 1.** [6 points] The curve  $y^2 = a x^2 + b x + 2$  is tangent to the line defined by  $8x + y = 14$  at the point  $(2, -2)$ . Find  $a$  and  $b$ .

Use implicit differentiation:

$$2y y' = 2ax + b \quad \text{we have } x=2, y=-2.$$

$$\text{Since } y \neq 0, \text{ we get } y' = \frac{2ax+b}{2y}$$

$$\text{The tangent line is } y = -8x + 14 \quad \text{so } \boxed{y' = -8}$$

$$\text{We get } -8 = \frac{2 \cdot a \cdot 2 + b}{4} \quad \text{so } \boxed{3b = 4a + b} \quad (1)$$

From the original equation we get:

$$(-2)^2 = a \cdot 2^2 + b \cdot 2 + 2 \quad \text{so } 4 = 4a + 2b + 2$$

$$2 = 4a + 2b$$

$$\boxed{1 = 2a + b} \quad (2)$$

We need to solve  $\begin{cases} 3b = 4a + b \\ 1 = 2a + b \end{cases}$

$$3b = 2a \quad \& \quad b = 1 - 2a = 1 - 3a = -3a$$

$$\boxed{\Delta: a = \frac{3t}{2} \quad \& \quad b = -3t.}$$

**Exercise 2. [12 points]**

a) Compute the following limits or show that they do not exist:

$$(1) \lim_{x \rightarrow \infty} \frac{x \sin(1/x)}{(1+x)^{1/3}}$$

$$(2) \lim_{x \rightarrow 1} \cos\left(\frac{\pi}{x-1} + \frac{\pi}{x^2-3x+2}\right)$$

(1) use the change of variables  $u = \frac{1}{x}$ . & rewrite the limit as

$$\lim_{u \rightarrow 0^+} \frac{\frac{1}{u} \sin u}{\left(1 + \frac{1}{u}\right)^{1/3}} = \lim_{u \rightarrow 0^+} \underbrace{\frac{\sin u}{u}}_1 \cdot \underbrace{\frac{1}{\left(1 + \frac{1}{u}\right)^{1/3}}}_0 = \boxed{0}.$$

$\downarrow$  (cont. function)  
 $\frac{1}{(1+\infty)} = 0$

$$(2) \lim_{x \rightarrow 1} \cos\left(\frac{\pi}{x-1} + \frac{\pi}{x^2-3x+2}\right) = ?$$

check if the angle has a limit:  $\lim_{x \rightarrow 1} \frac{1}{x-1} + \frac{1}{x^2-3x+2} = \lim_{x \rightarrow 1} \frac{1}{x-1} + \frac{1}{(x-1)(x-2)}$

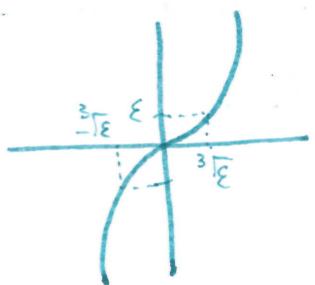
$$= \lim_{x \rightarrow 1} \frac{x-2+1}{(x-1)(x-2)} = \lim_{x \rightarrow 1} \frac{x-1}{(x-1)(x-2)} = \lim_{x \rightarrow 1} \frac{1}{x-2} = \frac{1}{1-2} = -1$$

Since  $\cos(\cdot)$  is cont:  $\lim_{x \rightarrow 1} \cos(\cdot) = \cos\left(\lim_{x \rightarrow 1} (\cdot)\right) = \cos(-\pi) = -1$

b) Using the  $\varepsilon/\delta$  method, prove that  $\lim_{x \rightarrow 0} x^3 = 0$ .

Given  $\varepsilon > 0$  we want to find  $\delta = \delta(\varepsilon)$  so that  $|x-0| < \delta$  ensures

$$|x-0| < \varepsilon. \quad \underline{\text{A: }} \delta = \sqrt[3]{\varepsilon} \text{ works.}$$



**Exercise 3.** [12 points] Consider the function  $f(x) = x^3 - 3x^2 - 24|x+3|$ .

- a) Does the function  $f$  have absolute maximum and minimum values on the interval  $[-4, 1]$ ? If so, find them.

*f is cont &  $[-4, 1]$  is closed & bounded so EVT applies.*

Note:  $f$  is not differentiable at  $-3$ , but it is outside.

$$x \neq -3 \quad f'(x) = 3x^2 - 6x - 24(|x+3|)' = \begin{cases} 3x^2 - 6x - 24 & x \geq -3 \\ 3x^2 - 6x + 24 & x < -3 \end{cases}$$

$$f'(x) = 0 \Rightarrow 0 = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) \quad x \geq -3$$

$$0 = 3x^2 - 6x + 24 = 3(x^2 - 2x + 8) \quad x < -3$$

$$\text{If } x > -3: \text{ roots are } \frac{2 \pm \sqrt{4 + 48}}{2} = \frac{2 \pm \sqrt{56}}{2} = \frac{2 \pm 6}{2} = 1 \pm 3 \rightarrow \begin{matrix} 4 \\ \nearrow \\ -2 \end{matrix}$$

Only consider  $x = -2$ .

If  $x < -3$ : No roots, ~~so~~ no critical pts  $\rightarrow$  MAX value

$$\text{Cut Values: } f(-2) = -8 - 12 - 24 = -48 \quad f(-3) = -27 - 27 = -54.$$

$$\text{End points: } f(-4) = -64 - 48 - 24 = -136 \rightarrow \text{MIN value}$$

$$f(1) = 1 - 3 - 24 = -26 \rightarrow \text{MAX value.}$$

- b) Does the function  $f$  have any inflection points on the interval  $[-4, 1]$ ? If so, find them.

For  $x \neq -3$  we can compute  $f''(x) = 6x - 6 \Rightarrow x = 1$  is a possible inflection pt



$\Rightarrow$  no change in direction of concavity anywhere. So there are no inflection points.

Recall that  $f(x) = x^3 - 3x^2 - 24|x+3|$ .

- c) Find the regions in  $[-4, 1]$  where  $f$  is increasing or decreasing and where it is concave up / down.

Viz (a) To check signs of  $f'$  &  $f''$ .

We know  $f$  is CD in  $[-4, 1]$  because  $f'' < 0$  everywhere (not defined at  $x = -3$ )

	-4	-3	-2	1
$f''$	+	-	-	-
$f'$	+	+	-	-

$f'$  has no roots here.

sign is constant.

$$f'(-4) = 48 + 24 + 24 > 0$$

so.  $\begin{cases} f \text{ is increasing in } [-4, -2] \\ f \text{ is decreasing in } (-2, 1] \end{cases}$

$$f' = \begin{cases} + & x < -2 \\ - & x > -2 \end{cases}$$

- d) Use the information from the previous items to sketch the graph of the function on the interval  $[-4, 1]$ .

To finish gathering info, we need to know  $x$  &  $y$ -intercepts.

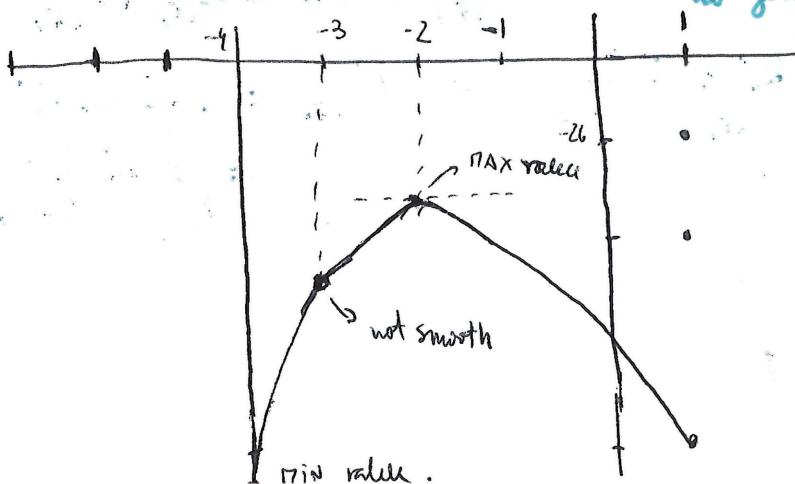
$$f(0) = -24 < 0$$

$$\bullet f(-4) = -54, \quad f(-2) = -44 \quad \text{no roots here.}$$

$$\bullet f(-2) = -44 \quad f(1) = -26 \quad f \text{ inc to } \infty \text{ no roots here.}$$

$$\lim_{x \rightarrow -3^+} f'(x) = +27 + 18 - 24 = +21$$

$$\lim_{x \rightarrow -3^-} f'(x) = +27 - 18 + 24 = +33$$



**Exercise 3.** [10 points]

- a) Find the height of the cylinder of maximum lateral area that can be inscribed in a sphere of radius  $R$ .

$$\text{Lateral } A = 2h \cdot \text{Length circle} = 2h \cdot 2\pi r^2 = 4\pi h r.$$



$$\text{constraint: } R^2 = r^2 + h^2 \Rightarrow h = \sqrt{R^2 - r^2} \quad \& \quad 0 \leq r \leq R.$$

Replace in  $A = 2\pi h \sqrt{R^2 - r^2}$  we know it has a max value in  $[0, R]$ .  
We compute crit pts:

$$A'(h) = 2\pi \left( \sqrt{R^2 - h^2} + h \frac{(-2h)}{\sqrt{R^2 - h^2}} \right) = \frac{2\pi(R^2 - h^2 - h^2)}{\sqrt{R^2 - h^2}} = \frac{2\pi(R^2 - 2h^2)}{\sqrt{R^2 - h^2}}$$

$$A'(h) = 0 \Rightarrow h = \frac{R}{\sqrt{2}}$$

$$A\left(\frac{R}{\sqrt{2}}\right) = 4\pi \frac{R}{\sqrt{2}} \frac{R}{\sqrt{2}} = 2\pi R^2$$

$$\hookrightarrow \text{find } r = \sqrt{R^2 - \frac{R^2}{2}} = \frac{R}{\sqrt{2}}$$

- b) Assume the sphere has an initial radius of 1 in and we let the radius of the sphere grow with time at a constant rate of 5 in/s, what is the rate of change of the maximum lateral area of the cylinder when the radius is 2 in.

Use related rates. From (a) we have  $h = \frac{R(t)}{\sqrt{2}} = r(t)$

$$\text{so } A(t) = 2\pi R(t)^2 \text{ so } A'(t) = 2\pi 2R(t)R'(t)$$

$$\text{When } R(t) = 2 \text{ in } \quad R'(t) = 5 \text{ in/s} \quad \text{so } A'(t) = 4\pi \cdot 2 \cdot 5 \text{ in}^2/\text{s} \\ = 80\pi \text{ in}^2/\text{s}$$

Exercise 5. [10 points] Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined as:

$$f(x) = \begin{cases} 2\sin(x) + 1 & \text{if } x \leq 0, \\ x^2 - 2x + 1 & \text{if } x \geq 0. \end{cases}$$

a) Show that  $f$  is continuous everywhere.

- For  $x \neq 0$ ,  $f(x)$  is cont because its either a polynomial.  
or  $2\sin(x) + 1$  & Both functions are continuous.

- For cont at  $x=0$  we use side limits

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^2 - 2x + 1 = 0^2 - 2 \cdot 0 + 1 = 1 = f(0)$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} 2\sin(x) + 1 = 2 \cdot \sin(0) + 1 = 1 = f(0)$$

- So  $f$  is cont at  $x=0$  ✓

b) Compute  $f'(x)$  for  $x \neq 0$ .

As in (a), each piece of  $f$  is differentiable so

$$\left\{ \begin{array}{ll} \text{If } x < 0 & f'(x) = (2\sin(x) + 1)' = 2\cos(x). \end{array} \right.$$

$$\left\{ \begin{array}{ll} \text{If } x > 0 & f'(x) = (x^2 - 2x + 1)' = 2x - 2 \end{array} \right.$$

c) Decide if  $f'(0)$  exists or not.

We use increments & side limits.

$$\lim_{\Delta x \rightarrow 0^+} \frac{f(\overbrace{0+\Delta x}^{>0}) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{(\Delta x)^2 - 2\Delta x + 1 - 1}{\Delta x} = 2 \cdot 0 - 2 = 2$$

$$\lim_{\Delta x \rightarrow 0^-} \frac{f(\overbrace{0+\Delta x}^{<0}) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{2\sin(\Delta x) + 1 - 1}{\Delta x} = 2 \lim_{\Delta x \rightarrow 0^-} \frac{\sin \Delta x}{\Delta x} = 2$$

Since the side limits are different, we conclude  $f'(0)$  does not exist.

This is the derivative of  $x^2 - 2x + 1$  at  $x=0$ !

**BONUS Problem:** Consider the function  $f$  from the previous exercise. Decide if  $f$  has absolute maximum and/or minimum values.

$f$  is unbounded but not bounded.

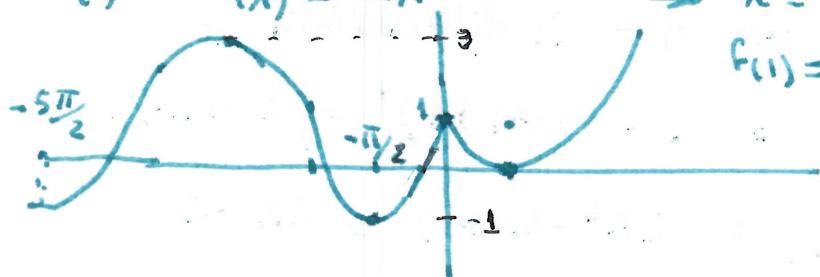
We divide our analysis to the 2 intervals  $(-\infty, 0]$  &  $[0, +\infty)$  of period  $2\pi$

NOTE: On  $[-\infty, 0]$   $f(x) = 2\sin(x) + 1$ . is periodic, so we know it has max & min values.  $\Rightarrow [-2\pi, 0]$  & so it has max & min values in  $(-\infty, 0]$  <sup>[by MVT]</sup>. Namely, at  $x = -\frac{\pi}{2}$ ,  $f(x) = -1$  the same.

We have a max value  $f(x) = 3$  & min value  $f(x) = -1$ .

On  $[0, +\infty)$   $f(x) = x^2 - 2x + 1$  so it behaves like a parabola.

Value of  $f(x)$ :  $f'(x) = 2x - 2 = 0 \Rightarrow x = 1$ .  $\Rightarrow$  min value in  $[0, +\infty)$



$$f(1) = 1 - 2 + 1 = 0 \text{ No max values since}$$

$$\lim_{x \rightarrow \infty} f(x) = +\infty.$$

- No abs-max value

- Infinitely many min values: at  $x = -\frac{\pi}{2}, -\frac{5\pi}{2}, -\frac{9\pi}{2}, \dots$

For Grader's use only:

1	2	3	4	5	TOTAL
a	b	a	b	c	d